CHAPTER 3

Monoidal Functors

The main goal of this chapter is to study appropriate notions of functors between monoidal categories.

Complementing the classical notions of lax and colax monoidal functors, we provide a definition of bilax monoidal functors between braided monoidal categories (Section 3.1). The Fock functors which occupy us throughout Part III of this monograph are all examples of bilax monoidal functors. Another very important example, this one of a classical nature, is the object of Chapter 5. A summary of these examples is given in Section 3.2. This section also discusses some other interesting examples.

An important property of bilax monoidal functors is that they preserve bimonoids. More generally, the composite of two bilax monoidal functors is again bilax monoidal. These properties are discussed in Sections 3.3 and 3.4. Additional properties of bilax monoidal functors are studied in Section 3.5; these involve a normalization condition with interesting consequences. Monoidal functors admit a strong version, in which the structure transformations are required to be invertible. These are studied in Section 3.6. Section 3.7 deals with Hopf lax functors. These are bilax monoidal functors with additional properties that ensure that they preserve Hopf monoids. We know that a bimonoid can be viewed as a monoid in a category of comonoids and vice versa. A similar interpretation for bilax monoidal functors is given in Section 3.8. We study adjunctions in the context of monoidal categories in Section 3.9. Section 3.10 discusses a construction which, in a certain context, allows us to dualize monoidal categories and functors.

In an abelian category, any morphism admits a monic-epi factorization. This idea can be expanded in various ways in the context of abelian monoidal categories. In particular, a morphism between (co, bi) lax monoidal functors can be similarly factorized. This gives rise to a new monoidal functor, the image of the given morphism. This important construction is the object of Section 3.11.

3.1. Bilax monoidal functors

Two kinds of morphisms one can consider between monoidal categories are the lax and colax monoidal functors introduced by Bénabou [36]. We follow the terminology of Kelly and Street [199, pp. 83–84]; see also Leinster [226, Definition 1.2.10] and Yetter [379, Definition 3.11]. We recall these notions below. Moreover, if the categories are braided, then one can define the notion of bilax monoidal functor, which appropriately combines the notions of lax and colax monoidal functors. This concept is very natural; however, it seems hard to find a reference where it is discussed. It is of central importance to our work in the rest of the monograph. We provide a definition in this section. The rest of the chapter is devoted to the study of basic properties of lax, colax and bilax monoidal functors.
Lax, colax and bilax monoidal functors may be regarded as analogues of monoids, comonoids and bimonoids, respectively. We make this analogy precise in Section 3.4.

**3.1.1. Lax, colax, and bilax monoidal functors.** Let \((\mathcal{C}, \bullet)\) and \((\mathcal{D}, \bullet)\) be two monoidal categories and \(\mathcal{F}\) be a functor from \(\mathcal{C}\) to \(\mathcal{D}\). We denote the unit object in both categories by \(I\) and write \(\mathcal{M}\) for the tensor product functors. Let

\[
\mathcal{F}^2 := \mathcal{M} \circ (\mathcal{F} \times \mathcal{F}) \quad \text{and} \quad \mathcal{F}_2 := \mathcal{F} \circ \mathcal{M};
\]

they are functors from \(\mathcal{C} \times \mathcal{C}\) to \(\mathcal{D}\). Let \(I\) be the one-arrow category and let

\[
\mathcal{F}^0 : I \to \mathcal{D} \quad \text{and} \quad \mathcal{F}_0 : I \to \mathcal{D}
\]

be the functors that send the unique object of \(I\) to \(I\) and \(\mathcal{F}(I)\) respectively.

**Definition 3.1.** We say that a functor \(\mathcal{F} : \mathcal{C} \to \mathcal{D}\) is **lax monoidal** if there is a natural transformation

\[
\varphi_{A,B} : \mathcal{F}(A) \bullet \mathcal{F}(B) \to \mathcal{F}(A \bullet B)
\]

from the functor \(\mathcal{F}^2\) to the functor \(\mathcal{F}_2\) and a map

\[
\varphi_0 : I \to \mathcal{F}(I)
\]

in \(\mathcal{D}\) such that the conditions below are satisfied. Observe that one may view \(\varphi_0\) as a natural transformation between \(\mathcal{F}^0\) and \(\mathcal{F}_0\).

**Associativity.** The transformation \(\varphi\) is associative, in the sense that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) & \xrightarrow{\text{id} \bullet \varphi_{B,C}} & \mathcal{F}(A) \bullet \mathcal{F}(B \bullet C) \\
\varphi_{A,B} \bullet \text{id} & \downarrow & \varphi_{A,B \bullet C} \\
\mathcal{F}(A \bullet B) \bullet \mathcal{F}(C) & \xrightarrow{\varphi_{A \bullet B, C}} & \mathcal{F}(A \bullet B \bullet C)
\end{array}
\]

**Unitality.** The transformation \(\varphi\) is left and right unital, in the sense that the following diagrams commute.

\[
\begin{array}{ccc}
\mathcal{F}(I) \bullet \mathcal{F}(A) & \xrightarrow{\mathcal{F}(\lambda_A)} & \mathcal{F}(A) \\
\varphi_{I,A} \bullet \text{id} & \downarrow & \mathcal{F}(\rho_A) \\
\mathcal{F}(I \bullet A) & \xrightarrow{\mathcal{F}(\lambda_A)} & \mathcal{F}(A \bullet I) \\
\end{array}
\]

The above three diagrams are the analogues of the associativity and unit axioms for a monoid.

**Definition 3.2.** We say that a functor \(\mathcal{F} : \mathcal{C} \to \mathcal{D}\) is **colax monoidal** if there is a natural transformation

\[
\psi_{A,B} : \mathcal{F}(A \bullet B) \to \mathcal{F}(A) \bullet \mathcal{F}(B)
\]

and a map

\[
\psi_0 : \mathcal{F}(I) \to I
\]
satisfying axioms dual to those in Definition 3.1. Namely, one replaces \( \varphi \) by \( \psi \) and reverses the arrows with those labels in diagrams (3.5) and (3.6).

**Definition 3.3.** Let \((C, \bullet, \beta)\) and \((D, \bullet, \beta)\) be two braided monoidal categories. We say that a functor \( F : C \rightarrow D \) is *bilax monoidal* if there are natural transformations \( \varphi \) and \( \psi \),

\[
\begin{array}{ccc}
F(A) \bullet F(B) & \xrightarrow{\varphi_{A,B}} & F(A \bullet B), \\
\downarrow & & \downarrow \\
F(A \bullet B \bullet C \bullet D) & \xrightarrow{\psi_{A,B} \bullet \psi_{C,D}} & F(A \bullet B) \bullet F(C \bullet D)
\end{array}
\]

between the functors \( F^2 \) and \( F_2 \) defined in (3.1), and morphisms

\[
\varphi_0 : I \rightarrow F(I) \quad \text{and} \quad \psi_0 : F(I) \rightarrow I
\]
in \( D \) such that \( (F, \varphi) \) is lax, \( (F, \psi) \) is colax and the conditions below are satisfied. Note that \( \varphi_0 \) and \( \psi_0 \) are natural transformations between the functors \( F^0 \) and \( F_0 \) defined in (3.2).

**Braiding.** The following hexagon commutes.

\[
\begin{array}{ccc}
F(A \bullet B) \bullet F(C \bullet D) & & F(A \bullet C \bullet B \bullet D) \\
\downarrow & & \downarrow \\
F(A \bullet B \bullet C \bullet D) & & F(A \bullet C) \bullet F(B \bullet D)
\end{array}
\]

where \( \beta \) denotes the braiding in either category.

**Unitality.** The following diagrams commute.

\[
\begin{array}{ccc}
I \xrightarrow{\varphi_0} F(I) & \xrightarrow{F(\lambda_I)} & F(I \bullet I) \\
\downarrow & & \downarrow \\
I \bullet I & \xrightarrow{\psi_I \cdot \varphi_0} & F(I) \bullet F(I)
\end{array}
\]

\[
\begin{array}{ccc}
I \leftarrow F(I) & \xleftarrow{\psi_0 \cdot \psi_0} & F(I) \bullet F(I) \\
\uparrow & & \uparrow \\
I \bullet I & \xleftarrow{\psi_0} & F(I) \leftarrow F(I \bullet I)
\end{array}
\]

\[
\begin{array}{c}
\varphi_0 \\
\downarrow
\end{array}
\quad
\begin{array}{c}
\psi_0
\end{array}
\]

\[
\begin{array}{c}
F(I)
\end{array}
\quad
\begin{array}{c}
F(I)
\end{array}
\]

(3.13)
The above four diagrams are the analogues of the compatibility axioms for a bimonoid given in Definition 1.10. In (3.12), we may write \( \rho_I \) instead of \( \lambda_I \), in view of (1.3).

**Notation 3.4.** For the sake of brevity we may sometimes omit the word “monoidal” and refer to the above classes of functors simply as lax, colax, or bilax. Suppose that \( \mathcal{F} \) is a lax functor with structure maps \( \varphi_{A,B} \) and \( \varphi_0 \), as in Definition 3.1. In order to denote this lax functor we may use \((\mathcal{F}, \varphi, \varphi_0)\), or \((\mathcal{F}, \varphi)\), or simply \( \mathcal{F} \), if the structure maps are understood. A similar convention applies to colax and bilax functors.

**Definition 3.5.** Let \( \mathcal{F}: \mathcal{C} \to \mathcal{D} \) be a functor, \( \varphi \) and \( \psi \) be transformations as in (3.3) and (3.7), and \( \varphi_0 \) and \( \psi_0 \) maps as in (3.4) and (3.8). We say that \((\mathcal{F}, \varphi)\) is **strong** if it is lax and \( \varphi \) and \( \varphi_0 \) are invertible. We say that \((\mathcal{F}, \psi)\) is **costrong** if it is colax and \( \psi \) and \( \psi_0 \) are invertible. We say that \((\mathcal{F}, \varphi, \psi)\) is **bistrong** if it is bilax and \( \varphi, \psi, \varphi_0 \) and \( \psi_0 \) are all invertible.

Note that \((\mathcal{F}, \varphi)\) is strong if and only if \((\mathcal{F}, \varphi^{-1})\) is costrong. Strong functors are studied in more depth in Section 3.6. In Proposition 3.45 we show that if \((\mathcal{F}, \varphi, \psi)\) is bistrong, then \( \varphi = \psi^{-1} \).

We turn to basic constructions involving monoidal functors.

**Proposition 3.6.** If \((\mathcal{F}, \varphi): \mathcal{C} \to \mathcal{D} \) and \((\mathcal{F}', \varphi'): \mathcal{C}' \to \mathcal{D}' \) are lax (resp. colax), then so is \((\mathcal{F} \times \mathcal{F}', \varphi \times \varphi'): \mathcal{C} \times \mathcal{C}' \to \mathcal{D} \times \mathcal{D}' \).

Further, if \((\mathcal{F}, \varphi, \psi): \mathcal{C} \to \mathcal{D} \) and \((\mathcal{F}', \varphi', \psi'): \mathcal{C}' \to \mathcal{D}' \) are bilax, then so is \((\mathcal{F} \times \mathcal{F}', \varphi \times \varphi', \psi \times \psi'): \mathcal{C} \times \mathcal{C}' \to \mathcal{D} \times \mathcal{D}' \).

The above result is a straightforward consequence of the definitions.

**Proposition 3.7.** Let \((\mathcal{C}^{op}, \bullet, \beta^{op})\) denote the opposite category of \((\mathcal{C}, \bullet, \beta)\). If \((\mathcal{F}, \varphi): \mathcal{C} \to \mathcal{D}\) is lax (resp. colax) monoidal, then \((\mathcal{F}, \varphi): \mathcal{C}^{op} \to \mathcal{D}^{op}\) is colax (resp. lax) monoidal. Further, if \((\mathcal{F}, \varphi, \psi): \mathcal{C} \to \mathcal{D}\) is bilax monoidal, then so is \((\mathcal{F}, \psi, \varphi): \mathcal{C}^{op} \to \mathcal{D}^{op}\).

**Proof.** For the first assertion, observe that reversing the arrows labeled \( \varphi \) in the diagrams for a lax functor yield the diagrams for a colax functor and viceversa. The diagrams for a bilax functor are preserved by switching \( \varphi \) with \( \psi \), and reversing the arrows with those labels. This proves the second assertion.

Thus passing to the opposite categories transforms a lax functor to a colax functor and viceversa, and preserves bilax functors.

**3.1.2. Morphisms between monoidal functors.**

**Definition 3.8.** Let \((\mathcal{C}, \bullet)\) and \((\mathcal{D}, \bullet)\) be two monoidal categories, and \((\mathcal{F}, \varphi)\) and \((\mathcal{G}, \gamma)\) be lax monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \). A morphism from \( \mathcal{F} \) to \( \mathcal{G} \) of lax monoidal functors is a natural transformation \( \theta: \mathcal{F} \Rightarrow \mathcal{G} \) such that both diagrams
Let \((\mathcal{F}, \psi)\) and \((\mathcal{G}, \delta)\) be colax monoidal functors from \(\mathcal{C}\) to \(\mathcal{D}\). A morphism from \(\mathcal{F}\) to \(\mathcal{G}\) of colax monoidal functors is a natural transformation \(\theta: \mathcal{F} \Rightarrow \mathcal{G}\) such that both diagrams below commute.

\[
\begin{array}{ccc}
\mathcal{F}(A \bullet B) & \xrightarrow{\varphi_{A,B}} & \mathcal{F}(A \bullet B) \\
\downarrow {\theta_{A \bullet B}} & & \downarrow {\theta_{A \bullet B}} \\
\mathcal{G}(A \bullet B) & \xrightarrow{\gamma_{A,B}} & \mathcal{G}(A \bullet B)
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{F}(I) & \xrightarrow{\varphi_0} & \mathcal{F}(I) \\
\downarrow {\theta_I} & & \downarrow {\theta_I} \\
\mathcal{G}(I) & \xrightarrow{\gamma_0} & \mathcal{G}(I)
\end{array}
\]

A morphism of (co)strong monoidal functors is a morphism of the underlying (co)lax monoidal functors.

**Definition 3.9.** Let \((\mathcal{C}, \bullet, \beta)\) and \((\mathcal{D}, \bullet, \beta)\) be two braided monoidal categories, and \(\mathcal{F}, \mathcal{G}: \mathcal{C} \to \mathcal{D}\) be bilax monoidal functors. A morphism from \(\mathcal{F}\) to \(\mathcal{G}\) is a natural transformation \(\theta: \mathcal{F} \Rightarrow \mathcal{G}\) such that diagrams (3.14) and (3.15) commute. In other words, a morphism of bilax functors is a morphism of the underlying lax and colax functors.

A morphism of bistrong monoidal functors is a morphism of the underlying bilax monoidal functors.

The following is straightforward.

**Proposition 3.10.** The composite of two morphisms of lax (colax, bilax) monoidal functors is again a morphism of lax (colax, bilax) monoidal functors.

Thus for two fixed monoidal categories, we have the categories of lax and colax functors between them. Similarly for two fixed braided monoidal categories, we have the category of bilax functors between them. We elaborate on this point in Section 3.3.3.

### 3.1.3. Braided bilax monoidal functors

We now define the notion that plays the role of commutativity for a bilax monoidal functor.

**Definition 3.11.** A lax (resp. colax) monoidal functor \((\mathcal{F}, \varphi)\) between two braided monoidal categories (resp. \((\mathcal{F}, \psi))\) is **braided** if the right-hand (resp. left-hand) diagram below commutes.

\[
\begin{array}{ccc}
\mathcal{F}(A \bullet B) & \xrightarrow{\psi_{A,B}} & \mathcal{F}(A) \bullet \mathcal{F}(B) \\
\downarrow {\beta} & & \downarrow {\varphi_{A,B}} \\
\mathcal{F}(B \bullet A) & \xrightarrow{\varphi_{B,A}} & \mathcal{F}(B) \bullet \mathcal{F}(A)
\end{array}
\]

(3.16)
A morphism of braided lax (colax) monoidal functors is a morphism of the underlying lax (colax) monoidal functors.

**Example 3.12.** The functor Hom in Example 3.17 is a braided lax monoidal functor from $\mathbf{C}^{\text{op}} \times \mathbf{C}$ to $\mathbf{Set}$, where the former category is endowed with the braiding $((\beta^{-1})^{\text{op}}, \beta)$.

**Definition 3.13.** A bilax monoidal functor $(\mathcal{F}, \varphi, \psi)$ is braided if both the diagrams in (3.16) commute, or equivalently, if $(\mathcal{F}, \varphi)$ and $(\mathcal{F}, \psi)$ are braided lax and colax monoidal functors, respectively.

A morphism of braided bilax monoidal functors is a morphism of the underlying bilax monoidal functors.

Thus for two fixed braided monoidal categories, we have the categories of braided lax, braided colax and braided bilax functors between them. They are full subcategories respectively of the categories of lax, colax and bilax functors.

**Definition 3.14.** Let $(\mathcal{C}, \bullet, \beta)$ and $(\mathcal{D}, \bullet, \beta)$ be two braided monoidal categories and $\varphi$ and $\psi$ be as in (3.9). Define natural transformations $\varphi^b$, $b\varphi$, $\psi^b$ and $b\psi$ as the following composites.

\[
\varphi^b : \mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\beta} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{B,A}} \mathcal{F}(B \bullet A) \xrightarrow{\mathcal{F}(\beta^{-1})} \mathcal{F}(A \bullet B),
\]

\[
b\varphi : \mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\beta^{-1}} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\varphi_{B,A}} \mathcal{F}(B \bullet A) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(A) \bullet \mathcal{F}(B),
\]

\[
\psi^b : \mathcal{F}(A \bullet B) \xrightarrow{\mathcal{F}(\beta)} \mathcal{F}(B \bullet A) \xrightarrow{\psi_{B,A}} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\beta^{-1}} \mathcal{F}(A) \bullet \mathcal{F}(B),
\]

\[
b\psi : \mathcal{F}(A \bullet B) \xrightarrow{\mathcal{F}(\beta^{-1})} \mathcal{F}(B \bullet A) \xrightarrow{\psi_{B,A}} \mathcal{F}(B) \bullet \mathcal{F}(A) \xrightarrow{\beta} \mathcal{F}(A) \bullet \mathcal{F}(B).
\]

We state analogues to Propositions 1.20 and 1.21. The proofs are straightforward.

**Proposition 3.15.** If $(\mathcal{F}, \varphi)$ (resp. $(\mathcal{F}, \psi)$) is a lax (resp. colax) monoidal functor from $(\mathcal{C}, \bullet)$ to $(\mathcal{D}, \bullet)$, then so are $(\mathcal{F}, \varphi^b)$ and $(\mathcal{F}, b\varphi)$ (resp. $(\mathcal{F}, \psi^b)$ and $(\mathcal{F}, b\psi)$).

**Proposition 3.16.** Let $(\mathcal{F}, \varphi, \psi)$ be a bilax monoidal functor from $(\mathcal{C}, \bullet, \beta)$ to $(\mathcal{D}, \bullet, \beta)$. Then

\[
(\mathcal{F}, \varphi^b, \psi^b) \quad \text{and} \quad (\mathcal{F}, b\varphi, b\psi)
\]

are bilax monoidal functors from $(\mathcal{C}, \bullet, \beta^{-1})$ to $(\mathcal{D}, \bullet, \beta^{-1})$. Therefore,

\[
(\mathcal{F}, \varphi^b, \psi^b) \quad \text{and} \quad (\mathcal{F}, b\varphi, b\psi)
\]

are bilax monoidal functors from $(\mathcal{C}, \bullet, \beta)$ to $(\mathcal{D}, \bullet, \beta)$.

In analogy with (1.15)–(1.16), we have equivalences among the four statements in each set below.

\[
\begin{align*}
\text{(3.17)} & \quad (\mathcal{F}, \varphi) \text{ is a braided lax monoidal functor;} \\
& \quad \text{id}: (\mathcal{F}, \varphi) \Rightarrow (\mathcal{F}, \varphi^b) \text{ is a morphism of lax monoidal functors;} \\
& \quad \text{id}: (\mathcal{F}, \varphi) \Rightarrow (\mathcal{F}, b\varphi) \text{ is a morphism of lax monoidal functors;} \\
& \quad \varphi = \varphi^b.
\end{align*}
\]

\[
\begin{align*}
\text{(3.18)} & \quad (\mathcal{F}, \psi) \text{ is a braided colax monoidal functor;} \\
& \quad \text{id}: (\mathcal{F}, \psi) \Rightarrow (\mathcal{F}, \psi^b) \text{ is a morphism of colax monoidal functors;} \\
& \quad \text{id}: (\mathcal{F}, \psi) \Rightarrow (\mathcal{F}, b\psi) \text{ is a morphism of colax monoidal functors;} \\
& \quad \psi = \psi^b.
\end{align*}
\]
3.1.4. The convolution comma category. Let $A$, $B$ and $C$ be monoidal categories,

\[(F, \psi) : (A, \bullet) \to (C, \bullet)\]

a colax monoidal functor, and

\[(G, \varphi) : (B, \bullet) \to (C, \bullet)\]

a lax monoidal functor. Consider the comma category $F \downarrow G$, as in Section A.5; its objects are triples $(A, \gamma, B)$ with

\[\gamma : F(A) \to G(B)\]

an arrow in $C$.

Given arrows $\gamma_i : F(A_i) \to G(B_i)$ in $C$, $i = 1, 2$, we may form the composite

\[F(A_1 \cdot A_2) \xrightarrow{\psi_{A_1,A_2}} F(A_1) \cdot F(A_2) \xrightarrow{\gamma_1 \cdot \gamma_2} G(B_1) \cdot G(B_2) \xrightarrow{\varphi_{B_1,B_2}} G(B_1 \cdot B_2).\]

We may also consider the composite

\[F(I_A) \xrightarrow{\psi_0} I_C \xrightarrow{\varphi_0} G(I_B),\]

where $I_A, I_B$ and $I_C$ are the unit objects of each category. This allows us to turn $F \downarrow G$ into a monoidal category, as follows. The tensor product on objects is

\[(A_1, \gamma_1, B_1) \cdot (A_2, \gamma_2, B_2) := (A_1 \cdot A_2, \varphi_{B_1,B_2}(\gamma_1 \cdot \gamma_2)\psi_{A_1,A_2}, B_1 \cdot B_2).\]

It is defined similarly on morphisms. The unit object is $(I_A, \varphi_0 \psi_0, I_B)$. Associativity follows from (3.5) and unitality from (3.6) (and the dual diagrams).

Suppose all the given data is braided (the monoidal categories $A, B$ and $C$, the colax monoidal functor $(F, \psi)$ and the lax monoidal functor $(G, \varphi)$). It then follows from (3.16) that the pair $(\beta_{A_1,A_2}, \beta_{B_1,B_2})$ defines a morphism from

\[(A_1, \gamma_1, B_1) \cdot (A_2, \gamma_2, B_2) \to (A_2, \gamma_2, B_2) \cdot (A_1, \gamma_1, B_1)\]

in the comma category $F \downarrow G$. It follows that in this situation the monoidal category $(F \downarrow G, \bullet)$ is braided.

3.2. Examples of bilax monoidal functors

In this section, we provide pointers to the main examples of bilax and bistrong monoidal functors and morphisms between them which are discussed in this monograph. We also provide some other basic examples.

3.2.1. Classical example from homological algebra. In Chapter 5, we discuss what may be the most classical bilax monoidal functor: the chain complex functor from simplicial modules to chain complexes. In this example, the transformations $\varphi$ and $\psi$ are the Eilenberg–Zilber and Alexander–Whitney maps. It turns out that the associated chain complex functor from simplicial modules to the homotopy category of chain complexes is bistrong.

The interested reader may enjoy going over Chapter 5 at this point; the discussion there uses some of the terminology developed so far and some results from later sections in this chapter.
3.2.2. The image of a morphism of bilax monoidal functors. A general procedure to factorize a morphism of bilax functors from a braided monoidal category to an abelian braided monoidal category is given in Section 3.11. In particular, this yields a new bilax functor, which is the image of the morphism. Our method makes use of the existence of monic-epi factorizations in abelian categories and a related bistrong functor called the image functor.

3.2.3. Examples related to species and Fock functors. The main examples of bilax monoidal functors in this monograph are the Fock functors from species to graded vector spaces. They are the object of study of Chapters 15 and 16. Decorated and colored versions of these functors are discussed in Chapters 19 and 20. Other examples of bilax monoidal functors include the Hadamard functor and the Hom functor on species. A summary is provided in Table 3.1.

The main examples of bistrong monoidal functors are summarized in Table 3.2. They include the duality functor on species and on graded vector spaces, the signature functor on species, and the bosonic and fermionic Fock functors from species to graded vector spaces.

Several morphisms of bilax monoidal functors play an important role in this monograph. The main ones are the morphisms $\mathcal{K} \Rightarrow \mathcal{K}$ and $\mathcal{K}^\vee \Rightarrow \mathcal{K}^\vee$ relating the full Fock functors with the bosonic Fock functors, similar morphisms with fermionic replacing bosonic, and the norm and half-twist transformations that relate the full Fock functors. These are summarized in Table 3.3. Generalizations of these morphisms are discussed in Chapters 19 and 20.

Table 3.1. Bilax monoidal functors.

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Table 3.2. Bistrong monoidal functors.

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### 3.2. Examples of Bilax Monoidal Functors

#### Table 3.3. Morphisms between bilax monoidal functors.

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#### 3.2.4. Some other examples.

In the remainder of this section, we discuss some other interesting examples.

**Example 3.17.** Let $(\text{Set}, \times)$ be the symmetric monoidal category of sets under Cartesian product, as in Example 1.3. The unit object is the one-element set $\{\emptyset\}$. For any monoidal category $(C, \bullet)$, the functor

$$\text{Hom}: C^{\text{op}} \times C \to \text{Set}$$

is a lax monoidal functor with the map

$$\varphi_{(A,C),(B,D)}: \text{Hom}(A,C) \times \text{Hom}(B,D) \to \text{Hom}(A \bullet B, C \bullet D)$$

which sends $(f, g)$ to $f \bullet g$, and the map

$$\varphi_0: \{\emptyset\} \to \text{Hom}(I, I)$$

which sends $\emptyset$ to the identity morphism from the unit object $I$ to itself.

If $(C, \bullet)$ is a linear monoidal category (Definition 1.6), then one similarly obtains a lax monoidal functor

$$(\text{Hom}, \varphi): C^{\text{op}} \times C \to \text{Vec}.$$

If $C$ is the category of finite-dimensional vector spaces under ordinary tensor product, then the map $\varphi$ is an isomorphism. By letting $\psi = \varphi^{-1}$, one obtains a colax functor $(\text{Hom}, \psi)$. Moreover, the two structures are compatible and we obtain a bilax functor $(\text{Hom}, \varphi, \psi)$, which is in fact bistrong by construction.

The situation concerning Hom is more delicate for graded vector spaces and for species; see Proposition 8.58, Proposition 8.64, and Remark 8.65.

**Example 3.18.** Let $C_N$ be the category whose objects are nonnegative integers and whose morphisms are

$$\text{Hom}_{C_N}(n, m) := \begin{cases} \text{id}_n & \text{if } n = m, \\ \emptyset & \text{if } n \neq m. \end{cases}$$

This is the **discrete category** on $\mathbb{N}$. It is a symmetric monoidal category under

$$n \otimes m := n + m \quad \text{and} \quad \beta_{n,m} = \text{id}_{n+m}.$$

In addition, $C_N$ is strict, that is, the associative and unit constraints are all identities.

Given a functor $F: C_N \to \text{Vec}$, let

$$|F| := \bigoplus_{n \in \mathbb{N}} F(n).$$

Then $|F|$ is a graded vector space. If $x \in F(n)$ we write $|x| = n$. 
Suppose $\mathcal{F}$ is a lax monoidal functor with structure maps
\[ \varphi_{n,m} : \mathcal{F}(n) \otimes \mathcal{F}(m) \to \mathcal{F}(n+m). \]
Defining
\[ x \cdot y := \varphi_{|x|,|y|}(x \otimes y) \]
on homogeneous elements $x$ and $y$ endows $|\mathcal{F}|$ with the structure of a graded algebra.

The unit element is $\varphi_0(1) \in \mathcal{F}(0)$.

If $\mathcal{F}$ is a colax monoidal functor, then we may similarly use the structure maps
\[ \psi_{n,m} : \mathcal{F}(n+m) \to \mathcal{F}(n) \otimes \mathcal{F}(m) \]
to turn $|\mathcal{F}|$ into a graded coalgebra; the coproduct is
\[ \Delta(x) = \sum_{n+m=|x|} \psi_{n,m}(x) \]
on homogeneous elements $x \in |\mathcal{F}|$ and the counit is defined in terms of $\psi_0$.

The above constructions define equivalences between the category of functors (resp. lax monoidal functors, colax monoidal functors) $\mathcal{C}_N \to \mathbf{Vec}$ and the category of graded vector spaces (resp. graded algebras, graded coalgebras).

One may expect a similar result for bilax monoidal functors and graded bialgebras, but it turns out that these two notions are not equivalent. Indeed, if $\mathcal{F} : \mathcal{C}_N \to \mathbf{Vec}$ is a bilax monoidal functor and we view $|\mathcal{F}|$ as an algebra and as a coalgebra as above, then diagram (3.11) leads to the following relation between the product and coproduct of $|\mathcal{F}|$:

\[ \psi_{n,m}(x \cdot y) = \psi_{a,b}(x) \psi_{c,d}(y) \tag{3.19} \]
whenever $a, b, c,$ and $d$ are related to $n, m,$ and the degrees of $x$ and $y$ by
\[ a + c = n, \quad b + d = m, \quad a + b = |x|, \quad c + d = |y|, \quad n + m = |x| + |y|. \tag{3.20} \]
On the other hand, the definition of graded bialgebra would instead require the following compatibility condition between the product and coproduct of $|\mathcal{F}|$:

\[ \psi_{n,m}(x \cdot y) = \sum_{a,b,c,d} \psi_{a,b}(x) \psi_{c,d}(y), \tag{3.21} \]
the sum being over all $a, b, c,$ and $d$ subject to the conditions (3.20). Conditions (3.19) and (3.21) are distinct in general. Specifically, consider the case of the graded bialgebra of polynomials $\mathbb{K}[t]$, where
\[ \Delta(t^n) = \sum_k \binom{n}{k} t^k \otimes t^{n-k}. \]
Take $x = t^i$ and $y = t^j$. The right-hand side of (3.19) is
\[ \binom{i}{a} \binom{j}{c} t^{a+c} \otimes t^{b+d}, \]
while the left-hand side of (3.19) is
\[ \binom{i+j}{n} t^n \otimes t^m. \]
(This agrees with the right-hand side of (3.21) by Vandermonde’s identity for binomial coefficients.)
3.2. EXAMPLES OF BILAX MONOIDAL FUNCTORS

In summary, a bilax monoidal functor $C \to Vec$ is equivalent to a graded vector space endowed with a structure of graded algebra and of graded coalgebra, linked by (3.19) (plus standard conditions involving the unit and counit), and this is not a graded bialgebra.

This flaw is rectified if one replaces graded vector spaces by species. In Proposition 8.35 we show how an analogous construction for species does lead to an equivalence between bimonoids in species and certain bilax monoidal functors.

There is a different way in which graded bialgebras may be seen as bilax monoidal functors. This in fact holds for bimonoids in an arbitrary braided monoidal category and is explained in Section 3.4.1.

**Example 3.19.** This example was proposed by George Janelidze. Throughout this discussion, we employ the terminology and notations of Section A.1.

Let $C$ and $D$ be categories with finite products. Consider the corresponding cartesian monoidal categories $(C, \times, J)$ and $(D, \times, J)$ as in Example 1.4.

Let $F: C \to D$ be an arbitrary functor. Given objects $A$ and $B$ of $C$, let

$$\psi_{A,B} := \left( F(\pi_A), F(\pi_B) \right): F(A \times B) \to F(A) \times F(B),$$

and let

$$\psi_0: \mathcal{F}(J) \to J$$

be the unique such arrow in $D$. Then $(F, \psi, \psi_0)$ is a colax monoidal functor. Indeed, both composites in the dual of diagram (3.5) coincide with the arrow

$$\left( F(\pi_A), F(\pi_B), F(\pi_C) \right).$$

The other diagrams in Definition 3.1 can be verified similarly. Note that the functor is costrong if and only if $F$ preserves products.

Thus, any functor between categories with finite products carries a canonical colax monoidal structure. Moreover, this structure is braided (Definition 3.11).

Dually, any functor $F: C \to D$ between cocartesian monoidal categories carries a canonical braided lax monoidal structure. The structure maps are

$$\varphi_{A,B} := \left( \frac{F(\iota_A \oplus C)}{F(\iota_B \oplus D)} \right): F(A) \amalg F(B) \to F(A \amalg B)$$

and

$$\varphi_0: I \to F(I).$$

Suppose now that $C$ has finite biproducts. Consider the corresponding bicartesian monoidal category $(C, \oplus, Z)$ of Example 1.4. Let $D$ be another such category. By the above, any functor $F: C \to D$ carries a canonical lax structure $\varphi$ and a canonical colax structure $\psi$. It turns out that

$$(F, \varphi, \psi): (C, \oplus, Z) \to (D, \oplus, Z)$$

is bilax monoidal. Indeed, the composites along both sides of diagram (3.11) are equal to

$$\begin{pmatrix}
F(\iota_A \oplus C), F(\pi_A \oplus B) \\
F(\iota_C \oplus D) \\
F(\iota_B \oplus D)
\end{pmatrix}
\begin{pmatrix}
F(\iota_B \oplus D), F(\pi_B \oplus D)
\end{pmatrix}
\begin{pmatrix}
F(\iota_C \oplus D)
\end{pmatrix}
\begin{pmatrix}
F(\iota_C \oplus D)
\end{pmatrix}
\begin{pmatrix}
F(A \oplus B) \oplus F(C \oplus D)
\end{pmatrix}
\to
F(A \oplus C) \oplus F(B \oplus D)$$

and so the braiding axiom is satisfied. The other axioms can be verified similarly.
Thus, any functor between bicartesian monoidal categories carries a canonical braided bilax monoidal structure. As an example, we may choose \( C = D = \text{Vec} \), the category of vector spaces (under direct sum), and
\[
\mathcal{F}(V) := V \otimes V.
\]
This functor is not bistrong.

### 3. Composites of bilax monoidal functors

Monoidal functors exhibit an interesting feature which is not visible for monoids. Namely, it is meaningful to ask whether the composite of monoidal functors is again monoidal. The first result in this direction is by Bénabou who showed that the composite of lax monoidal functors is again lax monoidal [36, Proposition 5].

In this section, we show that the same assertion also holds for bilax monoidal functors. This includes Bénabou’s result. We also briefly explain how this leads to 2-categories based on (co,bi)lax monoidal functors.

#### 3.3.1. Composites of bilax monoidal functors

**Definition 3.20.** Let \((\mathcal{F}, \varphi): (C, \bullet) \to (D, \bullet)\), where \(\varphi: \mathcal{F}^2 \Rightarrow \mathcal{F}_2\) and \(\varphi_0: I \to \mathcal{F}(I)\) are as in (3.3) and (3.4). Similarly, let \((\mathcal{G}, \gamma): (D, \bullet) \to (E, \bullet)\) with \(\gamma: \mathcal{G}^2 \Rightarrow \mathcal{G}_2\) and \(\gamma_0: I \to \mathcal{G}(I)\). Now let
\[
(\mathcal{G}\mathcal{F}, \varphi\gamma): (C, \bullet) \to (E, \bullet),
\]
where the functor \(\mathcal{G}\mathcal{F}: C \to E\) is the composite of \(\mathcal{F}\) and \(\mathcal{G}\), and the transformations
\[
\varphi\gamma: (\mathcal{G}\mathcal{F})^2 \Rightarrow (\mathcal{G}\mathcal{F})_2 \quad \text{and} \quad (\varphi\gamma)_0: I \to \mathcal{G}\mathcal{F}(I)
\]
are defined as follows.

\[
\begin{array}{c}
\mathcal{G}\mathcal{F}(A) \bullet \mathcal{G}\mathcal{F}(B) \quad \xrightarrow{\gamma_{\mathcal{F}(A), \mathcal{F}(B)}} \quad \mathcal{G}\mathcal{F}(A \bullet B) \\
\mathcal{G}((\varphi\gamma)_{A,B})
\end{array}
\]
\[
\begin{array}{c}
I \quad \xrightarrow{\gamma_0} \quad \mathcal{G}(I) \\
\mathcal{G}(\varphi_0)
\end{array}
\]

Similarly for the dual situation, given \((\mathcal{F}, \psi)\) and \((\mathcal{G}, \delta)\), we define \((\mathcal{G}\mathcal{F}, \delta\psi)\), where \(\delta\psi\) and \((\delta\psi)_0\) are obtained from the above by switching \(\varphi\) with \(\psi\), and \(\gamma\) with \(\delta\), and reversing the arrows. Combining the two situations, given \((\mathcal{F}, \varphi, \psi)\) and \((\mathcal{G}, \gamma, \delta)\), we define \((\mathcal{G}\mathcal{F}, \varphi\gamma, \delta\psi)\).

**Theorem 3.21.** If \((\mathcal{F}, \varphi): C \to D\) and \((\mathcal{G}, \gamma): D \to E\) are lax monoidal, then the functor \((\mathcal{G}\mathcal{F}, \varphi\gamma): C \to E\) is lax monoidal. Similarly, if \((\mathcal{F}, \psi)\) and \((\mathcal{G}, \delta)\) are colax, then so is \((\mathcal{G}\mathcal{F}, \delta\psi)\).

If \(\mathcal{F} \Rightarrow \mathcal{F}'\) is a morphism of \((\co\text{co})\)-lax monoidal functors, then the induced natural transformation \(\mathcal{G}\mathcal{F} \Rightarrow \mathcal{G}\mathcal{F}'\) is also a morphism of \((\co\text{co})\)-lax monoidal functors.

If \(\mathcal{G} \Rightarrow \mathcal{G}'\) is a morphism of \((\co\text{co})\)-lax monoidal functors, then the induced natural transformation \(\mathcal{G}\mathcal{F} \Rightarrow \mathcal{G}'\mathcal{F}\) is also a morphism of \((\co\text{co})\)-lax monoidal functors.

**Proof.** We prove the first statement (for lax functors). The statement for colax follows by passing to the opposite categories (Proposition 3.7) and the remaining assertions can be shown similarly.
The associativity axiom for \((G\mathcal{F}, \varphi\gamma)\) follows by the commutativity of the following diagram.

\[
\begin{array}{c}
G\mathcal{F}(A \bullet B) \bullet G\mathcal{F}(C) \xrightarrow{\gamma_{(A \bullet B), C}} G\mathcal{F}(A \bullet (B \bullet C)) \\
G\mathcal{F}(A) \bullet G\mathcal{F}(B) \bullet G\mathcal{F}(C) \xrightarrow{\gamma_{F, \mathcal{F}(B \bullet C)}} G\mathcal{F}(A \bullet \mathcal{F}(B \bullet C))
\end{array}
\]

The top left diagram is the associativity of \(\gamma\), while the bottom right diagram is the functor \(G\) applied to the associativity of \(\varphi\). The remaining two diagrams commute by the naturality of \(\gamma\).

The left unitality axiom for \((G\mathcal{F}, \varphi\gamma)\) follows by the commutativity of the following diagram.

\[
\begin{array}{c}
G(I) \bullet G\mathcal{F}(A) \xleftarrow{\gamma_{0, \mathcal{F}(A)}} G(I \bullet \mathcal{F}(A)) \\
G(I) \bullet G\mathcal{F}(A) \xrightarrow{\gamma_{I, \mathcal{F}(A)}} G(I \bullet \mathcal{F}(A))
\end{array}
\]

The oblique squares commute by the left unitality of \(\gamma\) and the functor \(G\) applied to the left unitality of \(\varphi\), while the third square commutes by the naturality of \(\gamma\).

The verification of the right unitality axiom is similar. This proves that the composite functor \((G\mathcal{F}, \varphi\gamma)\) is lax. 

\[\square\]

**Theorem 3.22.** If \((\mathcal{F}, \varphi, \psi) : C \to D\) and \((\mathcal{G}, \gamma, \delta) : D \to E\) are bilax monoidal, then so is \((G\mathcal{F}, \varphi\gamma, \delta\psi)\).

In addition, pre or post composing by a bilax monoidal functor preserves morphisms between bilax monoidal functors.

**Proof.** In view of Theorem 3.21, one only needs to prove the commutativity of diagrams (3.11), (3.12) and (3.13) for \((G\mathcal{F}, \varphi\gamma, \delta\psi)\).
The commutativity of (3.11) follows from that of the following diagram. For simplicity of notation, the tensor product symbol has been suppressed.

\[
\begin{array}{c}
\gamma_{F(AB),F(CD),F(FCD)} \\
\gamma_{F(AB),F(CD)F(FCD)} \\
\gamma_{F(AB),F(CD)F(FCD)} \\
\gamma_{F(AB),F(CD)F(FCD)} \\
\gamma_{F(AB),F(CD)F(FCD)}
\end{array}
\]

The squares commute by the naturality of \(\gamma\) and \(\delta\), while the hexagons commute by the braiding axiom (3.11) for \((G, \gamma, \delta)\) and the functor \(G\) applied to the same axiom for \((F, \phi, \psi)\).

The first axiom in (3.12) follows from the commutativity of the following diagram.

\[
\begin{array}{c}
I \xrightarrow{\gamma_0} G(I) \xrightarrow{G(\varphi_0)} G(F(I) \bullet I) \\
I \bullet I \xrightarrow{\gamma_0 \gamma_0} G(I) \bullet G(I) \xrightarrow{G(\phi_0 \cdot \phi_0)} G(F(I) \bullet F(I)) \\
\delta_{F(I),F(I)}
\end{array}
\]

The pentagons commute by the first axiom in (3.12) for \((G, \gamma, \delta)\) and the functor \(G\) applied to the same axiom for \((F, \phi, \psi)\). The square commutes by the naturality of \(\delta\).

The proof for the second axiom in (3.12) can be obtained from the above by reversing the appropriate arrows. Axiom (3.13) follows directly.

An alternative proof of Theorem 3.22 is given in Remark 3.78. It is also clear that if \(F, G\) and \(H\) are composable bilax monoidal functors, then

\[
H(GF) \cong (HG)F
\]

as bilax monoidal functors.
3.3.2. Composites of braided bilax monoidal functors. We now turn our attention to the interaction between composites and the constructions of Definition 3.14.

Proposition 3.23. We have

\[(\varphi\gamma)^b = \varphi^b\gamma^b \quad b(\varphi\gamma) = (b\varphi)(b\gamma) \quad (\delta\psi)^b = \delta^b\psi^b \quad b(\delta\psi) = (b\delta)(b\psi).\]

This is an easy consequence of the definitions. Thus, the composition of lax, colax, and bilax monoidal functors is compatible with conjugation by the braidings, and hence compatible with the constructions in Propositions 3.15 and 3.16. Combining Proposition 3.23 with (3.17) and (3.18) one obtains the following result.

Proposition 3.24. The composite of two braided lax (colax, bilax) monoidal functors is again braided lax (colax, bilax) monoidal.

Further, pre or post composing by a braided lax (colax, bilax) monoidal functor preserves morphisms between braided lax (colax, bilax) monoidal functors.

3.3.3. 2-categories arising from monoidal functors. The preceding results can be succinctly expressed using the notion of 2-category (Section C.1.1). Let \( \text{Cat} \) be the 2-category whose 0-cells, 1-cells, and 2-cells are respectively categories, functors, and natural transformations. Together with Proposition 3.10, the preceding results say that lax monoidal functors are the 1-cells of a 2-category whose objects are monoidal categories and whose 2-cells are morphisms of lax monoidal functors. We call this 2-category \( \text{lCat} \). The same construction with colax replacing lax yields \( \text{cCat} \).

Similarly, there is a 2-category whose 0-cells, 1-cells, and 2-cells are respectively braided monoidal categories, bilax monoidal functors, and their morphisms.

Further, there are braided versions of all these: Braided lax (colax, bilax) monoidal functors are the 1-cells of a 2-category whose objects are braided monoidal categories and whose 2-cells are morphisms of braided lax (colax, bilax) monoidal functors. We return to these ideas in Sections 6.11 and 7.9.

3.4. A comparison of bimonoids and bilax monoidal functors

The discussion in Section 3.1 reveals a parallel between the notions of monoid, comonoid, bimonoid, and those of lax, colax, and bilax monoidal functor, respectively. These notions are connected in this section in two different ways. First we show that any bimonoid may be seen as a special case of a bilax monoidal functor, then we deduce that the image of a bimonoid under a bilax monoidal functor is again a bimonoid. Similar results connect monoids to lax monoidal functors and comonoids to colax monoidal functors.

3.4.1. Bimonoids as bilax monoidal functors. Monoids may be viewed as lax monoidal functors. We recall this construction of Bénabou [38, Section 5.4.1] and then give the corresponding result for bimonoids.

Let \((A, \mu, \iota)\) be a monoid in a monoidal category \((C, \otimes)\) with unit object \(I\). Let \((I, \cdot)\) be the one-arrow category and let

\[ F_A : I \to C \]

be the functor that sends the unique object \(\ast\) of \(I\) to \(A\). Next, we define a transformation \(\varphi\) and a map \(\varphi_0\) in order to turn \(F_A\) into a lax monoidal functor (Definition 3.1). Since there is only one object and one morphism in the category \(I\), \(\varphi\)
consists of only one map, which is $\varphi_{*,*}$; also, $\varphi_0$ is a map $I \to F(*)$. We let
\[ F_A(*) \circ F_A(*) \xrightarrow{\varphi_{*,*} := \mu} F_A(* \circ *) \quad \text{and} \quad I \xrightarrow{\varphi_0 := I} F_A(*). \]

Then
\[(3.22) \quad (F_A, \varphi, \varphi_0): (I, \circ) \to (C, \bullet)\]
is a lax monoidal functor. Associativity of $\mu$ translates into associativity of $\varphi$ (3.5) and similarly for unitality (3.6).

Similarly, given a comonoid $(C, \Delta, \varepsilon)$ in $(C, \bullet)$, define a colax monoidal functor
\[(F_C, \psi, \psi_0)\]
by $F_C(*) := C$, $F_C(* \circ *) \xrightarrow{\psi_{*,*} := \Delta} F_C(*) \circ F_C(*)$ and $F_C(*) \xrightarrow{\psi_0 := \varepsilon} I$.

Proposition 3.25. The above construction defines an equivalence from the category of (co)monoids in $(C, \bullet)$ to the category of (co)lax monoidal functors from $(I, \circ)$ to $(C, \bullet)$.

Proof. Given a monoid $(A, \mu, \iota)$, we send it to the lax functor $(F_A, \varphi, \varphi_0)$ defined above. Conversely, given a lax functor $(F, \varphi, \varphi_0): (I, \circ) \to (C, \bullet)$, we send it to $(F(*), \varphi_{*,*}, \varphi_0)$. One can check directly that this is a monoid. These correspondences define the equivalence. The case of comonoids is similar. \qed

Combining the two situations above, given a bimonoid $(H, \mu, \iota, \Delta, \varepsilon)$ we construct
\[(3.23) \quad (F_H, \varphi, \varphi_0, \psi, \psi_0).\]
This is a bilax monoidal functor: the four compatibility diagrams for a bimonoid (Definition 1.10) correspond to the four compatibility diagrams for a bilax monoidal functor (Definition 3.3).

Proposition 3.26. Let $(C, \bullet, \beta)$ be a braided monoidal category. The above construction defines an equivalence from the category of bimonoids in $(C, \bullet, \beta)$ to the category of bilax monoidal functors from $(I, \circ, \beta)$ to $(C, \bullet, \beta)$.

Proof. As for Proposition 3.25. \qed

This discussion shows that a bilax monoidal functor need not be braided and a braided (co)lax monoidal functor need not be bilax.

3.4.2. Commutative monoids as braided lax monoidal functors. Recall that the (co)product of a (co)monoid can be twisted by the braiding to yield its opposite (co)monoid (Section 1.2.9). Recall that in much the same way, the structure of a (co)lax monoidal functor can be twisted to yield its conjugate (co)lax monoidal functor (Definition 3.14). We now make the analogy between these two constructions precise using the preceding discussion.

If the monoid $A = (A, \mu, \iota)$ corresponds to the lax monoidal functor $(F_A, \varphi)$, then the opposite monoid $A^{\text{op}} = (A, \mu\beta, \iota)$ corresponds to the conjugate lax monoidal functor $(F_A, \varphi^b)$, and similarly, $A^{\text{op}} = (A, \mu\beta^{-1}, \iota)$ corresponds to $(F_A, \iota^b \varphi)$. This is clear from the definitions.
Similarly, if the comonoid $C = (C, \Delta, \epsilon)$ corresponds to the colax monoidal functor $(F_C, \psi)$, then the opposite comonoid $C^{\text{co}} = (C, \beta^{-1} \Delta, \epsilon)$ corresponds to the conjugate colax monoidal functor $(F_C, \psi^{\text{b}})$, and similarly, $\text{co}^C = (C, \beta \Delta, \epsilon)$ corresponds to $(F_C, ^b \psi)$.

In addition, it is clear from (1.15) and (1.16), and (3.17) and (3.18) that

$A$ is a commutative monoid $\Leftrightarrow (F_A, \varphi)$ is a braided lax monoidal functor
$C$ is a cocommutative comonoid $\Leftrightarrow (F_C, \psi)$ is a braided colax monoidal functor.

The preceding statements can be phrased as follows.

**Proposition 3.27.** The category of (co)commutative (co)monoids in $(C, \bullet, \beta)$ is equivalent to the category of braided (co) lax monoidal functors from $(I, \bullet, \beta)$ to $(C, \bullet, \beta)$.

### 3.4.3. Bilax monoidal functors preserve bimonoids.

A significant property of lax, colax and bilax monoidal functors is that they preserve monoids, comonoids and bimonoids respectively. The assertions for lax (and colax) monoidal functors appear in [38, Proposition 6.1] and are also given below (Proposition 3.29).

**Definition 3.28.** Let $(F, \varphi): (C, \bullet) \to (D, \bullet)$, where $\varphi: F^2 \Rightarrow F_2$ and $\varphi_0: I \to F(I)$ are as in (3.3) and (3.4). Also consider $(A, \mu, \iota)$ where $A$ is an object and $\mu: A \bullet A \to A$ and $\iota: I \to A$ are morphisms in $C$. Then define the triple

$$(F(A), \mu \varphi, \iota \varphi_0),$$

where $\mu \varphi$ and $\iota \varphi$ are given by the following composites.

$$F(A) \bullet F(A) \xrightarrow{\varphi_{A,A}} F(A \bullet A) \xrightarrow{F(\mu)} F(A)$$

$$I \xrightarrow{\varphi_0} F(I) \xrightarrow{F(\iota)} F(A)$$

Similarly for the dual situation, given $(F, \psi)$ and $(C, \Delta, \epsilon)$, we define the triple

$$(F(C), \psi \Delta, \psi_0 \epsilon),$$

where $\psi \Delta$ and $\psi_0 \epsilon$ are given by the following composites.

$$F(C) \bullet F(C) \xleftarrow{\psi_{C,C}} F(C \bullet C) \xleftarrow{F(\Delta)} F(C)$$

$$I \xleftarrow{\psi_0} F(I) \xleftarrow{F(\iota)} F(C)$$

Combining the two situations, given $(F, \varphi, \psi)$ and $(H, \mu, \iota, \Delta, \epsilon)$, we can consider the quintuple

$$(F(H), \mu \varphi, \iota \varphi, \psi \Delta, \psi_0 \epsilon).$$

**Proposition 3.29.** If $F$ is a (co)lax monoidal functor from $(C, \bullet)$ to $(D, \bullet)$ and $H$ is a (co)monoid in $(C, \bullet)$, then $F(H)$ is a (co)monoid in $(D, \bullet)$ with the (co)product and (co)unit as in Definition 3.28.

Moreover, if $f: H \to H'$ is a morphism of (co)monoids in $(C, \bullet)$, then the induced morphism $F(f): F(H) \to F(H')$ is a morphism of (co)monoids in $(D, \bullet)$.

**Proof.** We explain the case of monoids. Recall that associated to a monoid $H$ there is the lax monoidal functor $F_H$ of (3.22). We have the following commutative
Since \( \mathcal{F} \) and \( \mathcal{F}_H \) are lax monoidal functors, so is \( \mathcal{F}_{\mathcal{F}(H)} \), by Theorem 3.21. Hence, by Proposition 3.25, \( \mathcal{F}(H) \) is a monoid in \((\mathcal{D}, \bullet)\), and further this monoid structure coincides with that in Definition 3.28. The assertion about morphisms follows similarly.

**Proposition 3.30.** A morphism of (co)lax monoidal functors \( \theta : \mathcal{F} \Rightarrow \mathcal{G} \) yields a morphism of (co)monoids \( \theta_H : \mathcal{F}(H) \rightarrow \mathcal{G}(H) \) in \((\mathcal{D}, \bullet)\), when \( H \) is a (co)monoid in \((\mathcal{C}, \bullet)\).

**Proof.** We explain the case of monoids. Let \( \mathcal{F}_H \) be the lax monoidal functor of (3.22). By precomposing \( \theta \) with \( \mathcal{F}_H \), as shown below

\[
\begin{array}{ccc}
I & \xrightarrow{\mathcal{F}_H} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{D}
\end{array}
\]

and applying Theorem 3.21, we obtain a morphism \( \mathcal{F}_{\mathcal{F}(H)} \rightarrow \mathcal{F}_{\mathcal{G}(H)} \) of lax monoidal functors. Equivalently, from Proposition 3.25, this yields the morphism \( \theta_H : \mathcal{F}(H) \rightarrow \mathcal{G}(H) \) of monoids.

The above results imply that lax and colax monoidal functors from \( \mathcal{C} \) to \( \mathcal{D} \) induce functors

\[
\text{Mon}(\mathcal{C}) \rightarrow \text{Mon}(\mathcal{D}) \quad \text{and} \quad \text{Comon}(\mathcal{C}) \rightarrow \text{Comon}(\mathcal{D})
\]

respectively, and that a morphism between two (co)lax monoidal functors yields a natural transformation between the induced functors on (co)monoids.

**Proposition 3.31.** If \( (\mathcal{F}, \varphi, \psi) \) is a bilax monoidal functor from \((\mathcal{C}, \bullet, \beta)\) to \((\mathcal{D}, \bullet, \beta)\), and \( H \) is a bimonoid in \((\mathcal{C}, \bullet, \beta)\), then \( \mathcal{F}(H) \) is a bimonoid in \((\mathcal{D}, \bullet, \beta)\) with structure maps as in Definition 3.28.

Moreover, if \( f : H \rightarrow H' \) is a morphism of bimonoids in \((\mathcal{C}, \bullet, \beta)\), then the induced morphism \( \mathcal{F}(f) : \mathcal{F}(H) \rightarrow \mathcal{F}(H') \) is a morphism of bimonoids in \((\mathcal{D}, \bullet, \beta)\).

**Proof.** Argue as in the proof of Proposition 3.29, using Theorem 3.22 and Proposition 3.26.

**Proposition 3.32.** A morphism of bilax monoidal functors from \( \theta : \mathcal{F} \Rightarrow \mathcal{G} \) yields a morphism of bimonoids \( \theta_H : \mathcal{F}(H) \rightarrow \mathcal{G}(H) \) in \((\mathcal{D}, \bullet)\) when \( H \) is a bimonoid in \((\mathcal{C}, \bullet)\).

**Proof.** This follows from Proposition 3.30.

**3.4.4. Braided lax functors preserve commutative monoids.** Braided lax monoidal functors preserve commutative monoids, and there are dual results for braided colax monoidal functors. These and related results are discussed next.

Recall the discussion in Section 3.4.2 which relates the opposite construction on (co)monoids to the conjugate construction on (co)lax monoidal functors. This used in conjunction with Proposition 3.23 yields the following.
Proposition 3.33. Let \((F, \varphi)\) be a lax monoidal functor. The images of a monoid \(A\) under the lax monoidal functors \((F, \varphi^b)\) and \((F, b\varphi)\) are respectively
\[
F(A)^{\text{op}} \quad \text{and} \quad \text{op}F(A^{\text{op}}).
\]
Let \((F, \psi)\) be a colax monoidal functor. The images of a comonoid \(C\) under the colax monoidal functors \((F, \psi^b)\) and \((F, b\psi)\) are respectively
\[
F(C)^{\text{cop}} \quad \text{and} \quad \text{cop}F(C^{\text{cop}}).
\]

Proposition 3.34. Let \((F, \varphi, \psi)\) be a bilax monoidal functor \((C, \bullet, \beta) \to (D, \bullet, \beta)\) and let \(H\) be a bimonoid in \((C, \bullet, \beta)\). The image of \(H^{\text{cop}}\) under the bilax monoidal functor \((F, \varphi^b, \psi^b)\) is
\[
F(H)^{\text{cop}}
\]
and the image of \(\text{op}H\) under the bilax monoidal functor \((F, b\varphi, \psi)\) is
\[
\text{op}F(H).
\]
The images of \(H\) under the bilax monoidal functors \((F, \varphi^b, \psi^b)\) and \((F, b\varphi, b\psi)\) are respectively
\[
F(H)^{\text{cop}} \quad \text{and} \quad \text{op, cop}F(H^{\text{cop}}).
\]

Proof. This follows from Proposition 3.33. Alternatively, it may also be deduced directly from Proposition 3.23. \(\square\)

We emphasize a small point here. For the bilax functor \((F, \varphi, \psi^b)\), the correct braiding to use on \(C\) is \(\beta^{-1}\) rather than \(\beta\). Hence in the statement above, this functor is applied to \(H^{\text{cop}}\) and not \(H\). A similar remark applies to the bilax functor \((F, b\varphi, \psi)\).

