CHAPTER I

HOLOMORPHIC FUNCTIONS

A. The Elementary Properties of Holomorphic Functions

The field of real numbers will be denoted by \( \mathbb{R} \), and the field of complex numbers by \( \mathbb{C} \); both are topological fields with the familiar structures. In studying the theory of functions of several complex variables, we are particularly interested in the space \( \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C} \), the Cartesian product of \( n \) copies of the complex plane. For the points of \( \mathbb{C}^n \) we shall use the notation \( z = (z_1, \ldots, z_n) \), where \( z_j = x_j + iy_j \in \mathbb{C} \) and \( x_j, y_j \) are real numbers (and \( i \) is a square root of \(-1\)). The absolute value of a complex number \( z \) will be denoted by \( |z| \), and for \( z \in \mathbb{C}^n \), we define

\[
|z| = \max \{|z_j|; 1 \leq j \leq n\}.
\]

An open polydisc (or open polycylinder) in \( \mathbb{C}^n \) is a subset \( \Delta(w; r) \subset \mathbb{C}^n \) of the form

\[
\Delta(w; r) = \Delta(w_1, \ldots, w_n; r_1, \ldots, r_n)
= \{z \in \mathbb{C}^n; |z_j - w_j| < r_j, 1 \leq j \leq n\};
\]

the point \( w \in \mathbb{C}^n \) is called the center of the polydisc, and

\[
r = (r_1, \ldots, r_n) \in \mathbb{R}^n, \quad (r_j > 0),
\]

is called the polyradius. The closure of \( \Delta(w; r) \) will be called the closed polydisc with center \( w \) and polyradius \( r \), and will be denoted by \( \overline{\Delta(w; r)} \). More generally, if \( D_j \subset \mathbb{C} \) are any subdomains (connected open subsets) of the complex plane, the product set \( D = D_1 \times \cdots \times D_n \subset \mathbb{C}^n \) will be called an open polydomain. A polydisc is the special case in which the sets \( D_j \) are discs; similarly, an open polysquare is the special case in which the sets \( D_j \) are open squares in the plane. The open polydiscs form a basis for the collection of open sets in the Cartesian product topology on \( \mathbb{C}^n \). Considered only as a topological space (or as a real vector space), \( \mathbb{C}^n \) is of course just the same as \( \mathbb{R}^{2n} \), the ordinary Euclidean space of \( 2n \) dimensions. Thus we can impose on \( \mathbb{C}^n \) in a natural manner any of the structures of \( \mathbb{R}^{2n} \); for
instance, the Lebesgue measure on $\mathbb{R}^{2n}$ becomes a measure on $\mathbb{C}^n$, which will be denoted by $dV$.

A complex-valued function $f$ on a subset $D \subset \mathbb{C}^n$ is merely a mapping from $D$ into the complex plane; the value of the function $f$ at a point $z \in D$ will be denoted by $f(z)$, as usual.

1. **Definition.** A complex-valued function $f$ defined on an open subset $D \subset \mathbb{C}^n$ is called **holomorphic in** $D$ if each point $w \in D$ has an open neighborhood $U$, $w \in U \subset D$, such that the function $f$ has a power series expansion

\[
f(z) = \sum_{v_1, \ldots, v_n = 0}^{\infty} a_{v_1, \ldots, v_n} (z_1 - w_1)^{v_1} \cdots (z_n - w_n)^{v_n}
\]

which converges for all $z \in U$. The set of all functions holomorphic in $D$ will be denoted by $\mathcal{O}_D$.

Notice that polynomials in the functions $z_1, \ldots, z_n$ are holomorphic in all of $\mathbb{C}^n$. It is a familiar result from elementary analysis that a power series expansion of the form (2) is absolutely uniformly convergent in all suitably small open polydiscs $\Delta(w; r)$ centered at the point $w$. A first consequence of this observation is that the function $f$ is continuous in such polydiscs $\Delta(w; r)$; and hence, any function holomorphic in $D$ is also continuous in $D$.

A second consequence is that the power series (2) can be rearranged arbitrarily and will still represent the function $f$. In particular, if the coordinates $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ are given any fixed values $a_1, \ldots, a_{j-1}$, $a_{j+1}, \ldots, a_n$, then this power series can be arranged as a convergent power series in the variable $z_j$ alone, for $z_j$ sufficiently close to $w_j$; and this holds for any values $a_i$ sufficiently near $w_i$. That is to say, the function $f$ is **holomorphic in each variable separately** throughout the domain in which it is analytic; thus the ordinary complex derivative with respect to one of the variables $z_j$ is well-defined, and will be denoted by $\partial / \partial z_j$. A converse to the boldface statement is also true, as follows.

2. **Theorem (Osgood’s Lemma).** If a complex-valued function $f$ is continuous in an open set $D \subset \mathbb{C}^n$, and is holomorphic in each variable separately, then it is holomorphic in $D$.

**Proof:** Select any point $w \in D$, and any closed polydisc $\bar{\Delta}(w; r) \subset D$. Since $f$ is holomorphic in each variable separately in an open neighborhood of $\Delta(w; r)$, a repeated application of the Cauchy integral formula for functions of one variable leads to the formula

\[
f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{|\zeta_1-z_1|=r_1} \frac{d\zeta_1}{\zeta_1 - z_1} \int_{|\zeta_2-z_2|=r_2} \frac{d\zeta_2}{\zeta_2 - z_2} \cdots \int_{|\zeta_n-z_n|=r_n} \frac{d\zeta_n}{\zeta_n - z_n} f(\zeta),
\]
for all $z \in \Delta(\omega; r)$. For any fixed point $z$, the integrand in (3) is continuous on the compact domain of integration; hence the iterated integral in (3) can be replaced by the single multiple integral

\[ f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{|w_j - \xi_j| = r_j} \frac{f(\xi) \, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}. \]

But now, again for a fixed point $z \in \Delta(\omega; r)$, the series expansion

\[ \frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{v_1, \ldots, v_n=0}^{\infty} \frac{(z_1 - w_1)^{v_1} \cdots (z_n - w_n)^{v_n}}{(\zeta_1 - w_1)^{v_1+1} \cdots (\zeta_n - w_n)^{v_n+1}} \]

is absolutely uniformly convergent for all points $\zeta$ on the domain of integration in (4); consequently, after substituting this expansion into (4) and interchanging the orders of summation and integration, it follows immediately that the function $f$ has a power series expansion of the form (2), with

\[ a_{v_1, \ldots, v_n} = \left( \frac{1}{2\pi i} \right)^n \int_{|w_j - \xi_j| = r_j} \frac{f(\xi) \, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - w_1)^{v_1+1} \cdots (\zeta_n - w_n)^{v_n+1}}. \]

Therefore $f$ is a holomorphic function, as desired.

**Remark:** The hypothesis that the function $f$ be continuous in $D$ is actually inessential; but this stronger theorem (Hartogs’ theorem) is surprisingly much more difficult. This result will not be needed in the present book, so we shall not include a proof; the reader interested in pursuing this question is referred to Bochner-Martin [46, VII].

Some of the observations made during the course of the preceding proof merit separating out for special attention. First, any function $f$ holomorphic in an open neighborhood of a closed polydisc $\Delta(\omega; r)$ has a **Cauchy integral representation** of the form (4); that formula is the natural generalization of the **Cauchy integral formula** for holomorphic functions of one complex variable. By differentiating (4), it follows that

\[ \frac{\partial^{k_1+\cdots+k_n} f(z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} = \left( \frac{k_1! \cdots k_n!}{(2\pi i)^n} \right) \int_{|w_j - \xi_j| = r_j} \frac{f(\xi) \, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{k_1+1} \cdots (\zeta_n - z_n)^{k_n+1}}. \]

Upon then comparing (5) and (6), it further follows that the coefficients in the power series expansion (2) of $f$ are given by

\[ (v_1!) \cdots (v_n!) a_{v_1, \ldots, v_n} = \frac{\partial^{v_1+\cdots+v_n} f}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (w). \]
As a further consequence of these observations, it follows that the power series expansion of a holomorphic function at \( w \) is uniquely determined by that function and converges within any polydisc \( \Delta(w; r) \) contained in the region of analyticity of that function; for the proof of Theorem 2 exhibited a power series expansion convergent within any fixed compact subset of \( \Delta(w; r) \), and by (7) all of these series expansions must actually coincide.

One corollary which can be drawn from Osgood's lemma is an extension of the familiar Cauchy-Riemann equations, as a criterion for analyticity. As a convenient notation, introduce the first-order linear partial differential operators

\[
\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),
\]

where \( x_j, y_j \) are the underlying real coordinates in \( \mathbb{C}^n \), and \( z_j = x_j + iy_j \). It should perhaps be remarked that the left-hand sides in (8) are defined by that equation, and have no separate meaning. However, note that

\[
\frac{\partial}{\partial z_j} z_j = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right)(x_j + iy_j) = 1,
\]

and hence that

\[
\frac{\partial}{\partial z_j} z_j^n = nz_j^{n-1};
\]

therefore, when applied to holomorphic functions, the operator \( \partial/\partial z_j \) coincides with the familiar complex derivative of a holomorphic function.

3. **Theorem (Cauchy-Riemann Criterion).** A complex-valued function \( f \), which is defined in an open subset \( D \subset \mathbb{C}^n \) and which is continuously differentiable in the underlying real coordinates of \( \mathbb{C}^n \), is holomorphic in \( D \) if and only if it satisfies the system of partial differential equations

\[
\frac{\partial}{\partial z_j} f(z) = 0, \quad j = 1, 2, \ldots, n.
\]

**Proof:** At any point of \( D \), consider \( f(z) \) as a function of the single variable \( z_j \), holding the other variables constant. Decomposing \( f \) into its real and imaginary parts by writing \( f(z) = u(z) + iv(z) \), note that

\[
2 \frac{\partial}{\partial z_j} f(z) = \left( \frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j} \right) + i \left( \frac{\partial u}{\partial y_j} + \frac{\partial v}{\partial x_j} \right).
\]

Therefore (9) is equivalent to the classical Cauchy-Riemann equations for each variable separately; and, as is well-known, this in turn is equivalent to the function \( f \) being holomorphic in each variable separately. The desired theorem then follows immediately from Osgood's lemma.
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The following facts follow easily from the Cauchy-Riemann criterion.

4. Theorem. Let $D$ be an open set in $\mathbb{C}^n$. Then:

(i) $\mathcal{O}_D$ is a ring under the operations $(f + g)(z) = f(z) + g(z)$, $(fg)(z) = f(z)g(z)$.

(ii) If $f$ is in $\mathcal{O}_D$ and is nowhere zero, then $1/f$ is in $\mathcal{O}_D$.

(iii) If $f$ is in $\mathcal{O}_D$, and is real-valued or has constant modulus, then $f$ is constant.

Proof: (i) By direct computation,

$$
\frac{\partial}{\partial z_j} (f + g) = \frac{\partial f}{\partial z_j} + \frac{\partial g}{\partial z_j},
$$

and

$$
\frac{\partial}{\partial z_j} (fg) = \frac{\partial f}{\partial z_j} g + f \frac{\partial g}{\partial z_j};
$$

hence the assertion (i) follows from Theorem 3.

(ii) Apply (10) with $g$ replaced by $f^{-1}$. We find that

$$
0 = f \cdot \frac{\partial f^{-1}}{\partial z_j}.
$$

(iii) If $f \in \mathcal{O}_D$ is real-valued, $\partial f/\partial x_j$ and $\partial f/\partial y_j$ are also real-valued. But $\partial f/\partial x_j = i \partial f/\partial y_j$, so both are zero for all $j$, $1 \leq j \leq n$. Thus $f$ is constant. If $f$ has constant modulus, then for any $w \in D$ we can write $f = \rho e^{i\theta(z)}$, where $\theta$ is a well-defined real-valued function in a neighborhood of $w$. Then, in $U$,

$$
0 = \frac{\partial f}{\partial z_j} = i f \cdot \frac{\partial \theta}{\partial z_j}.
$$

Thus $\theta$ is holomorphic, so is also constant.

One of the fundamental properties of holomorphic functions of one complex variable is that the composition of two holomorphic functions is also holomorphic; the Cauchy-Riemann criterion now permits us to extend this property to functions of several complex variables, as follows. Suppose that $D \subset \mathbb{C}^n$ and that $D' \subset \mathbb{C}^m$ are two open domains; the variables in $D$ will be written $z = (z_1, \ldots, z_n)$ and the variables in $D'$ will be written $w = (w_1, \ldots, w_m)$. Any mapping $G : D \to D'$ can be described by $m$ functions

$$
w_1 = g_1(z_1, \ldots, z_n), \ldots, w_m = g_m(z_1, \ldots, z_n).
$$

The mapping $G$ will be called a holomorphic mapping if the $m$ functions $g_1, \ldots, g_m$ are holomorphic functions in $D$. If $f(w_1, \ldots, w_m) = f(w)$ is any function defined in $D'$, the composite $f(G(z))$ is then a well-defined function in $D$. 

5. Theorem (Composition Theorem). If \( f(w) \) is a holomorphic function in \( D' \) and if \( G: D \rightarrow D' \) is a holomorphic mapping, then the composition \( f(G(z)) \) is a holomorphic function in \( D \).

Proof: Separate the functions (11) into their real and imaginary parts by writing \( g_j(z) = u_j(z) + iv_j(z) \). Since all the mappings involved are differentiable in the underlying real coordinates, the usual chain rule for differentiation can be applied as follows:

\[
\frac{\partial f(G(z))}{\partial z_j} = \sum_{k=1}^{m} \left( \frac{\partial f}{\partial u_k} \frac{\partial u_k}{\partial z_j} + i \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial z_j} \right)
\]

\[
= \sum_{k=1}^{m} \frac{1}{2} \left( \frac{\partial f}{\partial u_k} - i \frac{\partial f}{\partial v_k} \right) \frac{\partial g_k}{\partial z_j} + \sum_{k=1}^{m} \frac{1}{2} \left( \frac{\partial f}{\partial u_k} + i \frac{\partial f}{\partial v_k} \right) \frac{\partial g_k}{\partial z_j}
\]

\[
= \sum_{k=1}^{m} \left( \frac{\partial f}{\partial u_k} \frac{\partial g_k}{\partial z_j} + \frac{\partial f}{\partial v_k} \frac{\partial g_k}{\partial z_j} \right).
\]

(This is the complex form of the chain rule.) If the function \( f \) and the mapping \( G \) are both holomorphic, then \( \frac{\partial f}{\partial u_k} \frac{\partial g_k}{\partial z_j} = 0 \) and \( \frac{\partial f}{\partial v_k} \frac{\partial g_k}{\partial z_j} = 0 \) for all \( k \); so by the above formula, \( \frac{\partial f(G(z))}{\partial z_j} = 0 \) for all \( j \). It then follows from the Cauchy-Riemann criterion that the function \( f(G(z)) \) is holomorphic, as desired.

Many other familiar results from the theory of holomorphic functions of one complex variable also have easy extensions to functions of several complex variables.

6. Theorem (Identity Theorem). If \( f(z) \) and \( g(z) \) are holomorphic functions in a connected open set \( D \subset \mathbb{C}^n \), and if \( f(z) = g(z) \) for all points \( z \) in a non-empty open subset \( U \subset D \), then \( f(z) = g(z) \) for all points \( z \in D \).

Proof: Let \( E \) be the interior of the set consisting of all points \( z \) for which \( f(z) = g(z) \); thus \( E \) is an open subset of \( D \), and is nonempty since \( U \subset E \). It clearly suffices to show that \( E \) is relatively closed in \( D \) as well; for it will then follow from the connectedness of \( D \) that \( E = D \), and the theorem is therewith demonstrated. Therefore, consider any point \( w \in D \cap E \), where \( E \) is the point set closure of \( E \); and select a number \( r > 0 \) sufficiently small that the polydisc \( \Delta(w; r, \ldots, r) \subset D \). Since \( w \in E \), there must exist a point \( w' \) such that \( |w_j - w_j| < r/2 \) \( (j = 1, \ldots, n) \), and that \( w' \in E \); note that \( w \in \Delta(w'; r/2, \ldots, r/2) \). The function \( f(z) - g(z) \) is holomorphic in

\[
\Delta(w'; r/2, \ldots, r/2),
\]

hence has a power series expansion centered at \( w' \) and converging throughout this small polydisc. Now since \( w' \in E \), this function vanishes identically in
an open neighborhood of \( w' \), and so by (7) all the coefficients in this power series expansion are zero; but then \( f(z) - g(z) = 0 \) throughout

\[
\Delta(w' ; r/2, \ldots, r/2),
\]

and thus \( w \in E \). This shows that \( E \) is relatively closed in \( D \), as desired.

7. Theorem (Maximum Modulus Theorem). If \( f(z) \) is holomorphic in a connected open set \( D \subset \mathbb{C}^n \), and if there is a point \( w \in D \) such that \( |f(z)| \leq |f(w)| \) for all points \( z \) in some open neighborhood of \( w \), then \( f(z) \equiv f(w) \) for all points \( z \in D \).