Proposition 3.35. Let \(F\) be a braided lax (resp. colax) monoidal functor. Then for \(A\) a monoid (resp. \(C\) a comonoid), we have
\[
F(A)^{\text{op}} = \text{op}F(A) \quad \text{and} \quad F(A^{\text{op}}) = F(A)^{\text{op}}
\]
as monoids (resp.
\[
F(C)^{\text{cop}} = \text{cop}F(C) \quad \text{and} \quad F(C^{\text{cop}}) = F(C)^{\text{cop}}
\]
as comonoids).

Proposition 3.36. Let \(F\) be a braided bilax monoidal functor. Then for \(H\) a bimonoid, we have
\[
F(H^{\text{cop}}) = F(H)^{\text{cop}}, \quad F(H^{\text{op}}) = \text{op}F(H),
\]
\[
F(H)^{\text{cop}, \text{op}} = F(H)^{\text{op, cop}} \quad \text{and} \quad F(H^{\text{op}, \text{cop}}) = F(H)^{\text{op, cop}}
\]
as bimonoids.

The above results follow from Propositions 3.33 and 3.34.

Proposition 3.37. A braided (co)lax monoidal functor preserves (co)commutativity of (co)monoids and morphisms between (co)commutative (co)monoids.

Proof. The first assertion follows by combining Proposition 3.35 with (1.15) and (1.16). It may also be viewed as a special case of Proposition 3.24. The second assertion follows from the fact that the category of (co)commutative (co)monoids is a full subcategory of the category of (co)monoids. \(\square\)
The above result says that braided lax and braided colax monoidal functors induce functors
\[ \text{Mon}^{\text{co}}(C) \to \text{Mon}^{\text{co}}(D) \quad \text{and} \quad \text{coComon}(C) \to \text{coComon}(D) \]
respectively.

**Proposition 3.38.** A morphism of braided (co)lax monoidal functors \( \theta : \mathcal{F} \to \mathcal{G} \) yields a morphism of (co)commutative (co)monoids \( \theta_H : \mathcal{F}(H) \to \mathcal{G}(H) \) in \((D, \bullet)\) when \( H \) is a (co)commutative (co)monoid in \((C, \bullet)\).

**Proof.** This follows from Proposition 3.30. \( \square \)

### 3.4.5. The convolution monoid revisited

Recall the construction of the convolution monoid from Definition 1.13. This construction can be understood in terms of monoidal functors as follows.

A monoid \((C, A)\) in \(C^{\text{op}} \times C\) is the same as a comonoid \(C\) and a monoid \(A\) in \(C\). The convolution monoid \(\text{Hom}(C, A)\) then arises as the image of the monoid \((C, A)\) under the lax monoidal functor \(\text{Hom} : C^{\text{op}} \times C \to \text{Set}\) of Example 3.17. In fact, we saw in Example 3.12 that this functor is braided; so it preserves commutativity. A commutative monoid \((C, A)\) in \(C^{\text{op}} \times C\) is the same as a cocommutative comonoid \(C\) and a commutative monoid \(A\) in \(C\). Thus, in this case the convolution monoid \(\text{Hom}(C, A)\) is commutative.

The convolution monoid arises in yet another manner. Consider the functors
\[ \mathcal{F}_C : I \to C \] and \[ \mathcal{F}_A : I \to C, \]
as in Section 3.4.1. The former is colax monoidal and the latter is lax monoidal. We may thus consider the convolution comma category \(\mathcal{F}_C \downarrow \mathcal{F}_A\) of Section 3.1.4. This is a monoidal category. The objects are arrows \(C \to A\) in \(C\), and the only morphisms are identities. In other words, it is the discrete category corresponding to the set \(\text{Hom}(C, A)\). Further, the monoidal structure of \(\mathcal{F}_C \downarrow \mathcal{F}_A\) boils down to the monoid structure of \(\text{Hom}(C, A)\).

### 3.5. Normal bilax monoidal functors

In this section, we discuss normal bilax monoidal functors. The terminology is motivated by the example of the normalized chain complex functor, which is discussed in Section 5.4.

**Definition 3.39.** A bilax monoidal functor \((\mathcal{F}, \varphi, \psi)\) is called **normal** if
\[ \varphi_0 \psi_0 = \text{id}. \] (3.24)

Note that a bilax functor always satisfies the condition \(\psi_0 \varphi_0 = \text{id}\), by (3.13). Thus, if it is normal, then \(\varphi_0\) and \(\psi_0\) are inverse maps. The Fock functors we consider in Part III are normal. However, not every bilax functor is normal. An example of a class of bilax functors that are not normal is given below.

**Remark 3.40.** Let \(H\) be a bimonoid and \(\mathcal{F}_H\) be the corresponding bilax functor of (3.23). The bilax functor \(\mathcal{F}_H\) is normal if and only if \(\iota : I \to H\) and \(\epsilon : H \to I\) are inverse maps. Thus, \(\mathcal{F}_H\) is normal if and only if \(H\) is the trivial bimonoid. In particular, not every bilax functor is normal.

Normal bilax functors satisfy some interesting properties which we discuss next.
Proposition 3.41. Let \((\mathcal{F}, \varphi, \psi)\) be a normal bilax monoidal functor. Then the following properties hold for any objects \(A, B, C\).

1. The maps \(\varphi_{A,I}\) and \(\psi_{A,I}\) are inverse, and so are the maps \(\varphi_{I,A}\) and \(\psi_{I,A}\).
2. \(\psi_{A,B}\varphi_{A,B} = \text{id}\).
3. The following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{F}(A) \otimes \mathcal{F}(B) & \xrightarrow{\varphi_{A,B}} & \mathcal{F}(A \otimes B) \\
\bigtriangledown & & \bigtriangledown \\
\mathcal{F}(B) \otimes \mathcal{F}(A) & \xleftarrow{\psi_{B,A}} & \mathcal{F}(B \otimes A)
\end{array}
\]

This is equivalent to \(\psi^b \varphi = \text{id}\) and also to \((\psi^b) \varphi = \text{id}\).
4. The following diagrams commute.

\[
\begin{array}{ccc}
\mathcal{F}(A) \otimes \mathcal{F}(B \otimes C) & \xrightarrow{\text{id} \otimes \psi_{B,C}} & \mathcal{F}(A) \otimes \mathcal{F}(B) \otimes \mathcal{F}(C) \\
\bigtriangledown & & \bigtriangledown \\
\mathcal{F}(A \otimes B \otimes C) & \xleftarrow{\psi_{A,B \otimes C}} & \mathcal{F}(A \otimes B) \otimes \mathcal{F}(C)
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{F}(A \otimes B) \otimes \mathcal{F}(C) & \xrightarrow{\psi_{A,B} \otimes \text{id}} & \mathcal{F}(A) \otimes \mathcal{F}(B) \otimes \mathcal{F}(C) \\
\bigtriangledown & & \bigtriangledown \\
\mathcal{F}(A \otimes B \otimes C) & \xleftarrow{\text{id} \otimes \psi_{B,C}} & \mathcal{F}(A) \otimes \mathcal{F}(B \otimes C)
\end{array}
\]

Proof. The right diagram in (3.6) tells us that

\[\varphi_{A,I} = \mathcal{F}(\rho_A)^{-1} \mathcal{F}(\rho_A^{-1}) (\text{id} \otimes \varphi_0^{-1}),\]

while the dual diagram tells us that

\[\psi_{A,I} = (\text{id} \otimes \psi_0^{-1}) \rho_{\mathcal{F}(A)} \mathcal{F}(\rho_A)^{-1}\]

Since \(\varphi_0 = \psi_0^{-1}\), we have \(\varphi_{A,I} = \psi_{A,I}^{-1}\). One checks similarly that \(\varphi_{I,A}\) and \(\psi_{I,A}\) are inverse maps. This proves the first property.

The second property follows by the commutativity of the diagram below. For simplicity of notation, we omit the tensor product symbols. We write \(I\) for the unit object in both the source and the target category. The composite map on the top side of the big square is \(\psi_{A,B} \varphi_{A,B}\) and the composite obtained by following the
other three sides is the identity, since $\beta_{I,I} = \text{id}$.

The hexagon in the center commutes since $\mathcal{F}$ satisfies the braiding axiom (3.11). The other smaller diagrams, starting from the top left corner and going in counterclockwise direction, commute by the naturality of $\varphi$, the unitality of $\psi$, the naturality of $\beta$, the hypothesis $\varphi_0 \psi_0 = \text{id}$, the unitality of $\varphi$ and the naturality of $\psi$ respectively.

For the third property, we may proceed directly as in the above proof. Alternatively, we may use Proposition 3.16 to first deduce that $(\mathcal{F}, b^\varphi, \psi)$ and $(\mathcal{F}, \varphi, b^\psi)$ are bilax monoidal functors. Note that $\varphi_0$ and $\psi_0$ do not change during this construction. Now applying the second property to each of these functors, we obtain $\psi(b^\varphi) = \text{id}$ and $\varphi(b^\psi) = \text{id}$, which are both equivalent to the commutativity of (3.25).

For the fourth property, diagram (3.26) commutes by the commutativity of the diagram below. One can then use symmetry to deduce that diagram (3.27)
commutes as well.

The outside square is the diagram we want. The hexagon in the center commutes since \( F \) satisfies the braiding axiom (3.11). The other smaller diagrams commute by the naturality and unitality of \( \varphi \) and \( \psi \), the naturality of \( \beta \) and its compatibility with the unit (1.7), and the hypothesis \( \varphi_0 \psi_0 = \text{id} \).

We give an example below which shows that the converse to Proposition 3.41 is false. For a related result, see Proposition 3.46.

**Example 3.42.** Let \( G \) be a finite group and let \( \text{Mod}_G \) be the symmetric monoidal category of left \( G \)-modules. Consider the functor

\[
(-)^G : \text{Mod}_G \to \text{Vec}
\]

which sends a module \( M \) to \( M^G \), the space of \( G \)-invariants of \( M \) (Section 2.5.1). Define natural transformations \( \varphi \) and \( \psi \) as in (3.9) to be

\[
M^G \otimes N^G \xrightarrow{\varphi_{M,N}} (M \otimes N)^G;
\]

\[
M^G \otimes N^G \xrightarrow{\psi_{M,N}} (M \otimes N)^G.
\]
where \( \varphi \) is the natural inclusion and \( \psi \) is given by either of the following expressions
\[
\psi \left( \sum_i m_i \otimes n_i \right) = \frac{1}{|G|} \sum_{g \in G} \sum_i g \cdot m_i \otimes n_i
\]
\[
= \frac{1}{|G|} \sum_{h \in G} \sum_i m_i \otimes h \cdot n_i
\]
\[
= \frac{1}{|G|^2} \sum_{g, h \in G} \sum_i g \cdot m_i \otimes h \cdot n_i.
\]
Define the morphisms \( \varphi_0 \) and \( \psi_0 \) as in (3.10) to be the identity maps.

One has \( \varphi_0 \psi_0 = \text{id} \) and can show that \( \left( (-)^G, \varphi, \psi \right) \) satisfies all the properties in Proposition 3.41; however, it is not bilax. In fact, one may check that the braiding axiom (3.11) does not hold.

**Remark 3.43.** Functors \( (\mathcal{F}, \varphi, \psi) \) satisfying (3.26) and (3.27) have been considered in the literature; they are called Frobenius monoidal functors \cite{92, 265, 352, 353}. A Frobenius functor is said to be separable if in addition it satisfies
\[
\varphi_{A, B} \psi_{A, B} = \text{id}_{\mathcal{F}(A \bullet B)}.
\]
Note the difference with condition (ii) in Proposition 3.41. McCurdy and Street \cite[Proposition 3.10]{265} show that a separable Frobenius functor necessarily satisfies the braiding axiom (3.11). By contrast, the conditions in Proposition 3.41 do not suffice to imply (3.11), as Example 3.42 shows.

### 3.6. Bistrong monoidal functors

Strong, costrong and bistrong monoidal functors were introduced in Definition 3.5. In this section, we study these notions in more detail.

Recall that a strong monoidal functor is a lax monoidal functor \( (\mathcal{F}, \varphi) \) for which the transformation \( \varphi \) is invertible. In this case, the functor \( (\mathcal{F}, \varphi^{-1}) \) is colax, and so it is natural to wonder whether \( (\mathcal{F}, \varphi, \psi^{-1}) \) may be bilax. In Proposition 3.46 we show that this is the case if and only if the lax monoidal functor \( (\mathcal{F}, \varphi) \) is braided. This is an important difference with the general case, in which a braided lax monoidal functor may not be bilax, and a bilax monoidal functor may not be braided.

Recall that a bistrong monoidal functor is a bilax monoidal functor \( (\mathcal{F}, \varphi, \psi) \) for which \( \varphi \) and \( \psi \) are invertible. In Proposition 3.45 we show that in this case necessarily \( \psi = \varphi^{-1} \). It then follows that bistrong and braided strong are equivalent notions.

Another significant property of bistrong monoidal functors is that they preserve Hopf monoids. This is shown in Proposition 3.50. We mention that a general setup for dealing with the problem of preservation of Hopf monoids is considered in Section 3.7; see Proposition 3.60 and Remark 3.71 for the relevance to the present discussion.

#### 3.6.1. Strong and bistrong monoidal functors

Throughout this discussion, \( C \) and \( D \) are monoidal categories, \( \mathcal{F}: C \to D \) is a functor between them, \( \varphi: \mathcal{F}^2 \Rightarrow \mathcal{F}_{2} \) and \( \psi: \mathcal{F}_{2} \to \mathcal{F}^2 \) are natural transformations as in (3.3) and (3.7), and finally \( \varphi_0: I \to \mathcal{F}(I) \) and \( \psi_0: \mathcal{F}(I) \to I \) are maps in \( D \).

The following result is immediate from Definition 3.5.
Proposition 3.44. Assume that \( \varphi \) and \( \varphi_0 \) are invertible. Define \( \psi = \varphi^{-1} \) and \( \psi_0 = \varphi_0^{-1} \). Then

\[
(F, \varphi) \text{ is strong } \iff (F, \psi) \text{ is costrong.}
\]

Thus, strong and costrong are equivalent notions.

Recall the notion of normal bilax monoidal functors from Definition 3.39.

Proposition 3.45. Let \((F, \varphi, \psi)\) be a bistrong functor. Then \(\psi_0 = \varphi_0^{-1}\) and \(\psi = \varphi^{-1}\). In particular, \(F\) is normal. Conversely, if \((F, \varphi, \psi)\) is a normal bilax monoidal functor such that \(\varphi \psi = \text{id}\), then \((F, \varphi, \psi)\) is bistrong.

**Proof.** From (3.13) we know that \(\psi_0 \varphi_0 = \text{id}\). But since under the present hypothesis these maps are invertible (Definition 3.5), we have that \(\psi_0 = \varphi_0^{-1}\), that is, \(F\) is normal. It follows from Proposition 3.41 that \(\psi \varphi = \text{id}\), and again since these maps are invertible we have that \(\psi = \varphi^{-1}\).

Conversely, if \(F\) is normal, then \(\psi_0\) and \(\varphi_0\) are inverse maps. Further, Proposition 3.41 gives \(\psi \varphi = \text{id}\), which with the hypothesis says that \(\psi\) and \(\varphi\) are inverse maps. Hence \(F\) is bistrong. \(\square\)

Proposition 3.46. The following are equivalent.

(i) \((F, \varphi, \psi)\) is bistrong.

(ii) \((F, \varphi)\) is braided strong, \(\psi = \varphi^{-1}\), and \(\psi_0 = \varphi_0^{-1}\).

(iii) \((F, \psi)\) is braided costrong, \(\varphi = \psi^{-1}\), and \(\varphi_0 = \psi_0^{-1}\).

**Proof.** It is clear that braided strong is equivalent to braided costrong. The nontrivial part is to show the equivalence between bistrong and braided strong. Suppose \((F, \varphi, \psi)\) is bistrong. By Proposition 3.45, it is normal and \(\psi = \varphi^{-1}\). We may then use Proposition 3.41, part (iii), to deduce that diagram (3.25) commutes, which since \(\psi = \varphi^{-1}\) is equivalent to \((F, \varphi)\) being braided.

For the converse implication, we first note that for a braided strong functor \((F, \varphi)\), diagram (3.16) and the associativity of \(\varphi\) imply that diagrams (3.25), (3.26) and (3.27) commute. The braiding axiom (3.11) for \((F, \varphi, \psi)\) then follows from the commutativity of the diagram below (in which tensor product symbols are omitted).
The two triangles commute by construction, the hexagon commutes by the naturality of \( \varphi \) or \( \psi \) (here we use \( \psi = \varphi^{-1} \)), and the remaining five squares commute by diagrams (3.25), (3.26) and (3.27).

Since \( \psi = \varphi^{-1} \), the two diagrams in (3.12) are essentially the same. Their commutativity follows by setting \( A = I \) in the unitality axiom (3.6) for \((\mathcal{F}, \varphi)\). The commutativity of (3.13) follows since \( \psi_0 = \varphi_0^{-1} \). This verifies the unitality axioms for \((\mathcal{F}, \varphi, \psi)\) and completes the proof.

\[ \Box \]

**Remark 3.47.** The same proof essentially as above, but with a new perspective, is given in Chapter 6. It works as follows. There is an equivalent way to define a braided (co)lax functor which is closer to the definition of a bilax functor. In fact, by reversing some arrows, one can pass back and forth between the two definitions. The above result then follows, since all arrows are invertible. For more details, see Example 6.64. To summarize, in the strong situation, the distinction between braided lax and bilax disappears.

#### 3.6.2. Bistrong functors preserve Hopf monoids and antipodes

Below (Proposition 3.50) we show that the image of a Hopf monoid under a bistrong monoidal functor is again a Hopf monoid, and that the antipode of the former is the image of the antipode of the latter.

**Proposition 3.48.** A bilax monoidal functor \( \mathcal{F} : C \to D \) is bistrong if and only if the natural transformation

\[
\text{Hom}_C(-, -) \Rightarrow \text{Hom}_D(\mathcal{F}(-), \mathcal{F}(-))
\]

which sends \( f \) to \( \mathcal{F}(f) \) is a morphism of lax monoidal functors.

The second functor is the composite of the lax functors

\[
C^{\text{op}} \times C \xrightarrow{\mathcal{F} \times \mathcal{F}} D^{\text{op}} \times D \xrightarrow{\text{Hom}} \text{Set}.
\]

For the lax structure on \( \mathcal{F} \times \mathcal{F} \), one uses the colax structure of \( \mathcal{F} \) on the first component and the lax structure on the second component. The lax structure of \( \text{Hom} \) is described in Example 3.17.

**Proof.** The natural transformation

\[
\text{Hom}_C(-, -) \Rightarrow \text{Hom}_D(\mathcal{F}(-), \mathcal{F}(-))
\]

is a morphism of lax functors if and only if the following diagrams commute.

\[
\begin{array}{ccc}
\text{Hom}(A, C) \times \text{Hom}(B, D) & \xrightarrow{\biota} & \text{Hom}(\mathcal{F}(A), \mathcal{F}(C)) \times \text{Hom}(\mathcal{F}(B), \mathcal{F}(D)) \\
\downarrow & & \downarrow \\
\text{Hom}(A \bullet B, C \bullet D) & \xrightarrow{\varphi_{A, B}} & \text{Hom}(\mathcal{F}(A \bullet B), \mathcal{F}(C \bullet D))
\end{array}
\]

\[
\varphi_{C, D} \circ \psi_{A, B}
\]
This is equivalent to the commutativity of the diagrams

\[
\begin{array}{ccc}
\mathcal{F}(A \bullet B) & \xrightarrow{\mathcal{F}(f \circ g)} & \mathcal{F}(C \bullet D) \\
\psi_{A,B} & & \varphi_{C,D}
\end{array}
\quad
\begin{array}{ccc}
\mathcal{F}(I) & \xrightarrow{id} & \mathcal{F}(I)
\end{array}
\]

for any morphisms \( f: A \rightarrow C \) and \( g: B \rightarrow D \).

For the forward implication, we note that if \( \mathcal{F} \) is bistrong, then the left diagram commutes by the naturality of \( \psi \) or \( \varphi \) (since \( \varphi = \psi^{-1} \)) and the right diagram commutes since \( \varphi_0 = \psi_0^{-1} \).

For the backward implication, assume that the above diagrams commute. Setting \( A = C \) and \( B = D \) and \( f \) and \( g \) to be identities, we conclude that \( \varphi \psi = \text{id} \) and \( \varphi_0 \psi_0 = \text{id} \). Then Proposition 3.45 implies that \( \mathcal{F} \) is bistrong. \( \square \)

Let \( A \) be a monoid and \( C \) be a comonoid in \( \mathcal{C} \). As mentioned in Section 3.4.5, the convolution monoid \( \text{Hom}(C, A) \) arises as the image of the monoid \( (C, A) \) in \( \mathcal{C}^{\text{op}} \times \mathcal{C} \) under the lax monoidal functor \( \text{Hom}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set} \). Therefore, Propositions 3.30 and 3.48 imply:

**Proposition 3.49.** For \( \mathcal{F} \) a bistrong monoidal functor from \((\mathcal{C}, \bullet)\) to \((\mathcal{D}, \bullet)\) and \( C \) a comonoid and \( A \) a monoid in \((\mathcal{C}, \bullet)\), there is a morphism of convolution monoids

\[
\text{Hom}(C, A) \rightarrow \text{Hom}(\mathcal{F}(C), \mathcal{F}(A))
\]

which sends \( f \) to \( \mathcal{F}(f) \).

It follows that a bistrong monoidal functor preserves antipodes. In more detail, we have:

**Proposition 3.50.** If \( \mathcal{F} \) is a bistrong monoidal functor from \((\mathcal{C}, \bullet)\) to \((\mathcal{D}, \bullet)\) and \( H \) is a Hopf monoid in \((\mathcal{C}, \bullet)\) with antipode \( s: H \rightarrow H \), then \( \mathcal{F}(H) \) is a Hopf monoid in \((\mathcal{D}, \bullet)\) with antipode \( \mathcal{F}(s): \mathcal{F}(H) \rightarrow \mathcal{F}(H) \).

In addition, if \( f: H \rightarrow H' \) is a morphism of Hopf monoids in \((\mathcal{C}, \bullet)\), then \( \mathcal{F}(f): \mathcal{F}(H) \rightarrow \mathcal{F}(H') \) is a morphism of Hopf monoids in \((\mathcal{D}, \bullet)\).

Since a morphism of Hopf monoids is a morphism of the underlying bimonoids, Proposition 3.32 implies:

**Proposition 3.51.** A morphism of bistrong monoidal functors from \( \mathcal{F} \) to \( \mathcal{G} \) yields a morphism of Hopf monoids \( \mathcal{F}(H) \rightarrow \mathcal{G}(H) \) in \((\mathcal{D}, \bullet)\) when \( H \) is a Hopf monoid in \((\mathcal{C}, \bullet)\).

**Example 3.52.** Let \( k \) be a commutative ring. Consider the linearization functor

\[
\mathbb{k}(-): (\text{Set}, \times, \{\ast\}) \rightarrow (\text{Mod}_k, \otimes, k),
\]
which sends a set to the free $\mathbb{k}$-module with basis the given set. This functor is bistrong. Below we discuss three implications of this statement.

- Every set $X$ carries a unique comonoid structure in $(\text{Set}, \times, \{\ast\})$. The coproduct $\Delta : X \to X \times X$ is $\Delta(x) = (x, x)$ and the counit $\epsilon : X \to \{\ast\}$ is $\epsilon(x) = \ast$.

  It follows that $\mathbb{k}X$ is a coalgebra in which all elements of $X$ are group-like, that is, $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$ for $x \in X$. This is the coalgebra of a set [1, Example 2.1] or [350, Section 1.0, Example 1].

- If the set $X$ is a monoid (in the usual sense), then it is canonically a bimonoid in $(\text{Set}, \times, \{\ast\})$, and hence $\mathbb{k}X$ is a bialgebra.

- A monoid $X$ is a Hopf monoid precisely if $X$ is a group. (The antipode sends an element to its inverse.) Hence for any group $X$, the group algebra $\mathbb{k}X$ is a Hopf algebra. Its antipode is the linearization of the map of $x \mapsto x^{-1}$. See [191, Section III.3, Example 2] or [350, Section 4.0, Example 1].

3.7. Hopf lax monoidal functors

We know that a bilax monoidal functor preserves bimonoids. In addition, we have seen that bistrong monoidal functors preserve Hopf monoids and antipodes (Proposition 3.50). However, an arbitrary bilax monoidal functor need not preserve Hopf monoids or antipodes. In other words: If $H$ is a Hopf monoid in $C$ with antipode $s : H \to H$, and if $F$ is a bilax monoidal functor from $C$ to $D$, then $F(H)$ need not be a Hopf monoid, and even if it is, the antipode of $F(H)$ need not be $F(s)$.

We provide a simple example. Consider the one-arrow category $I$. Then its unique object $\ast$ is a Hopf monoid, whose antipode is the identity. Associated to any bimonoid $H$ there is the bilax monoidal functor $F_H$ of (3.23), and $F_H(\ast) = H$, which may be a Hopf monoid or not. Even when this is the case, the antipode of $H$ need not be the identity.

Numerous examples with these features appear in the later parts of the monograph. To give one concrete example, consider the Hopf monoid $L^\ast$ of linear orders in the category of species (Example 8.24). Call its antipode $s$. Applying the full Fock functor $K$ to $L^\ast$ yields the Hopf algebra of permutations $S\Lambda$ (Example 15.17), whose antipode is not $K(s)$.

The goal of this section is to show that there is an intermediate class of functors between bilax and bistrong that preserves Hopf monoids but modifies antipodes in a predictable manner, much as the rest of the structure is modified by a bilax monoidal functor. We call them Hopf lax monoidal functors. They have an interesting basic theory which we now present.

**Notation 3.53.** For $(F, \varphi)$ lax, we write

$$\varphi_{A,B,C} : F(A) \bullet F(B) \bullet F(C) \to F(A \bullet B \bullet C)$$

for the map obtained by following the two directions in diagram (3.5). Note that we are not specifying brackets here; the objects are to be interpreted as the unbracketed tensor products.

Similarly for $(F, \psi)$ colax, we write

$$\psi_{A,B,C} : F(A \bullet B \circ C) \to F(A) \bullet F(B) \bullet F(C).$$
Suppose \((\mathcal{F}, \varphi, \psi)\) and \((\mathcal{G}, \gamma, \delta)\) are composable functors, as in Definition 3.20. Then,
\[
\begin{align*}
(\varphi \gamma)_{A,B,C} &= \mathcal{G}(\varphi_{A,B,C}) \gamma_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C} \\
(\delta \psi)_{A,B,C} &= \delta_{\mathcal{F}A,\mathcal{F}B,\mathcal{F}C} \mathcal{G}(\psi_{A,B,C}).
\end{align*}
\]
These identities follow from the proof of Theorem 3.21.

### 3.7.1. Hopf lax monoidal functors

Let \((\mathcal{C}, \bullet, \beta)\) and \((\mathcal{D}, \cdot, \delta)\) be braided monoidal categories.

**Definition 3.54.** A *Hopf lax monoidal functor* \((\mathcal{F}, \varphi, \psi, \Upsilon)\) consists of a bilax monoidal functor \((\mathcal{F}, \varphi, \psi)\) from \(\mathcal{C}\) to \(\mathcal{D}\) and a natural transformation \(\Upsilon: \mathcal{F} \Rightarrow \mathcal{F}\) such that the following diagrams commute.

\[
\begin{align*}
\mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) & \xrightarrow{id_A \bullet \Upsilon_B \bullet id_C} \mathcal{F}(A) \bullet \mathcal{F}(B) \bullet \mathcal{F}(C) \\
\mathcal{F}(A \bullet B \bullet C) & \xrightarrow{id_{A \bullet B \bullet C}} \mathcal{F}(A \bullet B \bullet C) \\
\varphi_{A,B,C} & \xrightarrow{\mathcal{F}(A \bullet B \bullet C)} \varphi_{A,B,C}
\end{align*}
\]

\[
\begin{align*}
\mathcal{F}(I) \bullet \mathcal{F}(I) & \xrightarrow{id_I \cdot \Upsilon_I} \mathcal{F}(I) \bullet \mathcal{F}(I) \\
\mathcal{F}(I \bullet I) & \xrightarrow{\varphi_{I,I}} \mathcal{F}(I \bullet I) \\
\mathcal{F}(I) & \xrightarrow{\varphi_0 \cdot \psi_0} \mathcal{F}(I)
\end{align*}
\]

We say that \(\mathcal{F}\) is a Hopf lax monoidal functor with antipode \(\Upsilon\).

We give a reformulation of axiom (3.31). Recall that the unit object \(I\) of a braided monoidal category \(\mathcal{C}\) is a bimonoid. Suppose that \((\mathcal{F}, \varphi, \psi)\) is a bilax functor from \(\mathcal{C}\) to \(\mathcal{D}\) and \(\Upsilon: \mathcal{F} \Rightarrow \mathcal{F}\) is a natural transformation. Then \(\mathcal{F}\) preserves bimonoids by Proposition 3.31. By construction, the coproduct and product of \(\mathcal{F}(I)\) are the composites of the vertical maps in (3.31) and the counit and unit are \(\psi_0\) and \(\varphi_0\) respectively. In this situation,

\[
(3.32) \quad \mathcal{F} \text{ satisfies axiom (3.31)} \iff \mathcal{F}(I) \text{ is a Hopf monoid with antipode } \Upsilon_I.
\]

In particular, if \((\mathcal{F}, \varphi, \psi, \Upsilon)\) is Hopf lax, then \(\mathcal{F}(I)\) is a Hopf monoid with antipode \(\Upsilon_I\).
**Lemma 3.55.** The antipode of a Hopf lax functor $F$ is determined by its value on the unit object, in the sense that the following diagram commutes.

\[
\begin{array}{ccc}
F(I) & \xrightarrow{\phi_{I,A,I}} & F(A) \\
\psi_{I,A,I} & & \psi_{A,I} \\
F(I \bullet A \bullet I) & \cong & F(A) \\
\end{array}
\]

\[
\begin{array}{ccc}
F(I) \bullet F(A) \bullet F(I) & \xrightarrow{\varphi_{I,A,I}} & F(I) \bullet F(A) \bullet F(I) \\
\Upsilon_I \bullet \id_A \bullet \Upsilon_I & & \Upsilon_I \bullet \id_A \bullet \Upsilon_I \\
F(I \bullet A \bullet I) & \cong & F(A) \\
\end{array}
\]

**Proof.** The vertical maps near the bottom are the canonical isomorphisms constructed from the unit constraints. The diagram can be divided in two by inserting $\Upsilon_I \bullet A \bullet I$ in the middle. The top half commutes by (3.30) and the bottom half by naturality of $\Upsilon$. \qed

**Proposition 3.56.** The antipode of a Hopf lax functor is unique.

**Proof.** Let $F$ be a Hopf lax functor. We know from (3.32) that $F(I)$ is a Hopf monoid with antipode $\Upsilon_I$. Hence $\Upsilon_I$ is unique. Then $\Upsilon_A$ is determined by (3.33). \qed

**Definition 3.57.** Let $F$ be a Hopf lax functor. Define natural transformations $\upsilon: F \Rightarrow F$ and $\upsilon': F \Rightarrow F$ by

\[
\begin{array}{ccc}
F(A) \bullet F(I) & \xrightarrow{\id_A \bullet \Upsilon_I} & F(A) \bullet F(I) \\
\psi_{A,I} & & \psi_{A,I} \\
F(A \bullet I) & \cong & F(A) \\
\end{array}
\]

We say that $\upsilon$ and $\upsilon'$ are the convolution units associated to $F$.

There is an alternative way to define $\upsilon$ and $\upsilon'$; see Remark 3.67.

### 3.7.2. Morphisms of Hopf lax monoidal functors.

**Definition 3.58.** A morphism of Hopf lax monoidal functors is a morphism of the underlying bilax monoidal functors (Definition 3.9).

Next we show that such a morphism necessarily preserves antipodes and the associated convolution units, thus justifying the terminology.

**Proposition 3.59.** Let $(F, \varphi, \psi, \Upsilon)$ and $(G, \gamma, \delta, \Omega)$ be Hopf lax functors from $C$ to $D$. Let $\upsilon$ and $\upsilon'$ be the convolution units associated to $F$, as in (3.34), and $\omega$ and...
those associated to \(\mathcal{G}\). Let \(\theta: \mathcal{F} \rightarrow \mathcal{G}\) be a morphism of bilax functors. Then the following diagrams commute, for any object \(A\).

\[
\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\Upsilon_A} & \mathcal{F}(A) \\
\downarrow \theta_A & & \downarrow \theta_A \\
\mathcal{G}(A) & \xrightarrow{\Omega_A} & \mathcal{G}(A)
\end{array}
\quad
\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\nu_A} & \mathcal{F}(A) \\
\downarrow \theta_A & & \downarrow \theta_A \\
\mathcal{G}(A) & \xrightarrow{\omega_A} & \mathcal{G}(A)
\end{array}
\quad
\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\nu'_A} & \mathcal{F}(A) \\
\downarrow \theta_A & & \downarrow \theta_A \\
\mathcal{G}(A) & \xrightarrow{\omega'_A} & \mathcal{G}(A)
\end{array}
\]  

\[(3.35)\]

**Proof.** Applying the forward direction of (3.32), \(\mathcal{F}(I)\) and \(\mathcal{G}(I)\) are Hopf monoids with antipodes \(\Upsilon_I\) and \(\Omega_I\), respectively. By Proposition 3.32, \(\theta_I: \mathcal{F}(I) \rightarrow \mathcal{G}(I)\) is a morphism of bimonoids, and hence, by Proposition 1.16, it preserves the antipodes. Thus, the first diagram in (3.35) commutes when \(A = I\). The general case follows by using diagrams (3.14) and (3.15), since \(\Upsilon_A\) is determined by \(\Upsilon_I\) (3.33).

The commutativity of the other diagrams follows similarly, using that \(\nu_A\) and \(\nu'_A\) are determined by \(\Upsilon\) by means of (3.34). \(\square\)

### 3.7.3. Bistrong versus Hopf lax

Bistrong monoidal functors (Definition 3.5) are always Hopf lax. The converse implication holds provided the functor is in addition normal (Definition 3.39). In other words:

**Proposition 3.60.** Let \((\mathcal{F}, \varphi, \psi)\) be a bilax monoidal functor. Then

\[\mathcal{F}\] is bistrong \(\iff\) \(\mathcal{F}\) is Hopf lax and normal.

In this case, the antipode is \(\Upsilon = \text{id}\).

**Proof.** If \(\mathcal{F}\) is bistrong, the transformations \(\varphi_{A,B}\) and \(\psi_{A,B}\) are inverse. It follows that so are \(\varphi_{A,B,C}\) and \(\psi_{A,B,C}\). Hence diagrams (3.29)–(3.30) commute with \(\Upsilon = \text{id}\). Since \(\varphi_0\) and \(\psi_0\) are inverse, diagrams (3.31) commute too. Thus, \(\mathcal{F}\) is Hopf lax and normal.

Assume now that \(\mathcal{F}\) is Hopf lax and normal. Proposition 3.41 implies \(\psi\varphi = \text{id}\), so we only need to show \(\varphi\psi = \text{id}\).

Proposition 3.41 also tells us that \(\varphi_{I,I}\) and \(\psi_{I,I}\) are inverse maps. Therefore, diagrams (3.31) commute when \(\Upsilon_I\) is replaced by \(\text{id}_I\). By uniqueness of antipodes for the Hopf monoid \(\mathcal{F}(I)\), we must have \(\Upsilon_I = \text{id}_I\). Now considering diagram (3.29) with \(B = I\) we deduce

\[\varphi_{A,I,C}\psi_{A,I,C} = \text{id}.

But the diagram below shows that \(\varphi_{A,I,C}\) identifies with \(\varphi_{A,C}\) by composing with \(\varphi_0\):

\[
\begin{array}{cccccc}
\mathcal{F}(A) \bullet \mathcal{F}(I) \bullet \mathcal{F}(C) & \xrightarrow{\text{id} \bullet \varphi_{I,C}} & \mathcal{F}(A) \bullet \mathcal{F}(I \bullet C) & \xrightarrow{\varphi_{A,I \bullet C}} & \mathcal{F}(A \bullet I \bullet C) \\
\downarrow \varphi_{A,C} & & \downarrow \varphi_{A,I \bullet C} & & \downarrow \varphi_{A,I,C} \\
\mathcal{F}(A) \bullet I \bullet \mathcal{F}(C) & \xrightarrow{\text{id} \bullet \lambda_{\mathcal{F}(C)}} & \mathcal{F}(A) \bullet \mathcal{F}(C) & \xrightarrow{\varphi_{A,C}} & \mathcal{F}(A \bullet C).
\end{array}
\]
The diagram commutes by (3.5), (3.6), and the naturality of $\lambda$. Similarly, $\psi_{A,I,C}$ identifies with $\psi_{A,C}$ by composing with $\psi_0$. Therefore,

$$\varphi_{A,C} \psi_{A,C} = \text{id.}$$

The main examples of bilax functors discussed in this monograph that are not bistrong turn out not to be Hopf lax either. The reason is that they are normal.

3.7.4. Convolution of natural transformations. The antipode of a Hopf monoid is the inverse of the identity map in the convolution monoid (Definition 1.15). The situation is somewhat different for bilax monoidal functors; nevertheless, there is an operation between natural transformations that plays the role of convolution; it is defined in (3.37). The convolution identities in this context involve another operation that has no analogue for Hopf monoids; it is defined in (3.36).

Recall from (3.1) that if $F$ is a functor between monoidal categories $C$ and $D$, then $F_2 : C \times C \to D$ is the functor

$$F_2(A, B) := F(A \bullet B).$$

Suppose that $G : C \to D$ is another functor and $\theta : F \Rightarrow G$ is a natural transformation. We let

$$\theta^{(2)} : F_2 \Rightarrow G_2$$

be the natural transformation

(3.36) $\theta^{(2)}_{A,B} : F(A \bullet B) \xrightarrow{\theta_{A \bullet B}} G(A \bullet B).$

Now assume that $(F, \psi)$ is a colax monoidal functor and $(G, \gamma)$ is a lax monoidal functor. Given natural transformations $\sigma$ and $\tau : F \Rightarrow G$, we define their convolution

$$\sigma \ast \tau : F_2 \Rightarrow G_2$$

as the natural transformation

(3.37) $$(\sigma \ast \tau)_{A,B} : F(A \bullet B) \xrightarrow{\psi_{A,B}} F(A) \bullet F(B) \xrightarrow{\sigma_{A} \bullet \tau_{B}} G(A) \bullet G(B) \xrightarrow{\gamma_{A,B}} G(A \bullet B).$$

We study the behavior of morphisms of lax and colax functors (Definition 3.8) with respect to convolution of natural transformations.

**Proposition 3.61.** Let $(F, \psi)$ and $(F', \psi')$ be colax functors and $(G, \gamma)$ and $(G', \gamma')$ be lax functors, all from $C$ to $D$. Let

$$\theta : (F', \psi') \Rightarrow (F, \psi) \quad \text{and} \quad \kappa : (G, \gamma) \Rightarrow (G', \gamma')$$

be a morphism of colax functors and a morphism of lax functors, respectively. Then, for any natural transformations $\sigma, \tau : F \Rightarrow G$, we have

$$(\sigma \theta) \ast (\tau \theta) = (\sigma \ast \tau) \theta^{(2)} \quad \text{and} \quad (\kappa \sigma) \ast (\kappa \tau) = \kappa^{(2)}(\sigma \ast \tau).$$

This is the analogue of Proposition 1.14. The proof is straightforward.
3.7.5. Convolution identities. We now establish some familiar convolution identities in the context of Hopf lax functors. The transformations that play the role of the convolution unit are defined in (3.34). Here we take the most direct approach to establishing these identities in order to quickly build up to Theorem 3.70. A more in-depth study of convolution of natural transformations is carried out in Section D.4.

Throughout this discussion, \((\mathcal{F}, \varphi, \psi, \Upsilon)\) denotes a Hopf lax functor.

**Proposition 3.62.** We have \(\upsilon_I = \upsilon'_I = \varphi_0 \psi_0\).

**Proof.** This follows from definition (3.34) and axiom (3.31).

Let \(\text{id}_1\) denote the identity natural transformation of \(\mathcal{F}\) and \(\text{id}_2\) that of \(\mathcal{F}_2\).

**Proposition 3.63.** We have \(\upsilon \ast \text{id}_1 = \text{id}_1 \ast \upsilon' = \text{id}_2\).

**Proof.** The proof of the identity \(\upsilon \ast \text{id}_1 = \text{id}_2 \ast \upsilon'\) follows from the commutativity of the diagram below. The proof of the other identity is similar.

The hexagon commutes by the definition of \(\upsilon\), the square in the center commutes by axiom (3.29) and the remaining three squares commute by naturality of \(\psi\), \(\text{id}\), and \(\varphi\).

**Proposition 3.64.** We have \(\Upsilon \ast \upsilon = \upsilon' \ast \Upsilon = \Upsilon^{(2)}\).
Proof. The proof of the identity \( \upsilon' \ast \Upsilon = \Upsilon^{(2)} \) follows from the commutativity of the diagram below. The proof of the other identity is similar.

The hexagon commutes by the definition of \( \upsilon' \), the square in the center commutes by axiom (3.30) and the remaining three squares commute by naturality of \( \psi, \Upsilon \) and \( \varphi \).

\[ \square \]

**Proposition 3.65.** We have \( \text{id}_1 \ast \Upsilon = \Upsilon^{(2)} \) and \( \Upsilon \ast \text{id}_1 = \upsilon'^{(2)} \).

Proof. The proof of the identity \( \text{id}_1 \ast \Upsilon = \Upsilon^{(2)} \) follows from the commutativity of the diagram below. The proof of the other identity is similar.

The outer squares commute by the naturality of \( \psi, \Upsilon \) and \( \varphi \). The hexagon commutes by the definition of \( \upsilon \), the square in the center commutes by Proposition 3.63, the square above it commutes by Proposition 3.64, and the squares on its sides commute by the associativity of \( \psi \) and \( \varphi \).

\[ \square \]
Proposition 3.66. The following diagrams commute.

$$\begin{array}{ccc}
\mathcal{F}(I) \bullet \mathcal{F}(A) & \xrightarrow{(\text{id} \ast \Upsilon)} & \mathcal{F}(I) \bullet \mathcal{F}(A) \\
\psi_{I,A} & & \psi_{I,A} \\
\mathcal{F}(I \bullet A) & \xleftarrow{\varphi_{I,A}} & \mathcal{F}(I \bullet A) \\
\mathcal{F}(\lambda_A^{-1}) & \xrightarrow{\mathcal{F}(\lambda_A^{-1})} & \mathcal{F}(\lambda_A^{-1}) \\
\mathcal{F}(A) & \xrightarrow{\upsilon_A} & \mathcal{F}(A) \\
\mathcal{F}(\lambda_A) & \xleftarrow{\upsilon'_A} & \mathcal{F}(A)
\end{array}$$

(3.38)

PROOF. Proposition 3.65 implies

$$(\text{id} \ast \Upsilon)_I,A = \upsilon_I^{(2)} = \upsilon_{I,A}.$$ 

The naturality of $\upsilon$ then gives the result. The proof for $\upsilon'$ is similar. \hfill \Box

Remark 3.67. In defining $\upsilon$ and $\upsilon'$ by means of (3.34) we made an asymmetric choice: we decided to place the unit object $I$ on the right. In Proposition 3.66 we have arrived at the same diagrams with the unit object on the left. Thus, symmetry is recovered.

Proposition 3.68. We have $\upsilon \ast \upsilon = \upsilon^{(2)}$ and $\upsilon' \ast \upsilon' = \upsilon'^{(2)}$.

PROOF. This follows by a similar argument to those for Propositions 3.63, 3.64 and 3.65. \hfill \Box

3.7.6. A comparison of Hopf monoids and Hopf lax monoidal functors.

We complement the results of Section 3.4 by showing that any Hopf monoid can be viewed as a special case of a Hopf lax functor, and that Hopf lax functors preserve Hopf monoids.

Let $(1, \bullet, \beta)$ be the one-arrow category and let $*$ denote its unique object.

Proposition 3.69. The category of Hopf monoids in $(\mathcal{C}, \bullet, \beta)$ is equivalent to the category of Hopf lax functors from $(1, \bullet, \beta)$ to $(\mathcal{C}, \bullet, \beta)$.

PROOF. Given a Hopf monoid $(H, \mu, \iota, \Delta, \epsilon, s)$ in $(\mathcal{C}, \bullet, \beta)$, define a Hopf lax functor

$$(\mathcal{F}_H, \varphi, \psi, \Upsilon)$$

from $(1, \bullet, \beta)$ to $(\mathcal{C}, \bullet, \beta)$, where $(\mathcal{F}_H, \varphi, \psi)$ is defined as in (3.23) and $\Upsilon_*$ is defined to be $s$. We know from Proposition 3.26 that $(\mathcal{F}_H, \varphi, \psi)$ is bilax. Since the antipode $s$ is the inverse of the identity in the convolution monoid $\text{Hom}(H, H)$, we have

$$\text{id} \ast s \ast \text{id} = \text{id}, \quad s \ast \text{id} \ast s = s \quad \text{and} \quad \text{id} \ast s = s \ast \text{id} = \iota \epsilon.$$ 

Hence, axioms (3.29), (3.30) and (3.31) hold and $(\mathcal{F}_H, \varphi, \psi, \Upsilon)$ is Hopf lax.

Conversely, given a Hopf lax functor $(\mathcal{F}, \varphi, \psi, \Upsilon)$ from $1$ to $\mathcal{C}$, the object $\mathcal{F}(*)$ is a Hopf monoid with antipode $\Upsilon_*$ by applying the forward direction of (3.32). \hfill \Box

Bilax functors preserve bimonoids (Proposition 3.31) and the bimonoid structure maps get twisted by the structure maps of the functor, as in Definition 3.28. Hopf lax functors act similarly on Hopf monoids and their antipodes.
Theorem 3.70. Let \( F \) be a Hopf lax functor from \( C \) to \( D \) with antipode \( \Upsilon \). If \( H \) is a Hopf monoid in \( C \) with antipode \( s \), then \( F(H) \) is a Hopf monoid in \( D \) with antipode

\[
\Upsilon_H F(s) = F(s) \Upsilon_H. \tag{3.39}
\]

**Proof.** The equality in (3.39) holds by naturality of \( \Upsilon \). We only need to show that this map satisfies axioms (1.13). The first of these follows from the commutativity of the diagram below; the second axiom can be checked similarly.

\[
\begin{array}{ccc}
F(H)F(H) & \xrightarrow{\text{id}_H F(s)} & F(H)F(H) \\
\uparrow_{\psi_{H,H}} & & \uparrow_{\psi_{H,H}} \\
F(HH) & \xrightarrow{F(id_s)} & F(HH) \\
\uparrow_{F(\Delta)} & & \downarrow_{F(\mu)} \\
F(H) & \xrightarrow{F(\epsilon)} & F(I) \\
\downarrow_{\psi_0} & & \downarrow_{\varphi_0} \\
I & \xrightarrow{\varphi_0} & F(I)
\end{array}
\]

The squares commute by the naturality of \( \psi \) and \( \nu \), the antipode axiom for \( H \) and Proposition 3.65. The triangle commutes by Proposition 3.62. \( \square \)

**Remark 3.71.** Suppose \( F \) is a bistrong monoidal functor and \( H \) is a Hopf monoid with antipode \( s \). By Proposition 3.60, \( F \) is a Hopf lax monoidal functor with antipode \( \Upsilon = \text{id} \). Therefore, by Theorem 3.70, \( F(H) \) is a Hopf monoid with antipode \( F(s) \). This gives another proof of Proposition 3.50.

### 3.7.7. Composites of Hopf lax functors.

Consider two bilax monoidal functors \((F, \varphi, \psi) : C \to D \) and \((G, \gamma, \delta) : D \to E \). Their composite \((GF, \varphi \gamma, \delta \psi) \) (Definition 3.20) is also bilax, by Theorem 3.22. If \( \Upsilon : F \Rightarrow F \) and \( \Omega : G \Rightarrow G \) are natural transformations, we may define a new natural transformation \( \Omega \Upsilon : GF \Rightarrow GF \) by going around the diagram below.

The above diagram commutes by the naturality of \( \Omega \).

**Theorem 3.72.** If \((F, \varphi, \psi, \Upsilon) : C \to D \) and \((G, \gamma, \delta, \Omega) : D \to E \) are Hopf lax functors, then so is \((GF, \varphi \gamma, \delta \psi, \Omega \Upsilon) \).
3.8. AN ALTERNATIVE DESCRIPTION OF BILAX MONOIDAL FUNCTORS

Proof. We only need to check that $\Omega \Upsilon$ is the antipode of $\mathcal{G}\mathcal{F}$.
We first check axiom (3.31) for $\mathcal{G}\mathcal{F}$. The forward direction of (3.32) applied to $\mathcal{F}$ says that $\mathcal{F}(I)$ is a Hopf monoid with antipode $\Upsilon_I$. This along with Theorem 3.70 says that $\mathcal{G}\mathcal{F}(I)$ is a Hopf monoid with antipode

$$\mathcal{G}(\Upsilon_I) \Omega_{\mathcal{F}(I)} = \Omega_{\mathcal{F}(I)} \mathcal{G}(\Upsilon_I),$$

which by definition is $(\Omega \Upsilon)_I$. Further, the bimonoid structure on $\mathcal{G}\mathcal{F}(I)$ comes from the bilax structure of $\mathcal{G}\mathcal{F}$. Therefore by the backward direction of (3.32), $\mathcal{G}\mathcal{F}$ satisfies axiom (3.31).

We now check axiom (3.30) for $\mathcal{G}\mathcal{F}$. This follows from (3.28) and the commutativity of the following diagram.

The four squares commute by the naturality of $\gamma$ and $\Omega$, and axiom (3.30) applied to $\mathcal{G}$ and $\mathcal{F}$.

The verification of axiom (3.29) for $\mathcal{G}\mathcal{F}$ is similar.

Theorem 3.70 can be deduced from Theorem 3.72 by specializing $\mathcal{C}$ to the one-arrow category and using Proposition 3.69. The reason for writing these results in the opposite order is that we needed the former in the proof of the latter.

Remark 3.73. Theorem 3.72 can be used to supplement the discussion in Section 3.3.3: there is a 2-category whose 0-cells, 1-cells, and 2-cells are respectively braided monoidal categories, Hopf lax monoidal functors, and their morphisms.

3.8. An alternative description of bilax monoidal functors

We begin this section by studying the monoidal properties of the tensor product functor. This allows us to formulate an alternative description of bilax monoidal functors (Proposition 3.77). This result is the analogue of the description of a bimonoid as a monoid in a category of comonoids and vice versa.

3.8.1. The tensor product as a monoidal functor. Let $(\mathcal{C}, \bullet)$ be a monoidal category together with natural isomorphisms

$$\beta: A \bullet B \to B \bullet A.$$ 

We do not assume that $\beta$ is a braiding. Consider the tensor product functor

$$\mathcal{M}: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
which sends \((A, B)\) to \(A \bullet B\). Define natural transformations \(\varphi\) and \(\psi\) as in (3.9) to be
\[
\begin{array}{c}
\mathcal{M}(A, B) \bullet \mathcal{M}(C, D) \\
\xleftarrow{\varphi_{(A, B), (C, D)}}
\end{array}
\begin{array}{c}
\mathcal{M}((A, B) \bullet (C, D)) \\
\xrightarrow{\psi_{(A, B), (C, D)}}
\end{array}
\]
\[
A \bullet B \bullet C \bullet D \\
\xleftarrow{\text{id} \bullet \beta \bullet \text{id}}
\]
\[
A \bullet C \bullet B \bullet D.
\]
Define the morphisms \(\varphi_0\) and \(\psi_0\) as in (3.10) to be
\[
I \xrightarrow{\varphi_0} \mathcal{M}(I, I) \xrightarrow{\psi_0} I
\]
\[
I \xrightarrow{\cong} I \bullet I \xrightarrow{\cong} I.
\]

The following result describes the monoidal properties of the functor \(\mathcal{M}\) with respect to the structure maps \(\varphi\) and \(\psi\).

**Proposition 3.74 (Joyal and Street).** We have
\[
\beta \text{ is a braiding in } (\mathcal{C}, \bullet) \iff (\mathcal{M}, \varphi) \text{ is strong.}
\]
\[
\beta \text{ is a symmetry in } (\mathcal{C}, \bullet) \iff (\mathcal{M}, \varphi) \text{ is braided strong.}
\]

The first equivalence is [184, Proposition 5.2], and the second equivalence is [184, Proposition 5.4].

**Proposition 3.75.** We have
\[
(\mathcal{M}, \varphi) \text{ is braided strong} \iff (\mathcal{M}, \varphi, \psi) \text{ is bistrong}
\]
\[
\iff (\mathcal{M}, \psi) \text{ is braided costrong.}
\]

**Proof.** We explain the first equivalence. The backward implication follows from Proposition 3.46. For the forward implication: By Proposition 3.74, \(\beta\) is a symmetry and further by Proposition 3.46, \((\mathcal{M}, \varphi, \varphi^{-1})\) is bistrong. Since \(\beta\) is a symmetry, we have \(\varphi^{-1} = \psi\) which finishes the proof. \(\square\)

**3.8.2. An alternative description of bilax monoidal functors.** Let \((\mathcal{C}, \bullet)\) and \((\mathcal{D}, \bullet)\) be two monoidal categories and \(\mathcal{F}\) be a functor from \(\mathcal{C}\) to \(\mathcal{D}\). We denote the unit object in both categories by \(I\) and write \(\mathcal{M}\) for both tensor product functors. Let \(\mathcal{F}_0\), \(\mathcal{F}_0^0\), \(\mathcal{F}_2\) and \(\mathcal{F}_2^0\) be the functors defined in (3.1) and (3.2).

**Proposition 3.76.** If \(\mathcal{F}\) is (co)lax, then so are \(\mathcal{F}_0\) and \(\mathcal{F}_0^0\). If \(\mathcal{F}\) is (co)lax and \(\mathcal{C}\) and \(\mathcal{D}\) are braided, then the functors \(\mathcal{F}_2\) and \(\mathcal{F}_2^0\) are also (co)lax.

**Proof.** We explain the lax case, the colax case being similar. The assertions about \(\mathcal{F}_0\) and \(\mathcal{F}_0^0\) are clear (and may be seen as special cases of the construction of Section 3.4.1). When \(\mathcal{C}\) and \(\mathcal{D}\) are braided, the tensor product functors \(\mathcal{M}\) are lax (in fact, strong) by Proposition 3.74. Since \(\mathcal{F}\) is lax the functor \(\mathcal{F} \times \mathcal{F}\) is also lax by Proposition 3.6. Note that \(\mathcal{F}_2\) and \(\mathcal{F}_2^0\) are defined in terms of \(\mathcal{M}\), \(\mathcal{F}\) and \(\mathcal{F} \times \mathcal{F}\) via compositions. Hence the result follows from Theorem 3.21. \(\square\)
For \((\mathcal{F}, \psi)\) a colax monoidal functor, let \((\mathcal{F}_2, \psi_2)\) and \((\mathcal{F}^2, \psi^2)\) be the colax monoidal functors given by the above construction. Explicitly, they are as follows.

The lax structures on \(\mathcal{F}_2\) and \(\mathcal{F}^2\) induced from a lax structure on \(\mathcal{F}\) admit similar descriptions.

**Proposition 3.77.** Let \(\mathcal{F}\) be a lax and colax functor with structure maps \(\varphi\) and \(\psi\) respectively. The following statements are equivalent.

(i) \((\mathcal{F}, \varphi, \psi)\) is bilax;

(ii) \(\varphi : \mathcal{F}^2 \Rightarrow \mathcal{F}_2\) and \(\varphi_0 : \mathcal{F}^0 \Rightarrow \mathcal{F}_0\) are morphisms of colax monoidal functors;

(iii) \(\psi : \mathcal{F}_2 \Rightarrow \mathcal{F}^2\) and \(\psi_0 : \mathcal{F}_0 \Rightarrow \mathcal{F}^0\) are morphisms of lax monoidal functors.

**Proof.** We indicate how (i) and (ii) are equivalent. The equivalence between (i) and (iii) is similar.

From the explicit definitions of \(\psi_2\) and \(\psi^2\) given above, one sees that \(\varphi : \mathcal{F}^2 \Rightarrow \mathcal{F}_2\) being a colax morphism (Definition 3.8) is equivalent to the commutativity of the braiding diagram (3.11) and one of the unitality diagrams (3.12). Similarly, the condition that \(\varphi_0 : \mathcal{F}^0 \Rightarrow \mathcal{F}_0\) is a colax morphism is equivalent to the commutativity of the other two unitality diagrams in Definition 3.3. \(\square\)

**Remark 3.78.** The above result is the analogue of Proposition 1.11 for bimonoids. It may be used to obtain another proof of Theorem 3.22 as follows. Recall that morphisms of colax monoidal functors are the 2-cells of a 2-category (Section 3.3.3; this uses Theorem 3.21). The structure maps \(\varphi \gamma\) of the composite \(\mathcal{G} \mathcal{F}\) of two bilax functors (Definition 3.20) are obtained from \(\varphi\) and \(\gamma\) in terms of this 2-category.
structure. Now apply Proposition 3.77: \( \varphi \) and \( \gamma \) are morphisms of colax monoidal functors, hence so is \( \varphi \gamma \), and then \( \mathcal{G} \mathcal{F} \) is bilax. This approach is formalized in Section 6.11.