**Proof:** Following the pattern of one of the customary proofs of the maximum modulus theorem for functions of one complex variable, we begin by observing that as a consequence of the Cauchy integral formula (4), for any polydisc \( \Delta(w; r) \subset D \),

\[
V(\Delta) f(w) = \int_{\Delta(w; r)} f(\zeta) dV(\zeta),
\]

where \( dV(\zeta) \) is the Euclidean volume element and \( V(\Delta) = \int_{\Delta(w; r)} dV(\zeta) \) is the volume of \( \Delta(w; r) \). As a consequence of this formula,

\[
V(\Delta) |f(w)| \leq \int_{\Delta(w; r)} |f(\zeta)| dV(\zeta).
\]

Now select a polycylinder \( \Delta(w; r) \) such that \( |f(w)| - |f(z)| \geq 0 \) for all points \( z \in \Delta(w; r) \); then

\[
0 \leq \int_{\Delta(w; r)} (|f(w)| - |f(\zeta)|) dV(\zeta)
= V(\Delta) |f(w)| - \int_{\Delta(w; r)} |f(\zeta)| dV(\zeta) \leq 0,
\]

so that \( |f(w)| - |f(z)| = 0 \) for all \( z \in \Delta(w; r) \). Then, by Theorem 4, \( f \) must be constant in \( \Delta(w; r) \); indeed \( f(z) = f(w) \) for all \( z \in \Delta(w; r) \). The desired result follows immediately from the identity theorem.

Since the power series expansion (2) of a function holomorphic in a neighborhood of \( w \) converges absolutely, we may regroup terms into a series of homogeneous polynomials:

\[
f(z) = \sum_{k=0}^{\infty} \left( \sum_{v_1 + \cdots + v_n = k} a_{v_1} \cdots v_n (z_1 - w_1)^{v_1} \cdots (z_n - w_n)^{v_n} \right).
\]  
(13)
If \( f_k(z) \) is the homogeneous polynomial of lowest degree in this expansion which does not vanish identically, the function \( f \) is said to have **total order** \( k \) at the point \( w \); if \( f(z) = 0 \), so that all the homogeneous terms in (13) vanish identically, the function is said to be of total order \( \infty \). For example, the function \( f \) has total order 0 at \( w \) precisely when \( f(w) \neq 0 \); and it has total order 1 at \( w \) precisely when \( f(w) = 0 \) and \( (\partial f/\partial z_j)(w) \neq 0 \) for some \( j \).

**8. Theorem (Schwarz’s Lemma).** Let \( f \) be holomorphic in a neighborhood of \( \Delta(0; r) \), \( r = (r, \ldots, r) \), and suppose that \( f \) is of total order \( k \) at 0 and that \( |f(z)| \leq M \) for all \( z \in \Delta(0, r) \). Then \( |f(z)| \leq M \left| \frac{z}{r} \right|^k \) for all \( z \in \Delta(0; r) \).

**Proof:** The function \( f \) has the expansion in homogeneous polynomials
\[
f(z) = p_k(z) + p_{k+1}(z) + \cdots.
\]
Let \( z \in \Delta(0; r) \), \( z \neq 0 \), and define \( g(t) = t^{-k}f(t \in \mathbb{C}, |t| \leq r) \). Then \( g \) has the Taylor expansion \( g(t) = p_k(|z|^{-1}z) + p_{k+1}(|z|^{-1}z)t + \cdots \), and \( |g(t)| \leq Mr^{-k} \) for \( |t| = r \). Thus, by the maximum modulus theorem, \( |g(t)| \leq M r^{-k} \) for all \( t \), \( |t| \leq r \); so
\[
|z|^{-k}f(z)| = |g(z)| \leq Mr^{-k},
\]
and \( |f(z)| \leq M \left| \frac{z}{r} \right|^k \).

**9. Theorem (Jensen’s Inequality).** Let \( f \) be holomorphic in a neighborhood of \( \Delta(0; r) \subset \mathbb{C}^n \). Then
\[
\frac{1}{V_{\Delta(0; r)}} \int_{\Delta(0; r)} \log |f| \, dV \geq \log |f(0)|,
\]
(in the sense that if \( f(0) \neq 0 \), then \( \log |f| \) is integrable and the inequality holds). Here \( V_r \) is the volume of \( \Delta(0; r) \).

**Proof:** We assume the one-dimensional Jensen inequality: if \( f \) is holomorphic in a neighborhood of \( \Delta(0; r) \subset \mathbb{C}^1 \), then for \( \rho \leq r \),
\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})| \, d\theta \geq \log |f(0)|.
\]

Now let \( f \) be given as in the hypothesis, and suppose that \( f(0) \neq 0 \). Then, for \( \rho_1 \leq r_1 \),
\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho_1 e^{i\theta}, 0, 0, \ldots, 0)| \, d\theta_1 \geq \log |f(0)|.
\]
Thus \( \log |f(\rho e^{i\theta}, 0, \ldots, 0)| \) is finite for almost all \( \theta \), and for such \( \theta \),

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta}, \rho e^{i\theta}, 0, \ldots, 0)| \, d\theta \geq \log |f(\rho e^{i\theta}, 0, \ldots, 0)|
\]

for \( \rho \leq r \). Thus, putting these together and continuing, we ultimately find

\[
\log |f(0)| \leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \log |f(\rho_1 e^{i\theta_1}, \ldots, \rho_n e^{i\theta_n})| \, d\theta_1, \ldots, d\theta_n
\]

for all \( \rho_j \leq r_j \). Integrating with respect to \( \int_0^{r_1} \cdots \int_0^{r_n} (\rho_1 \, d\rho_1) \cdots (\rho_n \, d\rho_n) \), we obtain

\[
(15) \quad \log |f(0)| \leq \frac{1}{V_r} \int_0^{r_1} \cdots \int_0^{r_n} \int_0^{2\pi} \cdots \int_0^{2\pi}
\]

\[
\log |f(\rho_1 e^{i\theta_1}, \ldots, \rho_n e^{i\theta_n})| (\Pi_{\rho_i} \rho_1 \cdots d\theta_1 \cdots d\theta_n \, d\rho_1 \cdots d\rho_n).
\]

But the iterated integral on the right is the same as the volume integral of \( \log |f| \). For if \( L_n = \max (-n, \log |f|) \), then \(-n \leq L_n \leq |f|\), so \( L_n \) is integrable with respect to \( dV \); and by Fubini’s theorem, \( \int L_n \, dV \) is the iterated integral of \( L_n \). Since \( L_n \geq \log |f| \), by (15), \( \frac{1}{V_r} \int L_n \, dV \geq \log |f(0)| \). But \( L_n \) converges to \( \log |f| \) monotonically, so \( \log |f| \) is also integrable with respect to \( dV \), and thus the theorem is proved.

Jensen’s inequality in several variables gives some useful information which is trivial in one variable.

10. Corollary. Let \( f \) be holomorphic in the domain \( D \), and suppose \( f \) is not identically zero. Then the set \( V = \{ z \in D \mid f(z) = 0 \} \) has 2\( n \)-dimensional Lebesgue measure zero.

Proof: By the identity theorem, \( V \) has an empty interior. Thus \( D - V \) is dense in \( D \), so we can find a sequence of points \( \{x_n\} \subset D - V \) such that for suitable polydiscs \( \Delta_n \) about \( x_n \), \( \Delta_n = D \). By Jensen’s inequality,

\[
\int \log |f| \, dV > -\infty,
\]

so \( f \) cannot vanish in \( \Delta_n \) on a set of positive measure. Thus \( V = \bigcup_{n=1}^{\infty} (V \cap \Delta_n) \) has measure zero.
Much of the theory of normal families and many of the related convergence results can be extended to the case of several complex variables as well; the final portions of the present section will be devoted to such questions. It is more convenient, and perhaps clearer as well, to present these results in a slightly more abstract form than is customary in one complex variable.

For any open subset $D \subset \mathbb{C}^n$, the ring $\mathcal{C}_D$ of continuous, complex-valued functions in $D$ has the natural topology of a function space, under which $\mathcal{C}_D$ becomes a topological ring. The topology is defined as follows. For any compact subset $K \subset D$ and any real number $\varepsilon > 0$ let

$$U(K, \varepsilon) = \{f \in \mathcal{C}_D \mid |f(z)| < \varepsilon \text{ whenever } z \in K\};$$

the sets $U(K, \varepsilon)$, for all such $K$ and $\varepsilon$, are taken as a basis for the open neighborhoods of the zero element of $\mathcal{C}_D$, the function $f(z) \equiv 0$. It is a straightforward verification to show that these sets do determine a topology under which $\mathcal{C}_D$ becomes a topological ring; the details will be left to the reader.

It is also quite clear that a sequence of functions $f_n \in \mathcal{C}_D$ converges to a limit function $f \in \mathcal{C}_D$ if and only if the functions $f_n$ converge uniformly to $f$ on any compact subset of $D$; the topology we have introduced is sometimes called "the topology of uniform convergence on compact subsets," for this reason. The ring $\mathcal{C}_D$ is of course complete in this topology, as is familiar from elementary analysis.

This topology can also be introduced in the following equivalent, but rather more concrete, form. For any compact subset $K \subset D$ and any $f \in \mathcal{C}_D$, put

$$\|f\|_K = \sup_{z \in K} |f(z)|.$$  

The value $\|f\|_K$ is a well-defined, finite real number, since $f$ is continuous and $K$ is compact; and the mapping $f \mapsto \|f\|_K$ is a pseudo-norm on the ring $\mathcal{C}_D$, since clearly

$$\|f + g\|_K \leq \|f\|_K + \|g\|_K$$

and

$$\|fg\|_K \leq \|f\|_K \cdot \|g\|_K$$

for any elements $f, g \in \mathcal{C}_D$. Note that for a constant $c$,

$$\|cf\|_K = |c| \cdot \|f\|_K.$$  

The mapping is not a norm, since $\|f\|_K = 0$ does not imply that $f$ is the zero element of $\mathcal{C}_D$; the values of the function $f$ can be rather general on the subset $D - K$. However, $\|f\|_K = 0$ for every $K$ if and only if $f$ is the zero element.

The basic open neighborhoods of the zero element, in the topology on $\mathcal{C}_D$, can be described as

$$U(K, \varepsilon) = \{f \in \mathcal{C}_D \mid \|f\|_K < \varepsilon\}.$$
Now let $K_v \subseteq D$ be a sequence of compact subsets of $D$ such that

\begin{equation}
K_v \subseteq K_{v+1} \quad \text{and} \quad \bigcup_{v=1}^{\infty} K_v = D;
\end{equation}

and for any elements $f, g \in \mathcal{C}_D$, define

\begin{equation}
d(f, g) = \sum_{v=1}^{\infty} \frac{1}{2^v} \frac{\|f - g\|_{K_v}}{1 + \|f - g\|_{K_v}}.
\end{equation}

It is an easy exercise to show that (21) defines a metric on the ring $\mathcal{C}_D$, and that the topology on $\mathcal{C}_D$ is that given by this metric. It is clear that $\mathcal{C}_D$ is locally convex and complete in this metric; thus $\mathcal{C}_D$ is a Fréchet space (see Appendix B).

The subring $\mathcal{O}_D \subseteq \mathcal{C}_D$ inherits from $\mathcal{C}_D$ a topology under which it is also a topological ring. This topology is of fundamental importance in studying the ring $\mathcal{O}_D$, and is the only topology we shall impose on that ring. For emphasis, let it be understood that $\mathcal{O}_D$ will always be considered as a topological ring, with the topology of uniform convergence on compact subsets, as introduced above. Some of the simpler properties of $\mathcal{O}_D$ as a topological ring are as follows.

11. **Lemma.** $\mathcal{O}_D$ is a closed subring of $\mathcal{C}_D$, and thus a Fréchet space.

**Proof:** Let $f_j \in \mathcal{O}_D$ be a sequence of elements which converge to an element $f \in \mathcal{O}_D$. Let $w \in D$, and choose $r > 0$ so that $\Delta(w; r) \subseteq D$. Then by the Cauchy integral formula, for any point $z \in \Delta(w; r)$,

\begin{equation}
f_v(z) = \left( \frac{1}{2\pi i} \right)^n \int_{w_j - \zeta_j = r_j} \frac{f_v(\zeta) \ d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)},
\end{equation}

since $f_v \in \mathcal{O}_D$. But since $f_v$ converges to $f$ uniformly on $\Delta(w; r)$, it follows that

\begin{equation}f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{w_j - \zeta_j = r_j} \frac{f(\zeta) \ d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}.
\end{equation}

Thus, as in Theorem 2, $f$ is holomorphic in $\Delta(w; r)$. Since this can be done for every $w \in D, f \in \mathcal{O}_D$.

12. **Theorem (Generalized Vitali's Theorem).** Any bounded family of holomorphic functions on a domain $D \subseteq \mathbb{C}^n$ has a compact closure in $\mathcal{O}_D$.

**Proof:** The proof is essentially the same as the classical proof of Montel's theorem in one complex variable. For any constant $M > 0$, consider the set

\[ \mathcal{A} = \{ f \in \mathcal{O}_D \mid |f(z)| \leq M \quad \text{for all} \ z \in D \}; \]
it is clearly sufficient to show that this set $\mathcal{A}$ is compact. Let $\Delta = \Delta(w_i; r_i)$
be a sequence of compact polydiscs $\Delta_i \subset D$ such that $\bigcup_{i=1}^{\infty} \Delta_i = D$; and
for each such polydisc let $2\delta_i$ be the distance from $\Delta_i$ to $\mathbb{C}^n - D$, so that
$\Delta_i(w_i; r_i) \subset \Delta_i(w_i; r_i + \delta_i) \subset D$. Now for any $f \in \mathcal{A}$ and any point $z \in \Delta_i$,
it follows from formula (6) that
\[
\left| \frac{\partial f}{\partial z} (z) \right| = \left| \frac{1}{(2\pi i)^n} \int_{\Gamma = \partial \Delta} \frac{f(\zeta) d\zeta}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \right| \leq \frac{M}{\delta_i} \left( 1 + \frac{r_i}{\delta_i} \right)^n;
\]
the restrictions of the elements of $\mathcal{A}$ to each $\Delta_i$ thus determine uniformly equicontinuous families of functions. Then for any infinite sequence of elements $f_n \in \mathcal{A}$, and for any such polydisc $\Delta_i$, a subsequence can be selected which is uniformly convergent on $\Delta_i$. By the usual Cantor diagonalization procedure, a subsequence can actually be selected so that it converges uniformly on each such polydisc $\Delta_i$. The sequence is then convergent in the topology of $\mathcal{O}_D$, and the limit function is obviously contained in $\mathcal{A}$, thus proving the compactness of $\mathcal{A}$ as desired.

A reformulation of Vitali's theorem which is often of considerable use is the following. Suppose that $E \subset D \subset \mathbb{C}^n$ are open domains. For any function $f \in \mathcal{O}_D$, the restriction of $f$ to the subdomain $E$ is an element $f|_E \in \mathcal{O}_E$; the mapping $r_{ED}: \mathcal{O}_D \to \mathcal{O}_E$ which associates to each $f \in \mathcal{O}_D$ the restriction $r_{ED}(f) = f|_E$ is called the restriction mapping. It is clear that $r_{ED}$ is a homomorphism between $\mathcal{O}_D$ and $\mathcal{O}_E$, considered as topological rings or as topological vector spaces; that is to say, the mapping $r_{ED}$ is an algebraic homomorphism, and is continuous in the topologies introduced above. Under some simple restrictions, the mappings $r_{ED}$ have other topological properties as well. To state one such, recall that a linear mapping between two topological vector spaces is called compact (or completely continuous, in an older terminology) if some open neighborhood of the origin is mapped to a set with compact closure (see Appendix B).

13. Theorem. If $E$ and $D$ are domains in $\mathbb{C}^n$ such that $\overline{E}$ is compact and $\overline{E} \subset D$, then the restriction map $r_{ED}: \mathcal{O}_D \to \mathcal{O}_E$ is a compact mapping.

Proof: Recalling the definition of the topology on $\mathcal{O}_D$, the set
\[
U(\mathcal{O}, \varepsilon) = \{ f \in \mathcal{O}_D \mid |f(z)| < \varepsilon \text{ whenever } z \in \mathcal{E} \}
\]
is an open neighborhood of the origin in $\mathcal{O}_D$ for any fixed $\varepsilon > 0$. By Vitali's theorem the family $r_{ED} U(\mathcal{E}, \varepsilon)$ has compact closure in $\mathcal{O}_E$, hence $r_{ED}$ is a compact mapping.
B. Holomorphic Mappings and Complex Manifolds

The inverse function theorem, which is familiar from elementary calculus, can be extended to holomorphic functions and holomorphic mappings. These extensions can be derived by beginning with the known versions of the theorems, and then merely showing that the functions of the conclusion are holomorphic when the functions of the hypothesis are holomorphic. However, it is more instructive, and more useful, to derive all of these results from the very beginning, using function-theoretic methods. To begin, a few further properties of holomorphic functions are required.