**3.8.3. Monoidal properties of bilax functors on the category of (co)monoids.** We have seen that a (co)lax functor induces a functor on the category of (co)monoids. If the original functor is bilax, then more can be said about the induced functors, as follows.

**Proposition 3.79.** If \( (\mathcal{F}, \varphi, \psi): C \to D \) is a bilax monoidal functor, then

\[
(\mathcal{F}, \psi): \text{Mon}(C) \to \text{Mon}(D)
\]

is a colax monoidal functor and

\[
(\mathcal{F}, \varphi): \text{Comon}(C) \to \text{Comon}(D)
\]

is a lax monoidal functor.

**Proof.** We discuss the first assertion. Since \( \mathcal{F} \) is lax and \( C \) and \( D \) are braided, the functors \( \mathcal{F}_2, \mathcal{F}^2, \mathcal{F}_0 \) and \( \mathcal{F}^0 \) are all lax (Proposition 3.76). Further, since \( \mathcal{F} \) is bilax, \( \psi: \mathcal{F}_2 \to \mathcal{F}^2 \) and \( \psi_0: \mathcal{F}_0 \to \mathcal{F}^0 \) are morphisms of lax monoidal functors (Proposition 3.77). Now Proposition 3.30 implies that if \( A \) and \( B \) are monoids,

\[
\psi_{A,B}: \mathcal{F}(A \bullet B) \to \mathcal{F}(A) \bullet \mathcal{F}(B) \quad \text{and} \quad \psi_0: \mathcal{F}(I) \to I
\]

are morphisms of monoids. This finishes the proof of the first assertion. \( \square \)

A similar result for a braided (co)lax functor (whose proof we omit) is given below.

**Proposition 3.80.** If \( (\mathcal{F}, \varphi): C \to D \) is a braided lax monoidal functor, then

\[
(\mathcal{F}, \varphi): \text{Mon}(C) \to \text{Mon}(D)
\]

is a lax monoidal functor. Similarly, if \( (\mathcal{F}, \psi): C \to D \) is a braided colax monoidal functor, then

\[
(\mathcal{F}, \psi): \text{Comon}(C) \to \text{Comon}(D)
\]

is a colax monoidal functor.

Applying Proposition 3.29 to the lax and colax functors in the above results and using (1.14), one obtains an alternate proof of the facts that bilax functors preserve bimonoids and braided (co)lax functors preserve (co)commutative (co)monoids.

### 3.9. Adjunctions of monoidal functors

We now discuss the notion of adjunction between monoidal categories for various kinds of monoidal functors. We follow the notations of Section A.2.1, where some background information on adjunctions is also given. Throughout this section, \( C \) and \( D \) are monoidal categories and \( \bullet \) refers to their tensor products. Work of Kelly on adjunctions between categories with structure includes results on adjunctions between monoidal categories [195, Section 2.1]. We mention that Propositions 3.84 and 3.96 (which we prove directly) are special cases of [195, Theorems 1.2 and 1.4].

The results of this section play an important role in the universal constructions of Chapter 11. Interesting examples of adjunctions between monoidal functors can also be found in Propositions 18.4 and 18.18.
3.9. ADJUNCTIONS OF MONOIDAL FUNCTORS

3.9.1. Colax-lax adjunctions. Recall from Example 3.12 that

\[ \text{Hom} : C^{\text{op}} \times C \to \text{Set} \]

is a braided lax monoidal functor. If \( \mathcal{F} \) and \( \mathcal{G} \) are (braided) colax and (braided) lax monoidal functors respectively, then by Proposition 3.7, Theorem 3.21, and Proposition 3.24, the functors

\[ \text{Hom}_D(\mathcal{F}(-), -) \quad \text{and} \quad \text{Hom}_C(-, \mathcal{G}(-)) \]

are (braided) lax monoidal functors from

\[ C^{\text{op}} \times D \to \text{Set}. \]

**Definition 3.81.** Let \((\mathcal{F}, \psi)\) be a (braided) colax monoidal functor and \((\mathcal{G}, \gamma)\) a (braided) lax monoidal functor. We say that they form a pair of (braided) colax-lax adjoint functors if the bijection (A.2) is an isomorphism of (braided) lax functors \( C^{\text{op}} \times D \to \text{Set}. \)

In the above situation, we also say that the adjunction \((\mathcal{F}, \mathcal{G})\) is (braided) colax-lax. It is clear that \((\mathcal{F}, \mathcal{G})\) is braided colax-lax if \((\mathcal{F}, \mathcal{G})\) is colax-lax and both \( \mathcal{F} \) and \( \mathcal{G} \) are braided.

**Proposition 3.82.** Let \((\mathcal{F}, \mathcal{G})\) be a pair of adjoint functors between monoidal categories. Assume that \((\mathcal{F}, \psi)\) is a colax monoidal functor and \((\mathcal{G}, \gamma)\) is a lax monoidal functor. Then the following conditions are equivalent.

1. The adjunction \((\mathcal{F}, \mathcal{G})\) is colax-lax.
2. The following two diagrams commute.

\[
\begin{array}{ccc}
A \bullet B & \xrightarrow{\eta_A \bullet \eta_B} & \mathcal{G} \mathcal{F}(A \bullet B) \\
\downarrow \mathcal{G}(\psi_{A,B}) & & \downarrow \mathcal{G}(\psi_{0}) \\
\mathcal{G} \mathcal{F}(A) \bullet \mathcal{G} \mathcal{F}(B) & \xrightarrow{\gamma_{\mathcal{G}(A), \mathcal{F}(B)}} & \mathcal{G}(\mathcal{F}(A) \bullet \mathcal{F}(B))
\end{array}
\]

(3.40)

3. The following two diagrams commute.

\[
\begin{array}{ccc}
\mathcal{F}(\mathcal{G}(X) \bullet \mathcal{G}(Y)) & \xrightarrow{\mathcal{F}(\gamma_{X,Y})} & \mathcal{F} \mathcal{G}(X \bullet Y) \\
\downarrow \psi_{\mathcal{G}(X), \mathcal{G}(Y)} & & \downarrow \psi_0 \\
\mathcal{F} \mathcal{G}(X) \bullet \mathcal{F} \mathcal{G}(Y) & \xrightarrow{\xi_X \bullet \xi_Y} & X \bullet Y
\end{array}
\]

(3.41)

\[
\begin{array}{ccc}
\mathcal{F}(I) & \xrightarrow{\xi_I} & \mathcal{F} \mathcal{G}(I) \\
\downarrow \psi \downarrow \psi_0 & & \downarrow \psi_0 \\
\mathcal{F}(I) & \xrightarrow{\xi_I} & \mathcal{F} \mathcal{G}(I)
\end{array}
\]

**Proof.** The diagrams (3.40) say that for the bijection in (A.2), the map in one direction is a morphism of lax functors, while the diagrams (3.41) say that the map in the other direction is a morphism of lax functors. \( \square \)

**Proposition 3.83.** If \( \mathcal{F} \) and \( \mathcal{G} \) form a pair of colax-lax adjoint functors between the categories \( C \) and \( D \), then for \( C \) a comonoid in \( C \) and \( A \) a monoid in \( D \), the bijection (A.2)

\[ \text{Hom}_D(\mathcal{F}(C), A) \cong \text{Hom}_C(C, \mathcal{G}(A)) \]

is an isomorphism of convolution monoids.
Proof. Recall that the convolution monoid is the image of a certain monoid under the lax functor \( \text{Hom} \) (Section 3.4.5). The result then follows from Definition 3.81 and Proposition 3.30.

Proposition 3.84. Let \((\mathcal{F}, \mathcal{G})\) be a pair of adjoint functors. If \(\mathcal{F}\) is colax (resp. \(\mathcal{G}\) is lax), then there exists a unique lax structure on \(\mathcal{G}\) (resp. colax structure on \(\mathcal{F}\)) such that \((\mathcal{F}, \mathcal{G})\) is a colax-lax adjunction.

Proof. Let \((\mathcal{F}, \psi)\) be a colax monoidal functor. Use the adjunction (A.2) to define

\[
\mathcal{G}(X) \bullet \mathcal{G}(Y) \xrightarrow{\gamma_{X,Y}} \mathcal{G}(X \bullet Y)
\]
as the map that corresponds to

\[
\mathcal{F}((\mathcal{G}(X) \bullet \mathcal{G}(Y)) \xrightarrow{\psi_{\mathcal{G}(X), \mathcal{G}(Y)}} \mathcal{F}(\mathcal{G}(X) \bullet \mathcal{F}\mathcal{G}(Y)) \xrightarrow{\xi_X \bullet \xi_Y} X \bullet Y
\]

and \(\gamma_0 : I \rightarrow \mathcal{G}(I)\) as the map that corresponds to \(\psi_0 : \mathcal{F}(I) \rightarrow I\). In view of (A.5), \(\gamma\) and \(\gamma_0\) are the unique maps for which the diagrams in (3.41) commute. Hence, to complete the proof we only need to show that \((\mathcal{G}, \gamma)\) is indeed a lax monoidal functor.

The associativity (3.5) of \(\gamma\) follows from the commutativity of the diagram below.

The hexagon commutes by the associativity of \(\psi\). The other squares commute by the definition of \(\gamma\) and the naturality of \(\psi\) and \(\alpha\).
The unitality (3.6) of $\gamma$ follows from the commutativity of the diagram below.

The smaller diagrams commute by the naturality of $\xi$, $\lambda$ and $\psi$, by the definition of $\gamma$ and $\gamma_0$, and by the unitality of $\psi$.

**Proposition 3.85.** Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors. If $\mathcal{F}$ is braided colax (resp. $\mathcal{G}$ is braided lax), then the unique lax structure on $\mathcal{G}$ (resp. the unique colax structure on $\mathcal{F}$) afforded by Proposition 3.84 is braided.

Adjunctions can be composed [250, Theorem IV.8.1]; this operation preserves colax-lax adjunctions.

**Proposition 3.86.** Let $(\mathcal{F}, \mathcal{G})$ be a pair of adjoint functors between monoidal categories $\mathcal{C}$ and $\mathcal{D}$. Let $(\mathcal{F}', \mathcal{G}')$ be another pair of adjoint functors between $\mathcal{D}$ and another monoidal category $\mathcal{E}$. If both adjunctions are colax-lax, then so is the adjunction

$$(\mathcal{F}' \mathcal{F}, \mathcal{G} \mathcal{G}')$$

between $\mathcal{C}$ and $\mathcal{E}$.

### 3.9.2. Lax-lax and colax-colax adjunctions.

**Definition 3.87.** Let $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \gamma)$ be (braided) lax monoidal functors. We say that they form a pair of (braided) lax adjoint functors if the unit and counit $\eta$ and $\xi$ are morphisms of (braided) lax monoidal functors, where we view id as a braided lax functor with identity transformations. More explicitly, one requires that the following diagrams commute.

\[
\begin{array}{c}
A \bullet B \xrightarrow{\eta_A \bullet B} \mathcal{G}\mathcal{F}(A \bullet B) \\
\mathcal{G}\mathcal{F}(A) \bullet \mathcal{G}\mathcal{F}(B) \xrightarrow{\gamma_{\mathcal{F}(A), \mathcal{F}(B)}} \mathcal{G}(\mathcal{F}(A) \bullet \mathcal{F}(B))
\end{array}
\]

\[
\begin{array}{c}
I \xrightarrow{\eta_I} \mathcal{G}\mathcal{F}(I) \\
\mathcal{G}(\varphi_0) \xrightarrow{\eta_I \mathcal{G}(\varphi_0)} \mathcal{G}(\varphi_0)
\end{array}
\]

\[
(3.42)
\]
The diagrams in the first (resp. second) row say that $\eta$ (resp. $\xi$) is a morphism of lax monoidal functors.

**Definition 3.88.** Let $(F, \psi)$ and $(G, \delta)$ be (braided) colax monoidal functors. We say that they form a pair of (braided) colax adjoint functors if the adjunctions $\eta$ and $\xi$ are morphisms of (braided) colax monoidal functors. Explicitly, the necessary diagrams can be obtained from (3.42) and (3.43) by reversing the arrows labeled $\varphi$ and $\gamma$ and renaming them $\psi$ and $\delta$ respectively.

In the situation of Definitions 3.87 and 3.88, we also say that the adjunction is lax-lax and colax-colax, respectively. It is clear that an adjunction $(F, G)$ is braided lax-lax (colax-colax) if $(F, G)$ is lax-lax (colax-colax) and both $F$ and $G$ are braided.

**Remark 3.89.** A lax-lax adjunction is the same as an adjunction in the 2-category $lC_{at}$, in the sense of Section C.1.1. Similarly, a colax-colax adjunction is the same as an adjunction in the 2-category $cC_{at}$.

The above notions should not be confused with that of lax adjunctions, which pertain to the context of tricategories, as defined in [347].

**Example 3.90.** An adjunction between categories with finite products is always braided colax-colax, with the canonical braided colax structures of Example 3.19. Dually, an adjunction between categories with finite coproducts is always braided lax-lax.

**Proposition 3.91.** If $F$ and $G$ form a pair of lax adjoint functors between the categories $C$ and $D$, then they restrict to a pair of adjoint functors

$$\begin{align*}
\text{Mon}(C) & \xrightarrow{F} \text{Mon}(D) \\
\text{Mon}(D) & \xleftarrow{G} \text{Mon}(C)
\end{align*}$$

A similar result holds in the colax case.

**Proof.** We explain the lax case. One needs to check that a morphism of monoids maps to a morphism of monoids under the adjunction. So let $A$ and $X$ be monoids in $C$ and $D$ respectively and let $g: A \to G(X)$ be a morphism of monoids in $C$. Under the adjunction, this corresponds to the map given in (A.5). Since $F$ is lax, the first map in (A.5), namely $F(g)$, is a morphism of monoids. Since by assumption $\xi$ is a morphism of lax monoidal functors, the second map in (A.5) is also a morphism of monoids. This completes the check in one direction. For the other direction, which is similar, one uses that $\eta$ is a morphism of lax monoidal functors. $\square$

**Proposition 3.92.** If $F$ and $G$ form a pair of braided lax adjoint functors between the categories $C$ and $D$, then they restrict to a pair of adjoint functors

$$\begin{align*}
\text{Mon}^{co}(C) & \xrightarrow{F} \text{Mon}^{co}(D) \\
\text{Lie}(C) & \xleftarrow{G} \text{Lie}(D)
\end{align*}$$

A similar result holds in the colax case.
A similar result holds in the colax case.

3.9.3. Colax-lax as a generalization of lax-lax and colax-colax. We now derive some additional useful properties of the different types of adjunctions between monoidal functors that hold when one of the two functors is strong.

Proposition 3.93. Let \((\mathcal{F}, \mathcal{G})\) be a pair of adjoint functors.

1. Suppose \(\mathcal{G}\) is lax and \(\mathcal{F}\) is strong. View \(\mathcal{F}\) as a lax and colax functor as in Proposition 3.44. Then

\((\mathcal{F}, \mathcal{G})\) is a colax-lax adjunction \iff \((\mathcal{F}, \mathcal{G})\) is a lax-lax adjunction.

2. Suppose \(\mathcal{F}\) is colax and \(\mathcal{G}\) is strong. View \(\mathcal{G}\) as a lax and colax functor as in Proposition 3.44. Then

\((\mathcal{F}, \mathcal{G})\) is a colax-lax adjunction \iff \((\mathcal{F}, \mathcal{G})\) is a colax-colax adjunction.

Proof. We prove the first statement. If \(\mathcal{F}\) is strong, then by Proposition 3.44 it has a lax structure \(\varphi\) and a colax structure \(\psi\) such that \(\varphi = \psi^{-1}\) and \(\varphi_0 = \psi_0^{-1}\). In this situation, the diagrams in (3.42) and (3.43) become equivalent to the diagrams in (3.40) and (3.41), and the result follows.

Combining Propositions 3.93 and 3.84, we obtain:

Proposition 3.94. Let \((\mathcal{F}, \mathcal{G})\) be a pair of adjoint functors and \(\mathcal{F}\) (resp. \(\mathcal{G}\)) be strong. Then there exists a unique lax structure on \(\mathcal{G}\) (resp. colax structure on \(\mathcal{F}\)) such that \((\mathcal{F}, \mathcal{G})\) is a lax-lax (resp. colax-colax) adjunction.

Combining further with Proposition 3.85, we obtain:

Proposition 3.95. Let \((\mathcal{F}, \mathcal{G})\) be a pair of adjoint functors and \(\mathcal{F}\) (resp. \(\mathcal{G}\)) be braided strong. Then there exists a unique braided lax structure on \(\mathcal{G}\) (resp. braided colax structure on \(\mathcal{F}\)) such that \((\mathcal{F}, \mathcal{G})\) is a braided lax-lax (resp. braided colax-colax) adjunction.

Conversely, the existence of a lax-lax (resp. colax-colax) adjunction implies that the left (resp. right) adjoint is strong.

Proposition 3.96. Let \((\mathcal{F}, \mathcal{G})\) be a pair of adjoint functors. If the adjunction is lax-lax (resp. colax-colax), then \(\mathcal{F}\) (resp. \(\mathcal{G}\)) is strong.

Proof. We prove the first statement only. Assume \((\mathcal{F}, \varphi)\) and \((\mathcal{G}, \gamma)\) are lax functors and \((\mathcal{F}, \mathcal{G})\) is a lax-lax adjunction. The idea is to define a colax functor \((\mathcal{F}, \psi)\) using Proposition 3.84 and then show that \(\varphi\) and \(\psi\) are inverses. Accordingly, define

\[
\mathcal{F}(A \bullet B) \xrightarrow{\psi_{A,B}} \mathcal{F}(A) \bullet \mathcal{F}(B)
\]

as the map that corresponds to

\[
A \bullet B \xrightarrow{\eta_A \bullet \eta_B} \mathcal{G}\mathcal{F}(A) \bullet \mathcal{G}\mathcal{F}(B) \xrightarrow{\gamma_{\mathcal{F}(A),\mathcal{F}(B)}} \mathcal{G}(\mathcal{F}(A) \bullet \mathcal{F}(B))
\]

under the adjunction (A.2). Similarly, let \(\psi_0 : \mathcal{F}(I) \to I\) be the map that corresponds to \(\gamma_0 : I \to \mathcal{G}(I)\). We claim that \(\psi\) is the inverse of \(\varphi\) and \(\psi_0\) is the inverse of \(\varphi_0\).
Now consider the following diagram.

\[
\begin{array}{c}
\mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\varphi_{A,B}} \mathcal{F}(A \bullet B) \\
\mathcal{F}(\eta_A) \bullet \mathcal{F}(\eta_B) \xrightarrow{\varphi_{\mathcal{F}(A), \mathcal{F}(B)}} \mathcal{F}(\eta_A \bullet \eta_B) \\
\mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\xi_{\mathcal{F}(A), \mathcal{F}(B)}} \mathcal{F}(\mathcal{G}(A) \bullet \mathcal{G}(B)) \\
\mathcal{F}(A \bullet B) \xrightarrow{\psi_{A,B}} \mathcal{F}(\mathcal{G}(A) \bullet \mathcal{G}(B)) \\
\mathcal{F}(A \bullet B) \xrightarrow{\mathcal{F}(\gamma_{\mathcal{F}(A), \mathcal{F}(B)})} \mathcal{F}(\mathcal{G}(A) \bullet \mathcal{G}(B)) \\
\end{array}
\]

The top front square commutes by naturality of \( \varphi \). The bottom front square is a special case of (3.43), it commutes since the adjunction is lax-lax. The front triangle commutes because (A.4) and (A.5) are inverse correspondences. There are two faces on the back, a triangle on the left and a square on the right. The back square commutes by definition of \( \psi \). It follows that the back triangle commutes. This says that \( \psi_{A,B} \varphi_{A,B} = \text{id}_{\mathcal{F}(A) \bullet \mathcal{F}(B)} \).

Similarly, the diagram

\[
\begin{array}{c}
\mathcal{F}(A \bullet B) \xleftarrow{\mathcal{F}(\eta_A \bullet \eta_B)} \mathcal{F}(A) \bullet \mathcal{F}(B) \\
\mathcal{F}(\gamma_{\mathcal{F}(A), \mathcal{F}(B)}) \xleftarrow{\mathcal{F}(\gamma_{\mathcal{G}(A), \mathcal{G}(B)})} \mathcal{F}(\mathcal{G}(A) \bullet \mathcal{G}(B)) \\
\mathcal{F}(A) \bullet \mathcal{F}(B) \xleftarrow{\xi_{\mathcal{F}(A), \mathcal{F}(B)}} \mathcal{F}(\mathcal{G}(A) \bullet \mathcal{G}(B)) \\
\mathcal{F}(A) \xleftarrow{\psi_{A,B}} \mathcal{F}((\mathcal{G}(A) \bullet \mathcal{G}(B)) \\
\mathcal{F}(A \bullet B) \xleftarrow{\mathcal{F}(\gamma_{\mathcal{F}(A), \mathcal{F}(B)})} \mathcal{F}(\mathcal{G}(A) \bullet \mathcal{G}(B)) \\
\end{array}
\]

shows that \( \varphi_{A,B} \psi_{A,B} = \text{id}_{\mathcal{F}(A \bullet B)} \). Thus, \( \varphi \) and \( \psi \) are inverses.

A similar argument using the unital counterparts of the above diagrams shows that \( \varphi_0 \) and \( \psi_0 \) are inverses. This completes the proof.

**Proposition 3.97.** Let \((\mathcal{F}, \mathcal{G})\) be a pair of adjoint functors between monoidal categories \( \mathcal{C} \) and \( \mathcal{D} \). Let \((\mathcal{F}', \mathcal{G}')\) be another pair of adjoint functors between \( \mathcal{D} \) and another monoidal category \( \mathcal{E} \). If both adjunctions are either lax-lax, or colax-colax, then so is the adjunction

\[
(\mathcal{F}', \mathcal{G}')
\]

between \( \mathcal{C} \) and \( \mathcal{E} \).

**Proof.** We consider the lax-lax case. Proposition 3.96 implies that \( \mathcal{F} \) and \( \mathcal{F}' \) are both strong. Hence by Proposition 3.93, one can view \((\mathcal{F}, \mathcal{G})\) and \((\mathcal{F}', \mathcal{G}')\) as colax-lax adjunctions. Now applying Proposition 3.86, we see that the composite is also colax-lax, and further \( \mathcal{F}' \mathcal{F} \) is strong. Applying Proposition 3.93 in the opposite direction, we see that the composite is lax-lax. \( \square \)
3.10. The contragredient construction

In this section, we introduce the contragredient construction. Roughly speaking, it allows us to pass from a given situation to its dual situation. The “given situation” could be a monoidal category, or a monoidal functor, or some variation thereof. More general discussions, which build on this one, are given in Sections 6.12 and 7.10.

3.10.1. Contravariant monoidal functors. Let $\mathcal{F}: C \to D$ be a contravariant functor and let $C$ and $D$ be monoidal categories. We say that $\mathcal{F}$ is **contravariant strong** if

$$\mathcal{F}: C^{\text{op}} \to D,$$

or equivalently,

$$\mathcal{F}: C \to D^{\text{op}}$$

is strong. Now let $C$ and $D$ be braided monoidal categories. We say that $\mathcal{F}$ is **contravariant bilax** (bistrong) if

$$\mathcal{F}: (C^{\text{op}}, \bullet, \beta^{\text{op}}) \to (D, \bullet, \beta),$$

or equivalently,

$$\mathcal{F}: (C, \bullet, \beta) \to (D^{\text{op}}, \bullet, \beta^{\text{op}})$$

is bilax (bistrong). (The equivalence used in the second definition follows from Proposition 3.7.)

3.10.2. Contragredient of monoidal categories. In this discussion, $\ast$ stands for a contravariant functor, say from $C$ to $C'$. We assume that there is another functor from $C'$ to $C$, also called $\ast$, such that

$$C \begin{array}{c} \ast \end{array} \ast \begin{array}{c} \ast \end{array} C'$$

is an adjoint equivalence of categories.

If one of the categories, say $C$ for definiteness, is monoidal with product $\bullet$ and unit $I$, then it induces a monoidal structure on $C'$ by

$$A \ast \bullet B := (A^* \ast B^*)^*,$$

with the unit given by $I^* := I^\ast$. We say that $\bullet^\ast$ is the contragredient of $\bullet$, and that the monoidal category $(C', \bullet^\ast)$ is the contragredient of $(C, \bullet)$. We have that

$$(\bullet^\ast)^\ast \cong \bullet.$$

**Proposition 3.98.** The functors

$$(C, \bullet) \begin{array}{c} \ast \end{array} \ast \begin{array}{c} \ast \end{array} (C', \bullet^\ast)$$

are contravariant strong.

**Proof.** It follows from the definition of $\bullet^\ast$ that

$$A \bullet^\ast B)^* \cong A^* \bullet B^*, \quad (I^\ast)^* \cong I \quad \text{and} \quad (A \bullet B)^* \cong A^* \bullet^\ast B^*, \quad I^* \cong I^\ast$$

which implies that the $\ast$ functors are contravariant strong. □

Now further if $C$ is braided with braiding $\beta$, then so is $C'$ with braiding

$$\beta_{B,A}^\ast := \beta_{A^*,B^*}^\ast.$$

We say that $\beta^\ast$ is the contragredient of $\beta$ and that the braided monoidal category $(C', \bullet^\ast, \beta^\ast)$ is the contragredient of $(C, \bullet, \beta)$. We have that

$$(\beta^\ast)^\ast \cong \beta.$$
Proposition 3.99. The functors

\[(C, \bullet, \beta) \quad \overset{*}{\longrightarrow} \quad (C', \bullet^\vee, \beta^\vee)\]

are contravariant bistrong.

Proof. We saw in Proposition 3.98 that the \(\ast\) functors are contravariant strong. Similarly, it follows from the definition of \(\beta^\vee\) that the following diagrams commute.

\[
\begin{align*}
(A \bullet^\vee B)^* & \xrightarrow{\cong} A^* \bullet B^* & (A \bullet B)^* & \xrightarrow{\cong} A^* \bullet^\vee B^* \\
(B \bullet^\vee A)^* & \xrightarrow{\cong} B^* \bullet A^* & (B \bullet A)^* & \xrightarrow{\cong} B^* \bullet^\vee A^* \\
(\beta_{B,A}^\vee)^* & \quad \beta_{A^*,B^*}^\vee & \beta_{B^*,A}^\vee & \quad \beta_{A^*,B^*}^\vee
\end{align*}
\]

This implies that the \(\ast\) functors are contravariant braided strong, which is the same as contravariant bistrong.

\[\square\]

3.10.3. Contragredient of functors. Consider the following situation

\[(3.45) \quad \begin{array}{ccc}
C & \xrightarrow{\ast} & C' \\
\downarrow \beta_{B,A}^\vee & & \downarrow \beta_{A^*,B^*}^\vee \\
B \bullet^\vee A & \xrightarrow{\cong} & B^* \bullet A^*
\end{array} \quad \begin{array}{ccc}
D & \xrightarrow{\ast} & D' \\
\downarrow \beta_{B,A}^\vee & & \downarrow \beta_{A^*,B^*}^\vee \\
B \bullet A & \xrightarrow{\cong} & B^* \bullet^\vee A^*
\end{array}\]

where \(\mathcal{F}\) is a covariant functor and the functors \(\ast\) are as per the assumption (3.44). Let \(\mathcal{F}^\vee\) denote the above composite, namely

\[(3.46) \quad \mathcal{F}^\vee(-) := \mathcal{F}(-^\ast)^\ast.\]

We refer to \(\mathcal{F}^\vee\) as the contragredient of \(\mathcal{F}\). Observe that it is a covariant functor.

For a natural transformation \(\theta: \mathcal{F} \Rightarrow \mathcal{G}\), let \(\theta^\vee: \mathcal{G}^\vee \Rightarrow \mathcal{F}^\vee\) denote the induced natural transformation. Explicitly, it is given by

\[(3.47) \quad \mathcal{G}^\vee(A) = \mathcal{G}(A^\ast)^\ast \xrightarrow{(\theta_A^\ast)^\ast} \mathcal{F}(A^\ast)^\ast = \mathcal{F}^\vee(A).\]

We have that

\[(\mathcal{F}^\vee)^\vee \cong \mathcal{F},\]

where it is implicit that the appropriate adjoint \(\ast\) functors are used for defining the contragredient of \(\mathcal{F}^\vee\).

Example 3.100. In the context of the tensor, symmetric and exterior algebras, the isomorphisms in (2.69) are instances of the contragredient construction. We elaborate a little bit further to make this point clear.

Let \(\ast: \text{Vec} \rightarrow \text{Vec}\) be the duality functor which sends a vector space to its dual. This is an involutive contravariant bistrong functor on finite-dimensional vector spaces. Therefore, it maps (finite-dimensional) algebras to coalgebras and viceversa.

Let \(\mathbf{gAlg}\) and \(\mathbf{gCoalg}\) be the categories of (finite-dimensional) graded algebras and graded coalgebras. Consider the functor

\[\mathcal{T}: \text{Vec} \rightarrow \mathbf{gAlg}, \quad \mathcal{T}(V) := \bigoplus_{k \geq 0} V^\otimes k.\]
The object $\mathcal{T}(V)$ is the tensor algebra. It is graded by the number of tensor factors, and its product is given by concatenation. The contragredient of $\mathcal{T}$ is given by the composite

$$\mathcal{T}^\vee : \text{Vec} \longrightarrow \text{Vec} \longrightarrow \text{gAlg} \longrightarrow \text{gCoalg}.$$ 

It is clear that the coproduct on $\mathcal{T}^\vee$ is deconcatenation.

A duality functor similar to $*$ can be defined on the category of species (Section 8.6). The analogues of the tensor, symmetric and exterior algebras for species along with their contragredients are treated in Chapter 11.

**Example 3.101.** Another interesting instance of the contragredient construction is provided by the Fock functors which relate species to graded vector spaces. The duality functors on both species and vector spaces play a role here. See Section 15.1.2 for the simplest example of this kind.

### 3.10.4. Contragredient of monoidal functors.

The contragredient construction is compatible with monoidal functors. We illustrate this feature with some simple but important results. Given a functor $F : C' \to D'$ and a transformation $\varphi$ as in (3.3), consider its contragredient $F^\vee : C \to D$ (3.46) and define a transformation $\varphi^\vee$ by

$$\varphi^\vee_{A,B} : F^\vee(A \bullet^\vee B) = F(A^* \bullet B^*)^* \longrightarrow (F(A^*) \bullet F(B^*))^* = F^\vee(A) \bullet^\vee F^\vee(B).$$

Similarly, for $(F, \psi)$ with $\psi$ as in (3.7), one defines $(F^\vee, \psi^\vee)$.

**Proposition 3.102.** If $(F, \varphi) : C' \to D'$ is (braided) lax, then

$$(F^\vee, \varphi^\vee) : C \to D$$

is (braided) colax. Similarly, if $(F, \psi)$ is (braided) colax, then $(F^\vee, \psi^\vee)$ is (braided) lax, and if $(F, \varphi, \psi)$ is (braided) bilax, then so is $(F^\vee, \psi^\vee, \varphi^\vee)$.

Further, if $\theta : (F, \varphi) \Rightarrow (G, \gamma)$ is a morphism of lax (colax, bilax) functors, then

$$\theta^\vee : (G^\vee, \gamma^\vee) \Rightarrow (F^\vee, \varphi^\vee)$$

is a morphism of colax (lax, bilax) functors.

**Proof.** Let $(F, \varphi)$ be lax. The $*$ functors by Proposition 3.98 are contravariant strong. Then applying Theorem 3.21, the following composite of lax functors is lax.

$$\text{C}^{\text{op}} \longrightarrow C' \longrightarrow D' \longrightarrow D^{\text{op}}$$

Passing to the opposite categories and applying Proposition 3.7, we obtain the functor $F^\vee$ equipped with a colax structure. It is straightforward to check that the colax structure is given by $\varphi^\vee$.

The remaining claims are proved in a similar manner. $\square$

**Proposition 3.103.** If $(F, G)$ is a pair of adjoint functors, then so is $(G^\vee, F^\vee)$. In addition, if the adjunction $(F, G)$ is lax-lax (resp. colax-colax), then $(G^\vee, F^\vee)$ is colax-colax (resp. lax-lax), and if $(F, G)$ is colax-lax, then so is $(G^\vee, F^\vee)$. 
Proof. The proof is summarized in the following diagram.

\[
\begin{array}{ccc}
C' & \xymatrix{ \ar[r] \ar[d]^F & \ar[l] \ar[d]^G } & D' \\
& C^{\text{op}} & D^{\text{op}} \\
& \xymatrix{ \ar[r] \ar[d]^G & \ar[l] } & \xymatrix{ \ar[r] \ar[d]^F & \ar[l] } \\
& D & C.
\end{array}
\]

The content of the first implication is that adjunctions can be composed (Propositions 3.86 and 3.97). The second implication says that passing to the opposite categories switches left and right adjoints. This follows directly from the definition.

Since the contragredient construction \((\cdot)^{\lor}\) involves a passage to the opposite categories, it switches left and right adjoints, and lax and colax functors.

3.10.5. Self-duality. Now we specialize (3.44) to the situation where \(C = C'\) and the two \(*\) functors coincide. We say that an object \(V\) in \(C\) is \textit{self-dual} if \(V \cong V^*\).

Definition 3.104. A monoidal category \((C, \bullet)\) is \textit{self-dual} if \(\bullet^{\lor} \cong \bullet\), or more precisely, if

\[\text{id}: (C, \bullet) \to (C, \bullet^{\lor})\]

is a strong functor.

Similarly, a braided monoidal category \((C, \bullet, \beta)\) is \textit{self-dual} if \(\bullet^{\lor} \cong \bullet\) and \(\beta^{\lor} \cong \beta\), or more precisely, if

\[\text{id}: (C, \bullet, \beta) \to (C, \bullet^{\lor}, \beta^{\lor})\]

is a bistrong functor.

Definition 3.105. Let \(C\) and \(D\) be self-dual braided monoidal categories. A (bilax) functor \(F: C \to D\) is \textit{self-dual} if \(F^{\lor} \cong F\) as (bilax) functors.


The proof is straightforward.


Proof. Let \(F\) be a self-dual functor and let \(V\) be a self-dual object. Then by assumption,

\[F(V) \cong F(V^*) \cong F(V)^*\]

Hence \(F(V)\) is self-dual.

This result complemented with Proposition 3.106 yields the claim about self-dual bilax functors.

Definition 3.108. Let \(C\) and \(D\) be self-dual braided monoidal categories, and let \(F: C \to D\) be a (bilax) functor. A natural transformation \(\theta: F \Rightarrow F^{\lor}\) is \textit{self-dual} if \(\theta^{\lor} \cong \theta\).

Definition 3.109. A colax-lax adjunction \((F, G)\) is \textit{self-dual} if \(F \cong G^{\lor}\) as colax functors, \(G \cong F^{\lor}\) as lax functors, and these isomorphisms are compatible with the unit and counit of the adjunction.
3.10.6. Examples. The main examples of self-dual functors in this monograph are given below. A more elaborate summary is provided in Table 3.4.

- The functors $\mathcal{S}$, $\Lambda$ and $\mathcal{T}_0$ in Section 2.6.3 are self-dual. In Chapter 11, we construct analogues of these functors with species playing the role of graded vector spaces. It is interesting to note that in contrast to graded vector spaces, the functors $\mathcal{S}$ and $\Lambda$ for species are self-dual, regardless of the characteristic.

- The Hadamard functor on species is a self-dual bilax functor.

- Fock functors provide an important source of self-dual bilax functors. Their decorated and colored versions (not shown in the table) studied in Chapters 19 and 20 provide further examples.

The main examples of self-dual natural transformations are given in Table 3.5. These admit self-dual colored generalizations as well.

An example of a self-dual colax-lax adjunction is given in (8.81).

3.11. The image of a morphism of bilax monoidal functors

In an abelian monoidal category, a morphism of bimonoids has an image which is itself a bimonoid. Our main goal in this section is to obtain an analogous result for morphisms of bilax monoidal functors (Theorem 3.116). A nice proof of this fact can be given by viewing a morphism between two bilax monoidal functors as a bilax monoidal functor in an appropriate category. This is Proposition 3.111.

3.11.1. The category of arrows. Let $D$ be an arbitrary category. The category $D^{(2)}$ of arrows in $D$ has for objects the triples $(A, f, B)$ where $A$ and $B$ are objects

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<th>Self-dual functors</th>
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<tr>
<td>$\mathcal{S}$ and $\mathcal{S}^\vee$</td>
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<tr>
<td>Anyonic Fock functors $\mathcal{K}_q$</td>
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</table>

Table 3.5. Self-dual natural transformations.

<table>
<thead>
<tr>
<th>Self-dual natural transformations</th>
<th>Section</th>
</tr>
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<tbody>
<tr>
<td>norm map $\kappa: \mathcal{T} \to \mathcal{T}^\vee$</td>
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<tr>
<td>$q$-norm map $\kappa_q: \mathcal{T}_q \to \mathcal{T}_q^{\vee}$</td>
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<tr>
<td>norm map $\kappa: \mathcal{K} \to \mathcal{K}^\vee$</td>
<td>15.4</td>
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<tr>
<td>$q$-norm map $\kappa_q: \mathcal{K}_q \to \mathcal{K}_q^{\vee}$</td>
<td>16.2</td>
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of $D$ and $f: A \to B$ is a morphism in $D$. A morphism from $(A, f, B)$ to $(C, g, D)$ is a pair $(h, k)$ of morphisms in $D$ such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{k} \\
C & \xrightarrow{g} & D
\end{array}
$$

commutes. Composition and identities in $D^{(2)}$ are coordinatewise. The category $D^{(2)}$ is an example of a comma category; see Example A.22.

Suppose $(D, \bullet)$ is a monoidal category. Then so is $D^{(2)}$, with tensor product

$$(A, f, B) \bullet (A', f', B') := (A \cdot A', f \bullet f', B \bullet B').$$

The unit object is $(I, \text{id}_I, I)$, where $I$ is the unit object in $D$. If $D$ is braided, then so is $D^{(2)}$, with braiding

$$
(A, f, B) \bullet (A', f', B') \\
(\beta_{A,A'},\beta_{B,B'})
$$

where $\beta$ is the braiding in $D$.

**Proposition 3.110.** An object $(A, f, B)$ of $D^{(2)}$ is a monoid if and only if $A$ and $B$ are monoids in $D$ and $f: A \to B$ is a morphism of monoids. The same statement holds replacing monoids by comonoids or bimonoids (the latter if the category $D$ is braided).

The proof is straightforward.

### 3.11.2. Morphisms of monoidal functors as monoidal functors

Let $F$ and $G$ be functors from a category $C$ to a category $D$, and let $\theta: F \Rightarrow G$ be a natural transformation. Then one can define a functor

$$
H_\theta: C \to D^{(2)}
$$

by

$$
A \mapsto (F(A), \theta_A, G(A)), \quad f \mapsto (F(f), G(f)).
$$

Naturality of $\theta$ makes $H_\theta$ well-defined.

Now suppose that $C$ and $D$ are monoidal categories. Recall the notion of morphisms of monoidal functors (Definitions 3.8 and 3.9).

**Proposition 3.111.** The functor $H_\theta$ is lax monoidal if and only if the functors $F$ and $G$ are lax monoidal and $\theta$ is a morphism of lax monoidal functors. The same statement holds replacing lax for colax or bilax (the latter if the categories $C$ and $D$ are braided).
Proof. We explain the lax case. Suppose \((\mathcal{F}, \varphi)\) and \((\mathcal{G}, \gamma)\) are lax monoidal functors and \(\theta\) is a morphism of lax monoidal functors. Then we define \(\Phi\) by
\[
\Phi_{A,A'} (A \otimes A') = (\mathcal{F}(A) \otimes \mathcal{F}(A'), \theta_{A \otimes A'}, \mathcal{G}(A) \otimes \mathcal{G}(A')).
\]
This is a well-defined morphism in \(D^{(2)}\) in view of the commutativity of the diagram
\[
\begin{array}{ccc}
\mathcal{F}(A) \otimes \mathcal{F}(A') & \xrightarrow{\theta_{A \otimes A'}} & \mathcal{G}(A) \otimes \mathcal{G}(A') \\
\varphi_{A,A'} & \downarrow & \gamma_{A,A'} \\
\mathcal{F}(A \otimes A') & \xrightarrow{\theta_{A \otimes A'}} & \mathcal{G}(A \otimes A')
\end{array}
\]
which holds since \(\theta\) is a morphism of lax monoidal functors (3.14). We also set
\[
I_{D^{(2)}} = (I, \text{id}_I, I)
\]
which is well-defined in view of the second diagram in (3.14). The axioms in Definition 3.1 for \(\varphi\) and \(\gamma\) translate into the corresponding axioms for \(\Phi\). Conversely, if \(\Phi\) is a lax structure on the functor \(\mathcal{H}_{\theta}\), then its components define lax structures on \(\mathcal{F}\) and \(\mathcal{G}\) such that \(\theta\) is a morphism of lax functors.

Proposition 3.110 is the special case of Proposition 3.111 in which \(C\) is the one-object monoidal category as in Section 3.4.1.

3.11.3. The image of a morphism. Recall that in an abelian category [250, Section VIII.3], every morphism \(f: A \to B\) factors as
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
e & \searrow & m \\
& X & \\
\end{array}
\]
with \(e\) an epimorphism and \(m\) a monomorphism. This is called a monic-epi factorization of \(f\). The factorization is functorial in the following sense.

Proposition 3.112. Given a commutative diagram in an abelian category
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
h & \downarrow & k \\
A' & \xrightarrow{f'} & B'
\end{array}
\]
and monic-epi factorizations of \( f \) and \( f' \), there is a unique morphism \( j : X \to X' \) such that the diagrams below commute:

\[
\begin{array}{ccc}
A & \xrightarrow{e} & X & \xrightarrow{m} & B \\
\downarrow h & & \downarrow j & & \downarrow k \\
A' & \xrightarrow{e'} & X' & \xrightarrow{m'} & B',
\end{array}
\]

where the rows are the given monic-epi factorizations of \( f \) and \( f' \).

**Proof.** This is [250, Proposition VIII.3.1]. \( \square \)

It follows that if \( h \) and \( k \) are isomorphisms, then so is \( j \). In this sense, monic-epi factorizations are unique up to isomorphism. The maps \( m \) and \( e \) in (3.49) are called the image and coimage of \( f \) respectively. Sometimes, more loosely, the same terminology is applied to the object \( X \) (either term).

### 3.11.4. The image functor

Let \( D \) be an abelian category. We proceed to construct a functor

\[
\mathfrak{I} : D^{(2)} \to D.
\]

For each object \((A, f, B)\) of \( D^{(2)} \), we choose a monic-epi factorization as in (3.49) and we let

\[
\mathfrak{I}(A, f, B) := X
\]

where \( X \) is the middle object in the chosen factorization. Given a morphism \((h, k) : (A, f, B) \to (A', f', B')\) in \( D^{(2)} \), we let

\[
\mathfrak{I}(h, k) := j,
\]

where \( j \) is the unique morphism relating the chosen monic-epi factorizations of \( f \) and \( f' \) afforded by Proposition 3.112.

We refer to \( \mathfrak{I} : D^{(2)} \to D \) as the **image functor**. Its functoriality follows from Proposition 3.112.

**Lemma 3.113.** Let \((D, \bullet)\) be an abelian monoidal category (Definition 1.8) and let

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & B_1 \\
\downarrow e_1 & & \downarrow m_1 \\
X_1 & & \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
A_2 & \xrightarrow{f_2} & B_2 \\
\downarrow e_2 & & \downarrow m_2 \\
X_2 & & \\
\end{array}
\]

be monic-epi factorizations of two morphisms \( f_1 \) and \( f_2 \). Then

\[
\begin{array}{ccc}
A_1 \bullet A_2 & \xrightarrow{f_1 \bullet f_2} & B_1 \bullet B_2 \\
\downarrow e_1 \bullet e_2 & & \downarrow m_1 \bullet m_2 \\
X_1 \bullet X_2 & & \\
\end{array}
\]

is a monic-epi factorization of \( f_1 \bullet f_2 \).
Proof. We have to check that $e_1 \cdot e_2$ is a monomorphism and $m_1 \cdot m_2$ is an epimorphism. By exactness, the maps

$$e_1 \cdot \text{id}: A_1 \cdot A_2 \to X_1 \cdot A_2 \quad \text{and} \quad \text{id} \cdot e_2: X_1 \cdot A_2 \to X_1 \cdot X_2$$

are monomorphisms. Hence so is their composite $e_1 \cdot e_2$. For similar reasons, $m_1 \cdot m_2$ is an epimorphism.

Proposition 3.114. Let $(D, \cdot)$ be an abelian monoidal category. The functor

$$\mathcal{I}: (D^{(2)}, \cdot) \to (D, \cdot)$$

is strong. If $(D, \cdot)$ is braided, then $\mathcal{I}$ is bistrong.

Proof. We define structure maps $\varphi$ and $\varphi_0$ (Definition 3.1). Take two objects in $D^{(2)}$ and their chosen factorizations as in Lemma 3.113. Let also

$$A_1 \cdot A_2 \xrightarrow{e_{12}} X_{12} \xrightarrow{m_{12}} B_1 \cdot B_2$$

be the chosen factorization of $f_1 \cdot f_2$. Lemma 3.113 and uniqueness of factorizations (Proposition 3.112) allows us to define

$$\varphi(A_1,f_1,B_1) \cdot \varphi(A_2,f_2,B_2): \mathcal{I}(A_1,f_1,B_1) \cdot \mathcal{I}(A_2,f_2,B_2) \to \mathcal{I}(A_1 \cdot A_2,f_1 \cdot f_2,B_1 \cdot B_2)$$

as the unique isomorphism such that the following diagram commutes

$$
\begin{array}{ccc}
A_1 \cdot A_2 & \xrightarrow{e_{12}} & X_{12} \\
\downarrow \varphi & & \downarrow \varphi \\
A_1 \cdot A_2 & \xrightarrow{e_{12}} & X_{12} \\
A_1 \cdot A_2 & \xrightarrow{e_{12}} & X_{12} \\
\end{array}
$$

(3.50)

The identity of the unit object of $D$ can be factored through the unit object as $\text{id}_I = \text{id}_I \cdot \text{id}_I$. We let

$$\varphi_0: I \to \mathcal{I}(I, \text{id}_I, I)$$

be the isomorphism afforded by Proposition 3.112.

Now let $(A_3,f_3,B_3)$ be a third object of $D^{(2)}$. For ease of notation, assume the associativity constraints of $(D, \cdot)$ are identities. We use similar notations to the above for the chosen factorizations of $f_2 \cdot f_3$ and $f_1 \cdot f_2 \cdot f_3$. By definition of $\varphi$, the following diagram commutes.

For the same reasons, a similar diagram with the middle vertical maps being

$$X_1 \cdot X_2 \cdot X_3 \xrightarrow{\text{id} \cdot \varphi} X_1 \cdot X_{23} \xrightarrow{\varphi} X_{123}$$
commutes as well. Then, by uniqueness of factorizations,

\[ \varphi(\text{id} \bullet \varphi) = \varphi(\varphi \bullet \text{id}). \]

Thus, the associativity condition in Definition 3.1 holds. The unital condition can be verified similarly, and hence \((\mathcal{I}, \varphi)\) is a lax monoidal functor. Since \(\varphi\) is an isomorphism, it is strong.

If the category \(\mathcal{D}\) is braided, then the strong monoidal functor \((\mathcal{I}, \varphi)\) is braided. This follows from the commutativity of the diagram

\[ A_1 \bullet A_2 \xrightarrow{e_1 \bullet e_2} X_1 \bullet X_2 \xrightarrow{m_1 \bullet m_2} B_1 \bullet B_2 \]

\[ A_2 \bullet A_1 \xrightarrow{e_2 \bullet e_1} X_2 \bullet X_1 \xrightarrow{m_2 \bullet m_1} B_2 \bullet B_1, \]

which holds by naturality of \(\beta\). Hence, by Proposition 3.46, the functor \(\mathcal{I}\) is bistrong.

Let \(\mathcal{P}_1\) and \(\mathcal{P}_2: \mathcal{D}^{(2)} \to \mathcal{D}\) be the canonical projections, that is,

\[ \mathcal{P}_1(A, f, B) = A \quad \text{and} \quad \mathcal{P}_2(A, f, B) = B. \]

They are strong monoidal functors \(\mathcal{D}^{(2)} \to \mathcal{D}\) (bistrong if \(\mathcal{D}\) is braided). Moreover, there are transformations

(3.51) \[ \mathcal{P}_1 \Rightarrow \mathcal{I} \Rightarrow \mathcal{P}_2 \]

defined by

\[ \mathcal{P}_1(A, f, B) \xrightarrow{e} \mathcal{I}(A, f, B) \xrightarrow{m} \mathcal{P}_2(A, f, B), \]

where the bottom row is the chosen factorization of \(f\).

**Proposition 3.115.** The transformations (3.51) are morphisms of (bi)strong monoidal functors.

**Proof.** Naturality follows from the functoriality of factorizations (Proposition 3.112) and conditions (3.14) follow from the definition of \(\varphi\) in (3.50). \(\square\)

### 3.11.5. The image of a morphism of monoidal functors

We are now in position to prove the main result of this section. Let \(\mathcal{C}\) be an arbitrary monoidal category and \(\mathcal{D}\) an abelian monoidal category. Let \(F: \mathcal{C} \to \mathcal{D}\) and \(G: \mathcal{C} \to \mathcal{D}\) be two functors and \(\theta: F \Rightarrow G\) a natural transformation. Let \(\mathcal{I}_\theta\) denote the composite of functors

\[ \mathcal{C} \xrightarrow{\mathcal{H}_\theta} \mathcal{D}^{(2)} \xrightarrow{\mathcal{I}} \mathcal{D}, \]

where \(\mathcal{H}_\theta\) is the functor of Section 3.11.2 and \(\mathcal{I}\) is the image functor of Section 3.11.4. The functor \(\mathcal{I}_\theta\) sends an object \(A\) in \(\mathcal{C}\) to the image of the morphism \(\theta_A: F(A) \to G(A)\) in \(\mathcal{D}\).
Theorem 3.116. In the above situation, if $\mathcal{F}$ and $\mathcal{G}$ are lax monoidal functors and $\theta$ is a morphism of lax monoidal functors, then

$$\exists_\theta : C \to D$$

is a lax monoidal functor. Moreover, $\theta$ factors as a composite of morphisms of lax monoidal functors

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\theta} & \mathcal{G} \\
\downarrow & \Downarrow & \downarrow \\
\exists_\theta. & & \\
\end{array}$$

The same result holds replacing lax for colax or bilax (the latter if $C$ and $D$ are braided).

Proof. We explain the lax case. By construction $\exists_\theta$ is the composite of the lax monoidal functor $\mathcal{H}_\theta$ (Proposition 3.111) and the strong monoidal functor $\exists$ (Proposition 3.114), so Theorem 3.22 applies. Note that the composite of $\mathcal{H}_\theta$ and $\mathcal{P}_1$ is $\mathcal{F}$, and the composite of $\mathcal{H}_\theta$ and $\mathcal{P}_2$ is $\mathcal{G}$. The factors of $\theta$ are the compositions of the morphisms of Proposition 3.115 with the functor $\mathcal{H}_\theta$. They are morphisms of lax monoidal functors by Theorem 3.21.

Remark 3.117. The construction of the image functor $\exists$ involved a global choice of factorizations. Changing the choice leads to an isomorphic bistrong monoidal functor (again by uniqueness of factorizations). Suppose the category $D$ is the category of (graded) vector spaces, or more generally, the category of modules over a commutative ring. In such a case there are two canonical choices of monic-epi factorizations (3.49). Namely, one can choose the middle object $X$ as the classical image of $f$ (a subobject of $B$) or as the classical coimage of $f$ (a quotient of $A$). It follows that both choices lead to isomorphic monoidal functors $\exists_1^\theta$ and $\exists_2^\theta$. One thus obtains a diagram of morphisms of monoidal functors

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\theta} & \mathcal{G} \\
\downarrow & \Downarrow & \downarrow \\
\exists_1^\theta & \cong & \exists_2^\theta. \\
\end{array}$$

This is the situation encountered in Part III of the monograph; see Sections 15.4.3, 16.3.5, 19.2.1, 19.7.2 and 20.2.3. Further, in some of these situations, $\theta$ is given by symmetrization (an instance of the norm map in group theory). In that case, the image $\exists_2^\theta$ can be identified with invariants and the coimage $\exists_1^\theta$ with coinvariants, provided the field characteristic is 0.

Remark 3.118. While abelian monoidal categories constitute a natural context in which to formulate Theorem 3.116, this and the other results of this section hold in greater generality. In fact, all that is needed is the existence of functorial monic-epi factorizations in the category $D$ (as in Proposition 3.112), and the fact that the tensor product of $D$ preserves monomorphisms and epimorphisms. This holds not only in abelian monoidal categories (as in Definition 1.8), but also in $(\text{Set}, \times)$, and in many other situations: indeed, it holds in any topos viewed as a monoidal category under Cartesian product. This follows from [251, Propositions IV.6.1–2]; see also [60, Corollaries 5.9.2 and 5.9.4] and [59, Proposition 2.3.4]. Very general
3.11.6. Self-duality of the image. In order to be able to discuss issues related to self-duality, we combine the above setup with that of the contragredient construction (Section 3.10). Accordingly, we assume that C and D are equipped with contravariant $\ast$ functors as in (3.44) (taking $C' = C$). This induces a $\ast$ functor on the category of arrows $D^{(2)}$

$$(A, f, B)^\ast := (B^\ast, f^\ast, A^\ast).$$

Let $H_\theta$ be as in Section 3.11.2. Then

$$(H_\theta)^\vee = H_\theta^\vee,$$

with definitions as in (3.46) and (3.47). This follows from the following calculation.

$$(H_\theta)^\vee(A) = (\mathcal{F}(A^\ast), \theta_{A^\ast}, \mathcal{G}(A^\ast))^\ast$$

$$= (\mathcal{G}(A^\ast)^\ast, \theta_{A^\ast}^\ast, \mathcal{F}(A^\ast)^\ast)$$

$$= (\mathcal{G}^\vee(A), \theta_A^\vee, \mathcal{F}^\vee(A))$$

$$= H_\theta^\vee(A).$$

In particular, if $\theta$ is a self-dual transformation $F \Rightarrow F^\vee$, then $H_\theta$ is a self-dual functor (Definitions 3.105 and 3.108).

Now assume further that $D$ is an abelian category such that monic-epi factorizations in $D$ are compatible with the $\ast$ functor. Explicitly, this means that the dual of a monic-epi factorization of $f$ as in (3.49) yields a monic-epi factorization of $f^\ast$. With this assumption, it follows that the image functor $\mathfrak{I}$ of Section 3.11.4 is self-dual. Since $\mathfrak{S}_\theta$ is the composite of $H_\theta$ and $\mathfrak{I}$, it follows that

$$(\mathfrak{S}_\theta)^\vee = \mathfrak{S}_{\theta^\vee}.$$

In particular, if $\theta$ is a self-dual transformation $F \Rightarrow F^\vee$, then $\mathfrak{S}_\theta$ is a self-dual functor.

If, in addition, we assume that $C$ and $D$ are (braided) monoidal categories, then by employing Propositions 3.111 and 3.114, and Theorem 3.116, one sees that the above results generalize to that setting. Among these, we highlight the following result.

**Proposition 3.119.** Let $C$ be a self-dual braided monoidal category, $D$ be a self-dual braided abelian monoidal category, and let $F: C \to D$ be a bilax functor. If $\theta: F \Rightarrow F^\vee$ is a self-dual morphism of bilax functors, then the image $\mathfrak{S}_\theta$ is a self-dual bilax functor.

The image of the norm transformation between full Fock functors and its deformed, decorated and colored versions are examples of this kind. They are discussed in Part III of the monograph; see the sections cited in Remark 3.117.
The Coxeter Complex of Type $A$

Coxeter groups play an important role in many areas of mathematics. A concise introduction to Coxeter groups particularly relevant to the ideas presented here is given in [12, Chapter 1]. Supplementary material can be found in the books by Abramenko and Brown [3], Davis [89], Grove and Benson [156], Humphreys [174], Björner and Brenti [51] or Bourbaki [62]. Material related to the general context of hyperplane arrangements and oriented matroids can be found in [52, 257, 344, 286].

In this chapter, we only deal with the symmetric group, which is the Coxeter group of type $A$. This is because our main interest here is to tie Coxeter theory to species. The theory for the symmetric group can be understood explicitly; this makes our exposition fairly self-contained. We begin with a discussion of a number of combinatorial structures that play a fundamental role here and elsewhere in the monograph (Section 10.1). In Sections 10.2, 10.3, 10.4 and 10.5, we review some standard material, namely, the braid arrangement, faces and flats therein, the Coxeter complex of type $A$, Tits projection maps, the gallery metric, and the gate property. Of particular importance is a monoid structure carried by the set of faces. It is defined in terms of the projection maps and lifts the lattice structure of the set of flats. Sections 10.6 and 10.7 deal with shuffles (and their geometric meaning), and the descent and global descent maps. The action of faces on chambers by multiplication yields an embedding of the algebra of faces in the endomorphism algebra of the space of chambers. In Section 10.8 we explain how this relates to the notion of descents and Solomon’s descent algebra.