1. Definition. The holomorphic function \( f \) is said to be **regular of order** \( k \) in \( z_n \) at the point \( w \) if \( f(w_1, \ldots, w_{n-1}, z_n) \), considered as a holomorphic function of the single variable \( z_n \), has a zero of order \( k \) at the point \( z_n = w_n \). Equivalently, the condition can be stated as follows:

\[
\begin{align*}
1. \quad & f(w) = 0, \quad \frac{\partial f}{\partial z_n}(w) = 0, \quad \ldots, \quad \frac{\partial^{k-1} f}{\partial z_n^{k-1}}(w) = 0, \quad \frac{\partial^k f}{\partial z_n^k}(w) \neq 0.
\end{align*}
\]

2. Lemma. If \( f \) is a holomorphic function of total order \( k < \infty \) at a point \( w \), then after a suitable nonsingular linear change of coordinates in \( \mathbb{C}^n \) the function will be regular of order \( k \) in \( z_n \) at the point \( w \).

**Proof:** To simplify the notation we may assume that the point \( w \) is the origin in \( \mathbb{C}^n \). The function \( f \) then has an expansion in homogeneous polynomials of the form

\[
f(z) = \sum_{j=0}^{\infty} f_j(z), \quad \text{where } f_j(z) \neq 0.
\]

Select any point \( a = (a_1, \ldots, a_n) \neq 0 \) such that \( f_k(a_1, \ldots, a_n) \neq 0 \); since \( a \neq 0 \) there exist constants \( b_{ij} \) such that the linear change of coordinates

\[
z_i = a_i z_n + \sum_{j=1}^{n-1} b_{ij} z_j
\]

is nonsingular. In these new coordinates our function \( g(\zeta) = f(z(\zeta)) \) still has total order \( k \), and moreover \( g_k(0, \ldots, 0, 1) = f_k(a_1, \ldots, a_n) \neq 0 \); but then \( g \) is regular of order \( k \) in \( \zeta_n \) at the origin, as desired.

3. Lemma. If \( f \) is a function which is holomorphic in an open polydisc \( \Delta(w; r) \) and which is regular of order \( k \) in \( z_n \), then there is an open polydisc \( \Delta(\zeta; \delta) \subset \Delta(w; r) \) such that for every point \( (a_1, \ldots, a_{n-1}) \in \Delta(w; \delta_1, \ldots, \delta_{n-1}) \) the function \( f(a_1, \ldots, a_{n-1}, z_n) \), as a function of the single complex variable \( z_n \), has precisely \( k \) zeros (counting multiplicities) in the disc \( |z_n - w_n| < \delta_n \).

**Proof:** Again to simplify notation we may suppose that the point \( w \) is the origin in \( \mathbb{C}^n \). By hypothesis \( f(0, \ldots, 0, z_n) \) has a zero of order \( k \) at the
point \( z_n = 0 \). Since the zeros of a holomorphic function of one complex variable are isolated, there is a constant \( 0 < \delta_n < r_n \) such that \( f(0, \ldots, 0, z_n) \neq 0 \) for \( 0 < |z_n| \leq \delta_n \); set

\[
\inf_{|z_n| = \delta_n} |f(0, \ldots, 0, z_n)| = \varepsilon > 0.
\]

Since \( f(z) \) is continuous in an open neighborhood of the compact set \( z_1 = \cdots = z_{n-1} = 0, |z_n| = \delta_n \), there are constants \( \delta_j \) with \( 0 < \delta_j < r_j \), \( j = 1, \ldots, n-1 \), such that

\[
|f(z_1, \ldots, z_n) - f(0, \ldots, 0, z_n)| < \varepsilon
\]

for \( \begin{cases} |z_j| < \delta_j, & j = 1, \ldots, n-1; \\ |z_n| &= \delta_n. \end{cases} \)

Now for any point \((a_1, \ldots, a_{n-1})\) with \( |a_j| < \delta_j\) consider \( f(a_1, \ldots, a_{n-1}, z_n)\) as a function of \( z_n \) alone. By Rouche’s theorem from the theory of functions of one complex variable (Ahlfor's [A, p. 124]), it follows from (3) and (4) that \( f(a_1, \ldots, a_{n-1}, z_n)\) has the same number of zeros in \( |z_n| < \delta_n \) as has \( f(0, \ldots, 0, z_n)\), which by hypothesis is \( k \).

4. Theorem (Implicit Function Theorem). If \( f(z_1, \ldots, z_n) \) is holomorphic in an open polydisc \( \Delta(w; \delta) \subset \mathbb{C}^n \) and is regular of order 1 in \( z_n \) at the point \( w \), then in an open polydisc \( \Delta(w; \delta) \subset \Delta(w; r) \) there exists a unique holomorphic function \( \varphi(z_1, \ldots, z_{n-1}) \) such that

(i) \( \varphi(w_1, \ldots, w_{n-1}) = w_n \);

(ii) \( |\varphi(z_1, \ldots, z_{n-1}) - w_n| < \delta_n \) in \( \Delta(w; \delta_1, \ldots, \delta_{n-1}) \);

(iii) \( f(z_1, \ldots, z_{n-1}, z_n) = 0 \) at a point \( z \in \Delta(w; \delta) \) if and only if \( z_n = \varphi(z_1, \ldots, z_{n-1}) \).

Proof: Select an open polydisc \( \Delta(w; \delta) \subset \Delta(w; r) \) for which the conclusion of lemma 3 holds. Then for any point

\( (z_1, \ldots, z_{n-1}) \in \Delta(w; \delta_1, \ldots, \delta_{n-1}) \)

there is a unique value \( \varphi(z_1, \ldots, z_{n-1}) \in \Delta(w_n; \delta_n) \) such that

\( f(z_1, \ldots, z_{n-1}, \varphi(z_1, \ldots, z_{n-1})) = 0 \),

and necessarily \( \varphi(w_1, \ldots, w_{n-1}) = w_n \); we need merely show that this function \( \varphi \) is holomorphic. However, it follows from a familiar result in one complex variable (cf. Ahlfors [A, p. 124]) that

\[
\varphi(z_1, \ldots, z_{n-1}) = \frac{1}{2\pi i} \int_{|\zeta_n - w_n| = \delta_n} \frac{\overline{\partial}f(z_1, \ldots, z_{n-1}, \zeta_n)/\partial\zeta_n}{f(z_1, \ldots, z_{n-1}, \zeta_n)} d\zeta_n.
\]
By lemma 3 the function \( f(z_1, \ldots, z_{n-1}, \zeta_n) \) is never zero when
\[
(z_1, \ldots, z_{n-1}) \in \Delta(w; \delta_1, \ldots, \delta_{n-1})
\]
and \( |\zeta_n - w_n| = \delta_n \); consequently the integrand in (5) is a holomorphic function of \((z_1, \ldots, z_{n-1})\) in \( \Delta(w; \delta_1, \ldots, \delta_{n-1}) \), so the function \( \varphi \) is also holomorphic, as desired.

Recalling definition 1, the hypotheses in the implicit function theorem can be restated in the more familiar form
\[
f(w) = 0, \quad \frac{\partial f}{\partial z_n}(w) \neq 0;
\]
the form in which we have chosen to state the theorem will lead to a very important later generalization in a slightly different direction. At the moment, however, let us consider a straightforward extension of this result to holomorphic mappings.

5. Theorem (Implicit Mapping Theorem). If \( f_{k+1}, \ldots, f_n \) are holomorphic in an open polydisc \( \Delta(w; r) \subset \mathbb{C}^n \) and are such that:

(i) \( f_j(w) = 0, \quad j = k + 1, \ldots, n, \) and

(ii) \( \frac{\partial f_j}{\partial z_i}(w) = \delta_i^j, \quad i, j = k + 1, \ldots, n, \)

then in an open polydisc \( \Delta(w; \delta) \subset \Delta(w; r) \) there exist unique holomorphic functions \( \varphi_j(z_1, \ldots, z_k), \quad j = k + 1, \ldots, n, \) such that \( f_j(z_1, \ldots, z_n) = 0 \) for \( j = k + 1, \ldots, n \) if and only if \( z_j = \varphi_j(z_1, \ldots, z_k) \) for \( j = k + 1, \ldots, n \).

Proof: The proof will be by induction on the index \( n - k \) the number of functions in the hypothesis of the theorem. For the case in which \( n - k = 1 \), the result was demonstrated in theorem 4. For the induction step, assume that the result holds for \( < n - k \) functions; and consider \( n - k \) functions \( f_{k+1}, \ldots, f_n \) satisfying the hypothesis of the theorem. First, apply theorem 4 to the function \( f_n \) alone; so in some open polydisc \( \Delta(w; \delta') \subset \Delta(w; r) \) there will exist a unique holomorphic function \( \psi(z_1, \ldots, z_{n-1}) \) such that \( f_n(z) = 0 \) for \( z \in \Delta(w; \delta') \) if and only if \( z_n = \psi(z_1, \ldots, z_{n-1}) \). Now consider the \( n - k - 1 \) functions \( f_j'(z_1, \ldots, z_{n-1}) = f_j(z_1, \ldots, z_{n-1}, \psi(z_1, \ldots, z_{n-1})) \) in the polydisc \( \Delta(w; \delta') \), for \( j = k + 1, \ldots, n - 1 \). Observe that \( f_j(z) = 0 \) for \( z \in \Delta(w; \delta') \), \( j = k + 1, \ldots, n \), if and only if \( f_j'(z_1, \ldots, z_{n-1}) = 0 \) for \( j = k + 1, \ldots, n - 1 \) and \( z_n = \psi(z_1, \ldots, z_{n-1}) \). On the other hand, the \( n - k - 1 \) functions \( f_j'(z_1, \ldots, z_{n-1}) \) clearly satisfy the hypothesis of theorem 5, when considered as functions of the \( n - 1 \) variables \( z_1, \ldots, z_{n-1} \); so by
the induction hypothesis, in some polydisc \( \Delta(w; \delta) \subset \Delta(w; \delta') \) (where we put \( \delta_n = \delta'_n \)) there exist unique holomorphic functions

\[
\varphi_{k+1}(z_1, \ldots, z_k), \ldots, \varphi_{n-1}(z_1, \ldots, z_k)
\]
such that \( f_j'(z) = 0 \) for \( z \in \Delta(w; \delta) \) and \( j = k + 1, \ldots, n - 1 \) if and only if \( z_j = \varphi_j(z_1, \ldots, z_k) \), for \( j = k + 1, \ldots, n - 1 \). Upon putting

\[
\varphi_n(z_1, \ldots, z_k) = \varphi(z_1, \ldots, z_k, \varphi_{k+1}(z_1, \ldots, z_k), \ldots, \varphi_{n-2}(z_1, \ldots, z_k)),
\]
the desired result follows immediately.

Suppose that \( F : D \rightarrow \mathbb{C}^m \) is a holomorphic mapping, where \( D \) is a domain in \( \mathbb{C}^n \); the mapping is described by an \( m \)-tuple

\[
F(z) = (f_1(z), \ldots, f_m(z)),
\]
where \( f_j(z) \) are holomorphic in \( D \). The **Jacobian matrix** of the mapping \( F \) at a point \( w \in D \) is defined to be the matrix

\[
J_F(w) = \left( \frac{\partial f_j}{\partial z_j}(w) \right).
\]
The mapping \( F \) is said to be **nonsingular** at \( w \) if the rank of \( J_F(w) \) is maximal, that is, if rank \( J_F(w) = \min (m, n) \); and \( F \) is **nonsingular on \( D \)** if it is nonsingular at every \( w \in D \).

**6. Theorem.** Let \( n \geq m \), and let \( F \) be a nonsingular holomorphic mapping of a neighborhood of \( 0 \) in \( \mathbb{C}^n \) into \( \mathbb{C}^m \), such that \( F(0) = 0 \). There is a linear change of variables \( \omega_j = \sum_{i=1}^{n} a_{ij} \zeta_j \) in \( \mathbb{C}^n \), and there are functions

\[
\varphi_j(w_1, \ldots, w_{n-m}), \quad n - m + 1 \leq j \leq n,
\]
holomorphic in a polydisc \( \Delta(0; \delta) \), such that \( F(w_1, \ldots, w_n) = 0 \) if and only if \( w_j = \varphi_j(w_1, \ldots, w_{n-m}) \) in \( \Delta(0; \delta) \), for \( n - m + 1 \leq j \leq n \).

**Proof:** Let \( F = (f_{n-m+1}, \ldots, f_n) \). First, if \( (b_{ij}) \), \( n - m + 1 \leq i, j \leq n \), is any nonsingular \( m \times m \) matrix, and \( G = (g_{n-m+1}, \ldots, g_n) \), where \( g_i = \sum b_{ij} f_j \), then \( F(z) = 0 \) if and only if \( G(z) = 0 \). Thus we may replace \( F \) by any such nonsingular transform. Now, by assumption, the \( m \times n \) matrix

\[
J(0) = (\frac{\partial f_j}{\partial z_j}(0))
\]
has rank \( m \). Thus there exist nonsingular matrices \( B_{m \times m} \) and \( A_{n \times n} \) such that

\[
BJ(0)A^{-1} = (0, I_{m \times m}).
\]
Let \( B = (b_{ij}) \), \( n - m + 1 \leq i, j \leq n \); and let \( A = (a_{ij}) \), \( A^{-1} = (a'_{ij}) \), \( 1 \leq i, j \leq n \). Define \( g_i = \sum b_{ij} \frac{\partial f_j}{\partial w_j} \) and \( w_i = \sum a_{ij} \zeta_j \). Then

\[
\frac{\partial g_i}{\partial w_j} = \sum b_{ik} \frac{\partial f_k}{\partial w_j} = \sum b_{ik} \frac{\partial f_k}{\partial z_a} a'_{ij}.
\]
so
\[
\frac{\partial u_i}{\partial w_j}(0) = \sum_k b_{ik} \frac{\partial f_k}{\partial z_j}(0) a_{ij} = \delta_{ij}.
\]

Now theorem 5 applies to give the desired result.

7. **Theorem (Inverse Mapping Theorem).** Let \( F \) be a nonsingular holomorphic mapping of a neighborhood of 0 in \( \mathbb{C}^n \) into \( \mathbb{C}^n \) such that \( F(0) = 0 \). Then there is a polydisc \( \Delta(0; \delta) \) in which \( F \) is invertible. That is, there is a holomorphic mapping \( G \) of a neighborhood \( D \) of 0 onto \( \Delta(0; \delta) \) such that \( w = F(z) \) if and only if \( z = G(w) \).

**Proof:** Let \( J \) be the Jacobian matrix of \( F \). Since \( F \) is nonsingular, we can (by a change of variables both in the range and domain) assume that \( J(0) = I_{n \times n} \). Let \( w_1, \ldots, w_n; z_1, \ldots, z_n \) be these variables. Consider
\[
H(z, w) = w - F(z),
\]
a holomorphic mapping defined in a neighborhood of 0 in \( \mathbb{C}^{2n} \). \( H \) satisfies the hypotheses of theorem 6, so there is a mapping \( G(w_1, \ldots, w_n) \) defined in a polydisc \( \Delta(0; \epsilon) \) in \( \mathbb{C}^{2n} \), such that \( H(z, w) = 0 \) if and only if \( z = G(w) \). But this is just the desired conclusion.

Now let \( F \) be a mapping of a neighborhood of \( p \) in \( \mathbb{C}^n \) into \( \mathbb{C}^n \). If \( \det J_F(p) \neq 0 \), by the above theorem there is a neighborhood \( D \) of \( p \) in which \( F \) is an analytic isomorphism. If we write \( F \) in terms of coordinates as
\[
(6) \quad w_1 = w_1(z_1, \ldots, z_n), \quad \ldots \quad w_n = w_n(z_1, \ldots, z_n),
\]
this implies that a function defined at \( p \) is represented by a convergent power series in \( z_1, \ldots, z_n \) if and only if it can be represented as a convergent power series in \( w_1, \ldots, w_n \). Thus, as far as structure of analytic functions in a neighborhood of \( p \) is concerned, it doesn’t matter whether we consider functions in terms of the \( z \) variables or the \( w \) variables. We shall call any set of functions \( (6) \) a **coordinate set at \( p \)** if \( w_i(p) = 0, 1 \leq i \leq n \), and if they define an analytic isomorphism at \( p \), (i.e., in a neighborhood of \( p \)), or what is the same, the Jacobian matrix is nonsingular at \( p \).

8. **Definition.** A subset \( M \) of \( \mathbb{C}^n \) is a complex submanifold of \( \mathbb{C}^n \) if to every \( p \in M \) corresponds a neighborhood \( U \) and a mapping \( F : U \rightarrow \mathbb{C}^m \) (\( n \geq m \)) which is nonsingular at \( p \), such that \( M \cap U = \{ z \in U \mid F(z) = F(p) \} \).