Section 10.9 deals with directed faces and directed flats. Just as faces and flats carry a monoid structure, directed faces and directed flats carry a dimonoid structure. The inter-relationships between these algebraic objects are studied in Section 10.10. In Section 10.11, we discuss the break and join maps. These are natural companions to the projection maps and together they explain how various combinatorial and geometric objects compose and decompose. These ideas will be used to construct a number of Hopf monoids in Chapter 12.

In Section 10.12, we discuss a weighted version of the gallery metric. The starting data is an integer square matrix $A$ of size $r$. Letting $A = [1]$ recovers the usual gallery metric. Interesting distinctions occur when $A$ is symmetric or antisymmetric. We relate them to the unoriented and oriented cases which occur in integration theory. In Section 10.13, we return to the (weighted) Schubert statistic of Section 2.2 and relate it to the ideas of this chapter by interpreting it in terms of the (weighted) gallery metric. In Sections 10.14 and 10.15, we define some interesting bilinear forms on faces, directed faces and chambers (maximal faces) and study conditions under which they are nondegenerate. We will see later in Chapter 12 that the nondegeneracy of these forms implies (among other things) the self-duality of related Hopf monoids.
10.1. Partitions and compositions

Partitions and compositions are basic combinatorial structures. They play an important role in the theory of species. In this section we review these and some related structures.

10.1.1. Partitions and compositions of a number. Let \( n \) be a nonnegative integer. A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of \( n \) is a finite sequence of positive integers such that

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.
\]

A composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \) of \( n \) is a finite sequence of positive integers such that

\[
\alpha_1 + \alpha_2 + \cdots + \alpha_k = n.
\]

If the numbers \( \alpha_i \) are allowed to be nonnegative, we say that \( \alpha \) is a weak composition of \( n \).

We write \( \lambda \vdash n \) and \( \alpha \models n \) to indicate that \( \lambda \) is a partition of \( n \) and \( \alpha \) a composition of \( n \). The numbers \( \lambda_i \) and \( \alpha_i \) are the parts of \( \lambda \) and \( \alpha \). The empty sequence is the only partition (and composition) of 0; it has no parts.

Fix a positive integer \( k \). The number of partitions of \( n \) into \( k \) parts is denoted \( p_k(n) \). The generating function is

\[
\sum_{n \geq 1} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.
\]

The number of compositions of \( n \) into \( k \) parts is the binomial coefficient \( \binom{n-1}{k-1} \), with generating function

\[
\sum_{n \geq 1} \binom{n-1}{k-1}x^n = \frac{x^k}{(1-x)^k}.
\]

For \( n \geq k \geq 1 \), there is a bijection between compositions of \( n \) into \( k \) parts and subsets of \([n-1]\) of cardinality \( k-1 \) given by

\[
(\alpha_1, \alpha_2, \ldots, \alpha_k) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}.
\]

10.1.2. Partitions and compositions of a set. Let \( I \) be a finite set. A partition \( X \) of \( I \) is an unordered collection \( X \) of disjoint nonempty subsets of \( I \) such that

\[
I = \bigcup_{S \in X} S.
\]

A composition of \( I \) is an ordered sequence \( F = (F^1, \ldots, F^k) \) of disjoint nonempty subsets of \( I \) such that

\[
I = \bigcup_{i=1}^{k} F^i.
\]

If the subsets \( F^i \) are allowed to be empty, we say that \( F \) is a weak composition of \( I \).

When confusion with compositions and partitions of numbers may arise, we may use the terms set compositions and set partitions. The subsets \( S \) of \( I \) which belong to \( X \) and the subsets \( F^i \) in the sequence \( F \) are the blocks or parts of \( X \) and \( F \), respectively. We agree that there is only one composition and one partition of the empty set (with no blocks). We write \( X \vdash I \) and \( F \models I \) to indicate that \( X \) is a partition of \( I \) and \( F \) a composition of \( I \). We often write \( F = F^1|\cdots|F^k \) instead
of $F = (F^1, \ldots, F^k)$. For partitions, we may choose an arbitrary ordering of its blocks and write $X = \{X^1, \ldots, X^k\}$.

Decomposition is just another term for weak composition. In contexts where we are interested in compositions as a combinatorial structure, we stick to the latter terminology. In other contexts, we prefer to speak of decompositions $(S^1, \ldots, S^k)$ of a finite set $I$ and write

$$I = S^1 \sqcup \cdots \sqcup S^k.$$ 

Sometimes, for emphasis, we may add that the decomposition is disjoint and ordered, even though this is always assumed when using this notation.

Decompositions are called partages by Joyal [181, Section 2.1]. Set compositions are often called ordered set partitions, preferential arrangements [202, Exercise 5.3.1.3], [341, Example 3.15.10], or ballots [40].

Fix a positive integer $k$. The number of partitions of $[n]$ into $k$ blocks is denoted $S(n,k)$ and called the Stirling number of the second kind. The generating function is

$$\sum_{n \geq 1} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)}.$$ 

The number of compositions of $[n]$ into $k$ blocks is $k!S(n,k)$, with generating function

$$\sum_{n \geq 1} k!S(n,k)x^n = \frac{x}{1-x} \cdot \frac{2x}{1-2x} \cdots \frac{kx}{1-kx}.$$ 

Given a surjective function $f: I \rightarrow [k]$, the sequence of fibers $f^{-1}(1)|\cdots|f^{-1}(k)$ is a composition of $I$ into $k$ blocks. This sets up a bijective correspondence between compositions of $I$ into $k$ blocks and surjective functions $I \rightarrow [k]$.

### 10.1.3. Linear partitions and linear compositions of a set

Let $I$ be a finite set. A linear partition (composition) of $I$ is a partition (composition) of $I$ together with a linear order on each of its blocks. A disposition of $I$ is a weak composition of $I$ with a linear order on each block.

The terminology used here is that of Rota et al [180, 57]. Linear partitions are also called partitions into ordered blocks. In [12, Section 5.4.2] we used fully nested set partition (composition) for linear partition (composition).

We extend the notation employed for linear orders (Example 8.3) to these structures as follows. To specify a linear partition, we give the set of blocks and display the order in each block by listing the elements from left to right in increasing order, separated by bars. To specify a linear composition, we further indicate the order among blocks by listing them from left to right separated by long bars. For example,

$${n|a,i|r,k|h|s} \quad \text{and} \quad a|n|r|i|s|k|h$$

are respectively a linear partition and a linear composition of \{k, r, i, s, h, n, a\}. The former is equal to \{i|r, n|a, k|h|s\} but not to \{a|n, i|r, s|h|k\}.

Note that a linear composition may be equivalently described by a pair $(F,C)$, where $F$ is a composition of $I$ and $C$ is a linear order on $I$ which refines $F$, or even as a linear order on $I$ together with a composition of $|I|$. For example,

$$(na|ri|ksh, a|n|r|i|s|k|h) \quad \text{and} \quad (a|n|r|i|s|k|h, (2, 2, 3))$$
both correspond to the linear composition $a|n| r|i | s|k|h$ of $\{k,r,i,s,h,n,a\}$.

It follows that the number of linear compositions of $[n]$ into $k$ blocks and the number of linear partitions of $[n]$ into $k$ blocks are respectively

$$n!(\frac{n-1}{k-1}) \quad \text{and} \quad \frac{n!}{k!}(\frac{n-1}{k-1})$$

The latter is called the Lah number.

10.1.4. Refinement and partial orders. Let $X$ and $Y$ be partitions of $I$. We say that $Y$ refines $X$ if each block of $Y$ is contained in a block of $X$, or equivalently if each block of $X$ is a union of blocks of $Y$. In this case we write $X \leq Y$. This defines a partial order on the set of partitions of $I$ which is in fact a lattice. The top element is the partition into singletons and the bottom element is the partition whose only block is the whole set $I$.

Warning. Sometimes the opposite partial order on partitions is used in the literature.

Refinement is defined similarly for compositions of $I$ and for compositions of $n$. The bijection (10.1) defines an isomorphism between the poset of compositions of $n$ and the poset of subsets of $[n-1]$ (a Boolean poset).

We also define a partial order on the set of linear compositions of $I$ as follows. We view them as pairs consisting of a set composition and a finer linear order and declare $(F,C) \leq (G,D)$ if $C = D$ and $F \leq G$ ($G$ refines $F$). In the bar notation, this means that we go up in the partial order by turning some short bars into long bars.

We consider two partial orders on linear partitions. To this end, we make use of the notions of restriction, shuffle and concatenation of linear orders discussed in Examples 8.16 and 8.24.

Let $L$ and $M$ be two linear partitions of $I$. First, we write $L \leq' M$ if each ordered block of $M$ is a restriction of an ordered block of $L$, or equivalently if each ordered block of $L$ is a shuffle of ordered blocks of $M$. For instance,

$$\{l|a|k,s|h|m|i\} \leq' \{l|k,a,s|m,h|i\}.$$

Second, we write $L \leq M$ if the ordered blocks of $M$ are obtained by deconcatenating the ordered blocks of $L$, or equivalently, if each ordered block of $L$ is a concatenation of ordered blocks of $M$. For instance,

$$\{l|a|k,s|h|m|i\} \leq \{l|a,k,s|h,m|i\}.$$

Note that

$$L \leq M \implies L \leq' M.$$

10.1.5. Type, support, and base. The type of a composition $F$ of $I$ is the composition of $|I|$ whose parts are the sizes of the blocks of $F$. The type of a partition $X$ of $I$ is the partition of $|I|$ whose parts are the sizes of the blocks of $X$ (listed in decreasing order).

The support of a composition $F$ of $I$ is the partition supp($F$) of $I$ obtained by forgetting the order among the blocks. The support of a composition of $n$ is the partition of $n$ obtained by reordering the parts in decreasing order.
10.1. PARTITIONS AND COMPOSITIONS

The support and type maps commute with each other. This can be illustrated as follows.

\[
\begin{array}{c}
l|a|k|s|h|m|i \\
\downarrow \text{type} \\
(2, 3, 2) \downarrow \text{supp}
\end{array} \rightarrow \begin{array}{c}
\{l|a, k|s, h|m|i\} \\
\downarrow \text{type}
\end{array} \rightarrow \begin{array}{c}
(3, 2, 2)
\end{array}.
\]

The support of a linear composition \((F, C)\) is the linear partition \(\text{supp}(F, C)\) obtained by forgetting the order among the blocks (but keeping the order within each block).

The base of a linear composition (partition) is the composition (partition) obtained by forgetting the orders within each block, or equivalently, by removing the short bars.

The support and base maps commute with each other. This can be illustrated as follows.

\[
\begin{array}{c}
l|a|k|s|h|m|i \\
\downarrow \text{base} \\
la|ks|hm|i
\end{array} \rightarrow \begin{array}{c}
\{l|a, k|s, h|m|i\} \\
\downarrow \text{base}
\end{array} \rightarrow \begin{array}{c}
la|ks,hmi
\end{array}.
\]

10.1.6. Concatenation, restriction, shuffles and quasi-shuffles. Throughout this section, we fix an ordered disjoint decomposition \(I = S \sqcup T\) of a finite set \(I\).

Given a composition \(F\) of \(I\), the restriction \(F|_S\) is the composition of \(S\) whose blocks are the nonempty intersections of the blocks of \(F\) with \(S\). If \(F = F^1|\cdots|F^k\), we write

\[
F|_S = (F^1 \cap S) \cdots (F^k \cap S)\hat{\phantom{\cdot}}
\]

where the hat indicates that empty intersections are removed from the list.

Given compositions \(F = F^1|\cdots|F^k\) of \(S\) and \(G = G^1|\cdots|G^l\) of \(T\), their concatenation is the composition \(F \cdot G\) of \(I\) defined by

\[
F \cdot G := F^1|\cdots|F^k|G^1|\cdots|G^l.
\]

A quasi-shuffle of \(F\) and \(G\) is a composition \(H\) of \(I\) such that \(H|_S = F\) and \(H|_T = G\). It follows that each block of \(H\) is either a block of \(F\), or a block of \(G\), or a union of a block of \(F\) and a block of \(G\);

A shuffle of \(F\) and \(G\) is a quasi-shuffle \(H\) such that each block of \(H\) is either a block of \(F\) or a block of \(G\).

In other words, in a shuffle \(H\) the blocks \(F^i\) are listed in \(H\) in the same order as in \(F\), and similarly for the blocks of \(G\). A quasi-shuffle is obtained from a shuffle by substituting any number of pairs of blocks \((F^i, G^j)\) for \(F^i \sqcup G^j\), if they are adjacent in the shuffle.

For example,

\[
g|sh|i|au|ri|va \text{ is a shuffle of } sh|i|va \text{ and } g|au|ri,
\]

and

\[
vl|a|i|sh|ksh|mi|nu \text{ is a quasi-shuffle of } v|i|sh|nu \text{ and } la|ksh|mi.
\]

The notion of shuffle and quasi-shuffle can be extended to any finite number of set compositions.
Given a partition $X$ of $I$, the \textit{restriction} $X|_S$ is defined in the same manner as for compositions.

Given partitions $X$ of $S$ and $Y$ of $T$, their \textit{union} is the partition $X \sqcup Y$ of $I$ whose blocks are the blocks of $X$ and the blocks of $Y$. A \textit{quasi-shuffle} of $X$ and $Y$ is any partition of $I$ whose restriction to $S$ is $X$ and whose restriction to $T$ is $Y$. For example,

$$\{sh,i,va\}$$

is the union of $\{sh,i\}$ and $\{va\}$,

and

$$\{sh,i,va\}, \{sh,i,va\} \text{ and } \{shva,i\} \text{ are all their quasi-shuffles.}$$

Given a linear partition $L$ of $I$, the \textit{restriction} $L|_S$ is the linear partition of $I$ whose blocks are the nonempty intersections of the blocks of $L$ with $S$, ordered as in $L$.

Given linear partitions $L$ of $S$ and $M$ of $T$, their \textit{union} is the linear partition $L \sqcup M$ of $I$ whose ordered blocks are those of $L$ and those of $M$. A \textit{quasi-shuffle} of $L$ and $M$ of $I$ is any linear partition of $I$ each of whose ordered blocks is either an ordered block of $L$, or one of $M$, or a concatenation of one of $L$ followed by one of $M$. For example, the quasi-shuffles of $\{v|i|s,h\}$ and $\{n|u\}$ are

$$\{v|i|s,h,n|u\}, \quad \{v|i|s,hn|u\}, \quad \text{and } \{v|i|s|hn|u,h\}.$$  

\textbf{10.1.7. Factorials and related numbers.} The \textit{factorial} of a set partition $X$ is

\begin{equation}
X! := \prod_{S \in X} |S|!.
\end{equation}

It counts the number of ways of endowing each block of $X$ with a linear order. The \textit{cyclic factorial} of $X$ is

\begin{equation}
X^b := \prod_{S \in X} (|S| - 1)!.
\end{equation}

It counts the number of ways of endowing each block of $X$ with a cyclic order. Note that

\begin{equation}
(X \sqcup Y)! = X!Y! \quad \text{and} \quad (X \sqcup Y)^b = X^bY^b.
\end{equation}

The following relation between factorials and cyclic factorials is of importance.

\begin{equation}
\sum_{Y: X \leq Y} Y^b = X!.
\end{equation}

It may be derived as follows. Suppose $X$ has only one block $I$. Each permutation of $I$ determines a partition $Y$ of $I$ whose blocks are the cycles of the permutation. The left hand side counts the number of permutations of $I$ according to these cycle partitions, while the right hand side counts all permutations. The general case follows using (10.3).

The coefficients $X^b$ appear in the work of Brown [70, Theorem 1] in the general setting of left regular bands; also see [12, Sections 2.5.5 and 2.6.2]. In these references, the notations $n_X$ and $c_X$ are used instead of $X^b$ and $X!$.

Given set partitions $X$ and $Y$ with $Y$ refining $X$, let

\begin{equation}
(X : Y)! := \prod_{S \in X} (n_S)!,
\end{equation}

where $n_S$ is the number of blocks of $Y$ that refine the block $S$ of $X$. Note that if $Y$ is the unique partition into singletons, then $X! = (X : Y)!$. 

\textbf{10.1.7. Factorials and related numbers.} The \textit{factorial} of a set partition $X$ is

\begin{equation}
X! := \prod_{S \in X} |S|!.
\end{equation}

It counts the number of ways of endowing each block of $X$ with a linear order. The \textit{cyclic factorial} of $X$ is

\begin{equation}
X^b := \prod_{S \in X} (|S| - 1)!.
\end{equation}

It counts the number of ways of endowing each block of $X$ with a cyclic order. Note that

\begin{equation}
(X \sqcup Y)! = X!Y! \quad \text{and} \quad (X \sqcup Y)^b = X^bY^b.
\end{equation}

The following relation between factorials and cyclic factorials is of importance.

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It may be derived as follows. Suppose $X$ has only one block $I$. Each permutation of $I$ determines a partition $Y$ of $I$ whose blocks are the cycles of the permutation. The left hand side counts the number of permutations of $I$ according to these cycle partitions, while the right hand side counts all permutations. The general case follows using (10.3).

The coefficients $X^b$ appear in the work of Brown [70, Theorem 1] in the general setting of left regular bands; also see [12, Sections 2.5.5 and 2.6.2]. In these references, the notations $n_X$ and $c_X$ are used instead of $X^b$ and $X!$.

Given set partitions $X$ and $Y$ with $Y$ refining $X$, let

\begin{equation}
(X : Y)! := \prod_{S \in X} (n_S)!,
\end{equation}

where $n_S$ is the number of blocks of $Y$ that refine the block $S$ of $X$. Note that if $Y$ is the unique partition into singletons, then $X! = (X : Y)!$. 

Let $F$ be any set composition with support $X$. Then

\begin{equation}
(X : Y)! = |\{G \mid F \leq G, \supp(G) = Y\}|.
\end{equation}

The factorial of a set composition $F = F_1 \cdots F_k$ is

\begin{equation}
F! := \prod_{i=1}^{k} |F_i|!.
\end{equation}

This is the number of linear orders that refine $F$. The factorial of a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of an integer is

\begin{equation}
\alpha! := \prod_{i=1}^{k} \alpha_i!.
\end{equation}

Note that

$F! = (\supp F)! = (\text{type } F)!$.

\section*{10.2. Faces, chambers, flats and cones}

In this section, we discuss the braid arrangement, along with the basic notions of faces, flats and cones which are attached to it. These notions are closely related to the notions of partitions and compositions discussed in Section 10.1. More details on the braid arrangement can be found in [45, 46, 47, 72].

\subsection*{10.2.1. The braid arrangement.}

The \textit{braid arrangement} in Euclidean space $\mathbb{R}^n$ consists of the $\binom{n}{2}$ hyperplanes defined by

$x_i = x_j,$

where $1 \leq i < j \leq n$. The symmetric group $S_n$ acts on this arrangement by permuting the coordinates.

One may replace the set $[n]$ by any finite set $I$. Let $\mathbb{R}^I$ be the vector space consisting of all functions from $I$ to $\mathbb{R}$. The braid arrangement in $\mathbb{R}^I$ consists of the hyperplanes $H_{ij}$ defined by

$x_i = x_j,$

where $i \neq j$ range over the elements of $I$. Note that $H_{ij} = H_{ji}$. Let

$\text{Br}[I] := \{H_{ij} \mid i, j \in I, i \neq j\}$

denote the arrangement.

A bijection $I \cong J$ induces a linear isomorphism $\mathbb{R}^I \cong \mathbb{R}^J$ which sends $\text{Br}[I]$ to $\text{Br}[J]$. Thus, $\text{Br}$ is a set species, and so will be each of the objects associated to it throughout this chapter.

\subsection*{10.2.2. Faces and chambers.}

For each pair $(i, j) \in I^2$ with $i \neq j$, the subset of $\mathbb{R}^I$ defined by

$x_i \leq x_j$

is a \textit{half-space} of the braid arrangement. Its supporting hyperplane is $H_{ij}$. Note that each hyperplane supports two half-spaces.

Two points $x, y \in \mathbb{R}^I$ lie on opposite sides of a hyperplane $H$ if $x$ belongs to one half-space supported by $H$, $y$ belongs to the other half-space, and neither belongs to $H$. We say that two points are \textit{separated} if they lie on opposite sides of at least one hyperplane $H \in \text{Br}[I]$.
A face of the braid arrangement $\text{Br}[I]$ is a nonempty subset of $\mathbb{R}^I$ with the following two properties.

- If two points lie in the set, then they are not separated.
- If a point lies in the set, then any point that is not separated from it also lies in the set.

A face is defined by a system of equalities and inequalities which may be encoded by a composition of $I$: the equalities are used to define the blocks and the inequalities to order them. For example, for $I = \{a, b, c, d\}$,

$$x_a = x_c \leq x_b = x_d \quad \leftrightarrow \quad ac|bd.$$

Thus, faces correspond to compositions of the set $I$.

Let $\Sigma[I]$ denote the set of faces of the arrangement $\text{Br}[I]$. It is partially ordered by inclusion. The partial order on $\Sigma[I]$ corresponds to refinement of set compositions. The minimum element is the face

$$\{x \in \mathbb{R}^I \mid x_i = x_j \text{ for all } i, j \text{ in } I\}.$$ 

It corresponds to the composition with one block. The maximal faces are called chambers. They correspond to linear orders on $I$. For example,

$$x_a \leq x_c \leq x_b \leq x_d \quad \leftrightarrow \quad a|c|b|d.$$ 

Let $L[I]$ denote the set of chambers.

This defines the set species $\Sigma$ (of faces or set compositions) and $L$ (of chambers or linear orders). The linearized species are denoted $\bar{\Sigma}$ and $\bar{L}$. The latter is the species of Example 8.3.

Since the braid arrangement is central (all hyperplanes pass through the origin), every face has an opposite face. In terms of set compositions, the opposite $F$ of a face $F$ is obtained by reversing the order of the blocks: if $F = F_1|\cdots|F_l$, then

$$\overline{F} = F_l|\cdots|F_1.$$ 

The hyperplane $H_{ij}$ is called a wall of a chamber $C$ if $i$ and $j$ are consecutive in the linear order $C$.

In particular, $\Sigma[n]$ and $L[n]$ denote the sets of faces and chambers of the braid arrangement in $\mathbb{R}^n$. The action of $S_n$ on $\Sigma[n]$ corresponds to its obvious action on compositions of $[n]$. This action is simply transitive on $L[n]$. Hence, one may identify

$$S_n \to L[n] \quad w \mapsto wC_{(n)},$$

where $C_{(n)} := 1|\cdots|n$ is the canonical linear order on $[n]$. We refer to $C_{(n)}$ as the fundamental chamber.

Since there is no canonical order on $I$, for the arrangement $\text{Br}[I]$ there is no canonical choice of fundamental chamber. On the other hand, let $n$ be the cardinality of the set $I$ and let $\text{Bij}([n], I)$ be the set of bijections from $[n]$ to $I$. Then there is a bijection

$$\text{Bij}([n], I) \to L[I] \quad w \mapsto wC_{(n)},$$

where

$$C_{(n)} := 1|\cdots|n \quad \text{and} \quad wC_{(n)} = w(1)|w(2)|\cdots|w(n).$$

For $I = [n]$, this recovers (10.9).
10.2.3. Flats. A flat is a subspace obtained by intersecting some of the hyperplanes in the arrangement. Let $\Pi[I]$ denote the set of flats in the braid arrangement $\text{Br}[I]$. It is partially ordered by inclusion. The poset of flats is a lattice.

Flats correspond to partitions of $I$. For example, for $I = \{k, r, i, s, h, n, a\}$,

$$(x_k = x_i) \cap (x_r = x_i) \cap (x_n = x_a) \quad \rightarrow \quad \{kri, s, h, na\}.$$ 

We identify $\Pi[I]$ with the lattice of partitions of $I$.

This defines the set species $\Pi$ of flats, or equivalently, set partitions. The linearized species is denoted $\Pi$. The partial order on flats given by inclusion corresponds to the partial order on set partitions given by refinement (Section 10.1.4).

Let $\text{supp}: \Sigma[I] \rightarrow \Pi[I]$ be the map which sends a face to its linear span. Equivalently, $\text{supp}(F)$ is the intersection of the hyperplanes containing the face $F$. In combinatorial terms, this coincides with the support map which sends a set composition to its underlying set partition (Section 10.1.5).

10.2.4. Cones. A cone of the braid arrangement $\text{Br}[I]$ is defined to be an intersection of a subset of its half-spaces.

For example, for $I = \{a, b, c, d\}$,

$$\{x \in \mathbb{R}^I \mid x_a = x_c \leq x_b, x_d \leq x_b\}$$

is a cone. Note that a face of the arrangement is a cone. Similarly, a flat of the arrangement is also a cone.

A top-dimensional cone is a cone with a nonempty interior. In other words, it is a cone which contains a chamber. Note that a chamber of the arrangement is a top-dimensional cone and conversely a top-dimensional cone is the union of the chambers which belong to it. This defines the species of cones and the species of top-dimensional cones, both of whose $I$-components are posets under inclusion.

Observe that any flat in the braid arrangement inherits a hyperplane arrangement which is in fact isomorphic to a smaller braid arrangement. Now let $R$ be a cone. Define $X$ to be the flat obtained by intersecting all the hyperplanes which contain $R$. It follows that $R$ is a top-dimensional cone in the induced arrangement on $X$. Thus, every cone is a top-dimensional cone in some flat.

10.2.5. The spherical representation. Note that the intersection of all hyperplanes in the braid arrangement is the one-dimensional space where all coordinates are equal:

$$\bigcap_{i \neq j} H_{ij} = \{x \in \mathbb{R}^I \mid x_i = x_j \text{ for all } i, j\}.$$ 

Let

$$H_0 = \left\{x \in \mathbb{R}^I \mid \sum_{i \in I} x_i = 0\right\}$$

be the orthogonal complement. We intersect all hyperplanes $H_{ij}$ with $H_0$ and no information is lost. Then we intersect with the unit sphere in $H_0$ and we only lose the center of the arrangement. This is the spherical representation of the braid arrangement.

The procedure for $I = \{a, b, c\}$ is shown in Figure 10.1: $H_0$ is shown in perspective as a horizontal plane, with the 3 vertical hyperplanes $H_{ab}$, $H_{bc}$ and $H_{ac}$ cutting through it. The spherical representation is seen on the unit circle on the plane $H_0$. It is shown in more detail in Figure 10.2.
10.3. The Coxeter complex of type $A$

In this section, we associate a simplicial complex to the braid arrangement. This is the Coxeter complex of type $A$. We discuss this along with explicit low dimensional examples.

10.3.1. Simplicial complexes. We begin with a quick review of simplicial complexes. More information can be found in [3, Appendix A.1] and [340].

Let $I$ be a finite set and $2^I$ the set of subsets of $I$ ordered by inclusion:

$$S \leq S' \iff S \subseteq S'.$$

This is the Boolean poset.

Let $V$ be a finite set. A simplicial complex with vertex set $V$ consists of a nonempty collection $k$ of subsets of $V$ with the following properties:

- for each $v \in V$, the singleton $\{v\}$ belongs to $k$;
- if $K \in k$ and $J \subseteq K$, then $J \in k$.

The collection of all subsets of $V$ is a simplicial complex, called the simplex with vertex set $V$ and denoted $\Delta_V$.

Let $k$ be a simplicial complex. The subsets of $V$ which belong to $k$ are its faces. Note that the empty set is a face of any simplicial complex. If $K$ is a face of $k$, then the collection of subfaces of $K$ forms a simplicial complex, equal to the simplex $\Delta_K$.

If $k$ is a simplicial complex, the collection $k$ is partially ordered by inclusion and satisfies properties (10.11a)–(10.11c) below. Conversely, any poset satisfying these properties is isomorphic to the poset of faces of a unique simplicial complex [3, Exercise A.3].

(10.11a) The poset $k$ has a minimum element.

(10.11b) For any $K \in k$, the subposet $\{J \in k \mid J \leq K\}$ is isomorphic to a Boolean poset.

(10.11c) If $J, K \in k$ have an upper bound, then they have a least upper bound.

The simplex $\Delta_V$ corresponds in this manner to the Boolean poset $2^V$. 

![Figure 10.1. Euclidean and spherical representations.](image-url)
Let $k$ be a simplicial complex and $K \in k$ a face. The dimension of $K$ is one less than its cardinality. In particular, the dimension of the empty face is $-1$. The star of $K$ consists of the faces of $k$ which contain $K$:

$$\text{Star}_k(K) := \{ J \in k \mid K \subseteq J \}. $$

It is a simplicial complex whose vertices are the faces of $k$ in which $K$ has codimension 1.

The complex $k$ is pure of dimension $d$ if all maximal faces, called chambers, have the same dimension $d$.

The Cartesian product $k_1 \times k_2$ of two simplicial complexes $k_1$ and $k_2$ is another simplicial complex, called the join of $k_1$ and $k_2$. The empty face gives rise to a canonical embedding of each factor in the join. For instance,

$$k_1 \hookrightarrow k_1 \times k_2, \quad K \mapsto (K, \emptyset).$$

The vertex set of $k_1 \times k_2$ is the disjoint union of the vertex sets of $k_1$ and $k_2$.

A balanced simplicial complex is a pair $(k, \varphi)$ where $k$ is a simplicial complex and $\varphi : V \to [n]$ is a function that restricts to a bijection

$$C \cong [n]$$

for each maximal face $C$ of $k$. This implies that $k$ is pure of dimension $n-1$.

Balanced complexes are called colored complexes in [3] and labeled complexes in [68]. If we think of $\varphi(K)$ as a color assigned to a vertex $K \in k$, then the condition on $\varphi$ implies that all vertices in a face receive different colors.

A simplicial map $f : k \to k'$ between simplicial complexes with vertex sets $V$ and $V'$ is a map $f : V \to V'$ such that $f(K)$ is a face of $k'$ for every face $K$ of $k$. The simplicial map is nondegenerate if it preserves face dimensions. A simplicial isomorphism is necessarily nondegenerate. A nondegenerate simplicial map $f : k \to k'$ restricts to an isomorphism $\Delta_K \to \Delta_{f(K)}$ for each face $K$ of $k$.

### 10.3.2. The Coxeter complex.

Recall the poset of faces $\Sigma[I]$ associated to the braid arrangement in $\mathbb{R}^I$. One can easily check that it is the poset of faces of a simplicial complex. It admits the following (equivalent) descriptions:

- it is the reduced order complex of the Boolean poset $2^I$ (Example 13.21),
- it is the barycentric subdivision of the boundary of the simplex

$$\left\{ (x_i)_{i \in I} \in \mathbb{R}^I \mid \sum_{i \in I} x_i = 1, x_i \geq 0 \right\},$$

- it is the triangulation of the unit sphere in the spherical representation of the braid arrangement.

From now on, we will identify $\Sigma[I]$ with this simplicial complex.

A significant property of $\Sigma[I]$ is that it is a Coxeter complex. The theory of Coxeter complexes was developed by Tits [360]. We recall some important features of these complexes: A Coxeter complex is balanced, gallery-connected, and it satisfies the gate property. The star of any face in a Coxeter complex is again a Coxeter complex. The join of two Coxeter complexes is again a Coxeter complex. Further, the set of faces of a Coxeter complex is a monoid; the product is constructed using the projection maps of Tits.

We will discuss some of these properties explicitly for the Coxeter complex of type $A$, namely $\Sigma[I]$. For this example, these properties can be checked directly.
Hence familiarity with the general theory is not essential to follow the present discussion.

Let \( n := |I| \). Note that the simplicial complex \( \Sigma[I] \) is pure of dimension \( n - 2 \). Let \( \Delta_{[n-1]} \) denote the set of compositions of \( n \) (the simplex of dimension \( n - 2 \)).

Recall from Section 10.1.5 the type map

\[
\Sigma[I] \rightarrow \Delta_{[n-1]}
\]

which sends a composition \( F \) of \( I \) to the composition of \( n \) whose parts are the sizes of the blocks of \( F \). The type map is a nondegenerate simplicial map which turns \( \Sigma[I] \) into a balanced complex (also see Proposition 13.18).

10.3.3. Low dimensional examples. Figure 10.2 shows the simplicial complexes \( \Sigma[3] \) and \( \Sigma[\{a, b, c\}] \). The circle is the same as the one shown in Figure 10.1. The vertices are of two types, shown in black and white. The set composition \( abc \) (all elements in one block) indexes the center of the arrangement and does not show in the spherical representation.

The simplicial complex \( \Sigma[\{a, b, c, d\}] \) is shown in Figure 10.3. It has been essentially reproduced from the paper of Brown, Billera and Diaconis [47]. This complex triangulates the sphere into twenty four triangles, eighteen of which can be seen (either partly or completely) in the figure. The edges and vertices have not been labeled for space constraints. Observe that the vertex in the center of the figure has label \( abc|d \) and its star is isomorphic to the simplicial complex \( \Sigma[\{a, b, c\}] \) shown on the right in Figure 10.2. The vertices are of three types. Those shown in black are of type \((1,3)\), those in white are of type \((2,2)\), and the vertex in the center is of type \((3,1)\).

One can flatten the spherical representation so that all chambers except \( d|c|b|a \) are visible. This is shown in Figure 10.4. The six hyperplanes can be seen in full as the six ovals.

10.4. Tits projection maps and the monoid of faces

There is an operation on the set of faces of the Coxeter complex which turns this set into a monoid. The operation is given by the projection maps of Tits. This section discusses these notions from a combinatorial perspective. The underlying geometry is discussed later in Section 10.5.
10.4. TITS PROJECTION MAPS AND THE MONOID OF FACES

10.4.1. The monoid of faces. The set $\Sigma[I]$ has the structure of a monoid. We view faces as set compositions of $I$ and multiply two such by intersecting their blocks and ordering them lexicographically. More precisely, if $F = F^1 \mid \cdots \mid F^l$ and $G = G^1 \mid \cdots \mid G^m$, then

\[(10.13) \quad FG := (F^1 \cap G^1) \mid \cdots \mid (F^l \cap G^m),\]

where the hat indicates that any empty intersections should be deleted. For example,

\[(kri|shna)(s|khna|ri) = k|ri|s|hna.\]

It is clear that this product is associative. The set composition with one part serves as the unit. Thus, $\Sigma[I]$ is a monoid. It is not commutative. In fact,

\[(10.14) \quad FG = GF \iff F \text{ and } G \text{ are joinable},\]

where joinable means that there is a face which contains both $F$ and $G$.

**Proposition 10.1.** The product on $\Sigma[I]$ satisfies the following properties.

(i) $F \leq FG$.
(ii) $F \leq G \iff FG = G$.
(iii) If $G \leq H$, then $FG \leq FH$.
(iv) If $C$ is a chamber, then $CF = C$ and $FC$ is a chamber.
(v) If $FG = K$ and $F \leq H \leq K$, then $HG = K$.
(vi) If $H_1 F = K$ and $H_2 F = K$, then $(H_1 \land H_2)F = K$. 

---

**Figure 10.3.** The simplicial complex $\Sigma[\{a, b, c, d\}]$. 
Figure 10.4. The flattened simplicial complex $\Sigma[A,b,c,d]$. 

(vii) If $E$ and $F$ are subfaces of a face, that is, if $E$ and $F$ have an upper bound, then $H(E \vee F) = HE \vee HF$ for any face $H$.

(viii) $F \overline{F} = F$.

(ix) $FGF = F$.

(x) If $FPG = F\overline{P}G$, then $FP = F\overline{P} = F$.

(xi) If $F$ is a face and $D$ is a chamber, and $HF \neq D$ for any proper face $H$ of $D$, then $F \leq D$.

For any bijection $J \rightarrow I$, the corresponding map

\[(10.15) \quad \Sigma[J] \rightarrow \Sigma[I]\]

is both type and product preserving. In particular, the product of each $\Sigma[I]$ yields a morphism of set species

\[(10.16) \quad \Sigma \times \Sigma \rightarrow \Sigma,\]

where $\times$ denotes the Hadamard product on set species (8.36).
Remark 10.2. Property (ix) states that the monoid $\Sigma[I]$ is a left regular band (Section 8.7.7). Some of the properties listed above hold for all left regular bands.

10.4.2. The lattice of flats as a quotient of the monoid of faces. We have seen that the poset of flats $\Pi[I]$ is a lattice. We now view it as a commutative monoid with the product given by the join. The join $X \vee Y$ is the smallest common refinement of $X$ and $Y$. It is obtained by intersecting the parts of $X$ with the parts of $Y$ and deleting empty intersections. The similarity between the product in $\Sigma[I]$ and $\Pi[I]$ says that

$$\text{(10.17)}\quad \text{supp}(FK) = \text{supp}(F) \vee \text{supp}(K).$$

Thus, the support map is a morphism of monoids. Using this fact, one may view $\Pi[I]$ as a left module over $\Sigma[I]$ via

$$\text{(10.18)}\quad K \cdot X := \text{supp}(K) \vee X.$$

An alternative description of $\Pi[I]$ can be given as follows. Define an equivalence relation on $\Sigma[I]$: $F \sim G \iff FG = F$ and $GF = G$.

It follows from (10.17) that

$$F \sim G \iff \text{supp}(F) = \text{supp}(G).$$

Thus, equivalence classes can be identified with flats and the canonical quotient map which sends a face to its equivalence class is the support map.

We reformulate the preceding discussion in combinatorial terms. Let $F$ and $G$ be two set compositions. Then (10.13) implies that $FG = F$ and $GF = G \iff F$ and $G$ consist of the same blocks.

In other words, the equivalence class of $F$ consists of all its reorderings $G$. Thus, flats are identified with set partitions.

10.4.3. Shuffles, quasi-shuffles, and the product of faces. The monoid of faces is far from being a group. However, given faces $F$ and $H$ with $F \leq H$, there is always a face $G$ such that $FG = H$.

In fact, we may just choose $G = H$. We now discuss all solutions $G$ to this equation, from a combinatorial perspective. A related point is addressed in Section 10.7.5.

We view faces as set compositions and make use of the operations of concatenation, shuffle and quasi-shuffle of Section 10.1.6. The statements below are direct consequences of (10.13).

Let $(F, H)$ be a pair of set compositions with $F \leq H$. Write $F = F^1|\cdots|F^i$. Since $H$ refines $F$, it is the concatenation of a composition of $F^1$, followed by a composition of $F^2$, and so on. We refer to these compositions as the blocks of $(F, H)$. For example, if

$$F = 135|24789|6 \quad \text{and} \quad H = 3|15|7|48|29|6,$$

then the blocks of $(F, H)$ are

$$3|15, 7|48|29 \text{ and } 6.$$

Note that $H$ is a linear order if and only if the blocks of $(F, H)$ are linear orders.
Now let $G$ be another set composition. Then,

\[(10.20) \quad FG = H \iff G \text{ is a quasi-shuffle of the blocks of } (F, H).\]

In addition,

\[(10.21) \quad FG = H \text{ and } GF = G \iff G \text{ is a shuffle of the blocks of } (F, H).\]

In particular, let $(F, D)$ be a linear set composition (so that the linear order $D$ refines $F$) and let $C$ be another linear order. Then,

\[(10.22) \quad FC = D \iff C \text{ is a shuffle of the blocks of } (F, D).\]

10.4.4. **Tits projection maps.** Let $F$ be a fixed face. The map given by left multiplication by $F$,

\[(10.23) \quad p_F : \Sigma[I] \to \text{Star}_{\Sigma[I]}(F), \quad G \mapsto FG\]

is called the *Tits projection* [360, Section 2.30]. Properties (i) and (ii) in Proposition 10.1 imply that the image of $p_F$ is the star of $F$ and that $p_F$ is idempotent. We say that $FG$ is the projection of $G$ on $F$.

10.5. **The gallery metric and the gate property**

In this section, we introduce the gallery metric on chambers. The Tits projection of a chamber onto a face is the closest chamber in the star of the face. Its existence is guaranteed by the gate property of the gallery metric. This is the geometric meaning of the product of faces of Section 10.4. In addition to reviewing these facts, we discuss a distance function on faces which generalizes the one on chambers.

10.5.1. **A distance function on chambers. The gallery metric.** We say two chambers are *adjacent* if they have a common codimension 1 face. A *gallery* is a sequence of chambers such that consecutive chambers are adjacent. Its *length* is one less than the number of chambers in the sequence. We have remarked earlier that $\Sigma[I]$ is a gallery-connected simplicial complex. This means that for any two chambers $C$ and $D$, there is a gallery from $C$ to $D$. We then define the *gallery distance* $\text{dist}(C, D)$ to be the minimal length of a gallery connecting $C$ and $D$. Any gallery which achieves this minimum is called a *minimum gallery* from $C$ to $D$. This defines the *gallery metric* on $L[I]$. It verifies the familiar properties of a metric:

\[
\text{dist}(C, D) \geq 0, \text{ with equality if and only if } C = D,
\]

\[
\text{dist}(C, D) = \text{dist}(D, C),
\]

\[
\text{dist}(C, E) \leq \text{dist}(C, D) + \text{dist}(D, E),
\]

with equality if and only if there is a minimum gallery from $C$ to $E$ which passes through the chamber $D$. We use the notation $C - D - E$ for such a minimum gallery.

The gallery metric is natural in $I$: For any bijection $\sigma : I \to J$,

\[(10.24) \quad \text{dist}(C, D) = \text{dist}(\sigma C, \sigma D).\]

Further, it is compatible with the opposite map:

\[(10.25) \quad \text{dist}(C, D) = \text{dist}(\bar{D}, \mathcal{C}).\]
If we view $\Sigma[I]$ as the set of faces of the braid arrangement, then $\text{dist}(C,D)$ is the number of hyperplanes which separate $C$ and $D$. Let us make this more explicit. Write $C = C^1 \cdots |C^n$, where $n = |I|$. Define the inversion set of $(C,D)$ to be

$$\text{Inv}(C,D) := \{(i,j) \in [n] \times [n] \mid i < j \text{ and } C^i \text{ appears after } C^j \text{ in } D\}.$$  

Then

$$\text{dist}(C,D) = |\text{Inv}(C,D)|.$$  

Let us now relate this to the inversion set (2.19) and the number of inversions (2.20) of an appropriate permutation. From (10.10), there are unique bijections $u$ and $v$ from $[n]$ to $I$ such that $C = uC(n)$ and $D = vC(n)$. Then

$$\text{Inv}(C,D) = \text{Inv}(uC(n),vC(n)) = \text{Inv}(C(n),u^{-1}vC(n)) = \text{Inv}(v^{-1}u).$$  

Note that $w := u^{-1}v$ is a permutation. The second equality follows from naturality of the inversion set, and the last equality from the definitions (note that $w$ gets replaced by its inverse). It follows that

$$\text{dist}(C,D) = \text{inv}(v^{-1}u) = \text{l}(v^{-1}u).$$  

Recall from Section 2.2.3 that $\text{l}(w)$ denotes the length of $w$, which coincides with the number of inversions of $w$. Note that (2.25) can be seen as a consequence of the symmetry of the distance function.

It is convenient to define, with notation as above,

$$d(C,D) := u^{-1}v.$$  

This is known as the Weyl-valued distance between $C$ and $D$. It takes values in the symmetric group. In particular, for $I = [n]$, we obtain, for any permutation $\sigma$,

$$\text{inv}(\sigma) = \text{l}(\sigma) = \text{dist}(C(n),\sigma C(n)).$$  

It is clear that for any chambers $C$, $D$ and $E$,

$$d(C,E) = d(C,D) d(D,E).$$  

10.5.2. Gate property. There is a geometric way of describing Tits projections, and hence the product of $\Sigma[I]$, which we discuss briefly. It relies on the fact that $\Sigma[I]$ has the gate property.

**Proposition 10.3 (Gate property).** Let $F$ be a face and $D$ a chamber. Among the chambers containing $F$, there is a unique one that is closest to $D$ in the gallery metric. This unique chamber is $FD$.

In other words, $FD$, which is the projection of $D$ on $F$, is the gate of the star of $F$ viewed from $D$. This is illustrated in Figure 10.5 which shows the relevant portion of a simplicial complex of dimension two. The big dot is a vertex named $F$, and both $D$ and $FD$ are chambers, which in dimension two are triangles.

The product of two arbitrary faces turns out to be

$$FG = \bigwedge FD,$$  

where the meet is taken over all chambers $D$ which contain $G$.

The following is a consequence of Proposition 10.3.
Figure 10.5. The projection map at work.

**Proposition 10.4.** Let $C$ and $D$ be chambers and $F$ be a face of $C$. Then there exists a minimum gallery $C - FD - D$. In particular,

\[ \text{dist}(C, D) = \text{dist}(C, FD) + \text{dist}(FD, D). \]  

The gate property originated in the work of Tits [360, Section 3.19.6], and was abstracted later by Dress and Scharlau [320, 103]. It also appears in the work of Abels [2], M"uhlherr [281] and Mahajan [253] (to name a few references). Some basic information on this property can be found in [12, Section 1.1.1]. The poset of faces of any real hyperplane arrangement satisfies this property. This fact can be used to define a semigroup structure on the set of faces of any real hyperplane arrangement [12, Equation (1.1)]. If the arrangement is central, the semigroup is in fact a monoid.

**10.5.3. A distance function on faces.** For a face $F$, let $L_F$ denote the set of chambers containing $F$. It is straightforward [12, Lemma 2.2.1] to show that if faces $F$ and $G$ have the same support, then the projection

\[ p_G : L_F \rightarrow L_G \quad C \mapsto GC \]

is a bijection with inverse given by the projection $p_F : D \mapsto FD$.

Now let $F$ and $G$ be any two faces. Since $FG$ and $GF$ have the same support, the projection

\[ p_{GF} : L_{FG} \rightarrow L_{GF} \]

is a bijection, with inverse $p_{FG}$. Further, if $C$ is any chamber containing $FG$, then by using the compatibility of the symmetric group action with the distance function and the projection map, we see that $\text{dist}(C, p_{GF}(C))$ is independent of the particular choice of $C$.

This observation allows us to define the distance between any two faces $F$ and $G$:

\[ \text{dist}(F, G) := \text{dist}(C, p_{GF}(C)), \]

where $C$ is any chamber containing the face $FG$. Since $p_{GF}$ is a bijection with inverse $p_{FG}$, it follows that

\[ \text{dist}(F, G) = \text{dist}(p_{FG}(D), D), \]

where $D$ is any chamber containing $GF$. This shows that the distance function is symmetric. It is also clear that

\[ \text{dist}(F, G) = \text{dist}(FG, GF). \]

**Remark 10.5.** The above definition can in fact be made for the faces of any central hyperplane arrangement. The right-hand side of (10.33) is independent of
the particular choice of \( C \) and equals the number of hyperplanes which separate \( F \) and \( G \) (meaning that \( F \) and \( G \) lie on opposite sides of the hyperplane).

The distance function on faces does not define a metric. However, it does restrict to a metric on the set of faces with a fixed support. In particular,

\[
\text{(10.35)} \quad \text{dist}(F, G) = 0 \quad \text{and} \quad \text{supp} \ F = \text{supp} \ G \iff F = G.
\]

Since \( FG \) and \( GF \) have the same support, it follows from (10.34) and (10.14) that

\[
\text{(10.36)} \quad \text{dist}(F, G) = 0 \iff FG = GF \iff F \text{ and } G \text{ are joinable.}
\]

In particular, the distance between a face and a subface is always 0. It follows that the triangle inequality fails as well; so the distance function on faces is not a pseudometric either.

Let us make the distance function more explicit. We first deal with the case of equal support. Let \( F \) and \( G \) be faces with the same support. Write \( F = F^1 | \cdots | F^k \).

Then \( G \) is a set composition obtained by permuting the \( F^i \)'s in some order. Define the inversion set of \( (F, G) \) to be

\[
\text{Inv}(F, G) := \{(i, j) \in [k] \times [k] \mid i < j \text{ and } F^i \text{ appears after } F^j \text{ in } G\}.
\]

Then

\[
\text{(10.37)} \quad \text{dist}(F, G) = \sum_{(i,j) \in \text{Inv}(F,G)} |F^i| \ |F^j|.
\]

Note that if \( F \) and \( G \) are both chambers, then \( \text{dist}(F, G) = |\text{Inv}(F,G)| \) as noted in (10.26).

Now we go to the general case. Here, we have

\[
\text{(10.38)} \quad \text{dist}(F, G) = \sum_{i<k \atop j>l} |F^i \cap G^j| \ |F^k \cap G^l|,
\]

where \( i \) and \( k \) index the blocks of \( F \) while \( j \) and \( l \) index the blocks of \( G \).

### 10.6. Shuffle permutations

The set \( \text{Sh}(s, t) \) of \((s, t)\)-shuffle permutations was defined in (2.21). In this section, we extend this notion to any composition, and then relate it to the gallery metric and Tits projection maps.

#### 10.6.1. T-shuffle permutations and faces of type \( T \)

Let \( T = (t_1, \ldots, t_k) \) be a composition of \( n \). A permutation \( \zeta \in S_n \) is a \( T \)-shuffle if

\[
\zeta(1) < \cdots < \zeta(t_1), \; \zeta(t_1+1) < \cdots < \zeta(t_1+t_2), \ldots, \zeta(t_1+\cdots+t_{k-1}+1) < \cdots < \zeta(n).
\]

We now discuss the geometric meaning of this notion. The definitions imply:

**Proposition 10.6.** There is a canonical bijection between faces of type \( T \) in the Coxeter complex \( \Sigma[n] \) and \( T \)-shuffle permutations: For a face \( F \) of type \( T \), the corresponding \( T \)-shuffle permutation \( \zeta \) is determined by

\[
\text{(10.39)} \quad \text{FC}_{(n)} = \zeta C_{(n)}.
\]

The left-hand side is the projection of \( C_{(n)} \) on \( F \), while the right-hand side is the action of \( \zeta \) on \( C_{(n)} \).
A more general result is given in [12, Lemma 5.3.1]. As a special case, we note that \( \text{Sh}(s, t) \) can be identified with the set of vertices of type \((s, t)\) in \( \Sigma[n] \).

In view of Proposition 10.3, we note that under (10.9), \( T \)-shuffle permutations correspond to gates of the stars of faces of type \( T \). This is illustrated in Figure 10.6. The black dot is a face \( F \) of type \( T \), and the six triangles around it are the chambers in its star. The \( T \)-shuffle permutation that corresponds to \( F \) is \( \zeta = d(C_{(n)}, FC_{(n)}) \). It is shown as a vector pointing from \( C_{(n)} \) to \( FC_{(n)} \).

![Figure 10.6. Faces of type \( T \) correspond to \( T \)-shuffle permutations.](image)

**10.6.2. Shuffles as coset representatives.** Recall from (2.22) that \((s, t)\)-shuffle permutations are coset representatives for \( S_s \times S_t \) as a subgroup of \( S_n \). We now explain the geometric meaning of this decomposition.

**Proposition 10.7.** Let \( G \) denote the face of \( C_{(n)} \) of type \( T \). Any chamber in the Coxeter complex \( \Sigma[n] \) is uniquely determined by a \( T \)-shuffle permutation and a chamber in the star of \( G \).

**Proof.** Let \( C \) be any chamber. It has a unique face of type \( T \); call it \( F \). Let \( \zeta \) be the corresponding \( T \)-shuffle permutation given by Proposition 10.6. Then the action of \( \zeta \) maps the star of \( G \) bijectively to the star of \( F \). Thus, \( C \) is uniquely determined by \( \zeta \) and the chamber \( \zeta^{-1}C \) which belongs to the star of \( G \). \( \square \)

Let \( G \) be a vertex. Then the chambers in the star of \( G \) correspond under (10.9) precisely to those permutations which can be written in the form \( \sigma \times \tau \) for \( \sigma \in S_s \) and \( \tau \in S_t \). The notation is as in (2.23). This observation along with Proposition 10.7 yields the decomposition (2.22).

**10.7. The descent and global descent maps**

In this section, we discuss the descent and global descent maps which associate a face to a pair of chambers, and further relate them to the descent and global descent maps on permutations.

The descent map on permutations is classical. The notion of global descents is closely related to that of connected permutations, which is also classical. The order properties of the global descent map on permutations were studied in [14]. Both descent and global descent maps on pairs of chambers were introduced in [12, Chapter 5] in the generality of finite Coxeter groups.

**10.7.1. Descents and global descents of permutations.** A permutation \( w \) has a descent at position \( p \) if \( w(p) > w(p + 1) \). Let \( \text{Des}(w) \) denote the set of
10.7. THE DESCENT AND GLOBAL DESCENT MAPS

desents of \( w \). If \( w \) is a permutation on \( n \) letters, then \( \text{Des}(w) \) is a subset of \([n - 1]\), or equivalently by (10.1), it is a composition of \( n \). For example,

\[
\text{Des}(45132) = \{2, 4\} = (2, 2, 1).
\]

A permutation \( w \) has a \textit{global descent} at position \( p \) if for all \( i < p \) and \( j > p + 1 \), we have \( w(i) > w(j) \). Let \( \text{gDes}(w) \) denote the set of global descents of \( w \). It is clear that \( \text{gDes}(w) \subseteq \text{Des}(w) \), but these are not equal in general. For example,

\[
\text{gDes}(45132) = \{2\} = (2, 3).
\]

We view \( \text{Des} \) and \( \text{gDes} \) as maps from \( S_n \) to \( \Delta_{[n-1]} \), where the latter denotes the set of compositions of \( n \).

We let \( \text{des}(w) \) and \( \text{gdes}(w) \) stand for the number of descents and global descents of a \( w \). These are the cardinalities of \( \text{Des}(w) \) and \( \text{gDes}(w) \) respectively.

10.7.2. Descents and global descents of pairs of chambers. Let \( \mathbb{L}[I] \) be the set whose elements are pairs of linear orders on \( I \). For example,

\[
(k|r|i|s|h|n|a, n|a|r|i|k|s|h)
\]

is an element of \( \mathbb{L}[[k, r, i, s, h, n, a]] \). This defines the set species \( \mathbb{L} \). The linearized species is denoted \( \mathbb{IL} \). We now proceed to define morphisms of species

\[
\text{Des}: \mathbb{IL} \rightarrow \Sigma \quad \text{and} \quad \text{gDes}: \mathbb{IL} \rightarrow \Sigma.
\]

We will refer to these as the \textit{descent} and \textit{global descent maps}.

Let \( D \) be a linear order on \( I \). A subset \( S \) is called a \textit{segment} of \( D \) if all its elements appear contiguously in \( D \). For example,

\[
\{k, s, h\} \quad \text{is a segment of} \quad l|a|k|s|h|m|i.
\]

Now let \( C \) be another linear order. A segment of \( D \) is compatible with respect to \( C \) if the elements of that segment appear in the same order in \( C \) and \( D \). Partially order the set of compatible segments by inclusion. It is clear that the maximal compatible segments yield a partition of \( I \).

\textbf{Definition 10.8.} Let \( C \) and \( D \) be two linear orders on \( I \). Define \( \text{Des}(C, D) \) to be the face of \( D \) whose blocks are the maximal compatible segments of \( D \) with respect to \( C \).

For example,

\[
\text{Des}(m|k|s|i|h|l|a, l|a|k|s|h|m|i) = la|kshmi.
\]

In more geometric terms, \( \text{Des}(C, D) \) keeps track of those walls of \( D \) which separate \( C \) and \( D \).

\textbf{Definition 10.9.} Let \( C \) and \( D \) be two linear orders on \( I \). Define \( \text{gDes}(C, D) \) to be the maximal face \( F \) of \( D \) such that its opposite \( \overline{F} \) is a face of \( C \). In other words,

\[
\text{gDes}(C, D) = \overline{C} \land D.
\]

For example,

\[
\text{gDes}(m|k|s|i|h|l|a, l|a|k|s|h|m|i) = la|kshmi.
\]
Remark 10.10. Recall the descent cocycle from Section 9.7.2. The descent map on pairs of chambers is related to the descent cocycle as follows. Let $C$ be any chamber. Then the set $D_{S,T}(C)$ defined in (9.40) consists of those walls of $C$ which separate $C$ and $KC$. Therefore, $d_{S,T}(C)$ is the number of blocks of the face $\text{Des}(KC,C)$, where $K = S|T$.

In particular, by letting $C = C_{(n)}$ and using (10.39), we see that $d_{S,T}(C_{(n)})$ is the number of blocks of $\text{Des}(\zeta C_{(n)}, C_{(n)})$. This yields (9.42).

10.7.3. Relating the (global) descent maps. The (global) descent maps on pairs of chambers and on permutations are related by the following commutative diagrams.

\[
\begin{array}{ccc}
\mathbb{I}_L[n] & \xrightarrow{\text{Des}} & \Sigma[n] \\
\downarrow & & \downarrow \\
S_n & \xrightarrow{\Delta_{[n-1]}} & \Sigma[n]
\end{array}
\]

(10.40)

\[
\begin{array}{ccc}
\mathbb{I}_L[n] & \xrightarrow{\text{gDes}} & \Sigma[n] \\
\downarrow & & \downarrow \\
S_n & \xrightarrow{\Delta_{[n-1]}} & \Sigma[n]
\end{array}
\]

10.7.4. The weak order on permutations. Let $\text{Inv}(\sigma)$ be the set of inversions of a permutation $\sigma$ as in (2.19). Given permutations $\sigma$ and $\tau$, let

\[
\sigma \leq \tau \text{ if } \text{Inv}(\sigma) \subseteq \text{Inv}(\tau).
\]

This is the weak left Bruhat order on permutations. Equivalently, $\sigma \leq \tau$ if there is a minimum gallery $E - D - C$ such that $d(D,C) = u$ and $d(E,C) = v$. That is,

\[
\sigma \leq \tau \iff \tau^{-1}C_{(n)} - \sigma^{-1}C_{(n)} - C_{(n)}.
\]

The equivalence between the two definitions follows by noting that $\text{Inv}(\sigma)$ can be identified with the set of hyperplanes which separate $C_{(n)}$ and $\sigma^{-1}C_{(n)}$ by letting the pair $(i,j)$ correspond to the hyperplane $x_i = x_j$.

Figure 10.7, which is taken from [14], shows the weak left Bruhat order on $S_4$.

![Figure 10.7](image-url)
We now define a partial order on the set of pairs of chambers:

\[(C_1, D_1) \leq (C_2, D_2) \text{ if } D_1 = D_2 = D \text{ and } C_2 - C_1 - D,\]

where \(C_2 - C_1 - D\) is a minimum gallery from \(C_2\) to \(D\) passing through \(C_1\).

This partial order is illustrated in Figure 10.8 by a schematic two-dimensional picture. It stands for a minimal sequence of triangles starting with \(C_2\), ending at \(D\) and containing \(C_1\) such that adjacent triangles share a common edge.

It is clear that the Weyl-distance map \(d: \mathbb{L}[n] \to S_n\) is order-preserving. Further, (10.40) may now be viewed as commutative diagrams of posets.

**10.7.5. Descents and the product of faces.** The discussion here complements the one in Section 10.4.3.

Let \(H\) be a set composition and \(C\) and \(D\) two linear orders. It follows from the definition of the product of faces (10.13) that \(HC = D\) if and only if the blocks of \(H\) are compatible segments of \(D\) with respect to \(C\). In view of Definition 10.8, this may be expressed as follows:

\[(10.42) \quad HC = D \iff \text{Des}(C, D) \leq H \leq D.\]

In other words, \(\text{Des}(C, D)\) is the smallest face \(H\) of \(D\) such that \(HC = D\). This observation is due to Brown [70, Proposition 4], see also [12, Proposition 5.2.2].

We also note that \(\text{Des}(C, D) = D \iff C = D\).

To summarize:

**Proposition 10.11.** Let \(C\) and \(D\) be chambers. Then the set of solutions to the equation \(HC = D\) is a Boolean poset with minimal element \(\text{Des}(C, D)\) and maximal element \(D\). The solution is unique precisely if \(C = D\).

A more general result is given below.

**Proposition 10.12.** Let \(F\) and \(G\) be any faces, and consider the equation \(HF = G\). If \(GF \neq G\), then it has no solutions. If \(GF = G\), then the set of solutions is a Boolean poset with maximal element \(G\). A solution exists and is unique if and only if \(F \leq G\).

**Proof.** Suppose there is a \(H\) such that \(HF = G\). Multiplying by \(F\) on the right we deduce \(HF = GF\) (since \(FF = F\)), and hence \(G = GF\). This proves the first claim.

For the second claim: Suppose \(GF = G\). Let \(A\) be the nonempty set of solutions to the equation \(HF = G\). Using properties (v) and (vi) in Proposition 10.1, it follows that

- if \(H \in A\) and \(H \leq L \leq G\), then \(L \in A\),

\[\text{C}_2 \xrightarrow{} \text{C}_1 \xrightarrow{} D\]

**Figure 10.8.** The partial order on the set of pairs of chambers.
• if \( H_1, H_2 \in A \), then \( H_1 \wedge H_2 \in A \).

This shows that \( A \) is a Boolean poset under containment of faces (refinement of compositions). The maximal element is clearly \( G \).

For the third claim: Suppose \( \overline{F} \leq G \). Then \( G = GF \). Multiplying by \( F \) on the right and using property (viii), we deduce that \( GF = G \). So \( G \) is a solution. To show that it is unique, let \( HF = G \). Multiplying by \( \overline{F} \) on the left we deduce \( \overline{F}H = G \).

Since \( \overline{F} \) and \( H \) are joinable (both being faces of \( G \)), it follows from (10.14) that \( HF = H\overline{F} = G \). By property (x), we conclude that \( H = G \) proving uniqueness. Conversely, suppose a solution exists and is unique. So \( GF = G \) and \( HF \neq G \) for any proper face \( H \) of \( G \). One can then deduce from property (xi) that \( \overline{F} \leq G \). □

It is natural to ask whether the minimal element in the Boolean poset \( A \) can be interpreted using descents, as is the case for chambers. In this regard, note the following. If \( F \) and \( G \) have the same support, say \( X \), then the minimal element of \( A \) is precisely the face \( \text{Des}(F, G) \) obtained by applying Definition 10.8 to the complex \( \Sigma[X] \). In the general case, the description of the minimal element requires a more general notion of descents. We plan to explain this in a future work.

10.8. The action of faces on chambers and the descent algebra

Consider the species of faces \( \Sigma \) and the species of pairs of chambers \( \frak{L} \). Let \( \Sigma \) and \( \frak{L} \) denote their linearizations.

In this section, we equip \( \frak{L} \) with a product which is compatible with the product of \( \Sigma \) given by (10.13). Further, we show that by passing to invariants under the action of the symmetric group, the relation between \( \Sigma \) and \( \frak{L} \) recovers a well-known relation between Solomon’s descent algebra and the group algebra of the symmetric group.

10.8.1. Solomon’s descent algebra. Let \( \text{des}: S_n \to \Delta_{[n-1]} \) be the descent map on permutations as defined in Section 10.7.1. Let \( k \) be a field and let \( kS_n \) be the group algebra of \( S_n \) over \( k \). Solomon \cite{Solomon} showed that the subspace of \( kS_n \) linearly spanned by the elements

\[ d_T := \sum_{w: \text{des}(w) \leq T} w, \]

as \( T \) varies over subsets of \([n-1] \), is a subalgebra of \( kS_n \). This subalgebra is known as the descent algebra. A geometric formulation of the descent algebra was given by Bidigare \cite{Bidigare} and further clarified by Brown \cite[Section 9.6]{Brown}. An exposition is provided below. The relevant statement is given in Theorem 10.13.

10.8.2. The action on chambers. The set \( L[I] \) of chambers is a two-sided ideal of the monoid \( \Sigma[I] \) of faces. This follows from property (iv) in Proposition 10.1. The right action is trivial, while the left action is given by the Tits projection

\[ p_F: L[I] \to L[I], \quad D \mapsto FD. \]

Linearizing we obtain that \( L[I] \) is a left ideal in the algebra \( \Sigma[I] \). This gives rise to a morphism of algebras

\[ \Sigma[I] \to \text{End}_{\text{vec}}(L[I]), \quad F \mapsto p_F, \]
one for each finite set \( I \). Further, one can check that these morphisms are injective. These maps define an injective morphism of monoids

\[
(10.44) \quad \Sigma \rightarrow \mathcal{E}^\times(L)
\]

where \( \mathcal{E}^\times \) is as in (8.80). Here the monoids are with respect to the Hadamard product on species. In other words, both \( \Sigma \) and \( \mathcal{E}^\times(L) \) are species with values in the category of algebras.

We now point out the following canonical identifications.

\[
\mathcal{E}^\times(L) \cong L^* \times L \cong \mathbb{I}L
\]

The first one views a basis element \( D^* \otimes C \in L[I]^* \otimes L[I] \) as the endomorphism

\[
E \mapsto \begin{cases} C & \text{if } D = E, \\ 0 & \text{if not.} \end{cases}
\]

The second one identifies \( D^* \otimes C \) with the pair \( (D, C) \in L[I] \times L[I] \).

Under these identifications, the product of the algebra \( \mathbb{I}L[I] \) is as follows.

\[
(10.45) \quad (D_2, C_2)(D_1, C_1) = \begin{cases} (D_1, C_2) & \text{if } D_2 = C_1, \\ 0 & \text{otherwise.} \end{cases}
\]

The morphism of monoids (10.44) takes the form

\[
(10.46) \quad \Sigma \rightarrow \mathbb{I}L, \quad F \mapsto \sum_{(D, C): FD = C} (D, C).
\]

10.8.3. Invariant subalgebras. Since \( \Sigma \) and \( \mathbb{I}L \) are species, the components \( \Sigma[n] \) and \( \mathbb{I}L[n] \) are \( S_n \)-modules. Further, since these species are monoids with respect to the Hadamard product, the components are algebras whose products commute with the \( S_n \)-action. This yields subalgebras of \( S_n \)-invariants

\[
(\Sigma[n])^{S_n} \hookrightarrow \Sigma[n] \quad \text{and} \quad (\mathbb{I}L[n])^{S_n} \hookrightarrow \mathbb{I}L[n].
\]

We make these subalgebras explicit.

A basis for the subalgebra \( (\Sigma[n])^{S_n} \) is given by

\[
(10.47) \quad \sigma_T := \sum_{F: \text{type}(F) = T} F,
\]

as \( T \) ranges over all subsets of \( S \).

The subalgebra \( (\mathbb{I}L[n])^{S_n} \) can be identified with the opposite of the group algebra as follows.

\[
(kS_n)^{\text{op}} \xrightarrow{\cong} (\mathbb{I}L[n])^{S_n}, \quad w \mapsto \sum_{(D, C): \text{d}(D, C) = w} (D, C).
\]

The main assertion here is that this is a morphism of algebras. This is a consequence of (10.30) and (10.45).

Now consider the following commutative diagram of algebras.

\[
\begin{array}{c}
\Sigma[n] \xrightarrow{\cong} \mathbb{I}L[n] \\
(\Sigma[n])^{S_n} \xrightarrow{\cong} (kS_n)^{\text{op}}
\end{array}
\]
It follows from (10.42) and the first diagram in (10.40) that the bottom horizontal map sends $\sigma_T$ to $d_T$ as defined in (10.43) and (10.47). Hence the image of the bottom horizontal map is precisely the descent algebra. As a consequence:

**Theorem 10.13 (Bidigare).** The descent algebra is isomorphic to $\left((\Sigma[n])^{S_n}\right)^{op}$.

### 10.9. Directed faces and directed flats

Recall that there are two fundamental objects associated to the Coxeter complex of type $A$, namely, faces and flats, and they are related by the support map. In this section, we show that there is a parallel theory for directed faces and directed flats. In combinatorial terms, this means that we replace set compositions (partitions) by linear set compositions (partitions).

We mainly follow the exposition in [12, Section 2.3], where this theory is explained in the generality of left regular bands. In that work, directed faces are called *pointed faces* and directed flats are called *lunes*.

**10.9.1. Directed faces and directed flats.** A directed face of the complex $\Sigma[I]$ is a pair $(G, D)$ where $G$ is a face and $D$ is a chamber containing $G$.

Directed faces of $\Sigma[I]$ are the same as linear compositions of $I$: The face $G$ is a composition of the set $I$ and the chamber $D$ determines a linear order on each block of $G$. Since $D$ refines $G$, the pair $(G, D)$ can be recovered from the linear set composition.

Directed faces may be visualized as in Figure 10.9. The pair $(G, D)$ tells us to stand at the face $G$ and look in the direction of the chamber $D$.

Let $\overrightarrow{\Sigma[I]}$ denote the set of directed faces. This defines the set species $\overrightarrow{\Sigma}$ of directed faces, or equivalently, of linear set compositions. The linearized species is denoted $\overrightarrow{\Sigma}$.

Define an equivalence relation on $\overrightarrow{\Sigma[I]}$ as follows.

\[(10.48) \quad (G, D) \sim (F, C) \iff GF = G, \ GC = D, \ FG = F \text{ and } FD = C.\]

The equivalence classes are called *directed flats*.

It follows from (10.13) that $(G, D) \sim (F, C)$ if and only if the compositions $F$ and $G$ differ only in the ordering of the blocks, and the linear orders $C$ and $D$ agree on each of these blocks. Thus, directed flats are the same as linear partitions of $I$.

Let $\overrightarrow{\Pi[I]}$ denote the set of directed flats. This defines the set species $\overrightarrow{\Pi}$ of directed flats, or equivalently, of linear set partitions. The linearized species is denoted $\overrightarrow{\Pi}$.

![Figure 10.9. A directed face.](image-url)
10.9.2. The base and support maps. Let \( \text{supp}: \Sigma[I] \rightarrow \Pi[I] \) be the canonical quotient map and base: \( \Sigma[I] \rightarrow \Sigma[I] \) the projection on the first coordinate. From (10.19) and (10.48) we have that
\[
(G, D) \sim (F, C) \implies G \sim F.
\]
It follows that there is an induced map \( \Pi[I] \rightarrow \Pi[I] \), also denoted base, fitting in the commutative diagram below.

\[
\begin{array}{ccc}
\Sigma[I] & \xrightarrow{\text{base}} & \Sigma[I] \\
\downarrow \text{supp} & & \downarrow \text{supp} \\
\Pi[I] & \xrightarrow{\text{base}} & \Pi[I]
\end{array}
\]

These maps coincide with the support and base maps defined in Section 10.1.5. In particular, if \( \text{supp}(F, C) = L \), then the blocks of \( L \) are the blocks of \( F \) ordered according to \( C \).

10.9.3. Directed flats as top-dimensional cones. A directed flat may be visualized as follows. Let \((G, D)\) be a directed face. Imagine all hyperplanes containing \( G \) are opaque. Standing at \( G \) and looking in the direction of \( D \) one overlooks a portion of the ambient space. Two directed faces are equivalent under (10.48) if they overlook the same region. From this perspective, diagram (10.49) expresses the following fact: If two directed faces \((F, C)\) and \((G, D)\) overlook the same region, then the faces \( F \) and \( G \) have the same support.

This is illustrated in Figure 10.10. The directed faces \((F, C)\) and \((G, D)\) are equivalent: the oval in the figure is the region overlooked from either directed face. It is the intersection of the half-spaces (hemispheres) which contain \( C \) and whose supporting hyperplane (great circle) contains \( F \). Among these half-spaces, only those whose supporting hyperplanes are walls of \( C \) are essential to determine the intersection. There are two of these in this case. The support of either \( F \) or \( G \) is the set \( \{F, G\} \).

We now define the region overlooked from a directed face more precisely. To any directed face \((F, C)\), we associate a top-dimensional cone: intersect those half-spaces which contain \( C \) and whose supporting hyperplane contains \( F \). This cone is the region overlooked from \((F, C)\); we denote it by \( \Psi(F, C) \).

We now describe the set of faces contained in this cone.

**Proposition 10.14.** Let \((F, C)\) be a directed face. For any face \( K \),
\[
K \subseteq \Psi(F, C) \iff FK \leq C.
\]
In particular, for any chamber $D$,
\[ D \subseteq \Psi(F, C) \iff FD = C. \]

**Proof.** For simplicity we consider the second statement only. It follows from the definition of the cone that $D \subseteq \Psi(F, C)$ if and only if $C$ and $D$ lie on the same side of every hyperplane containing $F$. According to the description of chambers in Section 10.2.2, the latter is equivalent to the statement that if two elements $i$ and $j$ belong to the same block of $F$, then they appear in the same order in $C$ and $D$. In view of (10.13), this is in turn equivalent to $FD = C$. \qed

**Proposition 10.15.** Two directed faces yield the same cone if and only if they are equivalent under (10.48). Explicitly,
\[ (F, C) \sim (G, D) \iff \Psi(F, C) = \Psi(G, D). \]

In other words, the map $\Psi$ induces a bijection between the set of directed flats and the set of cones associated to directed faces. We continue to denote the induced map by $\Psi$. This bijection allows us to visualize directed flats as top-dimensional cones.

**Proof.** Using symmetry, it is enough to show that
\begin{equation}
GF = G \text{ and } GC = D \iff \Psi(F, C) \subseteq \Psi(G, D).
\end{equation}

Take a face $K$ in $\Psi(F, C)$. By Proposition 10.14, $FK \leq C$. If $GF = G$ and $GC = D$, then using property (iii) in Proposition 10.1 we have
\[ FK \leq C \implies GFK \leq GC = D \implies GK \leq D. \]
This shows that $K$ is in $\Psi(G, D)$.

Conversely, assume $\Psi(F, C) \subseteq \Psi(G, D)$. Since $C \subseteq \Psi(F, C)$, then $C \subseteq \Psi(G, D)$, and by Proposition 10.14, we have $GC = D$. In addition, since both $F$ and $\overline{F}$ are in $\Psi(F, C)$, they are also in $\Psi(G, D)$, and $GF \leq D$ and $G\overline{F} \leq D$. Now from property (ii) in Proposition 10.1 we derive $GFD = G\overline{F}D = D$, and from property (x) it follows that $GF = G$. \qed

Generalizations of Propositions 10.14 and 10.15 are given in [12, Lemmas 2.3.2 and 2.3.3].

Let $L$ be the support of the directed face $(F, C)$. Viewing $L$ as a linear set composition, the blocks of $L$ are the blocks of $(F, C)$ as in Section 10.4.3. It follows from Proposition 10.14 and (10.22) that
\[ D \subseteq \Psi(L) \iff D \text{ is a shuffle of the blocks of } L. \]

Let us now look at a specific example to illustrate the preceding discussion. Two directed flats in the simplicial complex $\Sigma[\{a, b, c, d\}]$ are shown in Figure 10.11. The first directed flat is bounded by the hyperplanes $x_a = x_d$ and $x_c = x_d$. As a linear set partition, it is given by $\{b, ad\}$. It is the cone associated to the directed face $(b|acd, b|a|d|c)$, or equivalently to $(acd|b, a|d|c|b)$. The four chambers that it contains are $b|a|d|c$, $a|b|d|c$, $a|d|b|c$, and $a|d|c|b$. These are precisely the shuffles of $b$ and $a|d|c$. The base of this directed flat is the set partition $\{b, ad\}$. This is the precisely the support of the vertices $b|acd$ and $acd|b$, which can be seen at the two corners of the directed flat.
10.9. DIRECTED FACES AND DIRECTED FLATS

The second directed flat is bounded by the hyperplanes $x_a = x_b$ and $x_c = x_d$. As a linear set partition, it is given by $\{a|b,c|d\}$. It is the cone associated to the directed face $(ab|cd,a|b|c|d)$, or equivalently to $(cd|ab,c|d|a|b)$. The six chambers that it contains are
\[a|b|c|d, a|c|b|d, a|c|d|b, c|a|b|d, c|a|d|b, \text{ and } c|d|a|b.\]
These are precisely the shuffles of $a|b$ and $c|d$. The base of this directed flat is the set partition $\{ab, cd\}$. This is the precisely the support of the vertices $ab|cd$ and $cd|ab$, which can be seen at the two corners of the directed flat.

These are two typical directed flats bounded by two hyperplanes, but general directed flats involve an arbitrary number of hyperplanes and range from the whole space and half-spaces at one end, to chambers at the other.

10.9.4. Left modules over faces. Given a face $K$ and a directed face $(G, D)$, define
\[(10.51)\]
\[K \cdot (G, D) := (KG, KD).\]
In view of properties (iii) and (iv) in Proposition 10.1, $(KG, KD)$ is a directed face. In this manner, the set of directed faces $\Sigma[I]$ is a left module over the monoid of faces $\Sigma[I]$. Note that (10.48) can be rewritten as:
\[(G, D) \sim (F, C) \iff G \cdot (F, C) = (G, D) \text{ and } F \cdot (G, D) = (F, C).\]
Observe that for any face $K$,
\[(G, D) \sim (F, C) \implies K \cdot (G, D) \sim K \cdot (F, C).\]
It follows that $\Pi[I]$ is also a left module over $\Sigma[I]$:
\[(10.52)\]
\[F \cdot M := \text{supp}(FG, FD),\]
where \((G,D)\) is any directed face whose support is \(M\). For example,
\[
(lak|shmi) \cdot \{s|h|k, l|a, m|i\} = \{l|a, k, s|h, m|i\}.
\]

By construction, the support map from directed faces to directed flats is a morphism of left modules.

**10.9.5. Partial orders.** Recall from Section 10.1.4 that we have a partial order on linear compositions, and two partial orders on linear partitions. We now phrase them in geometric terms.

The partial order on the set of directed faces is given by:
\[
(F,C) \leq (G,D) \quad \text{if} \quad C = D \text{ and } F \leq G.
\]

This is a disjoint union of Boolean posets, one for each chamber \(C\).

We now discuss the two partial orders on the set of directed flats.
\[
L \leq^\prime M \quad \text{if} \quad H \cdot L = M \text{ for some face } H,
\]

where \(\cdot\) denotes the left module structure of directed flats (10.52).

Let \((F,C)\) and \((G,D)\) be directed faces. Each has an associated cone as in Section 10.9.3. We have:
\[
\Psi(F,C) \subseteq \Psi(G,D) \iff GF = G \text{ and } GC = D \\
\iff G : (F,C) = (G,D) \\
\iff H \cdot (F,C) = (G,D) \text{ for some face } H \\
\iff \text{supp}(F,C) \leq^\prime \text{supp}(G,D).
\]

The first equivalence is (10.50). For the converse of the last implication, choose \(H'\) as in (10.54), and let \(H = GH'\). The remaining implications are straightforward.

**Proposition 10.16.** Let \(L\) and \(M\) be directed flats. We have
\[
L \leq^\prime M \iff \Psi(L) \subseteq \Psi(M).
\]

**Proof.** This follows by applying the above to any directed faces \((F,C)\) and \((G,D)\) with supports \(L\) and \(M\) respectively. \(\square\)

Figure 10.4 shows that the cone associated to \(\{a|d|b,c\}\) is contained in the cone associated to \(\{a|b,c,d\}\). So
\[
\{a|d|b,c\} \leq^\prime \{a|b,c,d\}.
\]

Let us now discuss the second partial order on directed flats.
\[
L \leq M \quad \text{if} \quad H \cdot L = M, \; H \subseteq \Psi(L) \text{ for some face } H,
\]

where \(\Psi(L)\) is the cone defined in Section 10.9.3 and \(\cdot\) denotes the left module structure of directed flats (10.52).

Observe that for any directed faces \((F,C)\) and \((G,D)\),
\[
\text{supp}(FG,C) = \text{supp}(G,D) \iff GF = G, \; GC = D, \; FD = C \\
\iff GF = G, \; GC = D, \; FG \leq C \\
\iff G : (F,C) = (G,D), \; FG \leq C \\
\iff H \cdot (F,C) = (G,D), \; FH \leq C \text{ for some face } H \\
\implies \text{supp}(F,C) \leq \text{supp}(G,D).
\]
Note that the backward implication on the last line may fail. What one can say instead is the following: $\text{supp}(F,C) \leq M$ if and only if there is a directed face $(G,D)$ with support $M$ such that $G \cdot (F,C) = (G,D)$ and $FG \leq C$ (or any of the above equivalent condition) holds. This further implies that $L \leq M$ if and only if there is a directed face with support $L$ which is less than a directed face with support $M$: Take directed faces $(F,C)$ and $(G,D)$ with supports $L$ and $M$ as above, and replace $(G,D)$ with $(FG,C)$.

**Proposition 10.17.** Let $(F,C)$ and $(G,D)$ be directed faces such that $G \cdot (F,C) = (G,D)$ and $FG \leq C$. Then for any face $H$,

$$H \cdot (F,C) = (G,D), \quad FH \leq C \iff HF = G.$$ 

**Proof.** The forward implication is clear. For the backward implication, we note that 

$$HC = HFC = GC = D \quad \text{and} \quad FH \leq FG \leq C.$$ 

Thus, $HC = D$ and $FH \leq C$ as required. \hfill $\square$

If $L \leq M$, then $L \leq' M$, and hence the cone associated to $L$ is contained in the cone associated to $M$. The converse, of course, is false since the two partial orders are distinct. For example,

$$\{a|d|b,c\} \leq' \{a|b,c,d\} \quad \text{but} \quad \{a|d|b,c\} \not\leq \{a|b,c,d\}.$$ 

### 10.10. The dimonoid of directed faces

The monoid structure of the set of faces of the Coxeter complex has played a central role in the preceding sections. The set of faces $\Sigma[I]$ is a monoid and the set of flats $\Pi[I]$ is a quotient monoid under the support map (Section 10.4). It is natural to ask whether there is a similar structure on the set of directed faces $\Sigma'[,I]$ and the set of directed flats $\Pi',[,I]$. It turns out that $\Sigma'[I]$ is a bimodule over $\Sigma[I]$ and $\Pi',[I]$ is a bimodule over $\Pi[I]$. Moreover, there is a finer structure of **dimonoid** on each of these bimodules.

#### 10.10.1. Dimonoids.

We now take a small detour to the world of dimonoids. This notion was introduced by Loday [238, Section 1].

**Definition 10.18.** A dimonoid is a set $D$ equipped with two binary operations $\sqsubset$ and $\sqsupset$ such that:

1. $x \sqsupset (y \sqsubset c) = (x \sqsupset y) \sqsubset z = x \sqsupset (y \sqsubset z)$,
2. $(x \sqsubset y) \sqsupset z = x \sqsubset (y \sqsupset z)$,
3. $(x \sqsupset y) \sqsubset z = x \sqsupset (y \sqsupset z) = (x \sqsupset y) \sqsupset z$.

A bar-unit is an element $1 \in D$ such that

$$1 \sqsubset x = x = x \sqsupset 1$$

for every $x \in D$.

Bar-units need not be unique.

A monoid $M$ can be viewed as a dimonoid by setting

$$x \sqsubset y := xy =: x \sqsupset y.$$
More generally, suppose $B$ is a bimodule over a monoid $M$, and there is a map $\delta : B \to M$ a bimodules. Then [238, Example 2.2.d] $B$ can be turned into a dimonoid by setting
\[ x \rhd y := \delta(x) \cdot y \quad \text{and} \quad x \lhd y := x \cdot \delta(y). \]
Every dimonoid arises in this manner from such a map $\delta$ [134, Proposition 1.6]. In fact, given a dimonoid $D$, let $M$ be the quotient by the dimonoid-ideal generated by the relations
\[ x \rhd y \equiv x \lhd y. \]
Then $M$ is a dimonoid in which $\rhd = \lhd$, and so it is a monoid. Let $\delta : D \to M$ denote the quotient map. Then
\[ \delta(x) \cdot y := x \rhd y \quad \text{and} \quad x \cdot \delta(y) := x \lhd y \]
yield a well-defined $M$-bimodule structure on $D$. The dimonoid associated to $\delta$ is the original one.

These constructions define a pair of adjoint functors between the category of dimonoids and a suitable category of maps $\delta$ as above. The functor from dimonoids is the left adjoint.

10.10.2. The dimonoids of directed faces and of directed flats. The set of directed faces $\Sigma[I]$ is a bimodule over $\Sigma[I]$. The left and right module structures are:
\[ F \cdot (G, D) := (FG, FD), \]
\[ (F, C) \cdot G := (FG, FGC). \]
The left module structure is the same as in Section 10.9.4. For the right module structure, note that $(FG, FGC)$ is a directed face in view of properties (i) and (iii) in Proposition 10.1. In addition, property (ix) guarantees that the right structure is associative. The fact that the two structures commute follows.

Recall the base and support maps from Section 10.9.2. The map
\[ \text{base} : \Sigma[I] \to \Sigma[I] \]
(which simply projects on the first coordinate) is a map of $\Sigma[I]$-bimodules.

It then follows that $\Sigma[I]$ is a dimonoid. The operations are
\[ (F, C) \rhd (G, D) := F \cdot (G, D) = (FG, FD), \]
\[ (F, C) \lhd (G, D) := (F, C) \cdot G = (FG, FGC). \]
Directed faces of the form $(\emptyset, C)$ are the bar-units of this dimonoid.

Similarly, the set of directed flats $\Pi[I]$ is a bimodule over $\Pi[I]$: \[ X \cdot M := \text{supp}(KG, KD), \]
\[ L \cdot X := \text{supp}(FK, FKC), \]
where $K$ is any face with support $X$, and $(F, C)$ and $(G, D)$ are directed faces with supports $L$ and $M$ respectively. The map
\[ \text{base} : \Pi[I] \to \Pi[I] \]
is a map of $\Pi[I]$-bimodules, and it follows that $\Pi[I]$ is a dimonoid.

Thus the monoids $\Sigma[I]$ and $\Pi[I]$ are dimonoids as well. One can now check that:
Proposition 10.19. **Diagram (10.49) is a commutative diagram of dimonoids.**

We observe that since $\Sigma[I] \to \Pi[I]$ is a morphism of monoids, $\overrightarrow{\Pi}[I]$ is also a bimodule over $\Sigma[I]$. Explicitly,

$$
K \cdot M := \text{supp}(KG, KD),
$$
$$
L \cdot K := \text{supp}(FK, FKC),
$$

where $(F, C)$ and $(G, D)$ are directed faces with supports $L$ and $M$ respectively.

As explained above, the dimonoids $\overrightarrow{\Sigma}[I]$ and $\overrightarrow{\Pi}[I]$ arise from the base maps $\Sigma[I] \to \Sigma[I]$ and $\Pi[I] \to \Pi[I]$ by means of the construction of Section 10.10.1. One can check that if one applies the left adjoint construction to these dimonoids one retrieves the monoids $\Sigma[I]$ and $\Pi[I]$ and the base maps.

10.10.3. **The Jacobson radical.** We now linearize the preceding discussion. The linearization of a dimonoid is a dialgebra. Proposition 10.19 yields the following commutative diagram of dialgebras.

![Diagram](10.56)

We discuss this diagram in more detail below. We pause to recall a basic fact. Let $A$ be an algebra and $J$ be a two-sided ideal. Then $A/J$ is an algebra and the quotient map $A \to A/J$ is a morphism of algebras. Now let $M$ be an $A$-bimodule. Then $M/JMJ$ is an $(A/J)$-bimodule and, in particular, an $A$-bimodule.

We proceed. Let $A$ be the algebra of faces $\Sigma[I]$, and let $J$ be its Jacobson radical. Bidigare [45] showed that $J$ is precisely the kernel of its support map. This result was generalized to left regular bands by Brown [70], also see [12, Lemma 2.5.5]. Thus, $A/J$ is the algebra of flats $\Pi[I]$, and the quotient map is the support map. Now let $M$ be the bimodule of directed faces $\overrightarrow{\Sigma}[I]$. One can check that $JMJ = JM = MJ$ and that this subbimodule is the kernel of the support map from directed faces to directed flats. Thus, $M/JMJ$ is the space of directed flats $\overrightarrow{\Pi}[I]$. It is a bimodule over $\Pi[I]$.

10.11. **The break and join maps**

We mentioned earlier that Coxeter complexes are closed under the star and join operations. In this section, we explain how this property can be used to define break and join maps on the faces, flats, directed faces and directed flats of the Coxeter complex of type $A$. These maps along with Tits projection maps will play an important role in the construction of Hopf monoids in species (Chapter 12).

10.11.1. **The break and join maps for faces.** Let $K = S|T$ be a vertex of $\Sigma[I]$. A face $F$ which contains $K$ consists of a composition of $S$ followed by a composition of $T$. This yields a canonical identification

$$
\text{Star}(S|T) \cong \Sigma[S] \times \Sigma[T]
$$
between the star of the vertex $S|T$ in $\Sigma[I]$ and the join of the complexes $\Sigma[S]$ and $\Sigma[T]$. We use

\begin{equation}
\text{Star}(S|T) \xrightarrow{b_{S|T}} \Sigma[S] \times \Sigma[T] \xleftarrow{j_{S|T}}
\end{equation}

to denote the inverse isomorphisms of simplicial complexes. We refer to $b_{S|T}$ and $j_{S|T}$ as the break and join maps, respectively. Explicitly, if $F = F^1 \cdot \cdot \cdot |F^n$ is a composition of $S$ and $G = G^1 \cdot \cdot \cdot |G^j$ is a composition of $T$, then

\[
j_{S|T}(F,G) = F^1 \cdot \cdot \cdot |F^n|G^1 \cdot \cdot \cdot |G^j.
\]

The star and join operations preserve Coxeter complexes; thus the break and join maps are simplicial isomorphisms between Coxeter complexes.

More generally, the break and join maps can be defined for any face: For $K = K^1|K^2|\cdot \cdot \cdot |K^j$, there are inverse isomorphisms of simplicial complexes

\begin{equation}
\text{Star}(K^1|K^2|\cdot \cdot \cdot |K^j) \xrightarrow{b_K} \Sigma[K^1] \times \Sigma[K^2] \times \cdot \cdot \cdot \times \Sigma[K^j] \xleftarrow{j_K}
\end{equation}

where $\text{Star}(K)$ is the star of the face $K$ in $\Sigma[I]$.

The following is a useful way to picture the break map; the figure illustrates the case when $K$ has three parts. The disc at the center is an apparatus which takes one input larger than $K$ and produces three ordered outputs.

\begin{center}
\begin{tikzpicture}
  \node (bK) at (0,0) {$b_K$};
  \node (K1) at (-2,-1) {$b_{K_1}$};
  \node (K2) at (0,-1) {$b_{K_2}$};
  \node (K3) at (2,-1) {$b_{K_3}$};
  \node (F) at (4,0) {$b_F$};
  \draw[->] (bK) -- (K1);
  \draw[->] (bK) -- (K2);
  \draw[->] (bK) -- (K3);
  \draw[->] (K1) -- (F);
  \draw[->] (K2) -- (F);
  \draw[->] (K3) -- (F);
\end{tikzpicture}
\end{center}

A similar picture can be drawn for the join map by reversing the arrows.

**10.11.2. Compatibilities.** The break and join maps are associative in the following sense: Let $K$ be a face of $F$, and let $b_K(F) = (F_1, \ldots, F_k)$. Then

\[
b_F = (b_{F_1} \times b_{F_2} \times \cdot \cdot \cdot \times b_{F_k}) \circ b_K.
\]

Equivalently, with the same setup,

\[
j_F = j_K \circ (j_{F_1} \times j_{F_2} \times \cdot \cdot \cdot \times j_{F_k}).
\]

An illustration for the associativity of the break map is provided below.

\begin{center}
\begin{tikzpicture}
  \node (bK) at (0,0) {$b_K$};
  \node (bF1) at (2,-1) {$b_{F_1}$};
  \node (bF2) at (4,-1) {$b_{F_2}$};
  \node (bF3) at (6,-1) {$b_{F_3}$};
  \node (bF) at (8,0) {$b_F$};
  \draw[->] (bK) -- (bF1);
  \draw[->] (bK) -- (bF2);
  \draw[->] (bK) -- (bF3);
  \draw[->] (bF1) -- (bF);
  \draw[->] (bF2) -- (bF);
  \draw[->] (bF3) -- (bF);
\end{tikzpicture}
\end{center}

The break and join maps are compatible with the projection maps: For faces $F$ and $G$ which contain $K$,

\[
b_K(FG) = b_K(F)b_K(G),
\]
the product on the right being taken componentwise. Equivalently, for \( K = K^1|K^2|\cdots|K^j \),
\[
j_K(F_1, \ldots, F_j) j_K(G_1, \ldots, G_j) = j_K(F_1 G_1, \ldots, F_j G_j),
\]
where \( F_i \) and \( G_i \) are compositions of \( K^i \), as \( i \) varies from 1 to \( j \).

The break and join maps are compatible with the distance function on faces, and hence in particular with the gallery metric on chambers.

\[
(10.59) \quad \text{dist} \left( j_K(F_1, \ldots, F_j), j_K(G_1, \ldots, G_j) \right) = \sum_{i=1}^j \text{dist}(F_i, G_i).
\]

Equivalently, for faces \( F \) and \( G \) which contain \( K \),

\[
(10.60) \quad \text{dist}(F, G) = \sum_{i=1}^j \text{dist}(F_i, G_i),
\]
where \( b_K(F) = (F_1, \ldots, F_j) \) and \( b_K(G) = (G_1, \ldots, G_j) \).

The compatibility of the distance function with projection maps (10.31), in conjunction with (10.60) yields the following important consequence.

Let \( I = S \sqcup T \) be a decomposition, and let \( K = S|T \). Further, let \( C \) and \( D \) be linear orders on \( I \), \( C_1 \) and \( D_1 \) be linear orders on \( S \), and \( C_2 \) and \( D_2 \) be linear orders on \( T \), such that \( b_K(KC) = (C_1, C_2) \) and \( b_K(D) = (D_1, D_2) \). Then

\[
(10.61) \quad \text{dist}(C, D) = \text{dist}(C_1, D_1) + \text{dist}(C_2, D_2) + \text{dist}(C, KC).
\]

It is clear that this can be generalized by replacing the vertex \( K \) by any face.

### 10.11.3. Relation with shuffles and quasi-shuffles.
Recall the notions of shuffles and quasi-shuffles from Section 10.1.6. We now explain how they fit into the framework of break and join maps.

Let \( K = S|T \) be a vertex of \( \Sigma[I] \). Let \( F, F_1 \) and \( F_2 \) be faces of \( \Sigma[I], \Sigma[S] \) and \( \Sigma[T] \) respectively. It follows from (10.20) that

\[
(10.62) \quad b_K(KF) = (F_1, F_2) \iff KF = j_K(F_1, F_2) \iff F \text{ is a quasi-shuffle of } F_1 \text{ and } F_2.
\]

For example, let \( I = \{l, a, k, s, h, m, i\} \), \( S = \{a, k, l\} \) and \( T = \{h, i, m, s\} \). If the vertex \( K \) is \( a k|s i h m \), then the set compositions

\[
F = l s h | m | a k i, \quad F_1 = l | a k, \quad \text{and} \quad F_2 = s h | m | i
\]
satisfy the conditions (10.62).

Going back to the general discussion, it follows from (10.21) that

\[
(10.63) \quad b_K(KF) = (F_1, F_2) \text{ and } FK = F \iff KF = j_K(F_1, F_2) \text{ and } FK = F \iff F \text{ is a shuffle of } F_1 \text{ and } F_2.
\]

With \( I, S, T \) and \( K \) as in the above example, the set compositions

\[
F = l s h | m | a k i, \quad F_1 = l | a k, \quad \text{and} \quad F_2 = s h | m | i.
\]
satisfy conditions (10.63).

Let \( C, C_1 \) and \( C_2 \) are linear orders on \( I \), \( S \) and \( T \) respectively and \( K = S|T \). It follows from (10.22) that

\[
(10.64) \quad b_K(KC) = (C_1, C_2) \iff KC = j_K(C_1, C_2) \iff C \text{ is a shuffle of } C_1 \text{ and } C_2.
\]
With $I$, $S$, $T$ and $K$ as before, the linear orders

$$C = l|s|h|m|a|k|i, \quad C_1 = l|a|k, \quad \text{and} \quad C_2 = s|h|m|i$$

satisfy conditions (10.64).

The above discussion can be generalized by replacing the vertex $K$ by any face.

10.11.4. The break and join maps for flats. We now discuss the analogues of the break and join maps for set partitions. For a partition $X$ of $I$, let $\text{Star}(X)$ denote the star of $X$ in $\Pi[I]$. It consists of those partitions of $I$ which refine $X$. For $K = K^1|K^2|\cdots|K^j$, there are inverse isomorphisms

$$\text{Star}(\supp(K)) \xrightarrow{b_K} \Pi[K^1] \times \Pi[K^2] \times \cdots \times \Pi[K^j].$$

These are the break and join maps for flats. Note that $\text{Star}(\supp(K))$ consists precisely of those flats $X$ for which $K \cdot X = X$, with the module structure as in (10.18).

10.11.5. The break and join maps for directed faces. We now discuss the analogues of the break and join maps for linear set compositions. For a set composition $K = K^1|K^2|\cdots|K^j$ of $I$, let

$$\text{Star}_{\Sigma[I]}(K)$$

denote the set of those linear set compositions $(G,D)$ for which $K \leq G$, or equivalently, $K \cdot (G,D) = (G,D)$, with the left module structure as in (10.51). In other words, it is the set of directed faces of the simplicial complex $\text{Star}(K)$. Note that an element of this set is a linear set composition of $K^1$, followed by a linear set composition of $K^2$, and so on. This observation yields inverse bijections

$$\text{Star}_{\Sigma[I]}(K) \xrightarrow{b_K} \Sigma[K^1] \times \Sigma[K^2] \times \cdots \times \Sigma[K^j].$$

These are the break and join maps for directed faces.

Note that these break and join maps are defined by using the break and join maps for both coordinates. For example, for a vertex $K$, if $b_K(F) = (F_1,F_2)$ and $b_K(D) = (D_1,D_2)$, then $b_K(F,D) = ((F_1,D_1),(F_2,D_2))$.

10.11.6. The break and join maps for directed flats. For a set composition $K = K^1|K^2|\cdots|K^j$ of $I$, let

$$\text{Star}_{\Pi[I]}(K)$$

denote the image of $\text{Star}_{\Sigma[I]}(K)$ under the support map. It consists of precisely those directed flats $M$ for which $K \cdot M = M$, with the left module structure as in (10.52). Alternatively, it is the set of directed flats of the simplicial complex $\text{Star}(K)$. More explicitly, an element of this set is a disjoint union of a linear set partition of $K^1$, a linear set partition of $K^2$, and so on. This yields inverse bijections

$$\text{Star}_{\Pi[I]}(K) \xrightarrow{b_K} \Pi[K^1] \times \Pi[K^2] \times \cdots \times \Pi[K^j].$$

These are the break and join maps for directed flats.
10.12. The weighted distance function

In this section we discuss the weighted version of the distance function on chambers and faces (Section 10.5). These come in two flavors: additive and multiplicative. They depend on a matrix of size \( r \) and a function \( f : I \to [r] \). We denote the matrix by \( A \) in the additive case, and by \( Q \) in the multiplicative case. The two cases can be related by (2.33).

10.12.1. The additive case. Let \( A \) be a fixed integer matrix of size \( r \) and \( f : I \to [r] \). To each half-space in the braid arrangement in \( \mathbb{R}^I \), we assign a weight as follows:

\[
 w^A_f(x_j \leq x_i) := a_{f(i)f(j)}.
\]

Given chambers \( C \) and \( D \), define the weighted additive distance from \( C \) to \( D \) by

\[
\text{dist}^A_f(C, D) := \sum w^A_f(H),
\]

where the sum is over all half-spaces \( H \) which contain \( C \) but do not contain \( D \).

Explicitly, if \( C = C^1 \cdots | C^n \), with \( n = |I| \), then (10.26) generalizes as follows.

\[
\text{dist}^A_f(C, D) = \sum_{(i,j) \in \text{Inv}(C,D)} a_{f(C^j) f(C^i)}.
\]

Note that if all entries of \( A \) are 1, then \( \text{dist}^A_f(C, D) \) is simply the gallery distance between \( C \) and \( D \).

Some basic properties of the weighted distance function are as follows. Let \( C \), \( D \) and \( E \) be chambers in \( \Sigma[I] \). Then

\[
\text{dist}^A_f(C, D) = \text{dist}^A_f(D, C),
\]

for any bijection \( \sigma : I \to J \),

\[
\text{dist}^A_f(C, D) = \text{dist}^A_f(\sigma C, \sigma D),
\]

if \( C - D - E \) is a minimum gallery, then

\[
\text{dist}^A_f(C, E) = \text{dist}^A_f(C, D) + \text{dist}^A_f(D, E).
\]

Further,

\[
\text{dist}^A_f(C, D) = \text{dist}^A_f(D, C).
\]

Interesting special cases are when \( A \) is symmetric or antisymmetric. We explain them briefly.

**Proposition 10.20.** Let \( A \) be a symmetric matrix with positive real entries. Then

\[
\text{dist}^A_f(C, C) = 0,
\]

\[
\text{dist}^A_f(C, D) = \text{dist}^A_f(D, C),
\]

\[
\text{dist}^A_f(C, D) + \text{dist}^A_f(D, E) \geq \text{dist}^A_f(C, E),
\]

with equality if \( C - D - E \) is a minimum gallery.

These are the familiar properties of “distance”.

Proposition 10.21. Let $A$ be an antisymmetric matrix. Then

$$\text{dist}^A_f(C, C) = 0,$$
$$\text{dist}^A_f(C, D) + \text{dist}^A_f(D, C) = 0,$$
$$\text{dist}^A_f(C, D) + \text{dist}^A_f(D, E) = \text{dist}^A_f(C, E).$$

These are the familiar properties of “displacement”.

**Proof.** We prove the last equality. For that, refer to Figure 10.12. We may assume that $C$, $D$ and $E$ are all distinct (the remaining cases are straightforward). Then there are three kinds of hyperplanes as shown in Figure 10.12 whose associated half-spaces may contribute to the weighted distances. The hyperplanes labeled 1 and 3 contribute once to both the left- and right-hand side via the half-space which contains $C$. The hyperplane labeled 2 does not contribute to the right-hand side and contributes twice to the left-hand side via the two half-spaces it supports. Since $A$ is antisymmetric, these contributions cancel. □

10.12.2. The multiplicative case. Let $Q$ be a matrix of size $r$ and $f: I \to [r]$ a function. Given chambers $C$ and $D$, define the *weighted multiplicative distance* from $C$ to $D$ by

\[
\text{dist}^Q_f(C, D) := \prod w^Q_f(H),
\]

where $w^Q_f(H)$ is as in (10.68) and the product is over all half-spaces $H$ which contain $C$ but do not contain $D$. Clearly, if the matrices $Q$ and $A$ are related by (2.33), then

\[
\text{dist}^Q_f(C, D) = q^{\text{dist}^A_f(C, D)}.
\]

The multiplicative analogue of (10.70) is the following.

\[
\text{dist}^Q_f(C, D) = \prod_{(i, j) \in \text{Inv}(C, D)} q^{f(C_j) - f(C_i)}.
\]
The multiplicative analogues of (10.71), (10.72) and (10.73) are as follows.

\[
\text{(10.77)} \quad \text{dist}_f^Q(C, D) = \text{dist}_f^{Q_\sigma}(D, C),
\]
for any bijection \(\sigma : I \to J\),

\[
\text{(10.78)} \quad \text{dist}_f^Q(C, D) = \text{dist}_{f_{\sigma^{-1}}}^Q(\sigma C, \sigma D),
\]
if \(C - D - E\) is a minimum gallery, then

\[
\text{(10.79)} \quad \text{dist}_f^Q(C, E) = \text{dist}_f^Q(C, D) \text{dist}_f^Q(D, E).
\]

Further,

\[
\text{(10.80)} \quad \text{dist}_f^Q(C, D) = \text{dist}_f^Q(D, C).
\]

The following is a multiplicative analogue of Proposition 10.21.

**Proposition 10.22.** Let \(Q\) be a log-antisymmetric matrix. Then

\[
\text{dist}_f^Q(C, C) = 1,
\]

\[
\text{dist}_f^Q(C, D) \text{dist}_f^Q(D, C) = 1,
\]

\[
\text{dist}_f^Q(C, D) \text{dist}_f^Q(D, E) = \text{dist}_f^Q(C, E).
\]

**10.12.3. Compatibility with breaks, joins and projections.** We now discuss compatibilities of the weighted distance function with the break, join and projection maps. To start with, Proposition 10.4 in conjunction with (10.73) in the additive case and (10.79) in the multiplicative case yields:

**Proposition 10.23.** Let \(C\) and \(D\) be chambers and \(F\) be a face of \(C\). Then there exists a minimum gallery \(C - FD - D\). In particular,

\[
\begin{align*}
\text{dist}_f^A(C, D) &= \text{dist}_f^A(C, FD) + \text{dist}_f^A(FD, D), \\
\text{dist}_f^Q(C, D) &= \text{dist}_f^Q(C, FD) \text{dist}_f^Q(FD, D).
\end{align*}
\]

The following are weighted analogues of (10.59) and (10.60) for the case of chambers. The notations are the same as before, so we do not repeat them here.

\[
\begin{align*}
\text{dist}_f^A(j_K(C_1, \ldots, C_j), j_K(D_1, \ldots, D_j)) &= \sum_{i=1}^j \text{dist}_f^A(C_i, D_i), \\
\text{dist}_f^Q(j_K(C_1, \ldots, C_j), j_K(D_1, \ldots, D_j)) &= \prod_{i=1}^j \text{dist}_f^Q(C_i, D_i),
\end{align*}
\]

where the \(f_i\)'s are appropriate restrictions of \(f\). Equivalently, for chambers \(C\) and \(D\) which contain \(K\),

\[
\begin{align*}
\text{dist}_f^A(C, D) &= \sum_{i=1}^j \text{dist}_{f_i}^A(C_i, D_i), \\
\text{dist}_f^Q(C, D) &= \prod_{i=1}^j \text{dist}_{f_i}^Q(C_i, D_i),
\end{align*}
\]

where \(b_K(C) = (C_1, \ldots, C_j)\) and \(b_K(D) = (D_1, \ldots, D_j)\).

By combining the above compatibilities, one obtains the following weighted analogue of (10.61).

Let \(I = S \sqcup T\) be a decomposition, and let \(K = S|T\). Let \(f : I \to [r]\) and let \(g\) and \(h\) be the restrictions of \(f\) to \(S\) and \(T\). Further, let \(C\) and \(D\) be linear orders
on $I$, $C_1$ and $D_1$ be linear orders on $S$, and $C_2$ and $D_2$ be linear orders on $T$, such that $b_K(KC) = (C_1, C_2)$ and $b_K(D) = (D_1, D_2)$. Then
\begin{equation}
\text{dist}^A_f(C, D) = \text{dist}^A_g(C_1, D_1) + \text{dist}^A_h(C_2, D_2) + \text{dist}^A_f(C, KC),
\end{equation}
\begin{equation}
\text{dist}^Q_f(C, D) = \text{dist}^Q_g(C_1, D_1) \text{dist}^Q_h(C_2, D_2) \text{dist}^Q_f(C, KC).
\end{equation}

10.12.4. Integration over galleries. We now view the weighted distance function on chambers as an integral. Since integration is traditionally defined using summations we formulate the discussion for the weighted additive distance function. The same discussion can be carried out for the multiplicative case by replacing sums by products.

Let $\mathcal{G}(C, D)$ stand for the following gallery starting at $C$ and ending at $D$:
\[
C = C_1 \overset{H_1}{\rightarrow} C_2 \overset{H_2}{\rightarrow} \cdots \overset{H_{n-1}}{\rightarrow} C_n = D.
\]
The chambers $C_i$ and $C_{i+1}$ are distinct and adjacent, that is, they share a codimension 1 face and $H_i$ is the half-space containing $C_i$ whose supporting hyperplane supports the common codimension 1 face of $C_i$ and $C_{i+1}$. We view $\mathcal{G}(C, D)$ as an oriented path in the complex. Now define
\begin{equation}
\int_{\mathcal{G}(C, D)} (A, f) := \sum_{k=1}^{n-1} w_f^A(H_k)
\end{equation}
with $w_f^A(H_k)$ as in (10.68). In other words, to integrate over a gallery we add the weights of all the half-spaces relevant to that gallery (in the above sense).

We now show that the weighted distance can be interpreted as an integral.

**Proposition 10.24.** We have
\[
\text{dist}^A_f(C, D) = \int_{\mathcal{G}(C, D)} (A, f)
\]
where $\mathcal{G}(C, D)$ is any minimum gallery from $C$ to $D$.

**Proof.** The proof follows from the following chain of equalities.
\begin{equation}
\text{dist}^A_f(C, D) = \sum_{i=1}^{n-1} \text{dist}^A_f(C_i, C_{i+1}) = \sum_{i=1}^{n-1} w_f^A(H_i) = \int_{\mathcal{G}(C, D)} (A, f)
\end{equation}
The first equality follows from (10.73), while the remaining two follow from the definitions. □

This result along with (10.71) implies that
\[
\int_{\mathcal{G}} (A, f) = \int_{-\mathcal{G}} (A^t, f),
\]
where $-\mathcal{G}$ denotes the gallery from $D$ to $C$ which traverses the chambers in the order opposite to that of $\mathcal{G}$.

We elaborate on the cases when $A$ is symmetric and antisymmetric.

If $A$ is symmetric, then one can assign a weight to each hyperplane using $f$ by:
\[
w_f^A(x_i = x_j) := a_{f(i)f(j)} = a_{f(j)f(i)}.
\]
It is clear that in this case the integral only depends on the path joining $C$ and $D$ and not on the orientation. That is, it does not matter whether one goes from $C$...
to $D$ or from $D$ to $C$. Hence in this case, one may view (10.85) as an integral over an unoriented domain.

If $A$ is antisymmetric, then one can assign a weight to each hyperplane using $f$ but only up to a sign. In other words, the weights of the two half-spaces supported by a hyperplane differ by a sign. Hence in this case, one may view $(A, f)$ as a “1-form” on the complex and (10.85) as an integral over an oriented domain. Further:

**Proposition 10.25.** If $A$ is antisymmetric, then

$$\text{dist}^A_f(C, D) = \int_{\mathcal{G}(C, D)} (A, f)$$

where $\mathcal{G}(C, D)$ is any gallery from $C$ to $D$.

**Proof.** This follows from (10.86); the first equality in that chain now holds due to Proposition 10.21. □

The above result says that the 1-form $(A, f)$ is in fact “exact”. By arbitrarily choosing the potential at a chamber, $(A, f)$ determines the potential at all other chambers and $\text{dist}^A_f(C, D)$ is then the potential difference between $C$ and $D$.

**10.12.5. A weighted distance function on faces.** So far in this section, we have been discussing the gallery metric on chambers. Now we turn our attention to the weighted versions of the distance function on faces. To avoid repetition, we will freely use the setup of Section 10.5.3.

The weighted additive distance between faces $F$ and $G$ is defined as follows.

$$\text{dist}^A_f(F, G) := \text{dist}^A_f(C, p_{FG}(C)) = \text{dist}^A_f(p_{FG}(D), D),$$

where $C$ is any chamber containing $FG$, and $D$ is any chamber containing $GF$. It is necessary to show (and one can show) that the definition is independent of the particular choice of $C$ or $D$. In fact, from (10.68) and (10.69), one can see that $\text{dist}^A_f(F, G)$ is the sum of the weights of the half-spaces $H$ which contain $FG$ but do not contain $GF$. It follows that

$$\text{dist}^A_f(F, G) = \text{dist}^A_f(FG, GF).$$

More explicitly, if $F$ and $G$ have the same support, with $F = F^1 | \cdots | F^k$, then (10.37) generalizes as follows.

$$\text{dist}^A_f(F, G) = \sum_{(i, j) \in \text{Inv}(F, G)} \sum_{t \in F^i} a_{f(t)} f(s),$$

In the general case, (10.38) generalizes as follows.

$$\text{dist}^A_f(F, G) = \sum_{i < k} \sum_{j > l} \sum_{t \in F^i \cap G^j} \sum_{s \in F^k \cap G^l} a_{f(t)} f(s),$$

where $i$ and $k$ index the blocks of $F$ while $j$ and $l$ index the blocks of $G$.

The weighted multiplicative distance between faces $F$ and $G$ is defined similarly as follows.

$$\text{dist}^Q_f(F, G) := \text{dist}^Q_f(C, p_{FG}(C)) = \text{dist}^Q_f(p_{FG}(D), D),$$

where $C$ is any chamber containing $FG$, and $D$ is any chamber containing $GF$. 
The remaining discussion works in an analogous manner: (10.88) holds, and (10.89) and (10.90) hold with sums replaced by products as follows. For equal supports,

$$\text{dist}^Q_f(F, G) = \prod_{(i, j) \in \text{Inv}(F, G)} \prod_{t \in F} q_{f(t)} f(s),$$

and in the general case,

$$\text{dist}^Q_f(F, G) = \prod_{i<k \ j>l} \prod_{s \in F \cap G} q_{f(t)} f(s).$$

10.13. The Schubert cocycle and the gallery metric

The Schubert statistic was introduced in Section 2.2. An equivalent formulation in terms of the Schubert cocycle was given in Section 9.7. We begin this section by relating them to the gallery metric by using projection maps.

A weighted version of the Schubert statistic was also discussed in Section 2.2. In this section, we give an equivalent formulation in terms of the weighted Schubert cocycle. We then relate these to the weighted gallery metric.

We conclude by introducing the Schubert cocycle on faces, along with its weighted version, and relating it to the distance function on faces. This generalizes the previous discussion.

10.13.1. The Schubert cocycle and the gallery metric. Let $I = S \sqcup T$ and let $C$ be a linear order on $I$. Also let $K = S|T$. Then it follows from the definitions that

$$\text{sch}_{S, T}(C) = \text{dist}(C, KC),$$

where the left-hand side is the Schubert cocycle (9.12). In particular, applying (9.13),

$$\text{sch}_n(S) = \text{dist}(C(n), KC(n)),$$

where the left-hand side is the Schubert statistic (2.13).

We illustrate how properties of the Schubert statistic or cocycle can be established using this geometric interpretation. For example, for (2.15) or (9.15), we note that

$$\text{dist}(KC, C) + \text{dist}(C, KC) = \text{dist}(KC, KC)$$

and the right-hand side, by (10.26), is equal to $st$, where $s = |S|$ and $t = |T|$.

Recall that $\omega_n$ is the permutation which sends $i$ to $n + 1 - i$ for each $i$. The following sequence of equalities establishes (2.16). The proof of (9.16) is contained in this argument.

$$\text{sch}_n(\omega_n(S)) = \text{dist}(C(n), \omega_n(K)C(n))$$

$$= \text{dist}(\omega_n(C(n)), \omega_n(KC(n)))$$

$$= \text{dist}(\omega_n(C(n)), KC(n))$$

$$= \text{dist}(C(n), KC(n))$$

$$= \text{sch}_n(T)$$
The first and last equalities follow from (10.95) (note that $\overline{K} = T|S$). The second equality uses the fact that the projection map commutes with the group action, and that $\omega_n$ switches $C_{(n)}$ and its opposite $\overline{C}_{(n)}$. The third equality follows from (10.24), and the fourth follows from (10.25).