9. **Theorem.** A subset \( M \) of \( \mathbb{C}^n \) is a complex submanifold if and only if to every point \( p \in M \) corresponds a coordinate set \( w_1, \ldots, w_n \) at \( p \) such that in some neighborhood \( U \) of \( p \),
\[
M \cap U = \{ z \in U \mid w_1(z) = 0, \ldots, w_m(z) = 0 \}.
\]
Proof: Clearly, if the condition of the theorem is fulfilled, the condition of the definition is satisfied with $F = (w_1, \ldots, w_m)$. Now conversely suppose $M$ is a complex submanifold and $p \in M$. Let $U, F = (f_1, \ldots, f_m)$ be as in the definition. We may assume $f_i(p) = 0$, $1 \leq i \leq m$. Then the vectors
\[
\left( \frac{\partial f_i}{\partial z_1}(p), \ldots, \frac{\partial f_i}{\partial z_m}(p) \right), \quad 1 \leq j \leq m,
\]
are independent. Let $(a_{j1}, \ldots, a_{j\delta})$, $m + 1 \leq j \leq n$, be $n - m$ vectors which expand these to a basis. Define $f_j = \sum a_{ji} z_i$ for $m + 1 \leq j \leq n$. Then the mapping $F = (f_1, \ldots, f_n)$ is nonsingular at $p$, so the functions $f_1, \ldots, f_n$ form a coordinate set at $p$, and $M \cap U = \{z \in U; f_1(z) = 0, \ldots, f_m(z) = 0\}$.

10. Theorem. A subset $M$ of $\mathbb{C}^n$ is a complex submanifold if and only if to every point $p \in M$ correspond a neighborhood $U$ of $p$, a polydisc $\Delta(0; \delta)$ in $\mathbb{C}^k$ ($k \leq n$), and a nonsingular holomorphic mapping $F: \Delta(0; \delta) \rightarrow \mathbb{C}^n$, such that $F(0) = p$, and
\[
M \cap U = F(\Delta(0; \delta)).
\]

Proof: Let $M$ be a complex submanifold, and $p \in M$. Then by theorem 9 there is a coordinate set $w_1, \ldots, w_n$ for a neighborhood $\Delta(0; r)$ of $p$ such that $M \cap U = \{w_1 = 0, \ldots, w_m = 0\}$. Let $k = n - m$ and
\[
\delta = (r_{m+1}, \ldots, r_n).
\]
Define $F: \Delta(0; \delta) \rightarrow \Delta(0; r)$, by $F(z_1, \ldots, z_k) = (0, \ldots, 0, z_1, \ldots, z_k)$.

Conversely, suppose there is such a mapping $F: \Delta(0; \delta) \rightarrow \Delta(0; r)$. Let $z_1, \ldots, z_n$ be coordinates for $\mathbb{C}^n$, $w_1, \ldots, w_k$ be coordinates for $\mathbb{C}^k$, and write $F$ as $z_i = f_i(w_1, \ldots, w_k)$, $1 \leq i \leq n$. Since $J_F(0)$ has rank $k$ the vectors
\[
\left( \frac{\partial f_1}{\partial w_1}(0), \ldots, \frac{\partial f_1}{\partial w_k}(0) \right), \quad 1 \leq j \leq k,
\]
are independent. Let us reorder the $z_i$'s so that these vectors, together with the vectors $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \ (1$ in the $f$th spot), $k + 1 \leq j \leq n$, are a basis. Define the mapping $F$ of $\Delta(0; r)$ into $\Delta(0; r)$ by
\[
z_i = f_i(w_1, \ldots, w_k), \quad 1 \leq i \leq k,
\]
\[
z_i = f_i(w_1, \ldots, w_k) + w_i, \quad k + 1 \leq i \leq n.
\]
Then $J_F(0)$ is nonsingular, so $w_1, \ldots, w_n$ form a coordinate set at $p$ in some neighborhood $\Delta(0; \varepsilon) \subset \Delta(0; r)$. Finally, $M \cap \Delta(0; \varepsilon) = F(\Delta(0; \delta)) \cap \Delta(0; \varepsilon) = \{z \in \Delta(0; \varepsilon) \mid w_{k+1}(z) = 0, \ldots, w_n(z) = 0\}$.

The number $k$ of theorem 10 is called the dimension of $M$ at $p$. It is independent of the mapping $F$. For if $F'$ is another mapping with the same properties, then $F^{-1} \circ F'$ must be an analytic isomorphism. Thus $J_{F^{-1}, F'}(0)$ is an invertible matrix, so it is necessarily a square $k \times k$ matrix.
C. Removable Singularities

One of the basic results in the theory of analytic functions of one complex variable is Riemann's theorem on removable singularities, or the Riemann extension theorem: a function which is bounded and holomorphic on a domain \( D \subset \mathbb{C} \) except for a discrete set of points, has a unique extension to a function holomorphic in all of \( D \). Our aim here is to discuss the form which this theorem takes in the case of holomorphic functions of several complex variables; in doing so, we shall note some pronounced differences between function theory in one variable and that in many variables.

1. Definition. Let \( D \) be a domain in \( \mathbb{C}^n \). A subset \( X \subset D \) is called thin if for every point \( z \in D \) there are an open polydisc \( \Delta(z; r) \subset D \) and a function \( f \) holomorphic and not identically zero in \( \Delta(z; r) \), such that \( f \) vanishes identically on \( X \cap \Delta(z; r) \).

By the identity theorem, the set where a nonzero holomorphic function vanishes is closed and has no interior; thus a thin set is nowhere dense (and in fact is of Lebesgue measure zero, by corollary A10). Clearly, the closure of a thin set is again thin.

2. Definition. Let \( D \) be a domain in \( \mathbb{C}^n \) and let \( X \) be a subset of \( D \). A function \( f \) defined on the set \( D - X \) is said to be locally bounded in \( D \) if to every point \( z \in D \) there is an open polydisc \( \Delta(z; r) \subset D \) such that the function \( f \) is bounded on \( \Delta(z; r) \cap (D - X) \).

3. Theorem (Riemann Extension Theorem, or Riemann Removable Singularity Theorem). Let \( X \) be a thin subset of a domain \( D \) in \( \mathbb{C}^n \), and let \( f \) be a holomorphic function on \( D - X \) which is locally bounded on \( D \). Then there is a unique function \( \hat{f} \) holomorphic on \( D \) and such that \( \hat{f}(z) = f(z) \) for \( z \in D - X \).

Proof: In view of the preceding definitions, it obviously suffices to prove the following special form of the theorem. Suppose that \( g \) is a nonzero holomorphic function in an open polydisc \( \Delta(w; r) \); and that \( X \) is the subset of \( \Delta(w; r) \) consisting of those points \( z \) at which \( g(z) = 0 \). Suppose moreover that \( f \) is holomorphic in \( \Delta(w; r) - X \) and is locally bounded in \( D \). Then in some open neighborhood of \( w \) there is a unique holomorphic function \( \tilde{f} \) such that \( \tilde{f}(z) = f(z) \) when \( z \notin X \).

We may suppose that \( w \in X \), since otherwise the result is trivial. By lemma B2 we can suppose that the coordinates in \( \mathbb{C}^n \) are so chosen that \( g \) is regular of some order \( k \) in \( z_n \) at the point \( w \). Then by lemma B3 there is an open polydisc \( \Delta(w; \delta) \) with \( \Delta(w; \delta) \subset \Delta(w; r) \), such that for \( (z_1, \ldots, z_{n-1}) \in \Delta(w_1, \ldots, w_{n-1}; \delta_1, \ldots, \delta_{n-1}) \) the function \( g(z_1, \ldots, z_n) \) has \( k \) zeros in
\[ |z| < \delta_n \text{ and is nonzero on } |z_n| = \delta_n. \] Now consider the function
\[ \hat{f}(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{|z_n - w_n| = \delta_n} \frac{f(z_1, \ldots, z_{n-1}, \zeta_n)}{\zeta_n - z_n} \, d\zeta_n \]
in \( \Delta(w; \delta) \). For any fixed \( \zeta_n \) in the domain of integration, the integrand is holomorphic in \( z_1, \ldots, z_{n-1} \) in \( \Delta(w; \delta) \), by the hypothesis and the above remarks; hence \( \hat{f}(z) \) is likewise holomorphic in \( z_1, \ldots, z_{n-1} \) in \( \Delta(w; \delta) \). On the other hand the function \( f \) is holomorphic in \( z_n \) from the definition; therefore, since \( \hat{f} \) is clearly continuous, by Osgood's lemma, \( \hat{f} \) is holomorphic in \( \Delta(w; \delta) \). Now for fixed \( z_1, \ldots, z_{n-1} \) the function \( f(z) \) itself is holomorphic in \( |z_n - w_n| < \delta_n \), except for finitely many points, and is locally bounded in the full disc; therefore, by the classical form of Riemann's theorem in one variable, it follows that \( \hat{f} = f \). The uniqueness is obvious, since \( X \) is nowhere dense, so the proof is completed.

4. Corollary. Let \( X \) be a thin subset of the connected open set \( U \) in \( \mathbb{C}^n \). Then \( U - X \) is connected.

Proof: If \( U - X \) is not connected, then neither is \( U - \overline{X} \); so we can write \( U - \overline{X} = D_1 \cup D_2 \) where \( D_1, D_2 \) are open sets and \( D_1 \cap D_2 = \emptyset \). Now the function \( f \) defined by \( f(x) = n \) for \( x \in D_n, n = 1, 2 \), is holomorphic in \( U - \overline{X} \). Since \( U \) is connected, there is a point \( x \in \overline{X} \cap D_1 \cap D_2 \). The function \( f \) clearly has no continuous extension through \( x \), and thus theorem 3 is contradicted.

The use of the Cauchy integral formula in theorem 3 has more profound consequences, some of which we shall now explore.

5. Theorem. Suppose that \( f \) is holomorphic in a connected neighborhood of the boundary of \( \Delta(0; r) \) in \( \mathbb{C}^n \), \( n \geq 2 \). Then there is a unique function \( \tilde{f} \) holomorphic in \( \Delta(0; r) \) such that \( \tilde{f} = f \) in their common domain of definition.

Proof: By assumption, \( f \) is holomorphic in a domain \( U \) such that \( U \supset \partial \Delta(0, r) \). Define, for \( z = (z_1, \ldots, z_n) \in \Delta(0, r) \),
\[ \hat{f}(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{|w| = r} \frac{f(z_1, \ldots, z_{n-1}, w)}{w - z_n} \, dw. \]
Again \( \hat{f} \) is holomorphic in \( z_1, \ldots, z_{n-1} \) for fixed \( z_n \); and since \( f \) is given by a Cauchy integral, it is holomorphic in \( z_n \) for fixed \( z_1, \ldots, z_{n-1} \). Hence \( \hat{f} \) is holomorphic in \( \Delta(0, r) \). But now, if \( z_1, \ldots, z_{n-1} \) are fixed so that \( |z_i| \) is close to \( r \), the disc \( |z_i| \leq r \) is contained in \( U \). Thus for such points \( (z_1, \ldots, z_n) \), \( \hat{f}(z_1, \ldots, z_n) = f(z_1, \ldots, z_n) \), and so \( \hat{f} = f \) in an open subset of \( U \cap \Delta(0, r) \);
then \( \tilde{f} = f \) in \( U \cap \Delta(0, r) \), by the identity theorem. The uniqueness of \( \tilde{f} \) follows from the same theorem, concluding the proof.

6. Corollary. Any isolated singularity of a holomorphic function of more than one variable is removable.

This result is in direct contrast with the situation in one complex variable. What is more important, we see that there are domains such as the boundary of the polycylinder for which all holomorphic functions extend to a larger domain. Such domains are not the natural domains of definition for any holomorphic function. However, it is the case in one variable that every domain is the natural domain of definition of a holomorphic function. Thus two problems appear in function theory of several variables: (1) to characterize those domains which are the natural domains of definition of holomorphic functions; (2) given any domain, to find the largest domain into which all holomorphic functions extend. We shall discuss these problems at greater length in Section G. For the present we shall derive another simple theorem in order to demonstrate that the solution to problem 2 consists in more than just an application of theorem 4 (i.e., filling in holes).

7. Theorem. Let \( f \) be holomorphic in a domain \( D \) in \( \mathbb{C}^n \) which contains \( \Delta(0, 1) \) and the set

\[
\{ z \in \mathbb{C}^n; |z_1| = 1, \ldots, |z_k| = 1, |z_{k+1}| < 1 + \varepsilon, \ldots, |z_n| < 1 + \varepsilon \}
\]

for some \( \varepsilon > 0 \), and integer \( k \). Then \( f \) has a unique extension \( \tilde{f} \) into \( D \cup \Delta(0, r) \), where \( r_i = 1 \) for \( 1 \leq i \leq k \), \( r_i = 1 + \varepsilon \) for \( k < i \leq n \).

**Proof.** Define

\[
\tilde{f}(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^k} \int_{|w_i|=1} \cdots \int_{|w_n|=1} \frac{f(w_1, \ldots, w_k, z_{k+1}, \ldots, z_n)}{(w_i - z_i) \cdots (w_k - z_k)} dw_1 \cdots dw_k.
\]

Note that \( \tilde{f} \) is well-defined and, as in the previous theorems, holomorphic in \( \Delta(0, r) \). But since \( f \) is holomorphic in \( \Delta(0, 1) \), the integral evaluates \( f \) in \( \Delta(0, 1) \). Thus \( \tilde{f} = f \) in \( \Delta(0, 1) \), so \( \tilde{f} \) extends \( f \), as desired.

As a result of the above theorem, we see that if \( D \) is a domain such that, for some boundary point \( w \) of \( D \), there are coordinates in a neighborhood of \( w \) in terms of which \( D \) satisfies theorem 7 and \( |z_j(w)| = 1, k + 1 \leq j \leq n \), then every function holomorphic in \( D \) extends to a neighborhood of \( w \).

Finally, we improve on the Riemann extension theorem in case the thin set has (in some sense yet to be defined) codimension 2.
8. Theorem. Suppose that $g_1$ and $g_2$ are holomorphic functions in an open neighborhood of a closed polydisc $\Delta(w; \delta)$ such that $g_1(z) \neq 0$ for $|z_1 - w_1| < \delta_1$, \ldots, $|z_{n-1} - w_{n-1}| < \delta_{n-1}$, $|z_n - w_n| = \delta_n$ and that $g_2(z) \neq 0$ for $|z_1 - w_1| < \delta_1$, \ldots, $|z_{n-2} - w_{n-2}| < \delta_{n-2}$, $|z_{n-1} - w_{n-1}| = \delta_{n-1}$, $|z_n - w_n| < \delta_n$. Let

$$V = \{z \in \Delta(w; \delta); g_1(z) = g_2(z) = 0\}.$$ 

If $f$ is holomorphic in $\Delta(w; \delta) - V$, then there is a unique function $\tilde{f}$ holomorphic in $\Delta(w; \delta)$ so that $f(z) = \tilde{f}(z)$ in their common domain of definition.

Proof: Note that

$$V \cap \{z \in \Delta(w; \delta) \mid |z_n - w_n| = \delta_n \text{ or } |z_{n-1} - w_{n-1}| = \delta_{n-1}\} = \emptyset.$$ 

The function

$$\tilde{f}(z_1, \ldots, z_n) = -\frac{1}{4\pi^2} \int_{|\zeta - w_n| = \delta_n} \int_{|\zeta - w_{n-1}| = \delta_{n-1}} f(z_1, \ldots, z_{n-2}, \zeta_{n-1}, \zeta_{n}) \frac{d\zeta_{n-1}}{(z_n - \zeta_n)(\zeta_n - w_n)} d\zeta_n$$

is therefore holomorphic throughout $\Delta(w; \delta)$. Now if $z_1, \ldots, z_{n-2}, \zeta_{n-1}$ are fixed with $|\zeta_{n-1} - w_{n-1}| = \delta_{n-1}$, the function $f(z_1, \ldots, z_{n-2}, \zeta_{n-1}, z_n)$ is holomorphic in $|z_n - w_n| \leq \delta_n$; hence

$$f(z_1, \ldots, z_{n-2}, \zeta_{n-1}, z_n) = \frac{1}{2\pi i} \int_{|z_n - w_n| = \delta_n} f(z_1, \ldots, z_{n-2}, \zeta_{n-1}, \zeta_n) d\zeta_n.$$

But further, for fixed $z_1, \ldots, z_{n-2}, z_n$ with $|z_n - w_n|$ close to $\delta_n$, the function $f(z_1, \ldots, z_{n-2}, z_{n-1}, z_n)$ is holomorphic in $|z_{n-1} - w_{n-1}| \leq \delta_{n-1}$; so that

$$f(z) = \frac{1}{2\pi i} \int_{|z_{n-1} - w_{n-1}| = \delta_{n-1}} \frac{f(z_1, \ldots, z_{n-2}, \zeta_{n-1}, \zeta_n)}{\zeta_{n-1} - z_{n-1}} d\zeta_{n-1}$$

for $|z_n - w_n|$ close to $\delta_n$. Upon combining (1) and (2) we see that $f(z) = \tilde{f}(z)$ in $\Delta(0; \delta)$ for $|z_n - w_n|$ close to $r_n$; from the identity theorem it then follows that $f(z) = \tilde{f}(z)$ in their common domain of definition, concluding the proof.