The cocycle condition (9.17) boils down to the following. For any decomposition $I = R \sqcup S \sqcup T$, and for any linear order $l$ on $I$,

$$\text{dist}(l, (R|S \sqcup T)l) + \text{dist}((R|S \sqcup T)l, (R|S[T])l) = \text{dist}(l, (R \sqcup S[T])l) + \text{dist}((R \sqcup S[T])l, (R|S[T])l).$$

(10.96)

This identity can be proved as follows. The gate property implies that $l - (R|S \sqcup T)l - (R|S[T])l$ are minimum galleries and hence both sides of the above identity equal $\text{dist}(l, (R|S[T])l)$.

The multiplicative property of the cocycle (9.18) boils down to the following. Consider a pair of decompositions $I = S \sqcup T = S' \sqcup T'$ and let $A, B, C, D$ be the resulting intersections, as in Lemma 8.7. Then

$$\text{dist}(l \cdot m, (S'|T')l \cdot m) = \text{dist}(l, (A|B)l) + \text{dist}(m, (C|D)m) + \text{dist}(B|C, C|B),$$

for any linear order $l$ on $S$, and linear order $m$ on $T$. This identity follows from the following sequence of equalities.

$$\text{dist}(l \cdot m, (S'|T')l \cdot m) = \text{dist}(l \cdot m, (A|C|B|D)l \cdot m)$$

$$= \text{dist}(l \cdot m, (A|B|(C|D)l \cdot m) + \text{dist}((A|B|C|D)l \cdot m, (A|C|B|D)l \cdot m)$$

$$= \text{dist}(l \cdot m, (A|B|C|D)l \cdot m) + \text{dist}(B|C, C|B)$$

$$= \text{dist}(l, (A|B)l) + \text{dist}(m, (C|D)m) + \text{dist}(B|C, C|B).$$

The first equality follows by noting that $(S'|T')l \cdot m = (A|C|B|D)l \cdot m.$

The gate property applied to the star of the face $A|B \sqcup C|D$ yields a minimum gallery

$$l \cdot m - (A|B|C|D)l \cdot m - (A|C|B|D)l \cdot m$$

which implies the second equality. The third equality uses the fact that to compute the distance between a face and its opposite it does not matter which chamber is used to do the computation. The last equality follows from the compatibility of the distance function with the join map (10.59).

The connection of the Schubert statistic with inversions (2.26) can be established as follows.

$$\text{sch}_n(S) = \text{dist}(C_{(n)}, KC_{(n)}) = \text{dist}(C_{(n)}, \zeta C_{(n)}) = l(\zeta) = \text{inv}(\zeta).$$

(10.98)

The first equality is (10.95). The $(s, t)$-shuffle permutation $\zeta$ is defined using (10.39) from which the second equality follows. Explicitly, $\zeta$ is the unique permutation which sends $[s]$ to $S$ and $[s + 1, s + t]$ to $T$ in an order-preserving manner. The third and fourth equalities follow from (10.29).
10.13.2. The weighted Schubert cocycle and the gallery metric. Let \( A \) be a square matrix of size \( r \). Let \( l \) be a linear order on a finite set \( I \), \( S \) a subset of \( I \), and \( f : I \to [r] \) a function. The \textit{weighted additive Schubert cocycle} is defined to be

\[
\text{sch}^A_{S,T,f}(l) := \sum_{(i,j) \in \text{Sch}_{S,T}(l)} a_{f(i)f(j)}
\]

where \( \text{Sch}_{S,T}(l) \) is as in (9.11). This is a reformulation of the weighted additive Schubert statistic (2.13). If \( I = [n] \) and \( C_{(n)} \) is the canonical linear order on \([n]\), then

\[
(10.100) \quad \text{Sch}_{S,T}(C_{(n)}) = \text{Sch}_n(S) \quad \text{and} \quad \text{sch}^A_{S,T,f}(C_{(n)}) = \text{sch}^A_n(S,f).
\]

Letting all the entries of \( A \) to be 1 recovers the Schubert cocycle.

We now relate the weighted additive Schubert cocycle to the weighted distance function (10.69). Let \( I = S \sqcup T \) and let \( C \) be a linear order on \( I \). Also let \( K = S|T \). Then

\[
(10.101) \quad \text{sch}^A_{S,T,f}(C) = \text{dist}^A_f(C,KC).
\]

This generalizes (10.94).

**Proof.** Let us look at the right-hand side. Note that the half-space \( x_j \leq x_i \) contains \( C \) precisely if \( j <_C i \), that is, if \( i \) is greater than \( j \) with respect to the linear order \( C \) on \( I \). In addition, the half-space \( x_j \leq x_i \) does not contain \( KC \) precisely if \( i \in S \) and \( j \in T \). Thus, the set of half-spaces which are used to define \( \text{dist}^A_f(C,KC) \) is in correspondence with the set \( \text{Sch}_{S,T}(C) \) which is used to define \( \text{sch}^A_{S,T,f}(C) \). One then checks that the corresponding weights match and the result follows. \( \square \)

We now go to the multiplicative case. Let \( Q \) be a square matrix of size \( r \). Let \( l \) be a linear order on a finite set \( I \), \( S \) a subset of \( I \), and \( f : I \to [r] \) a function. The \textit{weighted multiplicative Schubert cocycle} is

\[
(10.102) \quad \text{sch}^Q_{S,T,f}(l) := \prod_{(i,j) \in \text{Sch}_{S,T}(l)} q_{f(i)f(j)}
\]

where \( \text{Sch}_{S,T}(l) \) is as in (9.11). In terms of the weighted multiplicative distance function, this can be written as

\[
(10.103) \quad \text{sch}^Q_{S,T,f}(C) = \text{dist}^Q_f(C,KC).
\]

10.13.3. The braid coefficients. We now introduce the braid coefficients, which are closely related to those introduced in Section 2.2.7. The motivation for the terminology will become clear in Chapter 14 where we will use these coefficients to construct braiding on colored species.

Fix a decomposition \( I = S \sqcup T \). Let \( f : I \to [r] \) and let \( g \) and \( h \) be the restrictions of \( f \) to \( S \) and \( T \) respectively. Define

\[
(10.104) \quad \text{brd}^A_{S,T,f} := \sum_{s \in S, t \in T} a_{h(t)g(s)} \quad \text{and} \quad \text{brd}^Q_{S,T,f} := \prod_{s \in S, t \in T} q_{h(t)g(s)}.
\]

We refer to these as the additive and multiplicative braid coefficients respectively. If \( Q \) and \( A \) are related by (2.33), then

\[
\text{brd}^Q_{S,T,f} = q^{\text{brd}^A_{S,T,f}}.
\]
If further, \( r = 1 \), \( A = [1] \) and \( Q = [q] \), then
\[
\text{brd}^A_{S,T,f} = |S| |T| \quad \text{and} \quad \text{brd}^Q_{S,T,f} = q^{|S| |T|}.
\]

It also follows that
\[
(10.105) \quad \text{brd}^A_{S,T,f} = \text{brd}^A_{T,S,f} \quad \text{and} \quad \text{brd}^Q_{S,T,f} = \text{brd}^Q_{T,S,f}.
\]

It is convenient to view the braid coefficients as the result of a multistep process: a step consists of an interchange of an element of \( S \) and an element of \( T \). To such a step, we associate a weight depending on the colors of the elements involved and then look up the corresponding entry in the matrix \( A \) or \( Q \). To get the braid coefficient, we add or multiply the weights of all possible interchanges, as may be the case. It follows that
\[
(10.106) \quad \text{brd}^A_{S,T,f} = \text{brd}^A_{d(g),d(h)} \quad \text{and} \quad \text{brd}^Q_{S,T,f} = \text{brd}^Q_{d(g),d(h)}
\]
where the right-hand sides are the braid coefficients of \((2.36)\) and \( d(g) \) and \( d(h) \) are the multidegrees of the fibers of \( g \) and \( h \), as defined in \((2.38)\).

We now provide a geometric interpretation for the braid coefficients in terms of the weighted distance function. First recall that \( S|T \) and \( T|S \) are opposite vertices in this complex; let us call them \( K \) and \( \overline{K} \) for simplicity. Then \((10.107)\)
\[
(10.107) \quad \text{brd}^A_{S,T,f} = \text{dist}^f_A(KC, \overline{KC}) \quad \text{and} \quad \text{brd}^Q_{S,T,f} = \text{dist}^Q_f(KC, \overline{KC}),
\]
for any chamber \( C \), or equivalently,
\[
\text{brd}^A_{S,T,f} = \text{dist}^f_A(K, \overline{K}) \quad \text{and} \quad \text{brd}^Q_{S,T,f} = \text{dist}^Q_f(K, \overline{K}).
\]

**Proof.** The essential observation is that the set \( S \times T \) which is used to define the braid coefficients \((10.104)\) is in correspondence with the set of half-spaces which contain the vertex \( K \) but do not contain \( \overline{K} \), where \( K = S|T \) via
\[
(s, t) \mapsto x_s \geq x_t.
\]
The result then follows from the definitions. \( \square \)

The multistep process described for the braid coefficients is equivalent to a choice of a path from \( K \) to \( \overline{K} \). More precisely, it is a choice of a minimum gallery from \( KC \) and \( \overline{KC} \), where \( C \) is any chamber. Further, the weight associated to each step may be viewed as a weight associated to the corresponding half-space in the gallery. Thus, the braid coefficient can be viewed as an integral over a minimum gallery, see Proposition 10.24.

**10.13.4. Properties of the weighted Schubert cocycle.** We now record the weighted analogues of \((9.14)\)–\((9.18)\). They are reformulations of the properties of the weighted Schubert statistic \((2.39)\)–\((2.43)\). We also give the corresponding identities in terms of the gallery metric.

\[
(10.108) \quad \text{sch}^A_{I,\emptyset,f}(l) = \text{sch}^A_{\emptyset,I,f}(l) = 0, \quad \text{sch}^Q_{I,\emptyset,f}(l) = \text{sch}^Q_{\emptyset,I,f}(l) = 1.
\]
\[
(10.109) \quad \text{sch}^A_{tS,T,f}(l) + \text{sch}^A_{T,S,f}(l) = \text{brd}^A_{S,T,f}, \quad \text{sch}^Q_{tS,T,f}(l) \text{sch}^Q_{T,S,f}(l) = \text{brd}^Q_{S,T,f}.
\]
We see here an instance of how the braid coefficients relate to the Schubert cocycle. We will see another instance a little below. In geometric terms, the identities boil down to

\begin{align}
\text{dist}^A_f(C, KC) + \text{dist}^A_f(C, KC) &= \text{dist}^A_f(KC, KC), \\
\text{dist}^Q_f(C, KC) &\text{dist}^Q_f(C, KC) = \text{dist}^Q_f(KC, KC),
\end{align}

where \( K = S|T \). To prove either of these, apply (10.73) or (10.79) to the minimum gallery

\[ KC - C - KC \]

and then use (10.71) or (10.77).

\begin{align}
\text{sch}^A_{S,T,f}(l) &= \text{sch}^A_{T,S,f}(\bar{l}), \\
\text{sch}^Q_{S,T,f}(l) &= \text{sch}^Q_{T,S,f}(\bar{l}),
\end{align}

where \( \bar{l} \) is the linear order opposite to \( l \). This follows from either (10.74) or (10.80).

For any decomposition \( I = R \sqcup S \sqcup T \), and for any linear order \( l \) on \( I \), and \( f: I \rightarrow [r] \),

\begin{align}
\text{dist}_f^A(l, (R|S \cup T)|l) + \text{dist}_f^A((R|S \cup T)|l), (R|S|T)|l) \\
\quad = \text{dist}_f^A(l, (R \sqcup S|T)|l) + \text{dist}_f^A((R \sqcup S|T)|l), (R|S|T)|l),
\end{align}

\begin{align}
\text{dist}_f^Q(l, (R|S \cup T)|l) \text{dist}_f^Q((R|S \cup T)|l), (R|S|T)|l) \\
\quad = \text{dist}_f^Q(l, (R \sqcup S|T)|l) \text{dist}_f^Q((R \sqcup S|T)|l), (R|S|T)|l).
\end{align}

This can be proved the same way as (10.96).

Consider a pair of decompositions \( I = S \sqcup T = S' \sqcup T' \) and let \( A, B, C, \) and \( D \) be the resulting intersections, as in Lemma 8.7. Also, let \( f: I \rightarrow [r] \). Then, for any linear order \( l \) on \( S \), and linear order \( m \) on \( T \),

\begin{align}
\text{sch}^A_{S', T', f}(l \cdot m) &= \text{sch}^A_{A,B,f|_{S'}}(l) + \text{sch}^A_{C,D,f|_{T'}}(m) + \text{brd}^A_{B,C,f|_{S' \cup T'}}, \\
\text{sch}^Q_{S', T', f}(l \cdot m) &= \text{sch}^Q_{A,B,f|_{S'}}(l) \text{sch}^Q_{C,D,f|_{T'}}(m) \text{brd}^Q_{B,C,f|_{S' \cup T'}}.
\end{align}

This is the multiplicative property of the weighted Schubert cocycle. In geometric terms,

\begin{align}
\text{dist}_f^A(l \cdot m, (S'|T') \cdot m) &= \text{dist}_f^A(l, (A|B)|l) + \text{dist}_f^A(m, (C|D)m) \\
&\quad + \text{dist}_f^A_{|_{S' \cup T'}}(B|C, C|B), \\
\text{dist}_f^Q(l \cdot m, (S'|T') \cdot m) &= \text{dist}_f^Q(l, (A|B)|l) \text{dist}_f^Q(m, (C|D)m) \\
&\quad \text{dist}_f^Q_{|_{B \cup C}}(B|C, C|B).
\end{align}

This can be proved the same way as (10.97).
10.13.5. The weighted inversion statistic and the gallery metric. The weighted inversion statistic (2.44) and (2.45) is related to the weighted distance by:

\begin{align}
\text{inv}^A_f(\sigma^{-1}) &= \text{dist}^A_f(C(n), \sigma C(n)), \\
\text{inv}^Q_f(\sigma^{-1}) &= \text{dist}^Q_f(C(n), \sigma C(n)).
\end{align}

(10.116)

The connection of the weighted Schubert statistic and the weighted inversion statistic given in (2.46) can be established along the lines of (10.98): Use (10.101), (10.103) and (10.116).

The relation between the weighted inversion statistic of a permutation and its inverse given in (2.47) can be derived as follows.

\begin{align}
\text{inv}^Q_f(\sigma^{-1}) &= \text{dist}^Q_f(C(n), \sigma^{-1} C(n)) \\
&= \text{dist}^Q_{f \sigma}(\sigma^{-1} C(n), C(n)) \\
&= \text{dist}^Q_{f \sigma}(C(n), \sigma^{-1} C(n)) \\
&= \text{inv}^Q_{f \sigma}(\sigma).
\end{align}

The equalities follow from (10.77), (10.78) and (10.116).

Let us now establish (2.49). In view of (10.116), this identity is equivalent to

\begin{align}
\text{dist}^Q_f(C(n), \rho C(n)) &= \text{dist}^Q_{f \gamma}(C(n), \zeta C(n)) \text{ dist}^Q_{f \gamma}(C(s), \sigma C(s)) \text{ dist}^Q_{f \gamma}(C(t), \tau C(t)),
\end{align}

(10.117)

where $\gamma$ and $\bar{h}$ are as in (2.48). To prove this, we apply (10.84): Put $C = C(n)$ and $D = \rho C(n)$. Now note from Proposition 10.6 that $\zeta C(n) = KC(n)$, where $K = S|T$. Further, using the naturality of the weighted distance (10.78), it follows that

\begin{align}
\text{dist}^Q_{\gamma}(C_1, D_1) &= \text{dist}^Q_{\gamma}(C(s), \sigma C(s)) \quad \text{and} \quad \text{dist}^Q_{\bar{h}}(C_2, D_2) = \text{dist}^Q_{\bar{h}}(C(t), \tau C(t)).
\end{align}

The required identity now follows.

10.13.6. The Schubert cocycle on faces. Given a set composition $H \in \Sigma[I]$ and a decomposition $I = S \sqcup T$, let

\begin{align}
\text{Sch}_{S,T}(H) := \{(i, j) \in S \times T \mid i > j \text{ according to } H\},
\end{align}

(10.118)

where $i > j$ according to $H$ means that $i$ appears in a strictly later block of $H$ than $j$. Let

\begin{align}
\text{sch}_{S,T}(H) := |\text{Sch}_{S,T}(H)|.
\end{align}

(10.119)

For instance, if

\begin{align}
H = sh|iv|a, \quad S = \{i, s, a\}, \quad T = \{v, h\},
\end{align}

then

\begin{align}
\text{Sch}_{S,T}(H) = \{(i, h), (a, h), (a, v)\} \quad \text{and} \quad \text{sch}_{S,T}(H) = 3.
\end{align}

Alternatively,

\begin{align}
\text{sch}_{S,T}(H) = \sum_{1 \leq i < j \leq k} |H^i \cap T| |H^j \cap S|,
\end{align}

where $H = H^1 \cdots H^k$.

We view \text{sch}_{S,T} as an integer-valued function on $\Sigma[I]$ and refer to the family of maps $\text{sch}_{S,T}$ as the Schubert cocycle on faces. Its restriction to $L[I]$ is the usual
Schubert cocycle. The Schubert cocycle on faces can be interpreted using the distance function on faces:

\[ \text{sch}_{S,T}(H) = \text{dist}(H, K), \]

where \( K \) is the vertex \( S \upharpoonright T \). This follows from (10.38).

It satisfies the following generalizations of (9.14)–(9.18).

\[ \text{sch}_{I,\emptyset}(H) = \text{sch}_{\emptyset,I}(H) = 0. \]

\[ \text{sch}_{S,T}(H) + \text{sch}_{T,S}(H) + \sum_{i=1}^{k} |H^i \cap S||H^i \cap T| = |S||T|, \]

where \( H = H^1|\cdots|H^k \).

\[ \text{sch}_{S,T}(H) = \text{sch}_{T,S}(\overline{H}), \]

where \( \overline{H} = H^k|\cdots|H^1 \) denotes the face opposite to \( H \).

For any decomposition \( I = R \sqcup S \sqcup T \), and for any composition \( H \) of \( I \),

\[ \text{sch}_{R,S \sqcup T}(H) + \text{sch}_{S,T}(H|_{S \sqcup T}) = \text{sch}_{R \sqcup S,T}(H) + \text{sch}_{R,S}(H|_{R \sqcup S}). \]

This is the cocycle condition.

Consider a pair of decompositions \( I = S \sqcup T = S' \sqcup T' \) and let \( A, B, C \), and \( D \) be the resulting intersections, as in Lemma 8.7. Then

\[ \text{sch}_{S',T'}(F \cdot G) = \text{sch}_{A,B}(F) + \text{sch}_{C,D}(G) + |B||C|, \]

for any composition \( F \) of \( S \), and composition \( G \) of \( T \). Here \( F \cdot G \) stands for the concatenation of \( F \) and \( G \). This is the multiplicative property of the cocycle.

We now briefly consider the weighted versions of the Schubert cocycle on faces. The setup is as for chambers. The \textit{weighted additive Schubert cocycle on faces} is defined to be

\[ \text{sch}^A_{S,T,f}(H) := \sum_{(i,j) \in \text{Sch}_{S,T}(H)} a_{f(i)}f(j). \]

The \textit{weighted multiplicative Schubert cocycle on faces} is

\[ \text{sch}^Q_{S,T,f}(H) := \prod_{(i,j) \in \text{Sch}_{S,T}(H)} q_{f(i)}f(j). \]

Alternatively,

\[ \text{sch}^A_{S,T,f}(H) = \sum_{1 \leq i<j \leq k} \sum_{t \in F^i \cap T} \sum_{s \in F^j \cap S} a_{f(s)}f(t), \]

and

\[ \text{sch}^Q_{S,T,f}(H) = \prod_{1 \leq i<j \leq k} \prod_{t \in F^i \cap T} \prod_{s \in F^j \cap S} q_{f(s)}f(t), \]

where \( H = H^1|\cdots|H^k \). It follows from (10.90) and (10.93) that the relation to the weighted distance function on faces is given by

\[ \text{sch}^A_{S,T,f}(H) = \text{dist}^A_f(H, K) \quad \text{and} \quad \text{sch}^Q_{S,T,f}(H) = \text{dist}^Q_f(H, K). \]

One can also write down weighted analogues of (10.120)–(10.124). The cocycle condition, for example, takes the following form.
For any decomposition $I = R \sqcup S \sqcup T$, and for any composition $H$ of $I$, and $f: I \rightarrow [r]$, and $I^e = 0$.

\[ \text{sch}^A_{R,S \sqcup T,f}(H) + \text{sch}^A_{S,T,f|_{S \sqcup T}}(H|_{S \sqcup T}) = \text{sch}^A_{R \sqcup S,T,f}(H) + \text{sch}^A_{R,S,f|_{R \sqcup S}}(H|_{R \sqcup S}), \]

(10.127)

\[ \text{sch}^Q_{R,S \sqcup T,f}(H) \text{sch}^Q_{S,T,f|_{S \sqcup T}}(H|_{S \sqcup T}) = \text{sch}^Q_{R \sqcup S,T,f}(H) \text{sch}^Q_{R,S,f|_{R \sqcup S}}(H|_{R \sqcup S}). \]

This property may also be formulated in terms of the distance function; the details are omitted.


In this section, we review some work of Varchenko [367]. It involves the factorization of a bilinear form on the set of chambers of a hyperplane arrangement. Special cases of relevance to this monograph are also discussed. They pertain to the braid arrangement.

10.14.1. Weights on hyperplanes. Fix a central hyperplane arrangement, and let $L$ be its set of chambers. The bilinear form is defined on the linearization $\mathbb{k}L$, where $\mathbb{k}$ is assumed to have characteristic zero. It is as follows. Assign a weight to each hyperplane in the arrangement, and let $wtdist(C, D)$ be the product of the weights of the hyperplanes which separate $C$ and $D$. Define a symmetric bilinear form on $\mathbb{k}L$:

(10.128)

\[ \langle C, D \rangle := wtdist(C, D). \]

The determinant of this bilinear form [367, Theorem (1.1)] or [366, Theorem 2.6.2] is given by:

(10.129)

\[ \prod_X (1 - a(X)^2)^{l(X)}, \]

where the product is over all proper flats $X$ in the arrangement, $a(X)$ is the product of the weights of all hyperplanes that contain $X$, and $l(X)$ denotes the multiplicity of $X$ defined as follows. Pick any hyperplane $H$ which contains $X$. Then $l(X)$ is half the number of chambers $C$ which have the property that $X$ is the support of $C \cap H$.

Varchenko’s proof of the factorization (10.129) is geometric in nature and similar to the spirit of the present chapter. It makes crucial use of directed flats which he calls cones. (In our terminology, directed flats are examples of top-dimensional cones.) The use of minimum galleries and the gate property is also evident in his proof.

10.14.2. Weights on half-spaces. It is useful to work in a slightly more general setup, where instead of hyperplanes, one assigns weights to half-spaces. Details follow. Assign a weight to each half-space in the arrangement, and let $wtdist(C, D)$ be the product of the weights of the half-spaces which contain $C$ but do not contain $D$. Define a bilinear form on $\mathbb{k}L$:

(10.130)

\[ \langle C, D \rangle := wtdist(C, D). \]
It is not symmetric in general. In fact, it is symmetric precisely if for each hyperplane, the weights of the two half-spaces it supports are equal. The determinant of this more general bilinear form is given by:

\[(10.131) \prod_X (1 - b(X))^{l(X)},\]

where the product is over all proper flats \(X\) in the arrangement, \(b(X)\) is the product of the weights of all half-spaces whose supporting hyperplane contains \(X\), and \(l(X)\) is as in (10.129).

If for each hyperplane, the weights of the two half-spaces it supports are equal, then \(b(X) = a(X)^2\) and (10.131) reduces to (10.129). The factorization (10.131) can be established by generalizing Varchenko’s proof. We plan to provide the details in future work.

The following is an immediate consequence of (10.131). To keep the exposition self-contained, we give a direct proof following Varchenko.

**Lemma 10.26.** If \(b(X) \neq 1\) for any proper flat \(X\) in the arrangement, then the bilinear form (10.130) on \(kL\) is nondegenerate.

**Proof.** Let \(\gamma : kL \to kL^*\) be the map induced by the bilinear form. Explicitly,

\[\gamma(C) = \sum_D \text{wtdist}(C, D) D^* ,\]

where \(\text{wtdist}(C, D)\) is the product of the weights of all half-spaces whose supporting hyperplane contains \(X\), and \(l(X)\) is as in (10.129).

Assume that \(b(X) \neq 1\) for any proper flat \(X\) in the arrangement. We want to show that \(\gamma\) is an isomorphism, or equivalently that it is surjective.

For any directed face \((F, C)\), define

\[m(F, C) = \sum_{D \subseteq \Psi(F, C)} \text{wtdist}(C, D) D^* ,\]

where \(\Psi(F, C)\) is the cone associated to \((F, C)\) as in Proposition 10.14.

We claim that \(m(F, C)\) belongs to the image of \(\gamma\). The proof proceeds by backward induction on the dimension of \(F\). Note that \(m(C, C) = \gamma(C)\), so the claim holds if \(F\) has full dimension.

Let \(F\) be any face and let \(C\) and \(D\) be chambers opposite to each other in \(\text{Star}(F)\). Then

\[m(F, C) - (-1)^{\deg(C) - \deg(F)} \text{wtdist}(C, D) m(F, D) = \sum_{G : F \leq G \leq C, G \neq F} (-1)^{\deg(G) - \deg(F) + 1} m(G, C) .\]

This is a consequence of inclusion-exclusion. Rearranging the terms,

\[m(F, D) - (-1)^{\deg(D) - \deg(F)} \text{wtdist}(D, C) m(F, C) \]

also belongs to the image of \(\gamma\).

Let \(X\) denote the support of \(F\). Since by assumption \(b(X) \neq 1\), we have

\[1 - \text{wtdist}(C, D) \text{wtdist}(D, C) = 1 - b(X) \neq 0.\]
It follows that both \( m(F, C) \) and \( m(F, D) \) belong to the image of \( \gamma \). The claim follows.

To finish the proof, we note that for any chamber \( C \), we have \( m(\emptyset, C) = C^* \). By our claim, this is in the image of \( \gamma \). Hence, \( \gamma \) is surjective as required. \( \square \)

Let \( w(H) \) denote the weight of the half-space \( H \). View it as a variable. Note that for any proper flat \( X \), \( b(X) \) is a square free monomial in these variables. This yields the following.

**Lemma 10.27.** Let \( w(H) \) denote the weight of the half-space \( H \). If no square free monomial in the \( w(H) \)'s equals 1, then the bilinear form (10.130) on \( kL \) is nondegenerate.

This result will be required in this monograph for questions related both to Hopf monoids in species and to Fock functors.

### 10.14.3. Specialization: Equal weights

Let \( q \in k \) be any scalar. Define a symmetric bilinear form on the space \( kL \) indexed by chambers by:

\[
\langle C, D \rangle := q^{\text{dist}(C, D)}.
\]

This is a special case of (10.128) in which weights of all hyperplanes are equal to \( q \).

Note that a monomial in the hyperplane weights equals 1 if and only if \( q \) is a root of unity. Hence, applying Lemma 10.27, we obtain:

**Lemma 10.28.** If \( q \) is not a root of unity, then the bilinear form (10.132) on \( kL \) is nondegenerate.

A weaker result with a direct proof is given below.

**Lemma 10.29.** If \( q \) is not an algebraic integer, then the bilinear form (10.132) on \( kL \) is nondegenerate.

**Proof.** The determinant of the bilinear form on \( kL \) is a polynomial in \( q \) over \( Z \). The degree of this polynomial is the product of the number of chambers and the number of hyperplanes in the arrangement. Exactly one term in the determinant expansion gives the leading term: for each \( C \), take inner product with \( \overline{C} \). It follows that the coefficient of the leading term is either 1 or \( -1 \). Since \( q \) is not an algebraic integer, the value of the polynomial will not be zero. \( \square \)

**Example 10.30.** The bilinear form (10.132) for the braid arrangement in \( \mathbb{R}^n \) has been studied in several papers, including [96, 108, 159, 268, 380]. Its determinant is explicitly given by

\[
\prod_{i=2}^{n} \left( 1 - q^{i(i-1)} \right) \binom{n}{i-2}!(n-i+1)!
\]

This formula was first proved by Zagier [380, Theorem 2]. Zagier’s formula (10.133) holds in the polynomial ring \( Z[q] \), and hence over a field of any characteristic. It is a specialization of Varchenko’s formula (10.129). Details regarding this are given by Hanlon and Stanley [159]; additional information can be found in Krattenthaler’s surveys [208, Theorem 55] and [209, Section 5.7]. Varchenko’s bilinear form is also studied by Denham and Hanlon [97, 98, 96].

Let us explicitly look at the case \( n = 2 \). Then

\[ L[2] = \{1|2, 2|1\}, \]

...
and the matrix of the bilinear form (10.132) on \( kL \) is given by

\[
\begin{pmatrix}
1 & q \\
q & 1
\end{pmatrix}.
\]

Its determinant is \( 1 - q^2 \); so the roots are 1 and \(-1\). This agrees with (10.133).

### 10.14.4. Specialization: Power weights

Consider the more general case when all weights are a positive integral power of \( q \). Again, applying Lemma 10.27, we obtain:

**Lemma 10.31.** Let \( q \) be a scalar which is not a root of unity. If all the hyperplane weights are a positive integral power of \( q \), then the bilinear form (10.128) on \( kL \) is nondegenerate.

**Example 10.32.** Let \( X \) be a partition of \( I \). Let \( \Sigma[X] \) be the poset of faces associated to the braid arrangement in \( \mathbb{R}^X \). Similarly, let \( L[X] \) be the set of chambers. Observe that \( \Sigma[X] \) can be identified with those compositions of \( I \) whose support is less than \( X \), and \( L[X] \) with those compositions of \( I \) whose support is exactly \( X \).

Now consider the following bilinear form on \( L[X] \):

\[
\langle F, G \rangle := q^{\text{dist}(F,G)},
\]

where \( \text{dist}(F, G) \) is as given in (10.37). Note that \( \text{dist}(F, G) \) is not the gallery metric in \( \Sigma[X] \), rather it is the distance function between faces in \( \Sigma[I] \). Nevertheless, it can be interpreted as a weighted distance function between chambers of \( \Sigma[X] \) as follows.

Let \( X = \{X^1, \ldots, X^k\} \). Then hyperplanes in \( \Sigma[X] \) are given by \( X^i = X^j \) where \( i \) and \( j \) vary between 1 and \( k \). To each such hyperplane, we associate the weight \( q^{|X^i|\!-|X^j|} \).

Then observe that \( q^{\text{dist}(F,G)} \) is the product of the weights of the hyperplanes in \( \Sigma[X] \) which separate \( F \) and \( G \).

It follows from Lemma 10.31 that, for \( q \) not a root of unity, the bilinear form on \( L[X] \) is nondegenerate.

### 10.14.5. Specialization: Matrix weights

We work with the braid arrangement. Let \( Q \) be a square matrix of size \( r \), and let \( f: I \to [r] \). Define a bilinear form on \( kL \):

\[
\langle C, D \rangle := \text{dist}_f^Q(C, D),
\]

where the right-hand side is the weighted multiplicative distance (10.75). This bilinear form can be interpreted as follows. To the half-space \( x_j \leq x_i \), assign the weight \( q_{f(i)f(j)} \). Then \( \langle C, D \rangle \) is the product of the weights of the half-spaces which contain \( C \) but do not contain \( D \). Thus, (10.134) is a special case of (10.130). If all entries of \( Q \) are equal to say \( q \), then we recover (10.132).

Applying Lemma 10.27, we obtain:

**Lemma 10.33.** If no monomial in the \( q_{ij} \)'s equals 1, then the bilinear form (10.134) on \( kL \) is nondegenerate.
Example 10.34. We work with the setup of Example 10.32 and generalize it as follows. Let $Q$ be a square matrix of size $r$, and let $f : I \to [r]$. To the hyperplane $X^j \leq X^i$, we associate the weight
\[
\prod_{a \in X^1 \atop b \in X^j} q_{f(a)f(b)};
\]
where $q_{ij}$ denotes the $ij$-th entry of $Q$. Note that the weight is a monomial in the $q_{ij}$’s. Now consider the bilinear form on $L[X]$:
\[
\langle F, G \rangle := \prod_H w(H)
\]
where the product is over all half-spaces $H$ which contain $F$ but do not contain $G$ and $w(H)$ denotes the weight of $H$. If all entries of $Q$ are equal to (say) $q$, then we recover the bilinear form of Example 10.32.

It follows from Lemma 10.33 that, if no monomial in the $q_{ij}$’s equals one, then the bilinear form on $L[X]$ is nondegenerate.

10.15. Bilinear forms on directed faces and faces

In this section, we define bilinear forms on directed faces and faces, and study their nondegeneracy. This complements the discussion in Section 10.14. We continue to work in the setting of central hyperplane arrangements (though we are mainly interested in the braid arrangement).

Fix a central hyperplane arrangement, and let $\Sigma$ be its set of faces, and $\Sigma^\rightarrow$ be its set of directed faces. The bilinear forms are defined on the linearizations $k\Sigma^\rightarrow$ and $k\Sigma$, where $k$ is assumed to have characteristic zero. A key role is played by the gallery metric and Tits projection maps. These can be defined for a central hyperplane arrangement in the same manner as for the braid arrangement (Section 10.5).

10.15.1. A bilinear form on directed faces. Define a symmetric bilinear form on the space $k\Sigma^\rightarrow$ indexed by directed faces:

\[
\langle (F, C), (G, D) \rangle := \begin{cases} 
q^{\text{dist}(C, D)} & \text{if } GC = D \text{ and } FD = C, \\
0 & \text{otherwise}.
\end{cases}
\]

Lemma 10.35. If $q$ is not an algebraic integer, then the bilinear form on $k\Sigma^\rightarrow$ is nondegenerate.

Proof. We follow the pattern of the proof given for Lemma 10.29; the argument however is much more delicate. Firstly, $\overline{FC}$ belongs to the set $\Psi(F, C)$ (as in Proposition 10.14) and secondly, for any chamber $D$ in this set, there is a minimum gallery $C - D - \overline{FC}$ by [12, Fact 5.2.1]. Hence
\[
\text{dist}(C, D) \leq \text{dist}(C, FC).
\]

It follows that
\[
\langle (F, C), (G, D) \rangle = q^{\text{dist}(C, FC)} \quad \text{if } D = FC, \overline{F} \leq G \leq D.
\]

This is illustrated in Figure 10.13.

Further, for any other directed face $(G, D)$, the bilinear form evaluates either to 0 or a strictly smaller power of $q$. 
By a backward induction on the dimension of $F$, it follows that the determinant of the bilinear form on $\mathbb{k}\Sigma$ is a polynomial in $q$ of degree

$$\sum_{(F,C)\in \Sigma} \text{dist}(C, F^C)$$

and whose coefficient of the leading term is either 1 or $-1$. Exactly one term in the determinant expansion gives the leading term: for each $(F,C)$, take inner product with $(\overline{F}, \overline{FC})$. Since $q$ is not an algebraic integer, the value of this polynomial will not be zero.

\[\square\]

**Example 10.36.** Let $\Sigma[2]$ be the Coxeter complex of the symmetric group $S_2$. Then

$$\Sigma[2] = \{(12, 1|2), (1|2, 1|2), (12, 2|1), (2|1, 2|1)\}.$$  

The matrix of the bilinear form on $\mathbb{k}\Sigma[2]$, indexed in the above order, is

$$\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & q \\
0 & 0 & 1 & 1 \\
0 & q & 1 & 1
\end{pmatrix}.$$  

Its determinant is $-q^2$. Observe directly that exactly one summand in the determinant expansion yields this term; the rest cancel out.

**10.15.2. A bilinear form on faces.** Define a symmetric bilinear form on the space $\mathbb{k}\Sigma$ indexed by faces as follows:

$$\langle F, G \rangle := \sum_{C: FG \leq C} q^{\text{dist}(C, p_{GF}(C))}.$$  

Recall that $p_{GF}(C)$ is the projection of the chamber $C$ on the face $GF$. It follows from (10.33) (also see Remark 10.5) that

$$\langle F, G \rangle = |L_{FG}| q^{\text{dist}(F,G)},$$  

where $L_{FG}$ is the set of chambers containing $FG$. In view of (10.7), we have

$$\langle F, G \rangle = (FG)! q^{\text{dist}(F,G)}.$$  

Note that this bilinear form is induced from the one on $\mathbb{k}\Sigma$ by viewing $\mathbb{k}\Sigma$ as a subspace of $\mathbb{k}\Sigma$ via the map $F \mapsto \sum (F,C)$, where the sum is over all chambers $C$ containing $F$. Explicitly,

$$\langle F, G \rangle = \sum_{C,D: F \leq C, G \leq D} \langle (F,C), (G,D) \rangle.$$  

**Lemma 10.37.** If $q$ is not an algebraic number, then the bilinear form on $\mathbb{k}\Sigma$ is nondegenerate.
Proof. The proof is similar to that of Lemma 10.35; we explain it briefly. For $F$ fixed and $G$ varying, the highest power of $q$ which appears in $\langle F, G \rangle$ is $\text{dist}(F, G)$ and this happens precisely if $F \leq G$. By a backward induction on the dimension of $F$, it then follows that the determinant of the bilinear form on $k\Sigma$ is a polynomial in $q$ of degree

$$\sum_{G \in \Sigma} \text{dist}(G, \overline{G}),$$

with integer coefficients, and whose leading coefficient is

$$\prod_{G} G!.$$

Exactly one term in the determinant expansion gives the leading term: for each $G$, take inner product with $\overline{G}$. Since $q$ does not satisfy any polynomial over $\mathbb{Q}$, the result follows. □

Example 10.38. Let $\Sigma[2]$ be the Coxeter complex of the symmetric group $S_2$. Then

$$\Sigma[2] = \{12, 1|2, 2|1\}.$$

The matrix of the bilinear form on $k\Sigma[2]$, indexed in the above order, is

$$\begin{pmatrix}
2 & 1 & 1 \\
1 & 1 & q \\
1 & q & 1
\end{pmatrix}.$$ 

Its determinant is $-2q^2 + 2q$. Observe directly that exactly one summand in the determinant expansion contributes to the leading term $-2q^2$.

Question 10.39. For a central hyperplane arrangement, are there analogues of Varchenko’s theorem for the bilinear forms on faces and directed faces? In other words, can one describe how the determinants of these bilinear forms factorize?

The roots of these determinants are important since for such values of $q$, one may then take the quotient by the radical of the form to construct new objects. The only case considered in the literature seems to be $q = 1$ and we discuss it briefly.

For $q = 1$, the radical of the bilinear form on $k\Sigma$ is described in [12, Section 2.5.5]. The quotient by the radical is $k\Pi$ where $\Pi$ is the lattice of flats. The situation appears to be much more complicated for $k\overline{\Sigma}$. For $q = 1$, it is shown in the Coxeter case that the bilinear form on $k\overline{\Sigma}$ is degenerate in general [12, Section 2.5.1]. In particular, for the braid arrangement, the form is degenerate for $n \geq 3$. However, a description of the radical is not known, even in the case of the braid arrangement.

Remark 10.40. The bilinear forms on faces and directed faces for $q = 1$ were defined in the wider context of left regular bands in [12, Section 2.5]. Left regular bands, however, may be too general for defining these bilinear forms for general $q$, since this requires the notion of a distance.

10.15.3. A bilinear form on compositions and partitions. Let $W$ be a Coxeter group and let $S$ be the set of generators in its standard presentation. For the symmetric group, $S$ can be taken to be the set of adjacent transpositions. More information about Coxeter presentations can be found in the references cited at the beginning of the chapter.
Let $\Sigma$ be the Coxeter complex associated to $W$ and $S$. The action of $W$ on $\Sigma$ is type-preserving and further, faces with the same type are in the same $W$-orbit. Since types of faces correspond to subsets of $S$, it follows that a basis for $(k\Sigma)^W$, the space of $W$-invariants, is given by

$$\sigma_T := \sum_{F: \text{type}(F)=T} F,$$

as $T$ ranges over all subsets of $S$. For the symmetric group, these are the elements considered in (10.47).

Since the bilinear form on faces (10.136) commutes with the $W$-action, one obtains a bilinear form on $(k\Sigma)^W$. By Lemma 10.37, this form is nondegenerate if $q$ is not an algebraic integer. Explicitly, it is given by

$$\frac{1}{|W|} \langle \sigma_T, \sigma_U \rangle = \sum_{w \in W: \text{Des}(w) \leq T, \text{Des}(w^{-1}) \leq U} q^{l(w)},$$

where $\text{Des}(w)$ stands for the descent set of $w$ (Section 10.7.1). The main steps in the calculation are indicated below.

$$\langle \sigma_T, \sigma_U \rangle = \sum_{F,G: \text{type}(F)=T, \text{type}(G)=U} \langle F, G \rangle$$

$$= \sum_{(F,C),(G,D): \text{type}(F)=T, \text{type}(G)=U, FD=C, GC=D} q^{\text{dist}(C,D)}$$

$$= \sum_{(C,D): \text{Des}(C,D) \leq G, \text{Des}(D,C) \leq F} q^{\text{dist}(C,D)}$$

$$= |W| \sum_{w \in W: \text{Des}(w) \leq T, \text{Des}(w^{-1}) \leq U} q^{l(w)}.$$

The first step is the definition, the second step uses (10.137), the third step uses definition (10.135), the fourth step uses (10.42) ($F$ and $G$ are dropped from the summation index since they are determined by $C$ and $D$), the last step replaces $(C, D)$ by $w = d(C, D)$ and uses the first diagram in (10.40) (which is valid for any finite Coxeter group).

We define the $H$ basis for the space of invariants by setting $\sigma_T = \sqrt{|W|} H_T$. This normalization implies that

$$\langle H_T, H_U \rangle = \sum_{w \in W: \text{Des}(w) \leq T, \text{Des}(w^{-1}) \leq U} q^{l(w)}.$$

Now define the $K$ basis for the space of invariants by

$$H_T = \sum_{U \leq T} K_U.$$
A straightforward calculation shows that
\[ \langle K_T, K_U \rangle = \sum_{w \in W : \text{Des}(w) = T, \text{Des}(w^{-1}) = U} q^{l(w)}. \]

We note two special cases.

- For the Coxeter group of type $A$, the form (10.138) has been considered by Thibon and Ung [358, Formula (39)].
- The case of arbitrary finite Coxeter groups and $q = 1$ is discussed in detail in [12, Section 2.6]; some related references are [333, Theorem 3], [27], [139], [211, Corollary 3.11] and [70]. In this case, the bilinear form on $(k\Sigma)^W$ is degenerate. The quotient by the radical is $(k\Pi)^W$ where $\Pi$ is the lattice of flats.

Consider now the case of type $A$ and set $q = 1$. For each $n$, we obtain a bilinear form on the space spanned by all compositions of $n$, or equivalently, subsets of $[n-1]$ (10.1), given by

\[ (10.139) \quad \langle K_T, K_U \rangle = |\{ w \in S_n : \text{Des}(w) = T, \text{Des}(w^{-1}) = U \}|. \]

This form is degenerate and the quotient by the radical is the space spanned by all partitions of $n$. The resulting nondegenerate form can be identified with the standard inner product of symmetric functions, which we recall has a basis indexed by partitions. Under this identification, the quotient map sends $H_T$ to $h_\lambda$, where $\lambda$ is the underlying partition of $T$ and $h$ denotes the basis of complete symmetric functions [252, Section I.2], [343, Section 7.5]. The ribbon Schur functions are the spanning set of the space of symmetric functions obtained as the image of the $K$ basis [359, Section 2.2]. The ribbon Schur functions are a special kind of skew Schur functions (for which the skew shape is a ribbon, also called rim-hook or border strip). Formula (10.139) recovers a result of Gessel [144, Theorem 5]; also see [343, Corollary 7.23.8] and [358, Formula (39)].
CHAPTER 15

From Species to Graded Vector Spaces

Stover described how to construct graded Hopf algebras from Hopf monoids in species [346, Section 14]. These constructions were then discussed in more detail by Patras, Reutenauer, and Schocker [291, 292, 293]. In this chapter we formulate these constructions and study their properties in categorical terms.

In Section 15.1, we define four monoidal functors from species to graded vector spaces, namely, $\mathcal{K}$, $\mathcal{K}^\vee$, $\overline{\mathcal{K}}$, and $\overline{\mathcal{K}}^\vee$. We prove that the first two are bilax and the remaining two are bistrong. We refer to all of them collectively by the term Fock functors. For further distinction, we refer to $\mathcal{K}$ and $\mathcal{K}^\vee$ as full Fock functors, and $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ as bosonic Fock functors. As suggested by the terminology, there are also bistrong functors $\overline{\mathcal{K}}_{-1}$ and $\overline{\mathcal{K}}^\vee_{-1}$ called fermionic Fock functors. These will be introduced and studied in Chapter 16. The Fock functors with their notations are summarized in Table 15.1. The motivation for our terminology comes from classical Fock spaces. These are treated in Chapter 19; see in particular Tables 19.1 and 19.2.

The monoidal properties of the Fock functors imply that the image of a Hopf monoid in species under any Fock functor is a graded Hopf algebra. This is how the categorical framework relates to Stover’s constructions. This is explained in Section 15.2.

The categorical approach allows us to reduce the study of the relation between various properties of Hopf monoids and properties of the corresponding Hopf algebras, such as commutativity, duality, antipode, and primitive elements, to the study of general properties of the Fock functors. This is pursued in later sections of this chapter.

15.1. The Fock bilax monoidal functors

Recall that $(\mathbb{S}p, \cdot)$ and $(g\mathbb{V}ec, \cdot)$ denote respectively the categories of species and graded vector spaces under the Cauchy product. In this section, we first construct two bilax monoidal functors $\mathcal{K}$ and $\overline{\mathcal{K}}$ from $(\mathbb{S}p, \cdot)$ to $(g\mathbb{V}ec, \cdot)$ and a natural transformation between them which is compatible with the bilax monoidal structure. We also relate them in a different manner via the linear order species.

<table>
<thead>
<tr>
<th>Fock functor</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{K}$, $\mathcal{K}^\vee$</td>
<td>Full Fock functor</td>
</tr>
<tr>
<td>$\overline{\mathcal{K}}$, $\overline{\mathcal{K}}^\vee$</td>
<td>Bosonic Fock functor</td>
</tr>
<tr>
<td>$\overline{\mathcal{K}}<em>{-1}$, $\overline{\mathcal{K}}^\vee</em>{-1}$</td>
<td>Fermionic Fock functor</td>
</tr>
</tbody>
</table>
We also discuss two other bilax monoidal functors, namely $\mathcal{K}^\vee$ and $\mathcal{K}^\wedge$, which, as suggested by the notation, are related to $\mathcal{K}$ and $\mathcal{K}$ through duality.

### 15.1.1. The bilax monoidal functors $\mathcal{K}$ and $\mathcal{K}$

**Definition 15.1.** Let

$$\mathcal{K}, \bar{\mathcal{K}} : \text{Sp} \to \text{gVec}$$

be the functors defined by

$$\mathcal{K}(q) := \bigoplus_{n \geq 0} q[n] \quad \text{and} \quad \bar{\mathcal{K}}(q) := \bigoplus_{n \geq 0} q[n]_{S_n},$$

where $q[n]_{S_n}$ is the vector space of $S_n$-coinvariants of $q[n]$.

The quotient maps $\mathcal{K}(q) \to \bar{\mathcal{K}}(q)$ define a natural transformation $\mathcal{K} \Rightarrow \bar{\mathcal{K}}$, because a morphism of species $p \to q$ yields maps of $S_n$-modules $p[n] \to q[n]$, which therefore factor through coinvariants.

We proceed to turn $\mathcal{K}$ into a bilax monoidal functor with respect to the Cauchy product on graded vector spaces (2.2) and species (8.6), that is,

$$\mathcal{K} : (\text{Sp}, \cdot, \beta) \to (\text{gVec}, \cdot, \beta).$$

Define maps

$$\mathcal{K}(p) \cdot \mathcal{K}(q) \xrightarrow{\varphi_{p,q}} \mathcal{K}(p \cdot q) \xleftarrow{\psi_{p,q}}$$

as follows. On the degree $n$ components of these graded vector spaces, we define maps

$$\bigoplus_{s + t = n} p[s] \otimes q[t] \xrightarrow{\varphi_{p,q}} \bigoplus_{S \cup T = [n]} p[S] \otimes q[T]$$

as the direct sum of the following maps:

$$\varphi_{p,q} : p[s] \otimes q[t] \xrightarrow{p[id] \otimes q[\text{cano}]} p[s] \otimes q[s + 1, s + t]$$

$$\psi_{p,q} : p[S] \otimes q[T] \xrightarrow{p[\text{cano}] \otimes q[\text{cano}]} p[|S|] \otimes q[|T|],$$

with notations as defined in Notation 2.5. Note that the composite $\psi_{p,q} \varphi_{p,q}$ is the identity, but in general these maps are not invertible on the degree $n$ component.

It is also clear that $\mathcal{K}(1) = \mathbb{k}$; hence $\mathcal{K}$ takes the unit object in $(\text{Sp}, \cdot)$ to the unit object in $(\text{gVec}, \cdot)$. We define $\varphi_0$ and $\psi_0$ to be the identity maps

$$\mathbb{k} \xrightarrow{\varphi_0} \mathcal{K}(1).$$

We show in Theorem 15.3 below that $(\mathcal{K}, \varphi, \psi)$ is a bilax monoidal functor.
For the bilax structure of $\mathcal{K}$, we define the maps $\varphi$ and $\psi$ by the commutativity of the diagram below.

\[
\begin{array}{cccc}
\mathcal{K}(p) \cdot \mathcal{K}(q) & \xrightarrow{\varphi_{p,q}} & \mathcal{K}(p \cdot q) \\
\downarrow & & \downarrow \\
\mathcal{K}(p) \cdot \mathcal{K}(q) & \xrightarrow{\psi_{p,q}} & \mathcal{K}(p \cdot q)
\end{array}
\]

(15.1)

**Proposition 15.2.** The maps $\varphi$ and $\psi$ are well-defined and inverses of each other.

**Proof.** For the map $\psi$, we need to consider the diagram

\[
\bigoplus_{S \sqcup T = [n]} p[S] \otimes q[T] \xrightarrow{p[cano] \otimes q[cano]} \bigoplus_{s+t=n} p[s] \otimes q[t]
\]

and show that the bottom horizontal map is well defined.

Let $\sigma \in S_n$ be any permutation. For $S \sqcup T = [n]$, say $\sigma$ sends $S$ to $U$ and $T$ to $V$. This defines bijections $\sigma'_1 : S \to U$ and $\sigma'_2 : T \to V$. By standardizing the sets $S$, $T$, $U$ and $V$, we obtain two permutations $\sigma_1 \in S_s$ and $\sigma_2 \in S_t$, defined by the commutative diagrams

\[
\begin{array}{ccc}
S \xrightarrow{\text{cano}} [s] & \xrightarrow{\sigma'_1} & [s] \\
U \xrightarrow{\text{cano}} [s] & \xrightarrow{\sigma_1} & [s] \\
\downarrow & & \downarrow \\
T \xrightarrow{\text{cano}} [t] & \xrightarrow{\sigma'_2} & [t] \\
V \xrightarrow{\text{cano}} [t] & \xrightarrow{\sigma_2} & [t]
\end{array}
\]

where $s = |S| = |U|$ and $t = |T| = |V|$.

By functoriality, we then obtain a commutative diagram

\[
\begin{array}{ccc}
p[S] \otimes q[T] & \xrightarrow{p[cano] \otimes q[cano]} & p[s] \otimes q[t] \\
\downarrow & & \downarrow \\
p[S] \otimes q[t] & \xrightarrow{p[\sigma_1] \otimes q[\sigma_2]} & p[s] \otimes q[t] .
\end{array}
\]

This guarantees that $p[cano] \otimes q[cano]$ factors through coinvariants.

The argument for the map $\varphi$ is similar. From $\psi_{p,q} \varphi_{p,q} = \text{id}$ we deduce $\psi_{p,q} \varphi_{p,q} = \text{id}$. The map $\varphi_{p,q} \psi_{p,q}$ is given by permutation actions, so it induces the identity map on coinvariants. Thus, $\varphi$ and $\psi$ are inverses. \hfill $\square$
Theorem 15.3. The full Fock functor \((\mathcal{K}, \varphi, \psi)\) is bilax monoidal, the bosonic Fock functor \((\mathcal{K}, \varphi, \psi)\) is bistrong monoidal, and the transformation \(\mathcal{K} \Rightarrow \mathcal{K}\) is a morphism of bilax monoidal functors.

Proof. We start by showing that \((\mathcal{K}, \varphi)\) is lax and \((\mathcal{K}, \psi)\) is colax by checking that \(\varphi\) and \(\psi\) satisfy the conditions described in Definitions 3.1 and 3.2.

Naturality. Let \(f : p \to p'\) and \(g : q \to q'\) be morphisms of species. For the naturality of \(\psi\), the diagram

\[
\begin{array}{ccc}
p[S] \otimes q[T] & \xrightarrow{p[\text{cano}] \otimes q[\text{cano}]} & p[[S]] \otimes q[[T]] \\
fs \otimes gT & \downarrow & f[[S]] \otimes g[[T]] \\
p'[S] \otimes q'[T] & \xrightarrow{p'[\text{cano}] \otimes q'[\text{cano}]} & p'[[S]] \otimes q'[[T]]
\end{array}
\]

must commute. This follows from the naturality of \(f\) and \(g\). The argument for the naturality of \(\varphi\) is similar.

Associativity. If we follow the two directions in diagram (3.5) and its dual, the maps \(\varphi\) and \(\psi\) yield the following two unambiguous maps respectively

\[
p[S] \otimes q[t] \otimes r[u] \xrightarrow{p[\text{id}] \otimes q[\text{cano}] \otimes r[\text{cano}]} p[s] \otimes q[s + 1, s + t] \otimes r[s + t + 1, s + t + u]
\]

\[
p[S] \otimes q[T] \otimes r[U] \xrightarrow{p[\text{cano}] \otimes q[\text{cano}] \otimes r[\text{cano}]} p[[S]] \otimes q[[T]] \otimes r[[U]];
\]

hence they are associative.

Unitality. It is trivial to check that the unitality diagrams in (3.6) and their duals commute. All the maps in these diagrams are isomorphisms.

This shows that \((\mathcal{K}, \varphi)\) is lax and \((\mathcal{K}, \psi)\) is colax. We now check the braiding and unitality axioms in Definition 3.3.

Braiding. If we follow the two directions in diagram (3.11), the maps \(\varphi\) and \(\psi\) yield an unambiguous map

\[
\mathcal{K}(p \cdot q) \cdot \mathcal{K}(r \cdot s) \to \mathcal{K}(p \cdot r) \cdot \mathcal{K}(q \cdot s)
\]

defined as follows.

Let \(A, B, C\) and \(D\) be sets such that \(A \sqcup B = [m]\) and \(C \sqcup D = [n]\), and \(|A| = a, |B| = b, |C| = c\) and \(|D| = d\). The cano maps induce an isomorphism

\[
(p[A] \otimes q[B]) \otimes (r[C] \otimes s[D]) \to (p[a] \otimes r[a + 1, a + c]) \otimes (q[b] \otimes s[b + 1, b + d]).
\]

Varying \(A, B, C\) and \(D\), and then \(m\) and \(n\), and then taking direct sum gives the above map.

Unitality. It is trivial to check that the unitality diagrams in (3.12) and (3.13) commute. All the maps in these diagrams are isomorphisms.

This completes the proof that \((\mathcal{K}, \varphi, \psi)\) is bilax. The remaining claims follow immediately. The surjectivity of the vertical maps in diagram (15.1) and \((\mathcal{K}, \varphi, \psi)\) being bilax implies that \((\mathcal{K}, \varphi, \psi)\) is bistrong. Since \(\varphi\) and \(\psi\) are inverse isomorphisms, we moreover have that \((\mathcal{K}, \varphi, \psi)\) is bistrong. The fact that \(\mathcal{K} \Rightarrow \mathcal{K}\) is a morphism of bilax functors follows by construction. \(\square\)
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**Remark 15.4.** In Section 15.2 we employ the above result to construct graded Hopf algebras from Hopf monoids in \((Sp, \cdot)\). The functor \(K\) is not bistrong. Since it is normal (it satisfies \(\varphi_0 \psi_0 = \text{id}\)), it cannot be Hopf lax either, by Proposition 3.60. Nevertheless, the image of a Hopf monoid under \(K\) is always a graded Hopf algebra (Theorem 15.12).

15.1.2. The bilax monoidal functors \(K^\vee\) and \(K^\vee\). There are two other functors from species to graded vector spaces, namely the contragredients of \(K\) and \(\overline{K}\) (Section 3.10.4). We begin by describing them explicitly.

**Definition 15.5.** Let 
\[
K^\vee, K^\vee : Sp \to gVec
\]
be the functors defined by
\[
K^\vee(q) := \bigoplus_{n \geq 0} q[n] \quad \text{and} \quad K^\vee(q) := \bigoplus_{n \geq 0} q[n]^{S_n},
\]
where \(q[n]^{S_n}\) is the vector space of \(S_n\)-invariants of \(q[n]\).