D. The Calculus of Differential Forms

The calculus of exterior differential forms on an open domain in $\mathbb{C}^n$ proves to be a convenient and useful formal tool in several function-theoretic
questions. Since, in these applications, we are not interested in questions of invariance of forms under changes of coordinates, we do not need an intrinsic definition; so such forms can be introduced in the following explicit and elementary way. (Further details to accompany the following brief description can be found in [C, J].) Consider an open domain $D$ in the space $\mathbb{C}^n$, and let $\mathcal{C}_D^\infty$ be the ring of complex-valued functions in $D$ which possess partial derivatives of all orders with respect to the $2n$ real coordinates in $\mathbb{C}^n$. Denote by $\mathcal{E}^1(D)$ the free module over the ring $\mathcal{C}_D^\infty$ of rank $2n$, and let $dz^1, \ldots, dz^n, \bar{d}z^1, \ldots, \bar{d}z^n$ be a base for this module; any element $\varphi \in \mathcal{E}^1(D)$ can thus be written in a unique manner in the form

$$\varphi = \sum_{i=1}^{n} q_i \, dz^i + \sum_{i=1}^{n} \bar{q}_i \, \bar{d}z^i$$

where $q_i, \bar{q}_i \in \mathcal{C}_D^\infty$. The full exterior algebra $\mathcal{E}^\bullet(D)$ on the module $\mathcal{E}^1(D)$ is called the \textit{algebra of exterior differential forms} on the domain $D$. The algebra $\mathcal{E}^\bullet(D)$ can be described most explicitly as follows. Introduce formal expressions in the generators $dz^i, \bar{d}z^j$ of $\mathcal{E}^1(D)$ of the form $dz^{i_1} \wedge \cdots \wedge dz^{i_r} \wedge \bar{d}z^{j_1} \wedge \cdots \wedge \bar{d}z^{j_s}$, and assume that these expressions are fully skew-symmetric in their entries; this defines an operation, the \textit{wedge product}, on the generators $dz^i, \bar{d}z^j$, which is associative and skew-symmetric. For any indices $p, q$ let $\mathcal{E}^{p,q}(D)$ be the free module on the $\binom{n}{p} \binom{n}{q}$ elements $dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge \bar{d}z^{j_1} \wedge \cdots \wedge \bar{d}z^{j_q}$ over the ring $\mathcal{C}_D^\infty$; the elements $\varphi \in \mathcal{E}^{p,q}(D)$ are called \textit{exterior differential forms of bidegree} $(p, q)$ on the domain $D$; any such element can be written uniquely in the form

$$\varphi = \sum_{i_1}^{n} \cdots \sum_{i_p=1}^{n} \sum_{j_1}^{n} \cdots \sum_{j_q=1}^{n} q_{i_1 \ldots i_p j_1 \ldots j_q} \, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge \bar{d}z^{j_1} \wedge \cdots \wedge \bar{d}z^{j_q}$$

where $q_{i_1 \ldots i_p j_1 \ldots j_q} \in \mathcal{C}_D^\infty$ are skew-symmetric in all indices. In particular, $\mathcal{E}^{0,0}(D) = \mathcal{C}_D^\infty$. The direct sum

$$\mathcal{E}^\bullet(D) = \bigoplus_{p+q=0}^{n} \mathcal{E}^{p,q}(D)$$

inherits the wedge product from that product on its generators, and is hence an associative, skew-symmetric algebra over $\mathcal{C}_D^\infty$. The partial sums $\mathcal{E}^r(D) = \sum_{p+q=r}^{n} \mathcal{E}^{p,q}(D)$ are called the modules of differential forms of \textit{total degree} $r$. Note that if $\varphi \in \mathcal{E}^r(D)$ and $\psi \in \mathcal{E}^r(D)$, then $\varphi \wedge \psi = (-1)^{p \cdot q} \psi \wedge \varphi$.

The \textit{complex exterior derivations} $\partial, \bar{\partial}, d$ are maps

$$\partial: \mathcal{E}^{p,q}(D) \rightarrow \mathcal{E}^{p+1,q}(D), \quad \bar{\partial}: \mathcal{E}^{p,q}(D) \rightarrow \mathcal{E}^{p,q+1}(D)$$

$$d: \mathcal{E}^{r}(D) \rightarrow \mathcal{E}^{r+1}(D)$$
defined as follows: for a differential form \( \varphi \in \mathcal{C}^{p,q}(D) \) as in (1),

\[
\partial \varphi = \sum_{i, j, k} (\partial q_{i1}, \ldots, i_{j1}, \ldots, j_{k1}) (\partial z_k) 
\cdot d z_k^1 \wedge d z_1^1 \wedge \cdots \wedge d z_1^{i_1} \wedge d z_1^{j_1} \wedge \cdots \wedge d z_1^{j_{k_1}},
\]

(2) \[
\delta \varphi = \sum_{i, j, k} (\partial q_{i1}, \ldots, i_{j1}, \ldots, j_{k1}) (\partial z_k) 
\cdot d z_k^1 \wedge d z_1^1 \wedge \cdots \wedge d z_1^{i_1} \wedge d z_1^{j_1} \wedge \cdots \wedge d z_1^{j_{k_1}},
\]

\[
d \varphi = \partial \varphi + \delta \varphi.
\]

Clearly these mappings are linear over the subring of constant functions in \( \mathcal{C}^{\infty}_p(D) \), and for \( \varphi \in \mathcal{C}^{p,q}(D) \), \( \varphi \in \mathcal{C}^{r,n}(D) \)

\[
\partial (\varphi \wedge \psi) = \varphi \wedge \partial \psi + (-1)^{p+q+1} \varphi \wedge \delta \psi,
\]

(3) \[
\delta (\varphi \wedge \psi) = \varphi \wedge \delta \psi + (-1)^{p+q+1} \varphi \wedge \delta \psi.
\]

A straightforward calculation, which can be found in many places [G, J] [243], and hence will not be repeated here, shows that

\[
dd = 0,
\]

hence

\[
\partial \partial = 0, \quad \delta \delta = 0, \quad \partial \delta + \delta \partial = 0.
\]

1. **Theorem (Generalized Cauchy Integral Formula).** Let \( D \) be a region in the complex plane bounded by a rectifiable simple closed curve \( \gamma \); and let \( f \) be a complex-valued function in an open neighborhood of \( \overline{D} \) which has partial derivatives of all orders in the real coordinates. Then for any point \( z \in D \),

\[
2 \pi i f(z) = \int_{\gamma} f(\xi) \frac{d \xi}{\xi - z} - \int_{\partial D} \frac{\partial f(\xi)}{\partial \xi} \frac{d \xi \wedge d \overline{\xi}}{\overline{\xi} - z},
\]

(6) \[
2 \pi i f(z) = - \int_{\gamma} f(\xi) \frac{d \overline{\xi}}{\overline{\xi} - z} + \int_{\partial D} \frac{\partial f(\xi)}{\partial \xi} \frac{d \xi \wedge d \overline{\xi}}{\xi - z}.
\]

**Proof:** For any point \( z \in D \) select a disc \( \Delta(z, r) \) with closure contained in \( D \); and let \( \gamma_r \) be the boundary of \( \Delta(z, r) \), a circle of radius \( r \) centered at \( z \). Furthermore let \( D_r = D - \Delta(z, r) \), observing that this is an open region bounded by \( \gamma - \gamma_r \). Now note that

\[
\frac{\partial f(\xi)}{\partial \xi} \frac{d \xi \wedge d \overline{\xi}}{\xi - z} = \frac{\partial}{\partial \xi} \left( f(\xi) \frac{d \xi}{\xi - z} \right) d \xi \wedge d \overline{\xi} = d \left( f(\xi) \frac{d \xi}{\xi - z} \right).
\]
Therefore, by the familiar Stokes' theorem in the plane,

\[
\int_{D_r} \int_{D_r} \nabla f \cdot \frac{d\xi}{\xi - z} = \int_{D_r} \int_{D_r} \frac{df(\xi)}{\xi - z} \cdot \frac{d\xi}{\xi - z}
\]

Letting \( r \to 0 \) in (8), the surface integral over \( D_r \) clearly converges to the surface integral over \( D \) since \( (\xi - z)^{-1} \frac{d\xi}{\xi - z} \) is a bounded measure on the plane; and the integral over \( \gamma \) is of course unchanged. To investigate the behavior of the integral over \( \gamma_r \), introduce the parametrization \( \zeta = z + r e^{it}, 0 \leq t \leq 2\pi \), so that

\[
\int_{\gamma_r} f(\zeta) \frac{d\zeta}{\zeta - z} = \int_{t=0}^{2\pi} f(\zeta = z + r e^{it}) dt;
\]

then

\[
\lim_{r \to 0} \int_{\gamma_r} f(\zeta) \frac{d\zeta}{\zeta - z} = \int_{t=0}^{2\pi} f(z) dt = 2\pi i f(z).
\]

Therefore in the limit, formula (8) becomes

\[
\int_{D} \int_{D} \nabla f \cdot \frac{d\xi}{\xi - z} = \int_{\gamma} f(\zeta) \frac{d\zeta}{\zeta - z} - 2\pi i f(z),
\]

which is just formula (6). Formula (7) then follows by taking the complex conjugate in (6).

2. Lemma. Let \( D, \gamma, f \) be as in Theorem 1. Then there are \( C^\infty \) functions \( g, h \) in \( D \) such that

\[
\frac{\partial g(z)}{\partial \bar{z}} = f(z), \quad \frac{\partial h(z)}{\partial z} = f(z).
\]

Moreover, if \( f \) is \( C^\infty \) or holomorphic in some additional parameters, so are \( g \) and \( h \).

Proof: Introduce the function

\[
g(z) = \frac{1}{2\pi i} \int_{D} \int_{D} f(\zeta) \frac{d\xi}{\xi - z}.
\]

This function is clearly holomorphic or \( C^\infty \) in any additional parameters in which \( f \) is holomorphic or \( C^\infty \); so we need merely verify that it satisfies the appropriate partial differential equation. For any point \( z \in D \) select a disc
$\Delta(z; r)$ with closure contained in $D$, let $\gamma_r$ be the boundary of $\Delta(z; r)$, and let $D_r = D - \Delta(z; r)$. Noting that (as functions of $\zeta$ for $z$ fixed),

$$d \log |\zeta - z|^2 = d(\log (\zeta - z) + \log (\overline{\zeta} - \overline{z})) = \frac{d\zeta}{\zeta - z} + \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}},$$

it follows again from Stokes' theorem in the place that

$$\int_{\gamma_r} f(\zeta) \log |\zeta - z|^2 d\zeta = \int_{D_r} d[f(\zeta) \log |\zeta - z|^2]$$

$$= \int_{D_r} \frac{\partial f}{\partial \zeta} \log |\zeta - z|^2 d\zeta \wedge d\overline{\zeta} + \int_{D_r} f(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z}.$$

Again let $r \to 0$ in the above formula. Parametrizing $\gamma_r$ by $\zeta = z + re^{it}$, $0 \leq t \leq 2\pi$, and putting $M = \sup_{z \in D} |f(z)|$, we have

$$\lim_{r \to 0} \int_{\gamma_r} f(\zeta) \log |\zeta - z|^2 d\zeta = \lim_{r \to 0} \int_{t=0}^{2\pi} f(z + re^{it})2r(\log r)i e^{-it} dt\leq \lim_{r \to 0} 4\pi Mr \log r = 0.$$

Consequently, in the limit, equation (10) becomes

$$\int_{\gamma} f(\zeta) \log |\zeta - z|^2 d\zeta = \int_{D} \frac{\partial f}{\partial \zeta} \log |\zeta - z|^2 d\zeta \wedge d\overline{\zeta} + 2\pi i g(z),$$

upon recalling definition (9). Differentiation under the integral sign is permissible in the last equation, since the resulting integrand is still integrable; in particular, applying the operator $\partial / \partial \overline{z}$,

$$-\int_{\gamma} f(\zeta) \frac{d\overline{\zeta}}{\zeta - z} = -\int_{D} \frac{\partial f}{\partial \zeta} \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z} + 2\pi i \frac{\partial g}{\partial z}.$$

By the generalized Cauchy integral formula (Theorem 1), it follows that $f(z) = \partial g(z)/\partial \overline{z}$. Note that formula (11) shows that the function $g(z)$ is $C^1$, and the differential equation shows that $\partial g(z)/\partial \overline{z}$ is actually $C^\infty$. On the other hand, a similar calculation shows that

$$-\int_{\gamma} f(\zeta) \frac{d\overline{\zeta}}{\zeta - z} = -\int_{D} \frac{\partial f}{\partial \zeta} \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z} + 2\pi i \frac{\partial g}{\partial z}.$$
where we have now written \( \partial g(z) / \partial z \) as something which is clearly \( C^\infty \) (the contour integral) plus something which has the same form as \( g(z) \) in (9); so, by induction, \( \partial g(z) / \partial z \) is also \( C^\infty \). These observations show that \( g(z) \) is itself \( C^\infty \), as desired.

3. Theorem (Dolbeault Lemma). Let \( \tilde{\Delta} \subset \mathbb{C}^n \) be a compact polydisc, and let \( \omega \) be a \( C^\infty \) differential form of bidegree \( (p, q) \) in an open neighborhood of \( \tilde{\Delta} \). If \( q > 0 \) and if \( \tilde{\partial} \omega = 0 \), then there is a \( C^\infty \) differential form \( \eta \) of bidegree \( (p, q - 1) \) in \( \Delta \) such that \( \omega = \tilde{\partial} \eta \).

**Proof:** Let \( v \) be the least integer such that the explicit representation of \( \omega \) in coordinate form (1) involves only the conjugate differentials \( dz^1, \ldots, dz^v \); the proof will be by induction on \( v \). If \( v = 0 \) then necessarily \( \omega = 0 \), since by hypothesis \( \omega \) is of bidegree \( (1, q) \) with \( q > 0 \); the theorem holds trivially in this case. In general, for the induction step, write

\[
\omega = dz^v \wedge \alpha + \beta
\]

where \( \alpha, \beta \) are differential forms which involve only the conjugate differentials \( dz^1, \ldots, dz^{v-1} \). Since

\[
0 = \tilde{\partial} \omega = (dz^v \wedge \tilde{\partial} \alpha) + \tilde{\partial} \beta,
\]

it follows readily that the coefficients of the forms \( \alpha, \beta \) must be holomorphic in \( z_{v+1}, \ldots, z_n \); for the partial derivatives \( \partial / \partial z_{v+1}, \ldots, \partial / \partial z_n \) of any such coefficient are clearly all zero. Any coefficient \( f \) of the form \( \alpha \) is a \( C^\infty \) function of the variable \( z_v \) in an open neighborhood of \( \tilde{\Delta}_v \), where the original polydisc has the product decomposition \( \Delta = \Delta_1 \times \cdots \times \Delta_n \); and the function \( f \) is also a \( C^\infty \) function of \( z_1, \ldots, z_{v-1} \) and a holomorphic function of \( z_{v+1}, \ldots, z_n \) in the corresponding domains. By lemma 2 there exists a function \( g \) which is \( C^\infty \) in \( z_v \) in some open neighborhood of \( \tilde{\Delta}_v \), such that \( \partial g / \partial z_v = f \); \( g \) is also \( C^\infty \) in \( z_1, \ldots, z_{v-1} \) and holomorphic in \( z_{v+1}, \ldots, z_n \) in the same regions as \( f \) is. Replacing each coefficient \( f \) in the form \( \alpha \) by such a function \( g \) yields a new differential form \( \gamma \), which by this construction satisfies the equation

\[
\tilde{\partial} \gamma = dz^v \wedge \alpha + \delta
\]

for some differential form \( \delta \) involving only the conjugate differentials \( dz^1, \ldots, dz^{v-1} \). Now consider the differential form

\[
\varphi = \omega - \tilde{\partial} \gamma = \beta - \delta;
\]

note that \( \tilde{\partial} \varphi = \tilde{\partial} \omega - \tilde{\partial} \tilde{\partial} \gamma = 0 \), and that \( \varphi \) involves only the conjugate differentials \( dz^1, \ldots, dz^{v-1} \), since \( \beta \) and \( \delta \) do. The induction hypothesis shows that \( \varphi = \tilde{\partial} \psi \), and hence that \( \omega = \tilde{\partial} (\gamma + \psi) \), thus concluding the proof.
Remark: Upon replacing the form $\omega$ by its complex conjugate, it is evident that the analogue of the preceding theorem holds for the operator $\partial$ as well.