The inclusion maps \(K^\vee(q) \hookrightarrow K^\vee(q)\) define a natural transformation of functors \(K^\vee \Rightarrow K^\vee\).

We proceed to turn \(K^\vee\) and \(K^\vee\) into bilax monoidal functors \((Sp, \cdot, \beta) \to (gVec, \cdot, \beta)\). For the bilax structure of \(K^\vee\), we define maps

\[
\psi_{p,q}^\vee : \bigoplus_{s+t=n} p[s] \otimes q[t] \longrightarrow \bigoplus_{S \cup T = [n]} p[S] \otimes q[T]
\]

as follows. The lax structure map \(\psi_{p,q}^\vee\) is the direct sum of the following maps, one for each \(s, t\) and each summand in the target with \(|S| = s\) and \(|T| = t\):

\[
p[s] \otimes q[t] \xrightarrow{\psi_{p,q}^\vee} p[s+1, s+t] \otimes q[cano] \rightarrow p[s] \otimes q[t]
\]

The colax structure map \(\varphi_{p,q}^\vee\) is the direct sum of the following maps:

\[
p[s] \otimes q[s+1, s+t] \xrightarrow{\varphi_{p,q}^\vee} p[cano] \otimes q[cano] \rightarrow p[s] \otimes q[t]
\]

On the components for which \(S \neq [s]\) (and \(T \neq [s+1, s+t]\)), the map \(\varphi_{p,q}^\vee\) is zero. Note that the composite \(\varphi_{p,q}^\vee \psi_{p,q}^\vee\) is the identity but in general these maps are not invertible.

The structure maps of \(K^\vee\) restrict to invariants, as indicated below.

\[
K^\vee(p) \cdot K^\vee(q) \xleftarrow{\psi_{p,q}^\vee} K^\vee(p \cdot q) \xrightarrow{\varphi_{p,q}^\vee} K^\vee(p) \cdot K^\vee(q)
\]

\[
K^\vee(p) \cdot K^\vee(q) \xleftarrow{\psi_{p,q}^\vee} K^\vee(p \cdot q) \xrightarrow{\varphi_{p,q}^\vee} K^\vee(p) \cdot K^\vee(q)
\]

**Theorem 15.6.** The full Fock functor \((K^\vee, \psi^\vee, \varphi^\vee)\) is bilax monoidal, the bosonic Fock functor \((K^\vee, \psi^\vee, \varphi^\vee)\) is bistrong monoidal, and the transformation \(K^\vee \Rightarrow K^\vee\) is a morphism of bilax monoidal functors.
The proof is similar to that of Theorem 15.3. As for \( \mathcal{K} \), the functor \( \mathcal{K}^\vee \) is not Hopf lax.

**Remark 15.7.** A species \( q \) being a functor \( \text{Set}^X \to \text{Vec} \) and the category \( \text{Vec} \) being complete and cocomplete, one may consider the limit and colimit of the functor \( q \) (Section A.3.5). We have

\[
\text{colim } q = \bigoplus_{n \geq 0} q[n]_{S_n} \quad \text{and} \quad \lim q = \prod_{n \geq 0} q[n]_{S_n}.
\]

The former is the graded vector space \( \mathcal{K}(q) \) while the latter is a completion of the graded vector space \( \mathcal{K}^\vee(q) \). This “explains” why the functors \( \mathcal{K}(q) \) and \( \mathcal{K}^\vee(q) \) are better behaved than their counterparts \( \mathcal{K} \) and \( \mathcal{K}^\vee \) (as we will see).

**15.1.3. Relating the Fock functors.** We now show that in the finite-dimensional case, \( \mathcal{K}^\vee \) and \( \mathcal{K}^\vee \) are indeed the (bilax) contragredients of \( \mathcal{K} \) and \( \mathcal{K} \). This construction is discussed in Proposition 3.102. Note that \( p^\vee \) stands for the lax structure and \( p^\vee \) stands for the colax structure of \( \mathcal{K} \), as per the general notation in the contragredient construction.

**Proposition 15.8.** On finite-dimensional species, the bilax functors \( (\mathcal{K}^\vee, \psi^\vee, \varphi^\vee) \) and \( (\mathcal{K}^\vee, \overline{\psi}^\vee, \overline{\varphi}^\vee) \) are respectively isomorphic to the contragredients of \( (\mathcal{K}, \varphi, \psi) \) and \( (\mathcal{K}, \overline{\varphi}, \overline{\psi}) \).

**Proof.** The contragredient of the functors \( \mathcal{K} \) and \( \mathcal{K} \) are the composites

\[
\begin{align*}
\text{Sp} & \xrightarrow{(-)^*} \text{Sp} \xrightarrow{\mathcal{K}} \text{gVec} \xrightarrow{(-)^*} \text{gVec} \\
\text{Sp} & \xrightarrow{(-)^*} \text{Sp} \xrightarrow{\mathcal{K}} \text{gVec} \xrightarrow{(-)^*} \text{gVec}
\end{align*}
\]

where the arrows labeled \((-)^*\) denote the duality functors on species and graded vector spaces. First note that there are canonical isomorphisms

\[
(15.4) \quad \mathcal{K}^\vee(q) \cong \mathcal{K}(q^*)^* \quad \text{and} \quad \mathcal{K}^\vee(q) \cong \mathcal{K}(q^*)^*.
\]

given by the canonical identification \( q[n] \cong (q[n]^*)^* \). Under this identification, one easily checks that

\[
\psi_{p,q}^\vee = (\psi_{p^*,q^*})^*, \quad \varphi_{p,q}^\vee = (\varphi_{p^*,q^*})^*, \quad \overline{\psi}_{p,q}^\vee = (\overline{\psi}_{p^*,q^*})^*, \quad \text{and} \quad \overline{\varphi}_{p,q}^\vee = (\overline{\varphi}_{p^*,q^*})^*.
\]

The right-hand sides are precisely the bilax structures of the contragredients (Proposition 3.102). The result follows. \( \square \)

Note that the functors \( \mathcal{K}^\vee \) and \( \mathcal{K} \) coincide; however, their bilax structures are defined differently. We will see later that, in fact, they cannot be isomorphic as bilax functors (Example 15.17). In general, in the finite-dimensional case, properties of the functors \( \mathcal{K}^\vee \) and \( \mathcal{K} \) can be derived from the corresponding properties of \( \mathcal{K} \) and \( \mathcal{K} \) by means of Proposition 15.8 (and vice versa). For example, Theorems 15.3 and 15.6 imply each other.

Now let \( L \) be the Hopf monoid of linear orders (Example 8.16). The functor \( \mathcal{K} \) can be expressed in terms of the functor \( \mathcal{K} \) and the Hopf monoid \( \mathcal{L} \). To see this, consider the composite of functors

\[
(15.5) \quad \text{Sp} \xrightarrow{L \times (-)} \text{Sp} \xrightarrow{\mathcal{K}} \text{gVec}.
\]
Since $\mathbf{L}$ is a bimonoid, the functor $\mathbf{L} \times (-)$ can be viewed as a bilax monoidal functor, as in Proposition 8.66. Now, since both the above functors are bilax, so is the composite, by Theorem 3.22.

**Proposition 15.9.** There is an isomorphism of bilax monoidal functors

\[
\mathcal{K} \cong \overline{\mathcal{K}}(\mathbf{L} \times (-)).
\]

**Proof.** Given a species $\mathbf{p}$, define a map of graded vector spaces

\[
\mathcal{K}(\mathbf{p}) \rightarrow \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{p})
\]

with components

\[
\mathbf{p}[n] \rightarrow (\mathbf{L}[n] \otimes \mathbf{p}[n])_{S_n}, \quad x \mapsto \overline{C_{(n)} \otimes x},
\]

where $C_{(n)} = 1 \cdots n$ is the canonical linear order on $[n]$ and the overline denotes the projection to coinvariants.

Since $\mathbf{L}[n]$ is the regular representation of $S_n$, by Lemma 2.18, this defines a natural isomorphism of functors. We need to show that it is a morphism of bilax monoidal functors. We check below that the colax structures are preserved; the verification for the lax structures is similar.

According to Theorem 3.22, the colax structure of $\overline{\mathcal{K}}(\mathbf{L} \times (-))$ is given by the composite

\[
\overline{\mathcal{K}}((\mathbf{L} \times \mathbf{p}) \cdot (\mathbf{L} \times \mathbf{q})) \xrightarrow{\overline{\psi}} \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{p}) \cdot \overline{\mathcal{K}}(\mathbf{L} \times \mathbf{q}).
\]

The map $\Delta$ is the coproduct of $\mathbf{L}$ as in Example 8.16 and the vertical map is $\overline{\mathcal{K}}$ applied to the colax structure of the Hadamard functor as in (8.73).

Take $x \in \mathbf{p}[S]$ and $y \in \mathbf{q}[T]$ with $S \sqcup T = [n]$. Applying the above sequence of maps to the element $C_{(n)} \otimes x \otimes y$, we obtain

\[
\overline{C_{(n)} \otimes x \otimes y} \mapsto \sum_{S_1 \cup T_1 = [n]} C_{(n)}|_{S_1} \otimes C_{(n)}|_{T_1} \otimes x \otimes y \\
\mapsto \overline{C_{(n)} \otimes x \otimes C_{(n)} \otimes y} \\
\mapsto \overline{C_{(n)} \otimes \text{cano}(x) \otimes C_{(t)} \otimes \text{cano}(y)}.
\]

This matches the colax structure $\psi$ of $\mathcal{K}$.

**Proposition 15.10.** There is an isomorphism of bilax monoidal functors

\[
\mathcal{K}^\vee \cong \overline{\mathcal{K}}^\vee(\mathbf{L}^* \times (-)).
\]

This may be proved directly, as in Proposition 15.9. In the finite-dimensional case, it also follows by applying the contragredient construction to (15.6):

\[
\mathcal{K}^\vee \cong \overline{\mathcal{K}}^\vee((\mathbf{L} \times (-))^\vee) \cong \overline{\mathcal{K}}^\vee(\mathbf{L}^* \times (-)).
\]

The last isomorphism follows as in (8.82).
Remark 15.11. There is an additional relation between the Fock functors: over a field of characteristic 0, the bilax monoidal functors $K$ and $K^\vee$ are isomorphic. We show this in Proposition 15.21. However, over a field of positive characteristic they differ. The bilax monoidal functors $K$ and $K^\vee$ differ regardless of the field characteristic. These claims are justified in Section 15.3. There is nevertheless a natural transformation $\kappa: K \Rightarrow K^\vee$. This is studied in Section 15.4, where we explain how one may view the functors $K$ and $K^\vee$ as the coimage and the image of this transformation (when the field characteristic is 0); see Section 15.4.3. From this point of view, the functors $K$ and $K^\vee$ are determined by $K$ and $K^\vee$. On the other hand, (15.6) and (15.7) say that the latter are determined by the former also.

15.2. From Hopf monoids to Hopf algebras: Stover’s constructions

In this section, we provide a categorical framework for Stover’s constructions. We then explain how the different constructions relate to one another. Some elementary but illustrative examples will be given in the next section.

15.2.1. Evaluating the Fock functors on Hopf monoids. We apply general results on bilax functors to the bilax functors in Section 15.1 to construct graded Hopf algebras starting with Hopf monoids in species.

Theorem 15.12. If $h$ is a Hopf monoid in species, then $K(h), K(h), K^\vee(h),$ and $K^\vee(h)$ are graded Hopf algebras. This defines four functors from the category of Hopf monoids in species to the category of graded Hopf algebras.

Proof. According to Theorems 15.3 and 15.6, the Fock functors are bilax monoidal. Hence, Proposition 3.31 implies that their values on $h$ are graded bialgebras, and that the assignments are functorial. We need to check that they are Hopf algebras. The statement for $K$ and $K^\vee$ follows from Proposition 3.50, since they are bistrong monoidal functors. The functors $K$ and $K^\vee$ are not bistrong, not even Hopf lax; so a separate argument is needed. We give it below for $K$; the same argument applies to $K^\vee$.

Since $K(h)$ is a graded bialgebra, it is enough to show that $K(h)_0$ is a Hopf algebra (Section 2.3.2). Note that $K(h)_0$ is a subbialgebra of $K(h)$. Further, by definition, the product and coproduct on $K(h)_0$ are the $\emptyset$-component of the product and coproduct of $h$. In other words, $K(h)_0=h[\emptyset]$ as bialgebras. But recall that for any Hopf monoid $h$, $h[\emptyset]$ is a Hopf algebra (Proposition 8.10), so we are done. □

Let $\mu, \iota, \Delta,$ and $\epsilon$ be the structure maps of a Hopf monoid $h$ in species. According to Proposition 3.31, the graded Hopf algebra $K(h)$ of Theorem 15.12 has structure maps

\[
\begin{array}{c}
K(h) \cdot K(h) \xrightarrow{\varphi_{h,h}} K(h \cdot h) \xrightarrow{K(\mu)} K(h) \\
\kappa \\
\end{array}
\begin{array}{c}
K(h) \xrightarrow{K(\Delta)} K(h \cdot h) \xrightarrow{\psi_{h,h}} K(h) \cdot K(h) \\
K(h) \xrightarrow{K(\iota)} K(1) \xrightarrow{\psi_0} \kappa.
\end{array}
\]
More explicitly, the component of degree $n$ of the coproduct of $\mathcal{K}(\mathfrak{h})$ is the composite
\[
\mathfrak{h}[n] \xrightarrow{\sum \Delta_{S,T}} \bigoplus_{S \cup T = [n]} \mathfrak{h}[S] \otimes \mathfrak{h}[T] \xrightarrow{\sum \mathfrak{h}[\text{cano}] \otimes \mathfrak{h}[\text{cano}]} \bigoplus_{s + t = n} \mathfrak{h}[s] \otimes \mathfrak{h}[t]
\]
and similarly for the other structure maps. Recall the notions of shifting and standardization from Notation 2.5. The above expression shows that the coproduct of $\mathcal{K}(\mathfrak{h})$ always involves the notion of standardization. Similarly, the product of $\mathcal{K}(\mathfrak{h})$ always involve the notions of shifting. Thus we obtain a conceptual explanation for the occurrence of these combinatorial procedures in the definition of several Hopf algebras of prominence in combinatorics, including all the Hopf algebras in [12, Theorem 6.1.3].

This construction of the graded bialgebra $\mathcal{K}(\mathfrak{h})$ appears for the first time in work of Stover [346, Proposition 14.6.i], without reference to monoidal functors. Similarly, the other bialgebras in [346, Proposition 14.6.ii-iv] are the images of $\mathfrak{h}$ under the bilax monoidal functors $\mathcal{K}^\vee$, $\mathcal{K}$, and $\mathcal{K}^{\vee}$, with the structure afforded by Proposition 3.31.

Stover’s constructions have been taken up by Patras and Reutenauer [291] and Patras and Schocker [292, 293]. In these references, $\mathcal{K}(\mathfrak{h})$ and $\mathcal{K}^\vee(\mathfrak{h})$ are called the cosymmetrized and symmetrized bialgebras associated to $\mathfrak{h}$, respectively [292, Definition 8 and Proposition 15]. Stover mentions that these constructions admit two versions, one with signs and the other without [346, Section 14.7]. Patras et al consider the unsigned version (as we do presently). The signed case will be dealt in Chapter 16.

If $s$ is the antipode of $\mathfrak{h}$, then $\mathcal{K}(s)$ and $\mathcal{K}^\vee(s)$ are the antipodes of $\mathcal{K}(\mathfrak{h})$ and $\mathcal{K}^\vee(\mathfrak{h})$, according to Proposition 3.50. On the other hand, since the functors $\mathcal{K}$ and $\mathcal{K}^\vee$ are not bistrong (not even Hopf lax; see Remark 15.4), the antipodes of $\mathcal{K}(\mathfrak{h})$ and $\mathcal{K}^\vee(\mathfrak{h})$ are not directly related to that of $\mathfrak{h}$. Determining the antipodes of these Hopf algebras in explicit terms is often a challenging problem. We do not address this problem in this monograph.

15.2.2. Relating the values of the Fock functors. We now discuss various relations between the Hopf algebras constructed in Theorem 15.12.

Recall that $\mathbf{L}$ denotes the Hopf monoid of linear orders. For any species $\mathfrak{h}$, there are canonical identifications
\[
\mathfrak{h}[n] \cong (\mathbf{L}[n] \otimes \mathfrak{h}[n])_{\mathfrak{s}_n} \cong (\mathbf{L}^*[n] \otimes \mathfrak{h}[n])_{\mathfrak{s}_n}.
\]
Recall that the Hadamard product of Hopf monoids is another Hopf monoid (Corollary 8.59).

**Theorem 15.13.** Let $\mathfrak{h}$ be a Hopf monoid in species. There are natural isomorphisms of graded Hopf algebras
\[
\mathcal{K}(\mathfrak{h}) \cong \mathcal{K}(\mathbf{L} \times \mathfrak{h}) \quad \text{and} \quad \mathcal{K}^\vee(\mathfrak{h}) \cong \mathcal{K}^\vee(\mathbf{L}^* \times \mathfrak{h})
\]
given by the identifications $(15.8)$. The maps
\[
\mathcal{K}(\mathfrak{h}) \to \mathcal{K}(\mathfrak{h}) \quad \text{and} \quad \mathcal{K}^\vee(\mathfrak{h}) \to \mathcal{K}^\vee(\mathfrak{h})
\]
are natural morphisms of graded Hopf algebras. Further, if $\mathfrak{h}$ is finite-dimensional, there are natural isomorphisms of graded Hopf algebras
\[
\mathcal{K}^\vee(\mathfrak{h}) \cong \mathcal{K}(\mathfrak{h}^*)^* \quad \text{and} \quad \mathcal{K}^\vee(\mathfrak{h}) \cong \mathcal{K}(\mathfrak{h}^*)^*
\]
given by the canonical identification $\mathbf{h}[n] \cong (\mathbf{h}[n])^*$. 

PROOF. According to Theorems 15.3 and 15.6 and Proposition 15.8, the maps 

$$\mathcal{K} \Rightarrow \mathcal{K}^\vee, \quad \mathcal{K}^\vee \Rightarrow \mathcal{K}((-)^*)^*$$

are morphisms of bilax monoidal functors. Therefore, Proposition 3.32 implies that (15.11) and (15.10) are isomorphisms of graded bialgebras (hence also of Hopf algebras). Similarly, applying Proposition 3.32 to the isomorphisms (15.6) and (15.7) yields that (15.9) are isomorphisms of graded Hopf algebras. 

Recall from Remark 15.11 that over a field of characteristic 0, the bilax monoidal functors $\mathcal{K}$ and $\mathcal{K}^\vee$ are isomorphic. It follows from (15.11) that in this case 

$$\mathcal{K}(\mathbf{h})^* \cong \mathcal{K}^\vee(\mathbf{h})^*$$

as graded Hopf algebras. We come back to this point in more detail in Section 15.4.4.

In view of (15.11), the study of the general properties of the Hopf algebras $\mathcal{K}(\mathbf{h})$ and $\mathcal{K}^\vee(\mathbf{h})$ can be reduced to that of the Hopf algebras $\mathcal{K}(\mathbf{h})$ and $\mathcal{K}(\mathbf{h})^*$ (and vice versa).

### 15.3. Values of Fock functors on particular Hopf monoids

We illustrate Theorems 15.12 and 15.13 on the examples of Section 8.5. The findings in these examples will prove claims made in Remark 15.11. More elaborate examples are given in Chapter 17.

**Example 15.14.** Let $k[x]$ and $k\{x\}$ be the polynomial and divided power Hopf algebras in the variable $x$ (Example 2.3). Let $E$ be the Hopf monoid of Example 8.15. We claim that 

$$\mathcal{K}(E) \cong \mathcal{K}^\vee(E) \cong k[x] \quad \text{and} \quad \mathcal{K}^\vee(E) \cong \mathcal{K}(E) \cong k\{x\}.$$ 

Since $E$ is one-dimensional in each degree, it is clear that $\mathcal{K}(E) \cong \mathcal{K}^\vee(E)$ and $\mathcal{K}(E) \cong \mathcal{K}(E)$. We alter notation for purposes of this example and denote the element $*_{[n]} \in E[n]$ by $x^n$. The claim involving $k[x]$ follows from the computation below.

$$\begin{array}{c}
\mathcal{K}(E) \cdot \mathcal{K}(E) \xrightarrow{\varphi} \mathcal{K}(E \cdot E) \xrightarrow{\mathcal{K}(\mu)} \mathcal{K}(E) \\
\left. x^s \otimes x^t \right\} \xrightarrow{\varphi} x^{[s]} \otimes x^{[s+1,s+t]} \xrightarrow{\left. x^s \otimes x^t \right\} \mu} x^{s+t} \\
\mathcal{K}(E) \xrightarrow{K(\Delta)} \mathcal{K}(E \cdot E) \xrightarrow{\psi} \mathcal{K}(E) \cdot \mathcal{K}(E) \\
\left. x^n \right\} \xrightarrow{\sum_{S \sqcup T = [n]} x^{|S|} \otimes x^{|T|}} \xrightarrow{\sum_{s+t=n} \binom{n}{s}} x^s \otimes x^t
\end{array}$$

The claim involving $k\{x\}$ follows from a similar computation to the above using the structure maps $\psi^\vee$ and $\varphi^\vee$. Alternatively, one can deduce it from the first claim by applying duality as below.

$$\mathcal{K}^\vee(E) \cong \mathcal{K}(E^*)^* \cong \mathcal{K}(E)^* \cong k[x]^* \cong k\{x\}$$

For the first equality, we used (15.11) and for the second equality, we used the self-duality of $E$ (Example 8.22).
Over a field of positive characteristic, the graded Hopf algebras
\[ \mathcal{K}(E) \cong \mathbb{k}[x] \quad \text{and} \quad \mathcal{K}^\vee(E) \cong \mathbb{k}\{x\} \]
are not isomorphic; therefore, the bilax monoidal functors \( \mathcal{K} \) and \( \mathcal{K}^\vee \) are not isomorphic. For more in this direction, see Section 15.4.3.

**Example 15.15.** Let \( E^2 \) be the Hopf monoid of subsets of Example 8.17. Calculations similar to those in Example 15.14 show that
\[ \mathcal{K}(E^2) \cong \mathbb{k}(x, y) \quad \text{and} \quad \mathcal{K}^\vee(E^2) \cong \mathbb{k}\{x, y\}. \]
These are polynomial algebras in two variables. The square brackets indicate that the variables commute and the angle brackets that they do not. The coproduct is determined by declaring that \( x \) and \( y \) are primitive. The first isomorphism is defined by using the following fact. A monomial of degree \( n \) in the noncommutative variables \( x \) and \( y \) corresponds to a decomposition \( (S, T) \) of \([n]\): the positions of \( x \) and \( y \) define the subsets \( S \) and \( T \) respectively.

The canonical map \( \mathcal{K}(E^2) \to \mathcal{K}^\vee(E^2) \) sends a polynomial in two noncommuting variables to the same polynomial (with the understanding that the variables now commute). It is a morphism of Hopf algebras.

**Example 15.16.** We now generalize the previous examples. Let \( E_V \) be the Hopf monoid of Example 8.18. Then
\[ \mathcal{K}(E_V) = \mathcal{T}(V), \quad \mathcal{K}^\vee(E_V) = \mathcal{S}(V), \quad \mathcal{K}^\vee(E_V) = \mathcal{T}^\vee(V), \quad \text{and} \quad \mathcal{K}^\vee(E_V) = \mathcal{S}^\vee(V), \]
where the right-hand sides include the tensor algebra, the shuffle algebra and the symmetric algebra of \( V \) (Section 2.6.1). These are to be viewed as graded Hopf algebras with \( V \) belonging to the degree 1 component.

**Example 15.17.** Let \( L \) and \( L^* \) be the Hopf monoids of linear orders of Examples 8.16 and 8.24. Calculations similar to those in Example 15.14 show that
\[ \mathcal{K}(L) \cong \mathcal{K}^\vee(L) \cong \mathbb{k}[x] \quad \text{and} \quad \mathcal{K}(L^*) \cong \mathcal{K}^\vee(L^*) \cong \mathbb{k}\{x\} \]
as graded Hopf algebras.

We now turn our attention to the functors \( \mathcal{K} \) and \( \mathcal{K}^\vee \). The Hopf monoid \( L \) and the associated Hopf algebra \( \mathcal{K}(L) \) are studied by Patras and Reutenauer [291, Section 6]. The Hopf algebra \( \mathcal{K}(L) \) is cocommutative but not commutative and is as follows. The degree \( n \) component of \( \mathcal{K}(L) \) has the set of linear orders on \([n]\) for a linear basis. To describe the product and coproduct explicitly, we setup a notation.

Let \( I \to J \) be a bijection between finite sets, and let \( l \) be a linear order on \( I \). Then \( L[I \to J](l) \) denotes the linear order on \( J \) obtained by transporting \( l \) from \( I \) to \( J \) by means of the given bijection. In the situations we consider, \( I \) and \( J \) will be subsets of the integers and \( I \to J \) will be the unique order-preserving bijection between them. Further, if \( J = [n] \), then it is convenient to write
\[ \text{std}(l) := L[I \to [n]](l) \in L[n]. \]
We refer to it as the **standardization** of \( l \). Similarly if \( I = [n] \), then it is convenient to write
\[ \text{sft}_J(l) := L[[n] \to J](l) \in L[J]. \]
We refer to it as the **shifting** of \( l \) to \( J \).
We can now describe the product and coproduct of $\mathcal{K}(L)$. The coproduct is, for $l \in L[n]$,
\[
\Delta(l) = \sum_{[n]=S\cup T} \text{std}(l|_S) \otimes \text{std}(l|_T),
\]
where, given $S \subseteq I$, $l|_S$ denotes the restriction of $l$ to $S$. For example,
\[
\Delta(1|3|2) = ( \oplus l \otimes 1|3|2 + 2(1 \otimes 1|2|) + 1 \otimes 2(1|2 \otimes 1) + 2|1 \otimes 1 + 1|3|2 \otimes (),
\]
where $( )$ stands for the empty list. It is the unique linear order on the empty set. The product of $l_1 \in L[s]$ and $l_2 \in L[t]$ is the linear order
\[
l_1 \ast l_2 = l_1 \cdot (\text{sft}_{[s+1,s+t]}(l_2)) \in L[s+t]
\]
where $\cdot$ denotes concatenation of linear orders. In other words, $l_1 \ast l_2$ is obtained by adding $s$ to each entry of $l_2$ and then placing it to the right of $l_1$. For example,
\[
1|3|2 \ast 2|1 = 1|3|2|5|4.
\]
Using (15.11), we obtain
\[
\mathcal{K}^\vee(L^*) \cong \mathcal{K}(L)^*.
\]
We now describe the product and coproduct of the dual explicitly. For a linear order $l$ on $[n]$, let $l^* \in L^*[n]$ denote the dual basis element. Let $l_1$ and $l_2$ be linear orders on $[s]$ and $[t]$ respectively. Then the product is given by
\[
l_1^* \ast l_2^* = \sum_{[n]=S\cup T} \sum l^*,
\]
where the first sum is over all decompositions with $|S| = s$ and $|T| = t$, and the second sum is over all shuffles $l$ of the linear orders $\text{sft}_S(l_1)$ and $\text{sft}_T(l_2)$ on $S$ and $T$ respectively. The coproduct is given by
\[
\Delta(l^*) = \sum \text{std}(l^1| \cdots |l^s)^* \otimes \text{std}(l^{s+1}| \cdots |l^n)^*,
\]
where $l = l^1| \cdots |l^n \in L[n]$ and the sum is over those $s$ for which $l^1, \ldots, l^s$ are all less than $l^{s+1}, \ldots, l^n$. In other words, the sum is over all positions $s$ at which $l$ has a global ascent. Global ascents of $l$ correspond to global descents of the reverse linear order $\bar{l}$. The latter are defined in Section 10.7.1. It follows directly that the space of primitive elements is spanned by those linear orders which have no global ascents. For example,
\[
1^* \ast 2|1^* = (1|3|2^* + 3|1|2^* + 3|2|1^*) + (2|3|1^* + 3|2|1^* + 3|1|2^*) + (3|2|1^* + 2|3|1^* + 2|1|3^*)
\]
\[
= 1|3|2^* + 2(2|3|1^*) + 2|1|3^* + 2(3|1|2^*) + 3(3|2|1^*).
\]
\[
\Delta(1|3|2|5|4^*) = ( \oplus l \otimes 1|3|2|5|4^* + 1|3|2^* \otimes 2|1^* + 1|3|2|5|4^* \otimes ().
\]
On the other hand, we have
\[
\mathcal{K}(L^*) \cong SA \quad \text{and} \quad \mathcal{K}^\vee(L) \cong SA^* \cong SA,
\]
where $SA$ is the graded Hopf algebra of permutations of Malvenuto and Reutenauer [255, 256]. This Hopf algebra is self-dual, free and cofree, and neither commutative nor cocommutative. The fact that $\mathcal{K}(L^*) \cong SA$ was first pointed out by Patras and Reutenauer [291, Proposition 16]; the second fact then follows by (15.11) and self-duality. The self-duality appears in [255, Section 5.2] and [256, Theorem 3.3]. The freeness was established by Poirier and Reutenauer [297]. For related ideas, see the works of Reutenauer [311], Patras and Reutenauer [290], Loday and
Ronco [242, 243], Duchamp, Hivert and Thibon [106, 107], Foissy [131] and Aguiar and Sottile [13, 14, 15]. We now proceed to describe $S\Lambda$. Write $F_l$ for the basis element of $K(L^*)$ corresponding to $l^* \in L^*[n]$. The coproduct is

$$\Delta(F_l) = \sum_{s=0}^{n} F_{\text{std}}(l_{s}^{1}|\cdots|l_{s}^{r}) \otimes F_{\text{std}}(l_{s+1}^{1}|\cdots|l_{n}),$$

where $l = l^{1}|\cdots|l^{n} \in L[n]$, and the product is

$$F_{l_1} \ast F_{l_2} = \sum_{l} F_{l},$$

where $l_1 \in L[s], l_2 \in L[t]$, and the sum is over all linear orders $l \in L[s+t]$ obtained by adding $s$ to the entries of $l_2$ and then shuffling them with the entries of $l_1$. For example,

$$\Delta(F_{1|3|4|2}) = F_{()} \otimes F_{1|3|4|2} + F_{1} \otimes F_{2|3|1} + F_{1|2} \otimes F_{2|1} + F_{1|2|3} \otimes F_{1} + F_{1|3|4|2} \otimes F_{()}$$

and

$$F_{2|1} \ast F_{1|2} = F_{2|1|3|4} + F_{2|3|1|4} + F_{2|3|4|1} + F_{3|2|1|4} + F_{3|2|4|1} + F_{3|4|2|1}. $$

The primitive elements of $S\Lambda$ are harder to compute. The dimension of this space in degree $n$ is given by the number of permutations on $n$ letters with no global descents [14, 107, 297]. Thus, the dimension of the space of primitive elements of $K(L^*)$ and $K^\vee(L^*)$ is the same.

In view of (15.9), we have

$$K(L \times L^*) \cong K(L^*) \cong S\Lambda.$$

Thus, $S\Lambda$ can be viewed as the image of the Hopf monoid $L \times L^*$ under the functor $K$. This point of view is useful in light of the nice properties of $K$. For instance, the self-duality of $L \times L^*$ implies that of $S\Lambda$. These ideas are explained in more detail in Section 17.2.

The Hopf monoid $L$, as well as the associated Hopf algebra $K(L)$, are cocommutative. We show in Section 15.5 that, in general, the functor $K$ preserves cocommutativity. On the other hand, the Hopf monoid $L^*$ is commutative but the Hopf algebra $K(L^*)$ is not. Hence $K$ does not preserve commutativity. Exactly the reverse is true of $K^\vee$.

In addition, note that $K(L)$ and $K(L^*)$ cannot be dual Hopf algebras, since the first is cocommutative whereas the second in not commutative. A similar statement applies to $K^\vee$. Hence $K$ and $K^\vee$ do not preserve duality. This is further studied in Section 15.4.

The Hopf algebras $K(L)$ and $K^\vee(L)$ are not isomorphic, since the former is cocommutative, while the latter is not. Therefore, the bilax monoidal functors $K$ and $K^\vee$ are not isomorphic.

Recall the morphism of Hopf monoids $\pi^*: E^* \to L^*$ of (8.33). Its image under $K$ is the morphism of graded Hopf algebras

$$k[x] \to S\Lambda \quad \text{given by} \quad x^n \mapsto \sum_{l \in L[n]} F_{l}.$$
The canonical map $\mathcal{K}(L^*) \to \mathcal{K}^v(L^*)$ turns out to be the dual morphism of graded Hopf algebras

$$S\Lambda \to \mathbb{k}\{x\}.$$ 

15. FROM SPECIES TO GRADED VECTOR SPACES

15.4. The norm transformation between full Fock functors

In this section, we study a morphism between the full Fock functors $\mathcal{K}$ and $\mathcal{K}^\vee$. It is called the norm transformation and denoted $\kappa$. We explain how the bosonic Fock functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$ can be interpreted as the coimage and image of this morphism. Finally, we apply this circle of ideas to answer the questions on whether the Fock functors preserve duality.

15.4.1. Relating the structure transformations of the full Fock functors. Let $\text{Sh}(s,t)$ denote the set of $(s,t)$-shuffle permutations (2.21).

**Lemma 15.18.** The structure maps $\varphi$ and $\psi^\vee$, and $\psi$ and $\varphi^\vee$, are related by the formulas

\begin{align}
\psi^\vee(x \otimes y) &= \sum_{\zeta \in \text{Sh}(s,t)} \zeta(\varphi(x \otimes y)) \quad \text{for } x \in p[s], y \in q[t], \\
\psi(a \otimes b) &= \sum_{\zeta \in \text{Sh}(|S|,|T|)} \varphi^\vee(\zeta^{-1}(a \otimes b)) \quad \text{for } a \in p[S], b \in q[T].
\end{align}

In particular,

$$\psi^\vee(x \otimes y) = \varphi(x \otimes y) \quad \text{if } s = 0 \text{ or } t = 0,$$

$$\psi(a \otimes b) = \varphi^\vee(a \otimes b) \quad \text{if } S = \emptyset \text{ or } T = \emptyset.$$ 

**Proof.** We explain (15.13). Let $s = |S|$ and $t = |T|$. Among all $(s,t)$-shuffle permutations there is one such that $\zeta([s]) = S$ and $\zeta([s+1, s+t]) = T$. For this shuffle $\zeta$ we have

$$\varphi^\vee(\zeta^{-1}(a \otimes b)) = \psi(a \otimes b);$$

for all other shuffles $\zeta$ we have $\varphi^\vee(\zeta^{-1}(a \otimes b)) = 0$. \qed

15.4.2. The norm transformation between full Fock functors. We now define the norm transformation from $\mathcal{K}$ to $\mathcal{K}^\vee$ and show that it is a morphism of bilax monoidal functors. We then discuss the induced transformation between the functors $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}}^\vee$.

**Definition 15.19.** For any species $p$, let $\kappa_p : \mathcal{K}(p) \to \mathcal{K}^\vee(p)$ be the map of graded vector spaces given by

\begin{equation}
p[n] \to p[n] \quad \kappa_p(z) := \sum_{\sigma \in S_n} \sigma \cdot z,
\end{equation}

for any $z \in p[n]$. This defines a natural transformation $\kappa : \mathcal{K} \Rightarrow \mathcal{K}^\vee$ which we call the norm.

Thus, the degree $n$ component of $\kappa_p$ is the norm map $N_{p[n]}$ of Section 2.5. The naturality of $\kappa$ follows from that of $N$.

**Proposition 15.20.** The norm is a morphism of bilax monoidal functors

$$\kappa : \mathcal{K} \Rightarrow \mathcal{K}^\vee.$$
Proof. We first verify that norm is a morphism of colax functors. The diagrams in (3.15) are in this case

\[
\begin{array}{c}
\mathcal{K}(p \cdot q) \xrightarrow{\psi_{p,q}} \mathcal{K}(p) \cdot \mathcal{K}(q) \\
\kappa_{p,q} \downarrow \quad \kappa_{p,q} \\
\mathcal{K}^\vee(p \cdot q) \xrightarrow{\varphi_{p,q}} \mathcal{K}^\vee(p) \cdot \mathcal{K}^\vee(q)
\end{array}
\text{and}
\begin{array}{c}
\mathcal{K}(1) \xrightarrow{\psi_0} 1 \\
\kappa_1 \downarrow \quad \kappa_1 \\
\mathcal{K}^\vee(1) \xrightarrow{\varphi_0^\vee} 1
\end{array}
\]

The diagram on the right commutes trivially. For the diagram on the left, fix a decomposition \([n] = S \sqcup T\) and take \(a \otimes b \in p[S] \otimes q[T]\). Let \(s = |S|\) and \(t = |T|\). Using (15.13) and the naturality of \(\varphi^\vee\) we find

\[
(k_{p} \cdot k_{q}) \psi_{p,q}(a \otimes b) = \sum_{\sigma \in S_n, \tau \in Sh(s,t)} \sum_{\zeta \in Sh(s,t)} \varphi^\vee_{p,q}(\sigma \times \tau) \cdot \zeta^{-1} \cdot (a \otimes b).
\]

Taking inverses in (2.22) and replacing \(\sigma, \tau\) and \(\rho\) by their inverses, we deduce

\[
(k_{p} \cdot k_{q}) \psi_{p,q}(a \otimes b) = \sum_{\rho \in S_n} \varphi^\vee_{p,q}(\rho \cdot (a \otimes b))
\]

\[
= \varphi^\vee_{p,q} k_{p,q}(a \otimes b).
\]

Thus, the diagram on the left commutes and \(k\) is a morphism of colax functors. The proof can be summarized in the following commutative diagram

\[
\begin{array}{c}
p[S] \otimes q[T] \xrightarrow{p[c ano] \otimes q[c ano]} p[s] \otimes q[t] \\
p[\rho|S] \otimes q[\rho|T] \downarrow \quad \downarrow p[\rho|S] \otimes q[\rho|T]
\end{array}
\]

where \(\rho = (\sigma \times \tau) \cdot \zeta^{-1}\) and \(\zeta\) is the unique \((s,t)\)-shuffle permutation which sends \([s]\) to \(S\) and \([s + 1, s + t]\) to \(T\).

Similarly, using (15.12), one can show that \(k\) is a morphism of lax functors. On finite-dimensional species, this can be deduced from the above result plus self-duality of \(k\) (15.17), using Proposition 3.102.

The image of the norm map \(k_p : \mathcal{K}(p) \to \mathcal{K}^\vee(p)\) consists of invariant elements, so it is contained in \(\mathcal{K}^\vee\). Since the inclusion \(\mathcal{K}^\vee \hookrightarrow \mathcal{K}^\vee\) is a morphism of bilax monoidal functors, so is the resulting transformation

\[
\mathcal{K} \Rightarrow \bar{\mathcal{K}}^\vee.
\]

This transformation factors through coinvariants, giving rise to another morphism of bilax monoidal functors

\[
\bar{\pi} : \mathcal{K} \Rightarrow \bar{\mathcal{K}}^\vee.
\]
These fit in a commutative diagram as follows.

\[
\begin{array}{ccc}
\overline{\mathcal{K}} & \xrightarrow{\kappa} & \overline{\mathcal{K}}^\vee \\
\downarrow & & \downarrow \\
\overline{\mathcal{K}} & \xrightarrow{\pi} & \overline{\mathcal{K}}^\vee \\
\end{array}
\]

(15.15)

If the field characteristic is 0, then one can say more. Applying Lemma 2.20 we obtain:

**Proposition 15.21.** The morphism of bistrong monoidal functors

\[\overline{\pi}: \overline{\mathcal{K}} \to \overline{\mathcal{K}}^\vee\]

is an isomorphism if the field characteristic is 0. In addition, regardless of the field characteristic, if the species \(p\) consists of flat \(kS_n\)-modules \(p[n]\), then

\[\overline{\pi}_p: \overline{\mathcal{K}}(p) \to \overline{\mathcal{K}}^\vee(p)\]

is bijective.

Applying Proposition 3.32 we obtain that for any Hopf monoid \(h\), the diagram of graded Hopf algebras

\[
\begin{array}{ccc}
\mathcal{K}(h) & \xrightarrow{\kappa_h} & \mathcal{K}^\vee(h) \\
\downarrow & & \downarrow \\
\overline{\mathcal{K}}(h) & \xrightarrow{\pi_h} & \overline{\mathcal{K}}^\vee(h) \\
\end{array}
\]

(15.16)

commutes. In addition:

**Corollary 15.22.** If the species \(h\) consists of flat \(kS_n\)-modules \(h[n]\), then

\[\overline{\pi}_h: \overline{\mathcal{K}}(h) \to \overline{\mathcal{K}}^\vee(h)\]

is an isomorphism of graded Hopf algebras. This holds if the field characteristic is 0, for any \(h\).

**Example 15.23.** For the Hopf monoid \(E\) of Examples 8.15 and 15.14, we have that \(\kappa_E: \mathcal{K}(E) \to \mathcal{K}^\vee(E)\) is the map (2.11). It is an isomorphism of Hopf algebras in characteristic 0. More generally, for the Hopf monoid \(E_V\) of Examples 8.18 and 15.16, diagram (15.16) specializes to (2.66).

Suppose now that the species \(p\) is finite-dimensional. It follows from Proposition 15.8 that \(\kappa\) is related to its contragredient (3.47) as follows.

\[
\begin{array}{ccc}
\mathcal{K}^\vee(p)^* & \xrightarrow{(\kappa_p)^*} & \mathcal{K}(p)^* \\
\overline{\mathcal{K}}(p^*) & \xrightarrow{\pi_p} & \overline{\mathcal{K}}^\vee(p^*) \\
\end{array}
\]

(15.17)

This means that the norm transformation is self-dual (Definition 3.108). The same property holds for \(\overline{\pi}\). More generally, Lemma 2.22 yields:

**Proposition 15.24.** On finite-dimensional species, diagram (15.15) is self-dual.
15.4.3. The image of the norm. Let $\mathcal{F}$ denote the (co)image of the norm transformation $\kappa: \mathcal{K} \Rightarrow \mathcal{K}^\vee$, in the sense of Section 3.11. It is a bilax monoidal functor

$$\mathcal{F}: (\text{Sp}, \cdot, \beta) \to (\text{gVec}, \cdot, \beta).$$

Proposition 15.21 implies that, in characteristic 0, $\mathcal{K}$, $\mathcal{K}^\vee$, and $\mathcal{F}$ are isomorphic bistrong monoidal functors. Thus, in this situation the bilax monoidal functors $\mathcal{K}^\vee$ and $\mathcal{F}$ are naturally associated to the morphism $\kappa$ (they are the image and coimage of $\kappa$, see Remark 3.117).

In general, $\mathcal{K}$, $\mathcal{K}^\vee$, and $\mathcal{F}$ are three distinct bistrong monoidal functors related by morphisms of bistrong functors

$$\mathcal{K} \Rightarrow \mathcal{F} \Rightarrow \mathcal{K}^\vee.$$

The fact that $\mathcal{F}$ is bistrong follows from the fact that the first natural transformation is onto, or from the fact that the second one is into. The connection between all five functors is as in the following diagram.

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\kappa} & \mathcal{K}^\vee \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{\kappa} & \mathcal{K}^\vee
\end{array}
\]

On finite-dimensional species, this diagram is self-dual. In particular, $\mathcal{F}$ is a self-dual functor (regardless of the characteristic). This can be seen as a consequence of Proposition 3.119.

The distinction between $\mathcal{K}$ and $\mathcal{K}^\vee$ is illustrated in Example 15.23. It follows that, if the characteristic of $\mathbb{k}$ is $p$, then

$$\mathcal{F}(E) = \mathbb{k}[x]/(x^p).$$

Since a self-dual functor preserves self-dual objects (Proposition 3.107), it follows that $\mathcal{F}(E)$ is a self-dual Hopf algebra. This was noted in Example 2.3.

15.4.4. The Fock functors and duality. Consider the question of whether the functors $\mathcal{K}$ and $\mathcal{K}$ preserve duality of Hopf monoids. We know from Example 15.17 that for $\mathcal{K}$ the answer is negative. On the other hand, the functors $\mathcal{K}$ and $\mathcal{K}$ are related through duality to the functors $\mathcal{K}^\vee$ and $\mathcal{K}^\vee$, as given in Proposition 15.8. This may be rewritten as follows: if $p$ is a finite-dimensional species, then

$$\mathcal{K}(p^\ast) \cong \mathcal{K}^\vee(p)^\ast \quad \text{and} \quad \mathcal{K}(p^\ast) \cong \mathcal{K}^\vee(p)^\ast.$$

Therefore, the above question is closely related to whether $\mathcal{K}$ and $\mathcal{K}$, and $\mathcal{K}$ and $\mathcal{K}$, are isomorphic as bilax monoidal functors. This is a point we addressed in Section 15.4.2. When expressed in terms of duality, the answers take the following form.

Suppose $h$ is a finite-dimensional Hopf monoid. By applying (15.16) to $h^\ast$ together with the isomorphisms in (15.11), we obtain the commutative diagrams of
graded Hopf algebras below.

\[
\begin{align*}
\mathcal{K}(\mathbf{h}^*) &\xrightarrow{\kappa_{\mathbf{h}^*}} \mathcal{K}(\mathbf{h})^* \\
\mathcal{K}(\mathbf{h}^*) &\xrightarrow{\pi_{\mathbf{h}^*}} \mathcal{K}(\mathbf{h})^* \\
\mathcal{K}^\vee(\mathbf{h})^* &\xrightarrow{\kappa_{\mathbf{h}^*}} \mathcal{K}^\vee(\mathbf{h}^*) \\
\mathcal{K}^\vee(\mathbf{h})^* &\xrightarrow{\pi_{\mathbf{h}^*}} \mathcal{K}^\vee(\mathbf{h})^*
\end{align*}
\]

(The two diagrams are the same.) By applying Corollary 15.22 to \(\mathbf{h}^*\) we obtain:

**Corollary 15.25.** Let \(\mathbf{h}\) be a finite-dimensional Hopf monoid. If \(\mathbf{h}^*\) consists of flat \(kS_n\)-modules \(\mathbf{h}[n]^*\), then \(\pi_{\mathbf{h}^*}\) is an isomorphism of graded Hopf algebras. If in addition \(\mathbf{h}\) is a self-dual Hopf monoid, then \(\mathcal{K}(\mathbf{h})\) is a self-dual graded Hopf algebra.

Recall that over a field characteristic is 0 any \(S_n\)-module is flat.

### 15.5. The Fock functors and commutativity

In this section, we discuss whether the Fock functors preserve commutative monoids or cocommutative comonoids.

**15.5.1. Are the Fock functors braided?** In light of the discussion in Section 3.4.4, one essentially has to study whether \(\mathcal{K}\) and \(\mathcal{K}^\vee\) are braided viewed both as lax and colax functors.

**Proposition 15.26.** The functor \((\mathcal{K}, \psi)\) is braided colax, but the functor \((\mathcal{K}, \varphi)\) is not braided lax. On the other hand, the functor \((\mathcal{K}, \varphi, \psi)\) is braided bilax.

**Proof.** For the assertions about \(\mathcal{K}\), we have to show that the left-hand diagram below commutes while the right-hand diagram does not.

\[
\begin{align*}
\mathcal{K}(p \cdot q) &\xrightarrow{\psi_{p,q}} \mathcal{K}(p) \cdot \mathcal{K}(q) \\
\mathcal{K}(q \cdot p) &\xrightarrow{\psi_{q,p}} \mathcal{K}(q) \cdot \mathcal{K}(p)
\end{align*}
\]

We look at the degree \(n\) part of the above diagram. The relevant portion is shown below.

\[
\begin{align*}
p[S] \otimes q[T] &\xrightarrow{p[\text{cano}] \otimes q[\text{cano}]} p[s] \otimes q[t] \\
&\xrightarrow{p[\text{id}] \otimes q[\text{cano}]} p[s] \otimes q[s + 1, s + t]
\end{align*}
\]

The first diagram clearly commutes. The second diagram does not commute because following the two directions land us in

\[
q[t] \otimes p[t + 1, t + s] \quad \text{and} \quad q[s + 1, s + t] \otimes p[s],
\]

which are distinct components (unless \(s\) or \(t\) is zero).
This problem disappears for $\varphi$ because there is an element of $S_{s+t}$ which induces an isomorphism between the two components above, so commutativity is attained at the level of coinvariants. In other words, $\bar{K}$ is braided bilax. Alternatively, this can be deduced by noting that $\bar{K}$ is bistrong and applying Proposition 3.46. A third proof is given in Proposition 15.31.

Propositions 3.35, 3.36 and 3.37 yield:

**Corollary 15.27.** For any comonoid (Hopf monoid) $h$,

$$K(h^{\text{cop}}) = K(h)^{\text{cop}}$$

as comonoids (Hopf monoids). In particular, $K$ takes cocommutative comonoids to cocommutative coalgebras. The functor $\bar{K}$ preserves both commutativity and cocommutativity.

This has immediate implications for the contragredients $K^\vee$ and $K^\vee$. By Propositions 3.102 and 15.8:

**Proposition 15.28.** The functor $(K^\vee, \psi^\vee)$ is braided lax, but the functor $(K^\vee, \varphi^\vee)$ is not braided colax. On the other hand, the functor $(\bar{K}^\vee, \bar{\psi}^\vee, \bar{\varphi}^\vee)$ is braided bilax.

Hence $K^\vee$ takes commutative monoids to commutative algebras and $\bar{K}^\vee$ preserves both commutativity and cocommutativity. Proposition 4.13 gives that $K^\vee$ takes Lie monoids to graded Lie algebras. Similarly, $K$ takes Lie comonoids to graded Lie coalgebras, and $\bar{K}$ and $\bar{K}^\vee$ preserve both Lie monoids and Lie comonoids.

**15.5.2. The half-twist transformation from $K$ to itself.** Let $^b\varphi$ and $^b\psi$ denote the conjugates of $\varphi$ and $\psi$ as in Definition 3.14. Since the braidings are symmetries in the present case, the side on which the exponent $b$ is written does not matter.

We saw that the colax monoidal functor $(K, \psi)$ is braided, so $^b\psi = \psi$. On the other hand, the functor $K$ is not braided lax. So it may not take commutative monoids to commutative algebras. An example of this kind was given in Example 15.17. The fact that $K$ does not preserve commutativity can be stated formally by saying that the identity natural transformation

$$(K, ^b\varphi) \Rightarrow (K, \varphi)$$

is not a morphism of lax monoidal functors (3.17). For the same reason, given a monoid $p$, the identity map

$$K(p^{\text{op}}) \to K(p)^{\text{op}}$$

need not be a morphism of graded algebras. This opens the possibility of there being two distinct algebras $K(p^{\text{op}})$ and $K(p)^{\text{op}}$ associated to $p$. However, this is not the case: it turns out that there is a nontrivial isomorphism of algebras

$$K(p^{\text{op}}) \cong K(p)^{\text{op}}.$$

We explain this remarkable fact next.

**Definition 15.29.** For each $n$, let $\omega_n$ be the longest permutation in $S_n$. It sends $i$ to $n+1-i$ for each $i$. Let $\theta: K \Rightarrow K$ be the natural transformation defined by the maps

$$p[\omega_n]: p[n] \to p[n]$$
for each species \( p \) and each nonnegative integer \( n \). For \( n = 0, 1 \), this map is the identity.

The definition of morphism of species (8.1) guarantees that \( \theta \) is indeed a natural transformation. We call it the **half-twist transformation**.

**Proposition 15.30.** The half-twist transformation is an isomorphism of bilax monoidal functors

\[
\theta : (\mathcal{K}, b_\varphi, b_\psi) \Rightarrow (\mathcal{K}, \varphi, \psi).
\]

**Proof.** We first check that \( \theta \) is a morphism of colax monoidal functors. The second diagram in (3.15) commutes trivially while the first diagram takes the following form.

\[
\begin{array}{cccc}
\mathcal{K}(p \cdot q) & \xrightarrow{\beta^{-1}} & \mathcal{K}(q \cdot p) & \xrightarrow{\psi_{q,p}} & \mathcal{K}(q) \cdot \mathcal{K}(p) & \xrightarrow{\beta} & \mathcal{K}(p) \cdot \mathcal{K}(q) \\
\theta_{p \cdot q} & & \theta_{p \cdot q} & & \theta_{p \cdot q} & & \theta_{p \cdot q} \\
\mathcal{K}(p \cdot q) & & \mathcal{K}(p) \cdot \mathcal{K}(q). & & \\
\end{array}
\]

The commutativity of this diagram boils down to the following diagram, where \( S \sqcup T = [n] \) is a decomposition and \( S' = \omega_n(S), \, T' = \omega_n(T) \).

\[
\begin{array}{ccc}
p[S] \otimes q[T] & \xrightarrow{\beta^{-1}} & q[T] \otimes p[S] \\
(p \cdot q)[\omega_n] & & q[t] \otimes p[s] \xrightarrow{\beta} p[s] \otimes q[t] \\
p[S'] \otimes q[T'] & \xrightarrow{\omega_s \times \omega_t} & p[s] \otimes q[t] \\
\end{array}
\]

The composite along the top is \( p[\text{cano}] \otimes q[\text{cano}] \) (as encountered in the proof of Proposition 15.26). The commutativity of this diagram follows by functoriality from that of

\[
\begin{array}{ccc}
S \times T & \xrightarrow{\text{cano} \times \text{cano}} & [s] \times [t] \\
\omega_n|_S \times \omega_n|_T & & \omega_s \times \omega_t \\
S' \times T' & \xrightarrow{\text{cano} \times \text{cano}} & [s] \times [t].
\end{array}
\]

Similarly, to check that \( \theta \) is a morphism of lax monoidal functors, one needs to check the commutativity of the following diagram.

\[
\begin{array}{cccc}
\mathcal{K}(p) \cdot \mathcal{K}(q) & \xrightarrow{\beta^{-1}} & \mathcal{K}(q) \cdot \mathcal{K}(p) & \xrightarrow{\varphi_{q,p}} & \mathcal{K}(q \cdot p) & \xrightarrow{\beta} & \mathcal{K}(p \cdot q) \\
\theta_{p \cdot q} & & \theta_{p \cdot q} & & \theta_{p \cdot q} & & \theta_{p \cdot q} \\
\mathcal{K}(p) \cdot \mathcal{K}(q) & & \mathcal{K}(p) \cdot \mathcal{K}(q). \\
\end{array}
\]
This follows from that of
\[
\begin{array}{c}
[s] \times [t] \xrightarrow{\text{cano} \times \text{cano}} [t+1, t+s] \times [t] \\
\omega_s \times \omega_t \downarrow \downarrow \\
[s] \times [t] \xrightarrow{\text{cano} \times \text{cano}} [s+1, s+t].
\end{array}
\]

\[\square\]

**Proposition 15.31.** The following is a commutative diagram of morphisms of bilax functors.

\[
\begin{array}{ccc}
(\mathcal{K}, b_\varphi, b_\psi) & \xrightarrow{\theta} & (\mathcal{K}, \varphi, \psi) \\
\downarrow \downarrow & & \downarrow \downarrow \\
(\mathcal{K}, b_\varphi, b_\psi) & \xrightarrow{id} & (\mathcal{K}, \varphi, \psi)
\end{array}
\]

In particular, \((\mathcal{K}, \varphi, \psi)\) is braided bilax.

**Proof.** We first note that the above is a commutative diagram of natural transformations. In other words, \(\theta\) factors through the projection \(\mathcal{K} \Rightarrow \mathcal{\overline{K}}\) and gives rise to the identity natural transformation on \(\mathcal{\overline{K}}\). It then follows from Proposition 15.30 that the identity is a morphism of bilax functors and further that the diagram commutes as morphisms of bilax functors. \(\square\)

**Corollary 15.32.** For any Hopf monoid \(h\), the map

\[
\mathcal{K}(h^{\text{op}}) \rightarrow \mathcal{K}(h)^{\text{op}}
\]

whose degree \(n\) component is \(h[\omega_n]\) (Definition 15.29) is a natural isomorphism of Hopf algebras.

**Proof.** Propositions 15.30, 3.32, and 3.34 imply that

\[
\theta_h : (\mathcal{K}(h)^{\text{op}})^{\text{cop}} \rightarrow \mathcal{K}(h)
\]

is an isomorphism of Hopf algebras. By applying \((-)^{\text{op},\text{cop}}\) (which is the inverse to \(\text{op,\text{cop}}(-)\)), we obtain that

\[
\mathcal{K}(h^{\text{op},\text{cop}}) \rightarrow \mathcal{K}(h)^{\text{op},\text{cop}}
\]

is an isomorphism of Hopf algebras. The result now follows by replacing \(h\) by \(h^{\text{cop}}\), and using Corollary 15.27 and the fact that both braidings are symmetries. \(\square\)

**Example 15.33.** Consider the Hopf monoid \(L^*\) of Example 8.24. As explained in Example 15.17, we have \(\mathcal{K}(L^*) = S\Lambda\). Since \(L^*\) is commutative, we obtain an isomorphism of Hopf algebras

\[
SA \rightarrow S\Lambda^{\text{op}}
\]

given by \(F_{l_1|\ldots|l_n} \mapsto F_{n+1-l_1|\ldots|n+1-l_n}\).