Suppose that $D$ is any domain in the space $\mathbb{C}^n$, and consider the complex vector spaces $\mathcal{E}^{p,q}(D)$ of $C^\infty$ differential forms of various bidegrees $(p,q)$ in $D$. The lemma of Dolbeault shows that for any form $\varphi \in \mathcal{E}^{p,q}(D)$ such that $q > 0$ and $\partial \varphi = 0$, and for any point $z \in D$, then in some open neighborhood $U$ of $z$ the form $\varphi$ can be written $\varphi = \delta \psi$, where $\psi \in \mathcal{E}^{p,q-1}(U)$. That is to say, whenever $\partial \varphi = 0$ and $q > 0$, the form $\varphi$ is locally expressible as $\delta \psi$ for some form $\psi$. It is not true, in general, that there is a global form $\varphi \in \mathcal{E}^{p,q}(D)$ such that $\varphi = \delta \psi$; this global form of the assertion is true only for special classes of domains, and this matter is closely linked to the deeper function-theoretic properties of the domains. We shall return to this question again later. For the present, we shall demonstrate that this global assertion is true for the simplest classes of domains, the polydiscs; and we shall then draw some consequences from this theorem, to show the relevance of such considerations to function-theoretic questions.

Some additional notation and terminology is useful here. A differential form $\varphi \in \mathcal{E}^{p,q}(D)$ is called $\delta$-closed if $\delta \varphi = 0$; and the form is called $\delta$-exact if $\varphi = \delta \psi$ for some differential form $\psi \in \mathcal{E}^{p,q-1}(D)$. Since $\delta^2 = 0$, every $\delta$-exact form is also $\delta$-closed. Note that all forms of bidegree $(p,n)$ are $\delta$-closed; and that the trivial form $\varphi \equiv 0$ is the only form of bidegree $(p,0)$ which is $\delta$-exact, for any value $p = 0, 1, \ldots, n$. Considering the operator $\delta$ as a linear mapping $\delta: \mathcal{E}^{p,q}(D) \longrightarrow \mathcal{E}^{p,q+1}(D)$, the $\delta$-closed forms are the kernel of the mapping, and the $\delta$-exact forms are the image.

4. Definition. The Dolbeault cohomology groups of the domain $D$ are the complex vector spaces

$$
\mathcal{H}^{p,q}(D) = \frac{\{\text{\delta-closed forms of bidegree } (p,q) \text{ in } D\}}{\{\text{\delta-exact forms of bidegree } (p,q) \text{ in } D\}}.
$$

5. Theorem. Let $\Delta$ be any open polydisc in the space $\mathbb{C}^n$, not necessarily having compact closure. Then $\mathcal{H}^{p,0}(\Delta)$ is the space of differential forms of bidegree $(p,0)$ whose coefficients are holomorphic functions in $\Delta$; and $\mathcal{H}^{p,0}(\Delta) = 0$ for all $q \geq 1$.

Proof: By definition, $\mathcal{H}^{p,0}(\Delta)$ is just the space of $\delta$-closed forms of bidegree $(p,0)$ in $\Delta$. If

$$
\varphi = \sum_{i_1 \ldots i_p} \varphi_{i_1 \ldots i_p} \ dz^{i_1} \wedge \cdots \wedge dz^{i_p} \in \mathcal{H}^{p,0}(\Delta),
$$
then we have

\[ 0 = \delta q = \sum_{i_1 \cdots i_p} (\partial q_{i_1 \cdots i_p} / \partial z_i) \, dz^i \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p}; \]

and from the Cauchy-Riemann criterion, all the coefficients \( q_{ij_1 \cdots j_p} \) are holomorphic functions, as desired. In particular, note that \( \mathcal{S}^{0,0}(\Delta) = \mathcal{O}_\Delta \), the space of functions holomorphic in \( \Delta \). Turning then to the more interesting part of the theorem, we must show that every \( C^\infty \), \( \delta \) -closed form of bidegree \((p, q)\) in \( \Delta, q \geq 1 \) is exact. Therefore consider such a form \( q \in \mathcal{S}^{p,0}(\Delta) \), with \( \delta q = 0 \). The cases \( q > 1 \) and \( q = 1 \) will be treated separately. Select a sequence of polydiscs \( \Delta_n \) in the space \( \mathbb{C}^n \), which have the same center as \( \Delta \) and which satisfy the following two conditions: (i) \( \Delta = \bigcup_n \Delta_n \), and (ii) \( \Delta_n \subset \Delta_{n+1} \).

(a) Suppose, first of all, that \( q > 1 \). We shall show, by induction on \( r \), that there are differential forms \( \psi_r \), defined on open neighborhoods of \( \Delta \), such that \( \psi = \delta \psi_r \) and \( \psi_{r+1} \big| \Delta_n = \psi_r \), where \( \psi_{r+1} \big| \Delta_n \) denotes the restriction of the form \( \psi_{r+1} \) to the set \( \Delta_n \). For \( r = 1 \) we need only verify the first of these two conditions; and the existence of such a form \( \psi_1 \) follows immediately from the Dolbeault lemma (Theorem 3). Now assume that differential forms \( \psi_1, \ldots, \psi_r \) have already been defined, satisfying these two conditions. By the Dolbeault lemma again there is a \( C^\infty \) differential form \( \psi_{r+1} \) on an open neighborhood of \( \Delta_{r+1} \) such that \( \delta \psi_{r+1} = \psi_r \). On an open neighborhood of \( \Delta_r \) the form \( \psi_{r+1} \big| \Delta_r = \psi_r \) is then closed, since \( \delta(\psi_{r+1} \big| \Delta_r) = \psi_{r+1} \big| \Delta_r = \psi_r \). This form \( \psi_r \) is of bidegree \((p, q - 1)\), and \( q - 1 > 0 \) by hypothesis; so after yet another application of the Dolbeault lemma there will exist a \( C^\infty \) differential form \( \theta \) of bidegree \((p, q - 2)\) in an open neighborhood \( U_r \) of \( \Delta_r \) such that \( \psi_{r+1} = \psi_r \). Let \( \sigma \) be any \( C^\infty \) real-valued function defined in the space \( \mathbb{C}^n \), and such that \( \sigma(z) = 1 \) for \( z \in \Delta_r \) and that \( \sigma(z) = 0 \) for \( z \in \bigcup_n \Delta_n \); the existence of such functions is a familiar result (cf. Appendix A). The form \( \psi_{r+1} = \psi_{r+1} + \delta(\sigma \theta) \) is then a \( C^\infty \) form in the same region as \( \psi_{r+1} \); moreover, \( \delta \psi_{r+1} = \delta \psi_{r+1} = \delta \psi_{r+1} = \psi_r \), and \( \psi_{r+1} \big| \Delta_r = \psi_{r+1} + \delta \theta = \psi_{r+1} \), concluding the induction step.

To return to the proof of the theorem itself, as a result of the above construction there is a \( C^\infty \) form \( \psi \) defined in \( \Delta \) such that \( \psi \big| \Delta_n = \psi_n \); and this form then satisfies the equation \( \delta \psi = q \), which concludes the proof in this case.

(b) Next consider the case \( q = 1 \). The line of argument used above breaks down, but we can modify it as follows. We shall show, again by induction on \( r \), that there are \( C^\infty \) differential forms \( \psi_r \), defined on open neighborhoods of \( \Delta_n \) such that \( \delta \psi_r = q \), and that the differences \( \psi_{r+1} - \psi_r \) are holomorphic forms of bidegree \((p, 0)\) in open neighborhoods of \( \Delta_n \) and satisfy

\[ |\psi_{r+1,i_1 \cdots i_s}(z) - \psi_{r,i_1 \cdots i_s}(z)| < 2^{-r} \]
for all $z \in \Delta_v$ and all coefficients $(j_1, \ldots, j_p)$. For the case $v = 1$ we need only verify the first of these two conditions; and that is an immediate consequence of the Dolbeault lemma. Now assume that differential forms

$\psi_1, \ldots, \psi_v$

have been found, satisfying the above conditions. By the Dolbeault lemma there is a $C^\infty$ differential form $\psi'_{v+1}$ in an open neighborhood of $\Delta_{v+1}$ such that $\delta \psi'_{v+1} = \phi$. On an open neighborhood of $\Delta_v$ all the coefficients of the form $\psi'_{v+1} - \psi_v$ are holomorphic, since $\delta(\psi'_{v+1} - \psi_v) = \phi - \phi = 0$. Each coefficient has a power series expansion centered at the common center of all the polydiscs and converging uniformly in $\Delta_v$; hence, by choosing suitable partial sums, we find polynomial terms $p_{j_1} \ldots j_p(z)$ such that

$$|\psi'_{v+1,j_1 \ldots j_p}(z) - \psi_{v,j_1 \ldots j_p}(z) - p_{j_1} \ldots j_p(z)| < 2^{-v}$$

whenever $z \in \Delta_v$. Letting $p(z)$ be the differential form

$$p(z) = \sum p_{j_1} \ldots j_p(z) \, dz^{j_1} \wedge \cdots \wedge dz^{j_p},$$

define the differential form $\psi_{v+1} = \psi'_{v+1} - p$. Now $\psi_{v+1}$ is $C^\infty$ in the same domain as $\psi'_{v+1}$ is, and $\delta \psi_{v+1} = \delta \psi'_{v+1} = \phi$; the difference $\psi_{v+1} - \psi_v$ is holomorphic in an open neighborhood of $\Delta_v$, and satisfies the desired inequalities, thus concluding the induction step.

To complete the proof of the theorem itself, observe that the sequence of differential forms $\{\psi_v\}$ have coefficients which converge uniformly on any one of the polydiscs $\Delta_v$ to the coefficients of a differential form $\psi$. Since $\psi - \psi_v = \lim_{v \to \infty} (\psi_v - \psi_v)$ and the forms $\psi_v - \psi_v$ have coefficients holomorphic in $\Delta^*_v$, it follows that in $\Delta^*_v$, $\psi = \psi_v + \theta$, for some holomorphic form $\theta$; hence $\delta \psi = \delta \psi_v = \phi$ in each $\Delta_v$, which concludes the proof.

It is not true that $\mathcal{S}^q(D) = 0$ for all $q \geq 1$, for an arbitrary domain $D \subset \mathbb{C}^n$. As an example, let $D$ be the space of two complex variables $(z_1, z_2)$ with the origin deleted; we shall show that $\mathcal{S}^{0,1}(D) \neq 0$. Putting

$$r^2 = |z_1|^2 + |z_2|^2,$

note that $\frac{1}{z_1z_2} = \frac{z_2}{z_1r^2} + \frac{z_1}{z_2r^2}$ whenever $z_1z_2 \neq 0$; hence we can define a $\bar{\partial}$-closed differential form $\omega$ of type $(0, 1)$ in $D$ by putting

$$\omega = \begin{cases} \bar{\partial} \left( \frac{z_2}{z_1r^2} \right) & \text{when } z_1 \neq 0, \\ -\bar{\partial} \left( \frac{z_1}{z_2r^2} \right) & \text{when } z_2 \neq 0. \end{cases}$$

Suppose that there exists a $C^\infty$ function $f$ in $D$ such that $\bar{\partial}f = \omega$. The function $g = z_1f - \bar{z}_2r^2$ is then holomorphic for $z_1 \neq 0$, since $(1/z_1) \bar{\partial}g = \bar{\partial}f - \omega = 0$. Moreover, since $g$ is obviously locally bounded in $D$, it follows
from the Riemann extension theorem (theorem C3) that \( g \) is actually holomorphic in all of \( D \); and then, by corollary C6, the function \( g \) is holomorphic for all values of \( (z_1, z_2) \), including the origin. Letting \( z_1 \to 0 \), it then follows from the definition of the function \( g \) and its continuity that \( g(0, z_2) = 1/z_2 \), hence \( g \) cannot be holomorphic at the origin. This contradiction shows that the equation \( \omega = \bar{\partial}f \) can hold for no \( C^\infty \) functions \( f \) in \( D \), and hence \( \Omega^{1,0}(D) \neq 0 \). The interested reader can show, along the same lines, that \( \Omega^{0,1}(D) \) is not even a finite-dimensional complex vector space.

E. The Cousin Theorem

To illustrate the role of the Dolbeault cohomology groups in the theory of functions of several complex variables, we now turn to a problem investigated by Cousin [78]. In the present section we shall not attempt to give a complete and general treatment of this problem; rather, we shall examine some of the simpler cases in which the basic natures of the problem and of the solution are free of technical complications. We shall return to the same questions again in a later chapter after having developed considerably more machinery; the present example should then be kept in mind as motivation for the machinery and its application.

Let \( \Delta = \Delta(n; r) \) be an open polydisc in the space of \( n \) complex variables; this polydisc need not have compact closure, so that \( \Delta \) may be the entire space \( \mathbb{C}^n \), or may have some complex planes as factors. Furthermore, let \( \{U_i\} \) be an open covering of the polydisc \( \Delta \); that is to say, let \( U_i \subset \Delta \) be open subsets such that \( \bigcup_i U_i = \Delta \). By Cousin data for the covering \( \{U_i\} \) of \( \Delta \) we mean a collection of functions \( h_{ij} \) defined and holomorphic in the intersections \( U_i \cap U_j \) of pairs of open sets of the covering, and satisfying the following conditions:

1. \( h_{ij} + h_{ji} = 0 \) in \( U_i \cap U_j \);

2. \( h_{ij} + h_{jk} + h_{ki} = 0 \) in \( U_i \cap U_j \cap U_k \).

[Looking ahead to a more general situation, a collection of functions \( h_{ij} \) defined and holomorphic in the intersections \( U_i \cap U_j \) will also be called a one-cochain for the covering \( \{U_i\} \) with holomorphic functions as coefficients; and such a one-cochain will be called a one-cocycle if the functions satisfy conditions (1) and (2) above.]

1. Lemma. If \( \{h_{ij}\} \) are Cousin data for an open covering \( \{U_i\} \) of the polydisc \( \Delta \), then there are \( C^\infty \) functions \( f_i \) defined in the open sets \( U_i \) such that

3. \( f_i - f_j = h_{ij} \) in \( U_i \cap U_j \).
Proof: It is a familiar fact from point-set topology that any open covering \( \{ U_i \} \) of \( \Delta \) has a locally finite refinement. That is to say, there exists an open covering \( \{ V_a \} \) of \( \Delta \) such that each \( V_a \) is contained within one of the open sets \( U_i \), and such that each point of \( \Delta \) has an open neighborhood which meets only finitely many of the sets \( V_a \) (cf. [F]). It is also familiar that for any locally finite open covering \( \{ V_a \} \) of \( \Delta \) there exists an associated partition of unity. That is, there are \( C^\infty \) real-valued functions \( \{ \eta_a \} \) defined in \( \Delta \) such that \( \eta_a \) vanishes identically in an open neighborhood of \( \Delta - V_a \), and such that \( \sum a \eta_a = 1 \) (cf. Appendix A). Note that the sum \( \sum a \eta_a \) is well-defined, since in a neighborhood of any point of \( \Delta \) only finitely many of the functions \( \eta_a \) do not vanish identically. Now, for any fixed index \( a \), there exists a set \( U_i \) of the given covering such that \( V_a \subset U_i \). Then define functions \( \{ f_{a\beta} \} \) in the open sets \( \{ U_j \} \) by

\[
f_{a\beta}(z) = \begin{cases} \eta_a(z) h_{ij}(z) & \text{for } z \in V_a \cap U_j, \\ 0 & \text{for } z \in U_i - (V_a \cap U_j). \end{cases}
\]

Since \( \eta_a \) vanishes in an open neighborhood of \( U_j - (U_i \cap V_a) \), the function \( f_{a\beta} \) is \( C^\infty \) in all of \( U_j \). Furthermore, referring to the cocycle conditions (1) and (2), in each intersection \( V_a \cap U_j \cap U_k \) we have

\[
f_{a\beta} - f_{a\kappa} = \eta_a h_{jk},
\]

since both sides of the above equation are defined and vanish identically in \( U_j \cap U_k - V_a \cap U_j \cap U_k \), the equality (4) actually holds in all of \( U_j \cap U_k \). The functions \( f_i = \sum a f_{a\beta} \) are well-defined \( C^\infty \) functions in \( U_j \), since the sums are locally finite; and it results from (4) that in \( U_i \cap U_k \),

\[
f_j - f_k = \sum a \eta_a \cdot h_{jk} = h_{jk}.
\]

This then completes the proof.

2. Theorem (Theorem of Cousin). If \( \{ h_{ij} \} \) are Cousin data for an open covering \( \{ U_i \} \) of the polydisc \( \Delta \), then there are functions \( h_1 \) defined and holomorphic in the open sets \( U_i \) and such that

\[
h_i - h_j = h_{ij} \quad \text{in } U_i \cap U_j.
\]

Proof: It follows from lemma 1 that there are \( C^\infty \) functions \( f_i \) defined in the open sets \( U_i \) and satisfying the equation (5) in the pairwise intersections \( U_i \cap U_j \). For each set \( U_i \) consider the differential form \( \varphi_i = \delta f_i \in \mathcal{E}^{0,1}(U_i) \). In each intersection \( U_i \cap U_j \) note that \( \varphi_i = \delta (f_i + h_{ij}) = \varphi_j \), since the \( h_{ij} \) are holomorphic functions; hence there is a global differential form \( \varphi \in \mathcal{E}^{0,1}(\Delta) \) such that \( \varphi \mid U_i = \delta f_i \) for each set \( U_i \). Since clearly \( \delta \varphi = 0 \), it follows from theorem D5 that there is a \( C^\infty \) function \( g \in \mathcal{E}^{0,0}(\Delta) \) such that \( \varphi = \delta g \). Now define \( h_i = f_i - g \) in \( U_i \). The function \( h_i \) is holomorphic in
$U_i$, since $\bar{\partial}h_i = \bar{\partial}f_i - \psi = 0$; and in $U_i \cap U_j$ we have $h_i - h_j = f_i - f_j = h_{ij}$, as desired. This completes the proof.

**Note:** The reader will easily note that Cousin data are solvable in the form (5) for a domain $D$ if and only if $\mathcal{H}^0,1(D) = 0$; we shall return to this question later.

Let us consider in some detail one situation from among the many in which Cousin data arise naturally, and to which the Cousin theorem can be applied. Suppose that $M$ is an $(n-1)$-dimensional analytic submanifold of an open polydisc $\Delta$ in the space of $n$ complex variables. Recalling theorem B9, the set $M$ has the property that to every point $p \in M$ there correspond an open neighborhood $U$ of $p$ in $\Delta$ and a local coordinate system $(w_1, \ldots, w_n)$ in $U$ such that

$$M \cap U = \{z \in U \mid w_1(z) = 0\}. \quad (6)$$

It is useful to note in passing that if $f$ is a function holomorphic in $U$ such that $f \mid M \equiv 0$, then $f(z) = g(z) \cdot w_1(z)$ for some function $g(z)$ holomorphic in $U$. It is of course sufficient to prove this assertion in a neighborhood of an arbitrary point of $M$, since $f(z)/w_1(z)$ is clearly holomorphic in $U - M$; and for simplicity let us suppose that the point is the origin in the local coordinates $w_1, \ldots, w_n$. Considering the power series expansion $f(w_1, \ldots, w_n)$ of the function $f$ in these coordinates, it follows from the hypothesis $f(0, w_2, \ldots, w_n) \equiv 0$ that all monomials in that power series must contain $w_1$, and hence that we can write $f = g \cdot w_1$, as desired. In general, we shall say that a function $f(z)$ holomorphic in an open set $U \subset \mathbb{C}^n$ is a **defining function** for the submanifold $M$ in $U$ if $f \mid (M \cap U) = 0$ and if the quotient $g(z)/f(z)$ is holomorphic in $U$ whenever $g(z)$ is holomorphic in $U$ and $g \mid (M \cap U) = 0$; if $f$ and $g$ are both defining functions for $M$ in $U$, it is then clear that the quotient $g(z)/f(z)$ is holomorphic and nonvanishing in $U$.

Let $\{U_i\}$ be a collection of open subsets of $\Delta$ which cover the entire subset $M$, and in each of which there is a defining equation $f_i(z)$ for the submanifold $M$. Furthermore, let us adjoin enough additional open sets to secure an open covering of the entire polydisc $\Delta$; these new sets will also be written $U_i$, and we shall put $f_i(z) \equiv 1$ in $U_i$ whenever $U_i$ does not meet $M$. The submanifold $M$ has therefore the **local defining functions** $\{f_i\}$ in $\Delta$, in the sense that

$$M \cap U_i = \{z \in U_i \mid f_i(z) = 0\}; \quad (7)$$

and the question arises, whether the submanifold $M$ actually has **global defining functions** in the full set $\Delta$.

In each intersection $U_i \cap U_j$ the functions $f_i(z)$ and $f_j(z)$ are both defining functions for $M$, hence the quotients $f_{ij} = f_i/f_j$ and $f_{ji} = f_j/f_i$ are holomorphic
and nonvanishing functions in \( U_i \cap U_j \); moreover, these functions satisfy the conditions:

\[
(8) \quad f_{ij} \cdot f_{ji} = 1 \quad \text{in} \quad U_i \cap U_j;
\]

\[
(9) \quad f_{ij} \cdot f_{jk} \cdot f_{ki} = 1 \quad \text{in} \quad U_i \cap U_j \cap U_k.
\]

The functions \( \{ f_{ij} \} \) consequently form a multiplicative analogue of the Cousin data for the covering \( \{ U_i \} \) of the polydisc \( \Delta \). If there were a corresponding analogue of the Cousin theorem, it would assert the existence of holomorphic, nowhere vanishing functions \( a_i \) in each set \( U_i \) such that \( a_i = f_{ij}a_j \) in \( U_i \cap U_j \); the function \( f = f_i/a_i \) would then be globally defined in \( \Delta \), and would be the desired function. The natural suggestion here is to take the logarithms of the Cousin data, apply the Cousin theorem, and then take the exponential of the resulting solutions of the additive problem. The obvious difficulty arises in this approach since the logarithm is a multivalent function. We can, of course, choose single-valued branches of the functions \( h_{ij} = \log f_{ij} \), provided that the intersections \( U_i \cap U_j \) are simply connected, and we may assume that \( h_{ij} = -h_{ji} \); however, the functions \( \{ h_{ij} \} \) may not be admissible Cousin data, since from (9) we can only conclude that

\[
h_{ij} + h_{jk} + h_{ki} = 2\pi i v_{ijk}
\]

for some integer \( v_{ijk} \). Thus the question arises, whether it is possible to choose the branches of \( h_{ij} = \log f_{ij} \) in such a manner that all the integers \( v_{ijk} = 0 \). This is a purely topological question, and it is indeed always possible (after perhaps refining the given covering), since the polydisc is of a rather trivial topological nature. A proof of this assertion would be too great a digression at this point, and will be given in a later section; for the present, we shall finesse this difficulty by assuming that the manifold \( M \) has a global continuous defining function. That is to say, we shall assume that there is a continuous function \( g \) defined in \( \Delta \) such that in each set \( U_i \) the quotient \( f_i/g \) is continuous and nowhere vanishing; and for emphasis, we assert once again that this hypothesis is unnecessary.

3. Corollary. If \( M \) is an \( (n-1) \)-dimensional analytic submanifold of an open polydisc \( \Delta \subset \mathbb{C}^n \), and if \( M \) can be defined by a global, continuous function in \( \Delta \), then \( M \) can be defined by a global, holomorphic function in \( \Delta \).

\textbf{Proof:} We shall continue with the notation introduced in the discussion immediately preceding. By taking a refinement of the given covering, if necessary, we may assume that all the open subsets \( U_i \) and their intersections \( U_i \cap U_j \) are simply connected. Now the functions \( f_i/g \) are continuous and nowhere vanishing in \( U_i \), hence have well-defined logarithms; we choose any branch, and write it as \( \log (f_i/g) \). In each intersection \( U_i \cap U_j \) we choose
that branch of the logarithm $\log f_{ij}$ defined by
\[ \log f_{ij} = \log (f_i/g) - \log (f_j/g); \]
clearly the functions $\{\log f_{ij}\}$ are admissible Cousin data for the covering $\{U_i\}$. By Cousin’s theorem, there will exist holomorphic functions $h_i$ in $U_i$ such that $h_i - h_j = \log f_{ij}$ in $U_i \cap U_j$; the functions $a_i = \exp h_i$ are then holomorphic and nowhere vanishing in $U_i$, and $a_i = f_i a_j$ in $U_i \cap U_j$. Since $f_i/a_i = f_j/a_j$ in $U_i \cap U_j$, there is a function $f$ holomorphic in all of $\Delta$ such that $f | U_i = f_i/a_i$; and this is the desired function.

Suppose now that $g$ is a continuous function defined on the submanifold $M \subset \Delta$. Again, by theorem B9, recall that in a neighborhood $U$ of each point $p \in M$ there is a local coordinate system $(w_2, \ldots, w_n)$ such that $M \cap U$ is the hyperplane $w_1 = 0$; so that locally $g$ is defined and continuous as a function of $(w_2, \ldots, w_n)$ on the hyperplane $w_1 = 0$. We say that $g$ is holomorphic on $M$ if, locally, $g$ is a holomorphic function of the variables $(w_2, \ldots, w_n)$. We can of course extend $g$ to a function holomorphic on the entire neighborhood $U$, by defining $g$ to be independent of the variable $w_1$. Therefore, equivalently, $g$ is holomorphic on $M$ when there is a holomorphic function $h_U$ in some open neighborhood $U$ of any point $p \in M$ such that $h_U | M = g | (U \cap M)$. The obvious question at this point is whether there is a function $h$ holomorphic in all of $\Delta$ such that $h | M = g$.

4. Corollary. Let $M$ be an $(n - 1)$-dimensional analytic submanifold of an open polydisc $\Delta \subset \mathbb{C}^n$, and $f$ be a global defining function for $M$ in $\Delta$. Then for any function $g$ holomorphic on $M$ there exists a function $h$ holomorphic in $\Delta$ such that $h | M = g$.

Proof: Let $\{U_i\}$ be an open covering of $\Delta$, with the property that there is a function $f_i$ holomorphic in each $U_i$ such that
\[ f_i | (M \cap U_i) = g | (M \cap U_i); \]
if $M \cap U_i = \emptyset$, the function $f_i$ can of course be quite arbitrary. Note that in each intersection $U_i \cap U_j$ we have $(f_i - f_j) | (M \cap U_i \cap U_j) = 0$; hence, since $f$ is a defining function for $M$, we can write
\[ f_i - f_j = ff_{ij} \quad \text{in } U_i \cap U_j, \]
where $f_{ij}$ is holomorphic in $U_i \cap U_j$. It is clear that the functions $\{f_{ij}\}$ are admissible Cousin data for the covering $\{U_i\}$ of $\Delta$; therefore, by Cousin’s theorem, there are functions $h_i$ holomorphic in $U_i$ such that
\[ h_i - h_j = f_{ij} \quad \text{in } U_i \cap U_j. \]
Now since $f_i - fh_i = f_j - fh_i$ in $U_i \cap U_j$, there is a function $h$ holomorphic
throughout $\Delta$ such that $h \mid U_i = f_i - fh_i$; and since
\[
h \mid (M \cap U_i) = f_i \mid (M \cap U_i) = g \mid (M \cap U_i),
\]
this function $h$ is the desired solution.

With these samples in mind, it is tempting to try to generalize this approach—for instance, to consider analytic submanifolds of lower dimensions in $\Delta$. The reader who would care to attempt a straightforward extension would probably soon encounter considerable difficulties. We shall return to this question later, after developing enough additional tools to make this and many other generalizations possible, and indeed rather simple.

**F. Polynomial Approximations**

A familiar result from the classical theory of functions of one complex variable is Runge’s theorem: any function holomorphic in a simply connected open subset $D \subset \mathbb{C}$ is the limit, in the topology of $\mathcal{O}_D$, of a sequence of polynomials; that is, the function is the uniform limit on compact subsets of $D$ of a sequence of polynomials. In the case of several complex variables the situation is rather more complicated, and there is no purely topological characterization of the domains on which such an approximation theorem holds; in fact, there is not even an intrinsic analytic description, as an example below shows.

Clearly, we can approximate a holomorphic function in a polydisc by polynomials; indeed, we have already observed that any function holomorphic in an open polydisc $\Delta$ in $\mathbb{C}^n$ is the limit, in the topology of $\mathcal{O}_\Delta$, of the sequence of polynomials consisting of the partial sums of the power series expansion of the function at the center of the polydisc. The theorem of polynomial approximation is indeed true on any simply connected polydomain. The proof of this fact is the farthest we can go using one-variable methods.

**1. Lemma (Runge’s Theorem).** Let $K_0$ be a simply connected compact subset of $\mathbb{C}$, and $K_1$ be a compact subset of $\mathbb{C}^n$. Let $P$ be the class of polynomials of the form $\sum_{\lambda=0}^{k} f_\lambda w^\lambda$, where $f_\lambda$ is holomorphic in a neighborhood of $K_1$. Then any function holomorphic in a neighborhood of $K_0 \times K_1$ is uniformly approximable by functions in $P$.

**Proof:** Let $f$ be holomorphic in a neighborhood of $K_0 \times K_1$. Then there are neighborhoods $U_0$ of $K_0$ and $U_1$ of $K_1$ such that $f \in \mathcal{O}_{U_0 \times U_1}$. Using the Cauchy integral formula, for $z \in K_0$, $w \in K_1$,

\[
f(z, w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta, w)}{\zeta - z} d\zeta,
\]
where $\Gamma$ is a simple closed curve in $U_0$ whose interior contains $K_0$. Since the function $\zeta - z$ is bounded from below for $\xi \in \Gamma$, $z \in K$, it is easy to verify that the Riemann sums which approximate the integral on the right of (1) do so uniformly on $K_0 \times K_1$. Thus $f$ is approximable uniformly on $K_0 \times K_1$ by functions of the type $\sum_{i=1}^{p} \frac{f_i(w)}{\zeta_i - z}$. We need only prove that the functions $1/(\zeta - z)$ for $\zeta \notin K_0$ are approximable on $K$ by polynomials.

Let $\zeta_0 \notin K_0$, and let $Y$ be a curve in $\mathbb{C} - K_0$ parametrized by $\xi : [0, \infty) \rightarrow \mathbb{C}$, so that $\xi(0) = \zeta_0$ and $|\xi(t)| \rightarrow \infty$. Clearly the set

$$T = \{t \in [0, \infty) \mid (\xi(t) - z)^{-1} \text{ is approximable by polynomials}\}$$

is nonempty; for if $|\xi(t_0)| > \sup \{|z| ; z \in K_0\}$, then $(\xi(t_0) - z)^{-1}$ is holomorphic in a disc containing $K_0$, and represented by a power series. Finally, $T$ is open. Let $t_0 \in T$, and let $t$ be such that

$$|\xi(t_0) - \xi(t)| < \sup \{|\xi(t_0) - z| ; z \in K_0\}.$$ 

For simplicity, let $\zeta_0 = \xi(t_0)$, $\zeta = \xi(t)$. We shall write $(\zeta - z)^{-1}$ as a power series in $(\zeta_0 - z)^{-1}$ which converges uniformly on $K_0$, namely

$$(\zeta - z)^{-1} = (\zeta_0 - z)^{-1} \left(\frac{1}{1 + \frac{\zeta - \zeta_0}{\zeta_0 - z}}\right) = \sum_{k=0}^{\infty} \frac{(\zeta - \zeta_0)^k}{(\zeta_0 - z)^{k+1}}.$$ 

Thus, since $t_0 \in T$, $t \in T$ also. Since $T$ is open and closed and nonempty, $T = [0, \infty)$, and the theorem is proved.

2. Theorem. Let $D = D_1 \times \cdots \times D_n$ be a simply connected polydomain in $\mathbb{C}^n$. Then the polynomials are dense in $\mathcal{O}_D$.

Proof: Let $f \in \mathcal{O}_D$. Let $K$ be a compact subset of $D$, and $\varepsilon > 0$; we shall show that there is a polynomial $p$ such that $\|p - f\|_K < \varepsilon$. By enlarging $K$, we may assume that $K = K_1 \times \cdots \times K_n$, $K_i \subset D_i$, and that each $K_i$ is simply connected. The proof is by induction on $n$. The case $n = 1$ is contained in the preceding lemma. In the case of general $n$, by the preceding lemma, we can find a function of the type $\sum_{i=1}^{k} f_i(w)z^i = p_0(w, z)$ such that $f_i$ is holomorphic in a neighborhood of $K_1 \times \cdots \times K_n = K'$ and $\|f - p_0\|_K < \varepsilon/2$. Let $R = \sup \{|z| ; z \in K_i\}$. By the induction assumption, we can find polynomials $p_i(w)$ such that $\|f_i - p_i\|_{K'} < \varepsilon/(2R/k)$. Then
\[ p(z, w) = \sum_{i=0}^{k} p_i(w)z^i \] is a polynomial on \( \mathbb{C}^n \), and
\[
\| f - p \|_K \leq \| f - p_0 \|_K + \sum_{i=0}^{k} \| f_i - p_i \|_K \cdot \| z^i \|_K
\]
\[
\leq \frac{\varepsilon}{2} + \sum_{i=0}^{k} \frac{\varepsilon}{2R^i!} R_i = \varepsilon.
\]

**Example.** (Wermer [246]). We shall now construct a domain \( D \) in \( \mathbb{C}^3 \) which is analytically equivalent to a tricylinder but such that polynomial approximation does not take place on \( D \). Let \( F \) be the holomorphic mapping of \( \mathbb{C}^3 \) into \( \mathbb{C}^3 \) defined by
\[
F(z, w, t) = (z, zw + i, zw^2 - w + 2tw).
\]
The Jacobian of this mapping is \( J(z, w, t) = 1 - 2t \). Now, for sufficiently small \( b > 0 \), we shall show that \( F \) is one--one on \( \Delta_b = \Delta((0, 0, 0), (1 + b, 1 + b, b)). \) If not, there are points \((z_n, w_n, t_n), (z'_n, w'_n, t'_n) \in \Delta_b \) such that \( F(z_n, w_n, t_n) = F(z'_n, w'_n, t'_n) \). We can choose subsequences which converge to \((z_0, w_0, 0), (z'_0, w'_0, 0) \) respectively. Then \( F(z_0, w_0, 0) = F(z'_0, w'_0, 0) \), so \( z_0 = z'_0, z_0w_0 = z'_0w'_0 \) and \( w_0(z_0w_0 - 1) = w'_0(z'_0w'_0 - 1) \). It follows that \( z_0 = z'_0, w_0 = w'_0 \) Thus \( F \) is not one--one in any neighborhood of \((z_0, w_0, 0), \) contradicting the fact that \( J(z_0, w_0, 0) = 1 \).

Choose \( b < 1 \), so that \( F \) maps \( \Delta_b \) biholomorphically onto a domain \( D \) in \( \mathbb{C}^3 \). We shall show that polynomial approximation cannot take place on \( D \). Let \( z_1, z_2, z_3 \) be the coordinates of the range of \( F \). Let \( \Pi = \{(z_1, 1, 0), \Gamma = \{(z_1, 1, 0); |z_1| = 1 \}. \) Now \( F(\zeta, \zeta^{-1}, 0) = (\zeta, 1, 0), \) so \( \Gamma \) is contained in \( D \). Let \( p_0 = (z_1, 1, 0) \) with \(|z_1| < 1 \), be a point in \( D \cap \Pi \). If \( f \) is a polynomial in \( z_1, z_2, z_3 \), then by the maximum modulus principle on \( \Pi \),
\[
|f(p_0)| < \| f \|_\Pi.
\]
This must also be true for all functions approximable by polynomials. But \( z, w \) are holomorphic functions on \( D \), they are nonconstant on \( \Pi \cap D \), and \( z = w^{-1} \) there. So (2) cannot hold for both \( z, w \), hence one of these functions (namely \( w \), since \( z = z_1 \)) is not approximable by polynomials.

If we analyze the contradiction in the above example, we see that it lies essentially in the fact that there are a point \( p_0 \notin D \) and a compact set \( \Gamma \subset D \) such that (2) holds for all polynomials (e.g., choose \( p_0(0, 1, 0) \)). We shall now show that if this does not take place, polynomial approximation does hold.

3. **Definition.** A domain \( D \subset \mathbb{C}^n \) is polynomially convex if, for every compact subset \( K \) of \( D \), the set \( \hat{K} = \{z \in \mathbb{C}^n; |f(z)| \leq \| f \|_K \text{ for all polynomials } f \} \)
is contained in \( D \). \( \hat{K} \) is called the polynomially convex hull of \( K \). \( K \) is polynomially convex if \( K = \hat{K} \).

The most obvious examples of polynomially convex domains are the polynomial polyhedra: those domains defined by polynomial inequalities.

4. Definition. Let \( p_1, \ldots, p_r \) be polynomials in the \( n \) variables \( z_1, \ldots, z_n \). The polynomial polyhedron of polyradius \( \delta > 0 \) defined by these polynomials is the open subset

\[
P^n(p_1, \ldots, p_r; \delta) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \left| \begin{array}{c}
|z_i| < \delta, 1 \leq i \leq n \\
|p_j(z)| < \delta, 1 \leq j \leq r
\end{array} \right. \right\}.
\]

Before proving a polynomial approximation theorem on such sets, we must establish a few further analytical properties. Suppose that

\[
P^n(p_1, \ldots, p_r; \delta) \subset \mathbb{C}^n
\]

is a fixed, given polynomial polyhedron; and consider the family of related polynomial polyhedra

\[
P^{n+k}(p_{k+1}, \ldots, p_r; \delta)
\]

for \( 0 \leq k \leq r \). When \( k = 0 \), this set reduces to the given polyhedron \( P^n(p_1, \ldots, p_r; \delta) \); while when \( k = r \), this reduces to the open polydisc \( P^{n+r}(\cdot; \delta) = \Delta(0; \delta) \subset \mathbb{C}^{n+r} \). For any index \( k \) there is a mapping

\[
\mu_k: P^n(p_1, \ldots, p_r; \delta) \rightarrow P^{n+k}(p_{k+1}, \ldots, p_r; \delta)
\]

defined by

\[
\mu_k(z_1, \ldots, z_n) = (z_1, \ldots, z_n, p_1(z_1, \ldots, z_n), \ldots, p_k(z_1, \ldots, z_n)).
\]

The image of the mapping \( \mu_k \) is the analytic submanifold

\[
M_k \subset P^{n+k}(p_{k+1}, \ldots, p_r; \delta)
\]

defined by

\[
M_k = \{(z_1, \ldots, z_{n+k}) \in P^{n+k}(p_{k+1}, \ldots, p_r; \delta) \mid z_j = p_j(z_1, \ldots, z_n), n + 1 \leq j \leq n + k\}.
\]

It is obvious that \( \mu_k \) is a one-to-one and bicontinuous mapping from \( P^n(p_1, \ldots, p_r; \delta) \) onto the subset \( M_k \); and that \( M_k \) is a submanifold.

5. Theorem. Let \( \varphi \) be a \( C^\infty \) differential form of bidegree \( (p, q) \) defined on the polynomial polyhedron \( P^n(p_1, \ldots, p_r; \delta) \) and such that \( \tilde{\partial} \varphi = 0 \) and \( q > 0 \).
Then, for any \( \varepsilon < \delta \), there is a \( C^\infty \) differential form \( \psi \) of bidegree \((p, q - 1)\) defined on the polynomial polyhedron \( P^n(p_1, \ldots, p_r; \varepsilon) \), such that \( \varphi = \partial \psi \).

**Proof:** The proof of this assertion will be by induction on the number \( r \) of polynomials defining the polyhedron. When \( r = 0 \), the polyhedron reduces to an open polydisc \( \mathbb{C}^n \), and the desired result then follows from theorem D5. Next, for the induction step, assume that the result holds for all polyhedra defined by at most \( r - 1 \) polynomials; and consider the polyhedron (3). Let \( \sigma, \tau \) be any real numbers such that \( \varepsilon < \sigma < \tau < \delta \); thus

\[
P^n(p_1, \ldots, p_r; \varepsilon) \subset P^n(p_1, \ldots, p_r; \sigma) \subset P^n(p_1, \ldots, p_r; \tau) \subset P^n(p_1, \ldots, p_r; \delta),
\]

and there is therefore a \( C^\infty \) function \( h \) in \( \mathbb{C}^n \) such that \( h \equiv 1 \) in an open neighborhood of \( P^n(p_1, \ldots, p_r; \sigma) \), and that \( h \equiv 0 \) in the complement of \( P^n(p_1, \ldots, p_r; \tau) \). Let \( \tilde{\varphi} \) be the \( C^\infty \) differential form in \( \mathbb{C}^n \) defined by

\[
\tilde{\varphi}(z) = \begin{cases} h(z)\varphi(z) & \text{for } z \in P^n(p_1, \ldots, p_r; \delta), \\ 0 & \text{for } z \notin P^n(p_1, \ldots, p_r; \delta); \end{cases}
\]

and let \( \Phi \) be the \( C^\infty \) differential form in \( \mathbb{C}^{n+1} \) defined by

\[
\Phi(z_1, \ldots, z_{n+1}) = \tilde{\varphi}(z_1, \ldots, z_n).
\]

Recall that \( h \equiv 1 \) in an open neighborhood of \( P^n(p_1, \ldots, p_r; \sigma) \); and that the submanifold \( M_1 \subset P^{n+1}(p_2, \ldots, p_r; \sigma) \) defined by the equation \( z_{n+1} = p_1(z_1, \ldots, z_n) \) projects onto the set \( P^n(p_1, \ldots, p_r; \sigma) \) under the natural projection \( (z_1, \ldots, z_{n+1}) \mapsto (z_1, \ldots, z_n) \). Consequently, in a full open neighborhood of the submanifold \( M_1 \) in the set \( P^{n+1}(p_2, \ldots, p_r; \sigma) \), we have that

\[
\partial \Phi = \partial \tilde{\varphi} = \partial \varphi = 0;
\]

and therefore there is a \( C^\infty \) differential form \( \Theta \) in \( P^{n+1}(p_2, \ldots, p_r; \sigma) \) such that

\[
\partial \Phi = (z_{n+1} - p_1(z_1, \ldots, z_n))\Theta.
\]

Since \( \partial \Theta = 0 \) in the dense open subset \( P^{n+1}(p_2, \ldots, p_r; \sigma) - M_1 \), it follows that \( \partial \Theta = 0 \) throughout \( P^{n+1}(p_2, \ldots, p_r; \sigma) \). The induction hypothesis applies, so that for some real number \( \sigma' \) with \( \varepsilon < \sigma' < \sigma \), there is a \( C^\infty \) differential form \( \Omega \) in \( P^{n+1}(p_2, \ldots, p_r; \sigma') \) such that \( \partial \Omega = \Theta \). Now we have

\[
\partial[\Phi - (z_{n+1} - p_1(z_1, \ldots, z_n))\Theta] = 0 \text{ throughout } P^{n+1}(p_2, \ldots, p_r; \sigma');
\]

so by yet another application of the induction hypothesis, there exists a \( C^\infty \) differential form \( \Psi \) in \( P^{n+1}(p_2, \ldots, p_r; \varepsilon) \) such that

\[
\Phi - (z_{n+1} - p_1(z_1, \ldots, z_n))\Omega = \partial \Psi.
\]
Putting $\psi(z_1, \ldots, z_n) = \Psi(\pi_1(z_1, \ldots, z_n))$ and recalling that
$$\varphi(z_1, \ldots, z_n) = \Phi(\pi_1(z_1, \ldots, z_n))$$
in $P^n(p_1, \ldots, p_r; \varepsilon)$, it follows immediately that $\varphi = \delta \psi$ in $P^n(p_1, \ldots, p_r; \varepsilon)$, concluding the proof of the theorem.

6. Theorem. Let $\{h_{ij}\}$ be Cousin data for an open covering $\{U_i\}$ of the polynomial polyhedron $P^n(p_1, \ldots, p_r; \delta)$. Then for any positive number $\varepsilon < \delta$, there are functions $h_i$ defined and holomorphic in the open sets
$$U_i \cap P^n(p_1, \ldots, p_r; \varepsilon)$$
such that
$$h_i - h_j = h_{ij} \quad \text{on } U_i \cap U_j \cap P^n(p_1, \ldots, p_r; \varepsilon).$$

Proof: The proof is the obvious modification of the proof of the theorem of Cousin (theorem E2). It follows as in lemma E1 that there are $C^\infty$ functions $f_i$ defined in the open sets $U_i$ such that $f_i - f_j = h_{ij}$ in $U_i \cap U_j$; and hence there is a $C^\infty$ differential form $\varphi$ in the polyhedron such that $\varphi = \delta f_i$ in $U_i$. Since $\delta \varphi = 0$, it further follows from theorem 2 above that there is a $C^\infty$ function $g$ in the polyhedron $P^n(p_1, \ldots, p_r; \varepsilon)$ such that $\varphi = \delta g$. The functions $h_i = f_i - g$ are then the desired functions, to complete the proof.

7. Lemma. Let $f$ be a function holomorphic in the polynomial polyhedron $P^n(p_1, \ldots, p_r; \delta)$. For any positive number $\varepsilon < \delta$ and any index $k = 0, 1, \ldots, r$, there is a holomorphic function $h$ in the polyhedron $P^{n+k}(p_{k+1}, \ldots, p_r; \varepsilon)$ such that $h(\mu_k(z)) = f(z)$.

Proof: We shall prove this assertion by induction on the index $k$, noting that the case $k = 0$ is trivial. Actually it is sufficient just to prove the lemma for the case $k = 1$, since each induction step is precisely of this form for some polyhedron. Note that we can consider the function $f$ as a function on the submanifold $M_1 \subset P^{n+1}(p_0, \ldots, p_r; \delta)$, using the identification mapping $\mu_1$; and that the desired result can be reformulated as the assertion that there exists a function $h$ holomorphic in $P^{n+1}(p_0, \ldots, p_r; \varepsilon)$ such that $h|_{M_1} = f$. Since the submanifold $M_1$ has the global defining function $z_{n+1} - p_1(z_1, \ldots, z_n)$, this is just the analogue of corollary E4, but for polynomial polyhedra rather than for polydiscs. The proof is precisely the same as the proof of corollary E4, except that theorem 6 is used in place of Cousin’s theorem; hence no further details need be given here.

8. Theorem (Polynomial Approximation Theorem for Polynomial Polyhedra). Any function $f$ holomorphic in a polynomial polyhedron $P \subset \mathbb{C}^n$ is the limit, in the topology of $C_P$, of a sequence of polynomials.
Proof: Suppose that \( f \) is holomorphic in the polynomial polyhedron \( P = P^n(p_1, \ldots, p_r; \delta) \subset \mathbb{C}^n \); we wish to show that \( f \) is the uniform limit on compact subsets of \( P \) of a sequence of polynomials. Consider the sequence of subpolyhedra
\[
P_v = P^n(p_1, \ldots, p_r; \delta(1 - 2^{-v})) \subset P,
\]
for \( v = 1, 2, \ldots \); this is an increasing sequence of relatively compact subsets of \( P \), such that \( \bigcup_v P_v = P \). Applying lemma 7 to the function \( f \) in the polyhedron \( P \), for the case \( k = r \), it follows that there exist functions \( h_v \) holomorphic in the polydiscs \( P^{n+r}(\delta; (1 - 2^{-r})) \subset \mathbb{C}^{n+r} \), such that \( h_v(\mu_r(z)) = f(z) \) for all \( z \in P_v \). Now each of the functions \( h_v(z_1, \ldots, z_{n+r}) \) is the uniform limit on compact subsets of the polydisc \( P^{n+r}(\delta; (1 - 2^{-r})) \) of a sequence of partial sums of its power series expansion, as we have noted earlier; hence we can find polynomials \( H_v(z_1, \ldots, z_{n+r}) \) such that
\[
|H_v(z_1, \ldots, z_{n+r}) - h_v(z_1, \ldots, z_{n+r})| < 2^{-r} \quad \text{for } z \in P^{n+r}(\delta; (1 - 2^{-r-1})).
\]
Now let \( F_v(z) = H_v(\mu_r(z)) \), observing that \( F_v(z) \) is a polynomial since both \( H_v \) and \( \mu_r \) are given by polynomials. From the above observations it follows that
\[
|F_v(z) - f(z)| = |H_v(\mu_r(z)) - h_v(\mu_r(z))| < 2^{-r}
\]
for all points \( z \in P_{v+1} \), which suffices to prove the theorem.

9. **Corollary.** Let \( D \) be a polynomially convex domain. Any function \( f \in \mathcal{O}_D \) is approximable, uniformly on compact subsets of \( D \), by polynomials.

Proof: Let \( K \) be a compact subset of \( D \), and \( \varepsilon > 0 \) be given. We may replace \( K \) by \( \hat{K} \), thus assuming that \( K \) is polynomially convex. Let \( R > 0 \) be chosen so that \( K \subset \Delta(0; R) \). Let \( U \) be a neighborhood of \( K \) so that \( \bar{U} \) is compact, and \( \bar{U} \subset D \). For each \( p \in \partial U \), there is a polynomial \( P \) such that \( |P(p)| > \|P\|_K \). By multiplying by an appropriate constant, we may assume that
\[
|P(p)| > R > \|P\|_K.
\]
Let \( V_p \) be the open set
\[
\{z \in \mathbb{C}^n; |P(z)| > R\}.
\]
Choose finitely many such sets \( V_p \) which cover \( \partial U \), and let \( P_1, \ldots, P_r \) be the associated polynomials. Then the polynomial polyhedron \( \Delta = P(P_1, \ldots, P_r; R) \) has the following properties:

(i) \( \Delta \subset K \),

(ii) \( \Delta \cap \partial U = \emptyset \).

Let \( f' \in \mathcal{O}_\Delta \), \( f' = f \) in \( \Delta \cap U \), \( f' = 0 \) in \( \Delta - U \). By theorem 8, there is a polynomial \( p \) such that \( \|f' - p\|_K = \|f - p\|_K < \varepsilon \).