For example, \(F_{2|1|4|3} \mapsto F_{3|4|1|2}\).

**Example 15.34.** Consider the Hopf monoid \(\Sigma^*\) discussed in Section 12.4. Applying the functor \(\mathcal{K}\) yields a Hopf algebra indexed by set compositions. This is the Hopf algebra \(P\Pi\) considered in [12, Section 6.2.4]. More information regarding this is given in Section 17.3.
The Hopf monoid $\Sigma^*$ is commutative while the Hopf algebra $\mathcal{K}(\Sigma^*)$ is not. Hence, we obtain an isomorphism of Hopf algebras

$$\mathcal{K}(\Sigma^*) \rightarrow \mathcal{K}(\Sigma^*)^{\text{op}}$$

given by $M_F \mapsto M_{\omega_n(F)}$, where $F$ and $\omega_n(F)$ are both compositions of $[n]$, the latter obtained from the former by replacing $i$ by $n + 1 - i$. For example, $M_{13|5|24|6} \mapsto M_{46|2|35|1}$.

15.6. The Fock functors and primitive elements

Let $g\text{Hopf}$ and $g\text{Lie}$ be the categories of graded Hopf algebras and graded Lie algebras respectively. Recall the classical functor

$$\mathcal{P}: g\text{Hopf} \rightarrow g\text{Lie},$$

which sends a Hopf algebra to its Lie algebra of primitive elements. The analogue of this functor for species, namely

$$\mathcal{P}: \text{Hopf}(Sp) \rightarrow \text{Lie}(Sp),$$

was defined in (11.39). The functors $\mathcal{K}^\vee$ and $\mathcal{K}'^\vee$ are better behaved with respect to $\mathcal{P}$ than the functors $\mathcal{K}$ and $\mathcal{K}'$. One reason is that the functor $\mathcal{K}^\vee$ being braided lax preserves Lie monoids while the functor $\mathcal{K}$ does not. We consider the diagram

$$\text{Hopf}(Sp) \xrightarrow{\mathcal{P}} \text{Lie}(Sp)$$

$$\downarrow \quad \downarrow$$

$$g\text{Hopf} \xrightarrow{\mathcal{P}} g\text{Lie}$$

with the vertical functors being either $\mathcal{K}^\vee$ or $\mathcal{K}'^\vee$.

15.6.1. The main result. For any Hopf monoid $h$, $\mathcal{K}^\vee(h)$ can be viewed as a graded Lie algebra in two different ways. The first way is to view it as the image under $\mathcal{K}^\vee$ of the Lie monoid $h$. The second way is to view the Hopf algebra $\mathcal{K}^\vee(h)$ as a graded Lie algebra. One checks that the two Lie structures coincide, the key being that $\mathcal{K}^\vee$ is braided lax. Since $\mathcal{K}'^\vee$ and $\mathcal{K}$ are also braided lax, the same statement can be made for $\mathcal{K}'^\vee(h)$ and $\mathcal{K}(h)$.

**Proposition 15.35.** For any connected Hopf monoid $h$, we have the following diagram of graded Lie algebras.

$$\mathcal{K}^\vee(\mathcal{P}(h)) \subseteq \mathcal{P}(\mathcal{K}^\vee(h)) \subseteq \mathcal{K}^\vee(h)$$

PROOF. The inclusions in the second square are obvious. We check below the inclusion and equality in the first square (as graded vector spaces). Since all spaces involved are Lie subalgebras of $\mathcal{K}^\vee(h)$, the claim regarding the “Lie” part is automatic.
There are three coproducts one needs to keep track of; these are shown in the commutative diagram below.

\[
\begin{array}{ccc}
\mathfrak{h}[n] \xrightarrow{\Delta} & \bigoplus_{S \sqcup T = [n]} \mathfrak{h}[S] \otimes \mathfrak{h}[T] & \xrightarrow{\varphi^\vee} & \bigoplus_{s + t = n} \mathfrak{h}[s] \otimes \mathfrak{h}[t] \\
\mathfrak{h}[n]^{S_n} \xrightarrow{\Delta} & \left( \bigoplus_{S \sqcup T = [n]} \mathfrak{h}[S] \otimes \mathfrak{h}[T] \right)^{S_n} & \xrightarrow{\cong}{\varphi^\vee} & \bigoplus_{s + t = n} \mathfrak{h}[s]^{S_s} \otimes \mathfrak{h}[t]^{S_t}
\end{array}
\]

The coproduct on \( \mathfrak{h} \) is the map \( \Delta \) above, the coproduct on \( \mathcal{K}^\vee(\mathfrak{h}) \) is the composite \( \varphi^\vee \circ \Delta \) of the top horizontal arrows, and the coproduct on \( \overline{\mathcal{K}}^\vee(\mathfrak{h}) \) is the composite of the bottom horizontal arrows. It follows that

\[
(15.18) \quad \Delta_+ = \varphi^\vee \circ \mathcal{K}^\vee(\Delta_+) \quad \text{and} \quad \Delta_+ = \overline{\varphi}^\vee \circ \overline{\mathcal{K}}^\vee(\Delta_+),
\]

where the \( \Delta_+ \) in the right-hand sides refers to the positive part of the coproduct of \( \mathfrak{h} \) while the \( \Delta_+ \) in the left-hand sides refers to the positive part of the coproducts of \( \mathcal{K}^\vee(\mathfrak{h}) \) and \( \overline{\mathcal{K}}^\vee(\mathfrak{h}) \) respectively.

Recall that for a connected Hopf monoid \( \mathfrak{h} \) and \( \Delta_+ : \mathfrak{h}_+ \to \mathfrak{h}_+ \cdot \mathfrak{h}_+ \), we have \( \mathcal{P}(\mathfrak{h}) = \ker \Delta_+ \). The same result also holds for a connected graded Hopf algebra. The functoriality of \( \mathcal{K}^\vee \) and \( \overline{\mathcal{K}}^\vee \) and (15.18) now implies that

\[
\mathcal{K}^\vee(\mathcal{P}(\mathfrak{h})) \subseteq \mathcal{P}(\mathcal{K}^\vee(\mathfrak{h})) \quad \text{and} \quad \overline{\mathcal{K}}^\vee(\mathcal{P}(\mathfrak{h})) \subseteq \mathcal{P}(\overline{\mathcal{K}}^\vee(\mathfrak{h})).
\]

To complete the proof, we have to show that the second inclusion is an equality. Since the definition of the functor \( \overline{\mathcal{K}}^\vee \) is in terms of invariants, we know that it is left exact. We now claim the following chain of equalities from which the result follows.

\[
\mathcal{K}^\vee(\mathcal{P}(\mathfrak{h})) = \ker(\mathcal{K}^\vee(\Delta_+)) = \ker(\overline{\mathcal{K}}^\vee(\Delta_+)) = \mathcal{P}(\mathcal{K}^\vee(\mathfrak{h})).
\]

The second equality holds because \( \mathcal{K}^\vee \) is left exact, and the third equality because \( \overline{\varphi}^\vee \) is an isomorphism. □

Note that the functor \( \mathcal{K}^\vee \) is also left exact. However, the argument given for \( \overline{\mathcal{K}}^\vee \) fails for \( \mathcal{K}^\vee \) because the map \( \varphi^\vee \) is not an isomorphism.

**Proposition 15.36.** For any connected Hopf monoid \( \mathfrak{h} \), we have the following diagram of graded vector spaces.

\[
\begin{array}{ccc}
\mathcal{K}(\mathcal{P}(\mathfrak{h})) & \subseteq & \mathcal{P}(\mathcal{K}(\mathfrak{h})) \subseteq \mathcal{K}(\mathfrak{h}) \\
\overline{\mathcal{K}}(\mathcal{P}(\mathfrak{h})) & \subseteq & \mathcal{P}(\overline{\mathcal{K}}(\mathfrak{h})) \subseteq \overline{\mathcal{K}}(\mathfrak{h})
\end{array}
\]

The bottom horizontal row is an inclusion of graded Lie algebras. Moreover, if \( \mathbb{k} \) is a field of characteristic zero, then

\[
(15.19) \quad \mathcal{K}(\mathcal{P}(\mathfrak{h})) = \mathcal{P}(\overline{\mathcal{K}}(\mathfrak{h})).
\]
This result is similar to Proposition 15.35 and can be proved along the same lines. We note some differences. Since the functor $\mathcal{K}$ is not braided lax, the top horizontal row is only an inclusion of graded vector spaces. Since the definition of $\overline{\mathcal{K}}$ is in terms of coinvariants, this functor is right exact. Over a field of characteristic 0, it is also left exact, and (15.19) follows as in the proof of Proposition 15.35.

15.6.2. Examples. We illustrate the above results on some of our familiar examples. In particular, we will see that the results are optimal in the sense that all inclusions are strict in general. We write $kx$ for the one-dimensional subspace spanned by the variable $x$ inside the space of polynomials in $x$. For convenience, we also use it to denote a graded vector space which is $k$ in degree one and zero in all other components.

Example 15.37. Consider the Hopf monoid $E$. According to Examples 11.44 and 15.14, the functor $\overline{\mathcal{K}}(\mathcal{P}(E)) = \mathcal{P}(kx) = \mathcal{P}(\mathcal{K}(E))$.

Note that $\mathcal{P}(k\{x\})$ is always one-dimensional irrespective of the field characteristic. On the other hand, for the functor $\overline{\mathcal{K}}$, we get an inclusion $\overline{\mathcal{K}}(\mathcal{P}(E)) = kx \subseteq \mathcal{P}(k[x]) = \mathcal{P}(\mathcal{K}(E))$.

In characteristic 0, we have $kx = \mathcal{P}(k[x])$, but in characteristic $p$ the inclusion is strict: the primitive elements of $k[x]$ are spanned by the monomials $x^{p^e}$ where $e \geq 0$.

More generally, for any Lie algebra $g$, the space of primitive elements of the universal enveloping algebra $U(g)$ is the restricted Lie subalgebra of $U(g)$ generated by $g$. (To get the previous result, let $g := kx$.) This result is stated in the paper by Kharchenko [200, p. 69]. It also follows from [63, Exercises II.1.12 and II.3.4]. For the definition of restricted Lie algebras, see [175, Section V.7].

Example 15.38. Consider the Hopf monoid $L$. According to Examples 11.44 and 15.17, $\mathcal{K}(\mathcal{P}(L)) = \mathcal{K}(\mathcal{L}) \subseteq \mathcal{P}(S \Lambda) = \mathcal{P}(\mathcal{K}(L))$.

and the dimension of the degree $n$ component of $\mathcal{P}(S \Lambda)$ is the number of permutations in $S_n$ with no global descents. Now one can conclude that the left-hand side is a proper Lie subalgebra of the right-hand side. This can be seen from a dimension count: the number of permutations on $n$ letters with no global descents is in general greater than $(n-1)!$, which is the dimension of $\mathcal{L}[n]$.

On the other hand, the functor $\overline{\mathcal{K}}(\mathcal{P}(L)) = \bigoplus_n (\mathcal{L}[n])^{S_n} = \mathcal{P}(k[x]) = \mathcal{P}(\overline{\mathcal{K}}(L))$.

The first equality says that the space $\bigoplus_n (\mathcal{L}[n])^{S_n}$ carries the structure of a Lie algebra. We denote its bracket by $\ast$. It can be made explicit using the lax structure of $\overline{\mathcal{K}}$ and the bracket of the Lie monoid $\mathcal{L}$. The second equality says that this Lie algebra is abelian and further its component in degree $p^e$ for $e \geq 0$ is $k$, while all other components are zero (here $p$ is the field characteristic). This is a special case of a result of Fresse [136, Theorem 1.2.5]
and [137, Proposition 1.2.16] which implies that \( \mathcal{K}^\vee(\text{Lie}) \) is the free restricted Lie algebra on one generator. Fresse’s result implies more generally that \( \mathcal{K}_V^\vee \) is the free restricted Lie algebra on \( V \) (the functor \( \mathcal{K}_V^\vee \) is defined in Chapter 19).

For example, in characteristic 2, the invariants in degrees 1, 2 and 4 are spanned by
\[
\begin{align*}
[1], \quad [1 2] \quad \text{and} \quad [(1 2)[3 4]] + [(1 3)[2 4]] + [(1 4)[2 3]].
\end{align*}
\]
We explicitly compute the \( \ast \) product in two cases and check that it is zero.
\[
[1] \ast [1 2] = [1[2 3]] + [2[1 3]] + [3[1 2]] = 0.
\]

Note that for the product we shift up the indices of the second term and then sum over all shuffles. The middle term is the Jacobi identity and hence zero (this is why there are no degree 3 invariants).

\[
[1 2] \ast [1 2] = [(1 2)[3 4]] + [(1 3)[2 4]] + [(1 4)[2 3]] + [(2 3)[1 4]] + [(2 4)[1 3]] + [(3 4)[1 2]].
\]
The right-hand side is twice the degree 4 invariant and hence zero in characteristic 2.

For the functor \( \mathcal{K} \), we get an inclusion (strict in general):
\[
\mathcal{K}(\mathcal{P}(\text{L})) = \bigoplus_n (\text{Lie}[n])_{S_n} \subseteq \mathcal{P}(\mathbb{k}[x]) = \mathcal{P}(\mathcal{K}(\text{L})).
\]
The left-hand side is the free Lie algebra on one generator and is always one-dimensional, except in characteristic 2.

Now consider the Hopf monoid \( \text{L}^\ast \). We have
\[
\begin{align*}
\mathcal{K}(\mathcal{P}(\text{L}^\ast)) &= \mathbb{k}x \subseteq \mathcal{P}(\mathbb{k}\{x\}) = \mathcal{P}(\mathcal{K}(\text{L}^\ast)), \\
\mathcal{K}^\vee(\mathcal{P}(\text{L}^\ast)) &= \mathbb{k}x \subseteq \mathcal{P}(\mathcal{K}^\vee(\text{L}^\ast)).
\end{align*}
\]
Both inclusions are strict. The graded dimension of the spaces on the right in both statements is the same. The dimension of the component of degree \( n \) is the number of permutations in \( S_n \) with no global descents. For completeness, we also record the following.
\[
\begin{align*}
\mathcal{K}(\mathcal{P}(\text{L}^\ast)) &= \mathbb{k}x \subseteq \mathcal{P}(\mathcal{K}(\text{L}^\ast)), \\
\mathcal{K}^\vee(\mathcal{P}(\text{L}^\ast)) &= \mathbb{k}x = \mathcal{P}(\mathcal{K}^\vee(\text{L}^\ast)).
\end{align*}
\]
More examples are given in Sections 17.2.5 and 17.3.3.

**Remark 15.39.** The vector space \( \bigoplus_{n \geq 1} \text{Lie}[n] \) carries a structure of Lie algebra (Lie subalgebra of \( \mathcal{P}(\mathbb{S}\Lambda) \)) and also (a different structure) of twisted Lie algebra, as mentioned in Section 11.9.1. Both structures are discussed in [11, Section 5.3]. The connection between the two becomes now clear: the twisted Lie algebra structure is an equivalent formulation of the Lie monoid structure of the species \( \text{Lie} \), while the Lie algebra structure is the result of applying the functor \( \mathcal{K}^\vee \) to this Lie monoid.

### 15.7. The full Fock functors and dendriform algebras

In this section we look at further monoidal properties of the Fock functors. The main result is that \( \mathcal{K}^\vee \) is a Zinbiel-lax monoidal functor. We discuss some consequences involving dendriform algebra structures.
15.7.1. The operadic monoidal properties of $\mathcal{K}^\vee$. Let us restrict the full Fock functor $\mathcal{K}^\vee$ to the category of positive species (Section 8.9.2). Its image then lies in the category of positively graded vector spaces (Section 2.3.4):

$$\mathcal{K}^\vee: (Sp_+, \cdot) \to (gVec_+, \cdot).$$

We proceed to turn this functor into a Zinbiel lax monoidal functor, as in Definition 4.9. Note that the monoidal categories are nonunital, as in the situation of Notation 4.6. We first need to define a natural transformation

$$\gamma: \mathcal{K}^\vee(p) \cdot \mathcal{K}^\vee(q) \to \mathcal{K}^\vee(p \cdot q).$$

For this, we need maps

$$(15.20) \quad p[s] \otimes q[t] \to \bigoplus_{S \sqcup T = [n]} p[S] \otimes q[T]$$

where $s$ and $t$ are nonzero and $S$ and $T$ are nonempty. We define this to be the direct sum of the following maps, one for each summand in the target with $|S| = s$ and $|T| = t$ and $1 \in S$:

$$p[s] \otimes q[t] \otimes r[u] \xrightarrow{p[\text{cano}] \otimes q[\text{cano}] \otimes r[\text{cano}]} p[S] \otimes q[T] \otimes r[U],$$

one for each summand in the target with $|S| = s$, $|T| = t$, $|U| = u$ and $1 \in S$. The right-hand side of (4.5) consists of the sum of the same maps, split according to whether $s + 1 \in T$ or $s + 1 \in U$. \hfill \Box

Recall from (15.2) that the lax structure $\psi^\vee$ of $\mathcal{K}^\vee$ is given by a formula similar to that of $\gamma$, in which the sum is over all summands in the target with $|S| = s$ and $|T| = t$. Observe that $\gamma^b$, the conjugate of $\gamma$ by the braiding as given in Definition 3.14, has the same description as $\gamma$ except that the condition $1 \in S$ is replaced by $1 \notin S$. Therefore,

$$\psi^\vee = \gamma + \gamma^b.$$

Recall from Proposition 4.12 that associated to a Zinbiel-lax monoidal structure on a functor there is a braided lax monoidal structure on the same functor. It follows from the preceding observation that the braided lax monoidal structure on $\mathcal{K}^\vee$ associated to $\gamma$ is $\psi^\vee$. We thus recover (the nonunital version of) the result that $(\mathcal{K}^\vee, \psi^\vee)$ is braided lax monoidal (Proposition 15.28).
15.7.2. The functor $K^\vee$ and dendriform algebras. We combine the general results on transformation of monoids under monoidal functors (Sections 4.1.3 and 4.4.4) and the fact that $K^\vee$ is Zinbiel-lax in order to derive two constructions of graded dendriform and graded Zinbiel algebras.

**Proposition 15.41.** Let $p$ be a nonunital associative monoid in $(Sp_+, \cdot)$. Then $K^\vee(p)$ is a graded dendriform algebra. If $p$ is commutative, then $K^\vee(p)$ is in fact a graded Zinbiel algebra.

**Proof.** Both statements follow from Proposition 4.15. □

We mention that dual results hold for the full Fock functor $K$. If $p$ is a non-counital comonoid in $(Sp_+, \cdot)$, then $K(p)$ is a graded dendriform coalgebra, and if $p$ is cocommutative, then $K(p)$ is in fact a graded Zinbiel coalgebra.

We illustrate Proposition 15.41 with two well-known examples.

**Example 15.42.** The positive decorated exponential species $(E_V)_+$ is a nonunital commutative monoid in $Sp_+$ (Example 8.18). It follows from Proposition 15.41 that

$$T^\vee(V)_+ = K^\vee((E_V)_+)$$

is a (graded) Zinbiel algebra. As a nonunital associative algebra, it is the positive degree part of the shuffle algebra on $V$ (Section 2.6.1). The fact that $T^\vee(V)_+$ is a Zinbiel algebra is well-known; in fact, it is the free Zinbiel algebra on $V$, see [238, Section 7.1] and [325, p. 19].

**Example 15.43.** The positive linear order species $L_+$ (Example 8.16) is a nonunital monoid in $Sp_+$. It follows from Proposition 15.41 that

$$SA^*_+ = K^\vee(L_+)$$

is a graded dendriform algebra. Here $SA^*$ is the dual of the Malvenuto–Reutenauer Hopf algebra, as explained in Example 15.17. The dendriform structure was introduced by Loday and Ronco in [243, Definition 4.4].
In Chapters 15 and 16, we defined and studied various Fock functors and their deformations. We recall that these are bilax functors from species to graded vector spaces. In this chapter, we consider generalizations of these functors which depend on a vector space (the space of decorations). They are summarized in Table 19.1. When the vector space is the base field \( k \), we recover the earlier Fock functors. The earlier theory generalizes in a straightforward way to this more general setting.

In a sense, one may view the result of applying the functor \( \mathcal{K}_V \) (or its relatives) to a species \( p \) as a version of the graded vector space \( \mathcal{K}(p) \) (or its relatives) in which the given combinatorial structure determined by the species \( p \) has been decorated with elements of the vector space \( V \).

We begin by defining the decorated full Fock functors and the decorated bosonic Fock functors in Section 19.1 and show that they are bilax. We also explain how they can be constructed from their undecorated counterparts using the decorated exponential species. The discussion is continued in Section 19.2 where interrelationships between these functors are understood via the decorated norm transformation.

The values of the various decorated Fock functors on the exponential species are the tensor Hopf algebra, the shuffle Hopf algebra, the symmetric and exterior Hopf algebras, and their deformations (Section 2.6). The underlying vector spaces of these Hopf algebras are known as Fock spaces; the standard terminology of these spaces is summarized in Table 19.2. This constitutes our motivation for the terminology “Fock functors”.

We now turn to a feature which is new to this chapter. Graded vector spaces with creation-annihilation operators were discussed in Section 2.8. Fock spaces are examples of such spaces. Further, the creation-annihilation operators that they carry satisfy canonical commutation relations. The point of view of this chapter is

### Table 19.1. Decorated Fock functors.

<table>
<thead>
<tr>
<th>Fock functor</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{K}_V, \mathcal{K}_V^- )</td>
<td>Decorated full Fock functor</td>
</tr>
<tr>
<td>( \mathcal{K}<em>{V,q}, \mathcal{K}</em>{V,q}^- )</td>
<td>Deformed decorated full Fock functor</td>
</tr>
<tr>
<td>( \mathfrak{S}_V, \mathfrak{S}_V^- )</td>
<td>Decorated anyonic Fock functor</td>
</tr>
<tr>
<td>( \mathcal{K}_V, \mathfrak{S}_V, \mathcal{K}_V^- )</td>
<td>Decorated bosonic Fock functor</td>
</tr>
<tr>
<td>( \mathcal{K}<em>{V,-1}, \mathfrak{S}</em>{V,-1}, \mathcal{K}_{V,-1}^- )</td>
<td>Decorated fermionic Fock functor</td>
</tr>
<tr>
<td>( \mathcal{K}<em>{V,0}, \mathfrak{S}</em>{V,0}, \mathcal{K}_{V,0}^- )</td>
<td>Decorated free Fock functor</td>
</tr>
</tbody>
</table>
Table 19.2. Fock spaces.

<table>
<thead>
<tr>
<th>Fock spaces</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{K}<em>V(\mathbf{E})$, $\mathcal{K}<em>V^\vee(\mathbf{E})$, $\mathcal{K}</em>{V,q}(\mathbf{E})$, $\mathcal{K}</em>{V,q}^\vee(\mathbf{E})$</td>
<td>Full Fock space</td>
</tr>
<tr>
<td>$\mathfrak{S}_{V,q}(\mathbf{E})$</td>
<td>Anyonic Fock space</td>
</tr>
<tr>
<td>$\mathcal{K}<em>{V,-1}(\mathbf{E})$, $\mathfrak{S}</em>{V,-1}(\mathbf{E})$, $\mathcal{K}_{V,-1}^\vee(\mathbf{E})$</td>
<td>Bosonic Fock space</td>
</tr>
<tr>
<td>$\mathcal{K}<em>{V,0}(\mathbf{E})$, $\mathfrak{S}</em>{V,0}(\mathbf{E})$, $\mathcal{K}_{V,0}^\vee(\mathbf{E})$</td>
<td>Fermionic Fock space</td>
</tr>
<tr>
<td>$\mathcal{K}_{V,0}(\mathbf{E})$</td>
<td>Free Fock space</td>
</tr>
</tbody>
</table>

as follows. Species with up-down operators were discussed in Section 8.12, the basic example being that of the exponential species (Example 8.55). Now Fock functors convert up-down operators to creation-annihilation operators. This provides an explanation for the existence of such operators on Fock spaces. Further, we may now apply the Fock functors to other species which carry up-down operators leading to more general Fock spaces equipped with creation-annihilation operators. These ideas are due to Gută and Maassen [158] and Bożejko and Gută [64], and are explained in Sections 19.3 and 19.4.

Sections 19.5 and 19.6 deal with commutation relations. We introduce the notion of a species with balanced operators. This is a species with up-down operators where the up and down operators need to satisfy some compatibility relations. The exponential species is the basic example of a species with balanced operators. The main result here is that the Fock functors convert a species with balanced operators to a graded vector space with creation-annihilation operators which satisfy the canonical commutation relations.

The rest of the chapter deals with deformations. Deformations of the decorated full Fock functors, along with the fermionic and anyonic cases are treated in Section 19.7. The $q$-commutation relation is treated in Section 19.8. In Section 19.9, we consider a general situation in which the decorated Fock functors are deformed using a Yang–Baxter operator on $V$. The anyonic Fock space in this case is the Nichols algebra associated to $V$ (also called the quantum symmetric algebra).

There is another approach to combinatorial models for Fock spaces due to Baez and Dolan [29, Section 5]. It involves a generalization of the notion of species called 

stuff type.

We also point the reader to the related works [30] and [280]. We do not pursue the connections between this interesting approach and the ideas presented here.

We thank Roland Speicher for making us aware of [64, 158].

19.1. Decorated Fock functors

In this section, we define the decorated Fock functors along with their bilax structures. We explain how the decorated and undecorated Fock functors determine each other. We also address the behavior of the functors with respect to duality and the contragredient construction of Section 3.10. The connection to Schur functors is also explained.
19.1.1. Decorated Fock functors. Let $V$ be a vector space. For each $n \geq 0$, there is a left action of the symmetric group $S_n$ on $V^\otimes n$ given by

$$\sigma \cdot (v_1 \cdots v_n) := v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}.$$

For simplicity, we omit the tensor symbols between the $v_i$'s.

Let $p$ be a species. Then $S_n$ acts diagonally on $p[n] \otimes V^\otimes n$,

$$\sigma \cdot (x \otimes v_1 \cdots v_n) = p[\sigma](x) \otimes \sigma \cdot (v_1 \cdots v_n).$$

(19.1)

The spaces of invariants and of coinvariants for this action are respectively denoted by

$$p[n] \otimes^{S_n} V^\otimes n := (p[n] \otimes V^\otimes n)^{S_n}$$
and

$$p[n] \otimes_{S_n} V^\otimes n := (p[n] \otimes V^\otimes n)_{S_n}.$$

In other words, $p[n] \otimes^{S_n} V^\otimes n$ is the subspace of $p[n] \otimes V^\otimes n$ consisting of those tensors $\sum_i x_i \otimes v_i^1 \cdots v_i^n$, such that

$$\sum_i p[\sigma](x_i) \otimes \sigma \cdot (v_i^1 \cdots v_i^n) = \sum_i x_i \otimes v_i^1 \cdots v_i^n$$

for all $\sigma \in S_n$, and $p[n] \otimes_{S_n} V^\otimes n$ is the quotient of $p[n] \otimes V^\otimes n$ in which

$$x \otimes v_1 \cdots v_n = p[\sigma](x) \otimes \sigma \cdot (v_1 \cdots v_n)$$

for all $\sigma \in S_n$.

**Definition 19.1.** The decorated Fock functors

$$\mathcal{K}_V, \mathcal{K}_V^\vee, \bar{\mathcal{K}}_V, \bar{\mathcal{K}}_V^\vee : \text{Sp} \to \text{gVec}$$

are defined by

$$\mathcal{K}_V(p) := \mathcal{K}_V^\vee(p) := \bigoplus_{n \geq 0} p[n] \otimes V^\otimes n,$$

$$\bar{\mathcal{K}}_V(p) := \bigoplus_{n \geq 0} p[n] \otimes_{S_n} V^\otimes n,$$

$$\bar{\mathcal{K}}_V^\vee(p) := \bigoplus_{n \geq 0} p[n] \otimes^{S_n} V^\otimes n.$$

The quotient maps $\mathcal{K}_V(p) \twoheadrightarrow \bar{\mathcal{K}}_V(p)$ and the inclusions $\bar{\mathcal{K}}_V^\vee(p) \hookrightarrow \mathcal{K}_V^\vee(p)$ define natural transformations

$$\mathcal{K}_V \Rightarrow \bar{\mathcal{K}}_V \quad \text{and} \quad \mathcal{K}_V^\vee \Rightarrow \bar{\mathcal{K}}_V^\vee.$$

We refer to $\mathcal{K}_V$ and $\mathcal{K}_V^\vee$ as the **decorated full Fock functors** and to $\bar{\mathcal{K}}_V$ and $\bar{\mathcal{K}}_V^\vee$ as the **decorated bosonic Fock functors**.

We refer to $V$ as the space of **decorations**. Setting $V = k$ recovers the (undecorated) Fock functors of Definitions 15.1 and 15.5. The first thing we do below is to extend the bilax monoidal structure of these functors to the decorated context. It is this structure that distinguishes between $\mathcal{K}_V$ and $\mathcal{K}_V^\vee$ (as in the undecorated context). In the next section, we will see via the decorated norm transformation that the functors $\bar{\mathcal{K}}_V$ and $\bar{\mathcal{K}}_V^\vee$ are isomorphic in characteristic 0.
19.1.2. Schur functors. Let \( \mathbf{p} \) be a fixed species. The Schur functor associated to \( \mathbf{p} \) is [260, Definition 1.24]

\[
\mathcal{S}_\mathbf{p}: \text{Vec} \to \text{Vec}, \quad \mathcal{S}_\mathbf{p}(V) := \bigoplus_{n \geq 0} \mathbf{p}[n] \otimes_{S_n} V^\otimes n.
\]

The functor \( \mathcal{S}_\mathbf{p} \) is analytic. An intrinsic characterization of analytic functors is given by Joyal in [182, Théorème 1, Appendice]. Together with the results of [182, §2.0 and §4.1], this implies that if \( k \) is a field of characteristic 0, any analytic functor on \( \text{Vec} \) is the Schur functor of a unique species \( \mathbf{p} \). More precisely, there is an equivalence between the category of analytic functors on \( \text{Vec} \) and that of species. A related result is given by Fresse [137, Proposition 1.2.5].

When dealing with the decorated Fock functors \( \mathcal{S}_\mathbf{p} \to \mathfrak{g}\text{Vec} \), the vector space \( V \) is fixed and the species \( \mathbf{p} \) is varying. Thus

\[
\mathcal{K}_V(\mathbf{p}) = \mathcal{S}_\mathbf{p}(V).
\]

The same perspective can be adopted for the functor \( \mathcal{K}_V^\vee \). This leads to the divided power functor \( \Gamma_\mathbf{p}: \text{Vec} \to \text{Vec} \) defined by

\[
\Gamma_\mathbf{p}(V) := \mathcal{K}_V^\vee(\mathbf{p}).
\]

This functor is studied by Fresse in [136, Section 1] and [137, Section 1.2.12]. Additional information on Schur functors can be found in [137, Section 1.2].

19.1.3. Bilax structure of the decorated Fock functors. Let \( V \) be a fixed vector space. We proceed to endow the \( V \)-decorated Fock functors with a bilax monoidal structure.

Given species \( \mathbf{p} \) and \( \mathbf{q} \), we define morphisms of graded vector spaces

\[
\bigoplus_{s+t=n} (\mathbf{p}[s] \otimes V^\otimes s) \otimes (\mathbf{q}[t] \otimes V^\otimes t) \xrightarrow{\varphi_{\mathbf{p},\mathbf{q}}} \bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[t] \otimes V^\otimes (s+t)
\]

as follows. The degree \( n \) components of these maps

\[
\bigoplus_{s+t=n} (\mathbf{p}[s] \otimes V^\otimes s) \otimes (\mathbf{q}[t] \otimes V^\otimes t) \xleftarrow{\varphi_{\mathbf{p},\mathbf{q}}} \bigoplus_{s+t=n} \mathbf{p}[s] \otimes \mathbf{q}[t] \otimes V^\otimes (s+t)
\]

are the direct sum of the following maps:

\[
\mathbf{p}[s] \otimes V^\otimes s \otimes \mathbf{q}[t] \otimes V^\otimes t \xrightarrow{\varphi_{\mathbf{p},\mathbf{q}}} \mathbf{p}[s] \otimes \mathbf{q}[s+1, s+t] \otimes V^\otimes n
\]

\[
x \otimes v_1 \cdots v_s \otimes y \otimes w_1 \cdots w_t \longrightarrow x \otimes \mathbf{q}[\text{cano}](y) \otimes v_1 \cdots v_sw_1 \cdots w_t,
\]

\[
\mathbf{p}[s] \otimes \mathbf{q}[T] \otimes V^\otimes s \otimes \mathbf{q}[t] \otimes V^\otimes t
\]

\[
x \otimes y \otimes v_1 \cdots v_n \longrightarrow \mathbf{p}[\text{cano}](x) \otimes v_1 \cdots v_s \otimes \mathbf{q}[\text{cano}](y) \otimes v_{j_1} \cdots v_{j_t},
\]

where we have written \( S = \{i_1 < \cdots < i_s\} \) and \( T = \{j_1 < \cdots < j_t\} \) and \( \text{cano} \) denotes the canonical order-preserving maps, as in Notation 2.5. Thus \( \varphi \) and \( \psi \) act on the species part as in the undecorated case (Section 15.1.1), while on tensors \( \varphi \) concatenates and \( \psi \) deshuffles. Note that the composite \( \psi_{\mathbf{p},\mathbf{q}} \varphi_{\mathbf{p},\mathbf{q}} \) is the identity, but in general these maps are not invertible on the degree \( n \) component, as before.
Note that $\mathcal{K}_V(1) = 1$, the graded vector space of (2.7). We let $\varphi_0$ and $\psi_0$ be the identity maps

$$1 \xrightarrow{\varphi_0} \mathcal{K}_V(1).$$

We proceed similarly for the functor $\mathcal{K}_V^\vee$. The maps

$$\bigoplus_{s+t=n} (p[s] \otimes V^{\otimes s}) \otimes (q[t] \otimes V^{\otimes t}) \xrightarrow{\psi_{p,q}^\vee} \left( \bigoplus_{S \sqcup T = [n]} p[S] \otimes q[T] \right) \otimes V^{\otimes n}$$

as follows. The lax structure map $\psi_{p,q}^\vee$ is the direct sum of the following maps, one for each $s,t$ and each summand in the target with $|S| = s$ and $|T| = t$:

$$p[s] \otimes V^{\otimes s} \otimes q[t] \otimes V^{\otimes t} \rightarrow p[S] \otimes q[T] \otimes V^{\otimes n}$$

$$x \otimes v_1 \cdots v_s \otimes y \otimes w_1 \cdots w_t \mapsto p[\text{cano}](x) \otimes q[\text{cano}](y) \otimes u_1 \cdots u_n.$$

Here, we write $S = \{i_1 < \cdots < i_s\}$ and $T = \{j_1 < \cdots < j_t\}$ and define

$$u_h := \begin{cases} v_k & \text{if } h = i_k \in S, \\ w_k & \text{if } h = j_k \in T. \end{cases}$$

In other words, the tensor $u_1 \cdots u_n$ is the result of shuffling the tensors $v_1 \cdots v_s$ and $w_1 \cdots w_t$ according to the shuffle determined by $S$ and $T$.

The colax structure map $\varphi_{p,q}$ is the direct sum of the following maps:

$$p[s] \otimes q[s+1, s+t] \otimes V^{\otimes n} \rightarrow p[s] \otimes V^{\otimes s} \otimes q[t] \otimes V^{\otimes t}$$

$$x \otimes y \otimes v_1 \cdots v_n \mapsto x \otimes v_1 \cdots v_s \otimes q[\text{cano}](y) \otimes v_{s+1} \cdots v_{s+t}.$$

On the components for which $S \neq [s]$ (and hence $T \neq [s+1, s+t]$), the map $\varphi_{p,q}^\vee$ is zero.

We let $\varphi_0^\vee$ and $\psi_0^\vee$ be the identity maps of $\mathcal{K}_V^\vee(1) = 1$.

The structure maps of $\mathcal{K}_V^\vee$ descend to coinvariants and those of $\mathcal{K}_V^\vee$ restrict to invariants, as indicated below.

The proofs of these statements are similar to those in the undecorated case; see the proof of Proposition 15.2 for the coinvariant case.

We now state the main result of this section.

**Theorem 19.2.** The functors

$$(\mathcal{K}_V, \varphi, \psi), (\mathcal{K}_V^\vee, \psi^\vee, \varphi^\vee) : (Sp, \cdot, \beta_q) \rightarrow (gVec, \cdot, \beta_q)$$

are bilax monoidal. The functors

$$(\mathcal{K}_V, \varphi, \psi^\vee), (\mathcal{K}_V^\vee, \psi^\vee, \varphi^\vee) : (Sp, \cdot, \beta_q) \rightarrow (gVec, \cdot, \beta_q)$$
are bistrong monoidal. The natural transformations $\mathcal{K}_V \Rightarrow \mathcal{K}_V$ and $\mathcal{K}_V^\vee \Rightarrow \mathcal{K}_V^\vee$ are morphisms of bilax monoidal functors.

The proofs are again similar to those in the undecorated case; see Theorems 15.3 and 15.6. We recall here that $\beta_q$ are the deformed braidings on species (9.1) and vector spaces (2.50). Due to the similarity in their definitions, the parameter $q$ plays a passive role in the proof.

Alternatively, the above result can be deduced from the following result (whose proof is straightforward) used in conjunction with Theorem 3.22.

**Proposition 19.3.** The functor $\mathcal{K}_V$ is the following composite of bilax functors:

$$
(\text{Sp}, \cdot, \beta_q) \xrightarrow{(-) \times \mathbf{E}_V} (\text{Sp}, \cdot, \beta_q) \xrightarrow{\mathcal{K}} (\mathbf{gVec}, \cdot, \beta_q).
$$

The same result holds for the other decorated Fock functors as well; they are obtained by precomposing their undecorated counterparts with $(-) \times \mathbf{E}_V$.

Here, $\mathbf{E}_V$ is the bimonoid of the decorated exponential species discussed in Example 8.18, and $(-) \times \mathbf{E}_V$ is the bilax functor associated to it as in Proposition 8.66. This functor is in fact bistrong. Strictly speaking, the latter functor was studied for the case $q = 1$, but the same can be done for a general $q$ by using Proposition 9.5.

To summarize, the undecorated and decorated Fock functors determine each other. The former is the special case of the latter in which $V = \mathbb{k}$, while the latter can be obtained from the former by precomposing with $(-) \times \mathbf{E}_V$.

**Example 19.4.** We apply the decorated Fock functors to the Hopf monoid $\mathbf{E}$. In view of Proposition 19.3 and Example 15.16, we obtain

$$
\mathcal{K}_V(\mathbf{E}) = \mathcal{K}(\mathbf{E}_V) = \mathcal{T}(V) \quad \text{and} \quad \mathcal{K}_V^\vee(\mathbf{E}) = \mathcal{S}(V),
$$

the tensor and symmetric Hopf algebras on $V$ (the elements of $V$ have degree one). Similarly,

$$
\mathcal{K}_V^\vee(\mathbf{E}) = \mathcal{T}^\vee(V) \quad \text{and} \quad \mathcal{K}_V(\mathbf{E}) = \mathcal{S}^\vee(V),
$$

the shuffle Hopf algebra and its symmetric Hopf subalgebra. These examples have been considered by Fresse [136, Section 1.2.11].

Thus, one may view the result of applying the functor $\mathcal{K}_V$ (or its relatives) to a Hopf monoid $\mathbf{p}$ in species as a decorated version of the Hopf algebra $\mathcal{K}(\mathbf{p})$ (or its relatives). Hence, every Hopf algebra of the form $\mathcal{K}(\mathbf{p})$ admits a decorated version in this sense.

**19.1.4. Duality between decorated Fock functors.** Let $V$ be a finite-dimensional vector space. This implies a natural isomorphism of $S_n$-modules

$$
\mathbf{p}[n]^* \otimes (V^*)^\otimes n \cong (\mathbf{p}[n] \otimes V^\otimes n)^* \quad \text{for any finite-dimensional species } \mathbf{p} \text{ and } n \geq 0,
$$

and hence an isomorphism of functors

$$
\mathcal{K}_V^\vee(\mathbf{p}^*) \cong \mathcal{K}_V(\mathbf{p})^*.
$$
If \( p \) and \( q \) are finite-dimensional species, then the bilax structures of \( \mathcal{K}_V \) and \( \mathcal{K}_V^\vee \) are related through duality as expressed by the following commutative diagrams.

\[
\begin{array}{ccc}
\mathcal{K}_V^\vee (p^\ast) \cdot \mathcal{K}_V^\vee (q^\ast) & \xrightarrow{\psi^\vee} & \mathcal{K}_V^\vee (p^\ast \cdot q^\ast) \\
\downarrow & & \downarrow \\
\mathcal{K}_V (p)^\ast \cdot \mathcal{K}_V (q)^\ast & \xrightarrow{\psi^\ast} & \mathcal{K}_V (p \cdot q)^\ast
\end{array}
\]

This means that the decorated full Fock functors \( \mathcal{K}_V \) and \( \mathcal{K}_V^\vee \) are contragredient, in the sense of Section 3.10. A similar statement holds for the decorated bosonic Fock functors. The above discussion can be summarized as follows.

**Proposition 19.5.** Let \( V \) be a finite-dimensional vector space. On finite-dimensional species, the bilax functors \( (\mathcal{K}_V^\vee, \psi^\vee, \varphi^\vee) \) and \( (\mathcal{K}_V^\vee, \overline{\psi}, \overline{\varphi}) \) are respectively isomorphic to the contragredients of \( (\mathcal{K}_V, \varphi, \psi) \) and \( (\mathcal{K}_V, \overline{\varphi}, \overline{\psi}) \).

This result can also be viewed more conceptually as a consequence of earlier results of a similar nature. For example, the fact that the contragredient of \( \mathcal{K}_V \) is \( \mathcal{K}_V^\vee \) can be deduced as follows.

\[
(\mathcal{K}_V)^\vee (-) \cong \mathcal{K}^\vee \left( ((-)^\ast \times \mathbf{E}_V) \right) \cong \mathcal{K}^\vee ((-)^\ast \times \mathbf{E}_V^\ast) \cong \mathcal{K}_V^\vee.
\]

The first isomorphism follows by applying the contragredient construction to (19.2), and noting that the contragredient of \( \mathcal{K} \) is \( \mathcal{K}^\vee \) (Proposition 15.8). The middle isomorphism follows from the self-duality of the Hadamard functor (Proposition 8.60), and noting that the dual of \( \mathbf{E}_V \) is \( \mathbf{E}_V^\ast \) (Example 8.23). The last isomorphism follows from Proposition 19.3 applied to \( \mathcal{K}_V^\vee \).

### 19.2. The decorated norm transformation

Let \( V \) be a vector space. The norm transformation (Definition 15.19) can be extended to the decorated context.

**Definition 19.6.** For any species \( p \), let \( \kappa_p : \mathcal{K}_V (p) \rightarrow \mathcal{K}_V^\vee (p) \) be the map of graded vector spaces given by

\[
\kappa_p(x \otimes v_1 \cdots v_n) := \sum_{\sigma \in S_n} \sigma \cdot (x \otimes v_1 \cdots v_n),
\]

for any \( x \in p[n], v_i \in V \).

The action of \( S_n \) on \( p[n] \otimes V^\otimes n \) is as in (19.1). Each homogeneous component of \( \kappa_p \) is an instance of the norm map of Section 2.5. It follows that \( \kappa : \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee \) is a natural transformation, which we call the decorated norm. Note that the dependence of \( \kappa \) on \( V \) is not manifest in the notation.

**Proposition 19.7.** The decorated norm is a morphism of bilax monoidal functors

\[
\kappa : \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee.
\]
Proof. This can be proved directly in the same way as Proposition 15.20. Alternatively, it may also be deduced from it, using Proposition 19.3 and noting that the decorated norm $\kappa: \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee$ is the composition of the undecorated norm $\kappa: \mathcal{K} \Rightarrow \mathcal{K}^\vee$ with the bilax functor $(-) \times E_V$. \hfill \Box

The decorated norm map $\kappa_p: \mathcal{K}_V(p) \rightarrow \mathcal{K}_V^\vee(p)$ factors through coinvariants and its image consists of invariant elements (see Section 2.5). It therefore gives rise to a morphism of bilax monoidal functors

$$\bar{\kappa}: \bar{\mathcal{K}}_V \Rightarrow \bar{\mathcal{K}}_V^\vee$$

fitting in the commutative diagram below.

(19.3)

$$\begin{array}{ccc}
\mathcal{K}_V & \xrightarrow{\kappa} & \mathcal{K}_V^\vee \\
\downarrow & & \downarrow \\
\bar{\mathcal{K}}_V & \xrightarrow{\bar{\kappa}} & \bar{\mathcal{K}}_V^\vee
\end{array}$$

Proposition 19.8. If $k$ is a field of characteristic 0, then the morphism of bistrong monoidal functors

$$\bar{\kappa}_p: \bar{\mathcal{K}}_V(p) \rightarrow \bar{\mathcal{K}}_V^\vee(p)$$

is an isomorphism. More generally, for any commutative ring $k$, if the species $p$ consists of flat $kS_n$-modules $p[n]$ and $V$ is a flat $k$-module, then

$$\bar{\kappa}_p: \bar{\mathcal{K}}_V(p) \rightarrow \bar{\mathcal{K}}_V^\vee(p)$$

is invertible.

Proof. If $V$ is flat as a $k$-module, then so is $V^\otimes n$. This and the flatness of $p[n]$ as a $kS_n$-module imply that $p[n] \otimes V^\otimes n$ is flat as a $kS_n$-module, according to [69, Exercise III.0.1]. The result then follows from Lemma 2.20. \hfill \Box

19.2.1. The image of the decorated norm. Let $\mathfrak{Z}_V$ denote the (co)image of the decorated norm transformation $\kappa: \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee$, in the sense of Section 3.11. It follows from Proposition 19.8 that, in characteristic 0,

$$\bar{\mathcal{K}}_V, \ \mathfrak{Z}_V, \ \text{and} \ \bar{\mathcal{K}}_V^\vee$$

are isomorphic bistrong monoidal functors. In general, these are three distinct bistrong monoidal functors related by morphisms of bistrong functors

$$\bar{\mathcal{K}}_V \Rightarrow \mathfrak{Z}_V \Rightarrow \bar{\mathcal{K}}_V^\vee.$$
19.3. CLASSICAL CREATION-ANNIHILATION OPERATORS

Suppose now that $V$ and $p$ are finite-dimensional. It follows from Proposition 19.5 that $\kappa$ is related to its dual as follows.

$$
\begin{align*}
\mathcal{K}_V^\vee(p)^* & \xrightarrow{(\kappa p)^*} \mathcal{K}_V(p)^* \\
\mathcal{K}_V(p^*) & \xrightarrow{\kappa p^*} \mathcal{K}_V^\vee(p^*)
\end{align*}
$$

This means that the contragredient (3.47) of the $V$-decorated norm $\kappa: \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee$ is the $V^*$-decorated norm $\kappa: \mathcal{K}_{V^*} \Rightarrow \mathcal{K}_{V^*}^\vee$. A similar relation holds for $\bar{\kappa}: \mathcal{K}_V \Rightarrow \mathcal{K}_V^\vee$ and $\bar{\kappa}: \mathcal{K}_{V^*} \Rightarrow \mathcal{K}_{V^*}^\vee$. More generally, Lemma 2.22 yields:

**Proposition 19.9.** Let $V$ be a finite-dimensional vector space. On finite-dimensional species, the contragredient of diagram (19.3) for $V$ is diagram (19.3) for $V^*$.

We also note from Proposition 3.119 and the discussion preceding it that

$$(\mathcal{S}_V)^\vee = \mathcal{S}_{V^*},$$

and hence the image $\mathcal{S}_V$ is self-dual (regardless of the characteristic).

**Remark 19.10.** Recall the connection between decorated Fock functors and Schur functors from Section 19.1.2. The norm transformation has been considered by Fresse in the context of Schur functors in [136, Section 1.1.14] and [137, Section 1.2.12].

19.3. Classical creation-annihilation operators

The mathematical context for creation-annihilation operators is that of graded vector spaces. This was discussed in Section 2.8. We now briefly review the main motivating example, namely that of creation-annihilation operators on Fock spaces. The terminology that we are using in this chapter is borrowed from this example.

Bosons and fermions are commonly used terms in particle physics; very roughly, they stand for classes of particles which behave like $+1$ and $-1$ respectively. They are named after the physicists Enrico Fermi and Satyendra Bose. Fock spaces are used in quantum mechanics to describe quantum states with a variable number of particles. They are named after the physicist V. A. Fock. The terms bosonic Fock space and fermionic Fock space are used depending on whether the particles are bosons or fermions. A creation operator acts on bosonic or fermionic Fock space by increasing the number of particles by 1. Similarly, an annihilation operator decreases the number of particles by 1. A discussion of these ideas can be found in the books by Merzbacher [272, Chapter 20] and Landau and Lifshitz [220, Chapter IX]. For the original work of Fock, see [127, paper 32-2].

To relate to the notation below, $V$ stands for the quantum states of a single particle. It is customary in physics textbooks to choose a basis for $V$ and proceed from there; however this is not necessary for our purposes. The canonical commutation relations between creation and annihilation operators are stated here without proof. They can be checked directly and will also follow from the generalities of subsequent sections.
19.3.1. Classical Fock spaces. Let $V$ be a vector space. The \textit{algebraic Fock spaces} associated to $V$ are the underlying spaces of the tensor and symmetric algebras of $V$ \cite[Examples 1.3.3]{368}. More precisely, \textit{full Fock space} is

$$\mathcal{T}(V) = \bigoplus_{n \geq 0} V^\otimes n,$$

the underlying space of the tensor algebra, and \textit{bosonic Fock space} is

$$S(V) = \bigoplus_{n \geq 0} (V^\otimes n)_{S_n},$$

the underlying space of the symmetric algebra.

Suppose there is given a bilinear form $\langle \ , \rangle$ on $V$. We may extend it to full Fock space by

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_m \rangle := \begin{cases} \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle & \text{if } n = m, \\ 0 & \text{otherwise}. \end{cases}$$

In this context, the following operators on full Fock space are of interest \cite[Example 1.5.3]{368}: the (left) \textit{annihilation operator} associated to $v \in V$,

$$a(v) : \mathcal{T}(V) \to \mathcal{T}(V), \quad a(v)(1) = 0, \quad a(v)(v_1 \otimes \cdots \otimes v_n) = \langle v, v_1 \rangle \otimes v_2 \cdots \otimes v_n,$$

and the (left) \textit{creation operator} associated to $v$,

$$c(v) : \mathcal{T}(V) \to \mathcal{T}(V), \quad c(v)(1) = v, \quad c(v)(v_1 \otimes \cdots \otimes v_n) = v \otimes v_1 \otimes \cdots \otimes v_n.$$

The operators $a(v)$ and $c(v)$ are adjoint with respect to the above bilinear form on $\mathcal{T}(V)$, in the sense that

$$\langle a(v)(\xi), \eta \rangle = \langle \xi, c(v)(\eta) \rangle$$

for every $\xi, \eta \in \mathcal{T}(V)$.

Note that full Fock space is also the underlying space of the shuffle algebra $\mathcal{T}^\vee(V)$. Thus one may view the creation-annihilation operators as acting on either the tensor algebra or the shuffle algebra. It turns out that, from an algebraic point of view, it is more natural to let creation act on $\mathcal{T}(V)$ and annihilation on $\mathcal{T}^\vee(V)$. Indeed, it is easy to see that each annihilation operator is a derivation for the product of $\mathcal{T}^\vee(V)$ (shuffle), and each creation operator is a coderivation for the coproduct of $\mathcal{T}(V)$ (deshuffle). In addition, the creation operators descend to coinvariants and give rise to well-defined operators on bosonic Fock space. Dually, the annihilation operators restrict to invariants and give rise to well-defined operators on $S^\vee(V)$. This is shown below.

If the characteristic of the base field is 0, bosonic Fock space may be identified with the underlying space of $S^\vee(V)$ by means of the transformation $\bar{\kappa}$. In this situation, creation-annihilation operators act on bosonic Fock space and can therefore
be composed. The following commutation relations hold on bosonic Fock space.

\[ \tilde{c}(w)\tilde{c}(v) = \tilde{c}(v)\tilde{c}(w), \]
\[ \bar{a}(w)\bar{a}(v) = \bar{a}(v)\bar{a}(w), \]
\[ \tilde{a}(w)\tilde{c}(v) - \tilde{c}(v)\tilde{a}(w) = \langle v, w \rangle \text{id} \]

These are identities of operators on \( \mathcal{S}^{\vee}(V) \). To keep the notation straight, we have written \( \tilde{c}(v) \) for the operator corresponding to \( \tilde{c}(v) \).

We will be working with a more general formulation of these relations given in (19.4).

19.3.2. The one-dimensional case. Let \( V = k \), the base field, equipped with the canonical inner product \( \langle 1, 1 \rangle = 1 \). In this case, full Fock space and bosonic Fock space coincide and equal the space of polynomials in one variable:

\[ \mathcal{T}(V) = \mathcal{S}(V) = k[x] \quad \text{and} \quad \mathcal{T}^{\vee}(V) = \mathcal{S}^{\vee}(V) = k\{x\}. \]

These are the polynomial and divided power Hopf algebras of Example 2.3. Since \( V \) is one-dimensional, up to a scalar there is only one creation and one annihilation operator (corresponding to \( v = 1 \)). These are given by

\[ x : k[x] \to k[x] \quad x^n \mapsto x^{n+1} \quad \text{and} \quad k\{x\} \to k\{x\} \quad x^{(n)} \mapsto x^{(n-1)}, \]

with the convention that \( x^{(-1)} = 0 \). The former is a coderivation, while the latter is a derivation.

In characteristic 0, the norm map \( x^n \mapsto n!x^{(n)} \) provides an isomorphism of Hopf algebras from \( k[x] \) to \( k\{x\} \). The annihilation operator when viewed as an operator on \( k[x] \) via this isomorphism is the derivative operator

\[ \frac{d}{dx} : k[x] \to k[x] \quad x^n \mapsto nx^{n-1}. \]

It is well-known or one verifies directly that the creation-annihilation operators satisfy:

\[ \frac{d}{dx}x - x \frac{d}{dx} = 1. \]

This is the simplest instance of the commutation relation on bosonic Fock space mentioned above.

19.3.3. Generalized Fock spaces. Guță and Maassen [158] and Bożejko and Guță [64] work with the assumption that \( V \) is a Hilbert space. We limit our attention to the algebraic aspects of their constructions. This allows us to work in a slightly more general setting in which a bilinear form on \( V \) is not required. In this setting, there is a creation operator for each \( v \in V \) (as in the classical setting), and an annihilation operator for each functional \( f \in V^* \) (rather than for each \( v \in V \)).

The commutation relations on bosonic Fock space then take the following form.

\[ \tilde{c}(w)\tilde{c}(v) = \tilde{c}(v)\tilde{c}(w), \]
\[ \bar{a}(g)\bar{a}(f) = \bar{a}(f)\bar{a}(g), \]
\[ \bar{a}(g)\tilde{c}(v) - \tilde{c}(v)\bar{a}(g) = f(v)\text{id} \]

These are also called canonical commutation relations (usually abbreviated C.C.R).

As noted in Example 19.4, the classical Fock spaces are the values of the decorated Fock functors on the exponential species: \( \mathcal{K}_V(E) = \mathcal{T}(V) \) and \( \mathcal{K}_V(E) = \mathcal{S}(V) \).

In the next sections of this chapter, following [158, 64] we present a generalization of
these constructions in which the classical Fock spaces are replaced by the values of the decorated Fock functors on species with up-down operators. We refer to these as generalized Fock spaces. We show that if the operators are balanced, then the commutation relations (19.4) continue to hold on generalized bosonic Fock spaces (Proposition 19.27).

19.3.4. Fermionic Fock spaces. The emphasis of this chapter is on bosonic Fock spaces. However, in the final sections, we do touch upon fermionic, and more generally, anyonic Fock spaces. The fermionic Fock space is

$$\Lambda(V) = \bigoplus_{n \geq 0} (V \otimes n)_{S_n},$$

the underlying space of the exterior algebra; the invariants are taken with respect to the signed action of $S_n$:

$$V \otimes^k \mapsto V \otimes^k, \quad v_1 \otimes \cdots \otimes v_k \mapsto (-1)^{\text{inv } \sigma} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)},$$

where $\text{inv } \sigma$ denotes the number of inversions of $\sigma$ (2.20). This is the usual action tensored with the sign representation.

The following commutation relations hold on fermionic Fock space.

$$\tilde{c}(w)\tilde{c}(v) = -\tilde{c}(v)\tilde{c}(w),$$

$$\tilde{a}(g)\tilde{a}(f) = -\tilde{a}(f)\tilde{a}(g),$$

$$\tilde{a}(f)\tilde{c}(v) + \tilde{c}(v)\tilde{a}(f) = f(v) \text{id}.$$  

We show that these relations continue to hold on generalized fermionic Fock spaces that arise from species with balanced operators (Proposition 19.39).

19.4. The generalized Fock spaces of Gută and Maassen

We present a construction of generalized Fock spaces; these are termed combinatorial Fock spaces by Gută and Maassen [158]. We formulate it in functorial terms, in agreement with the main ideas in this monograph. More precisely, generalized Fock spaces are the values of the decorated Fock functors on species with up-down operators. In particular, we add to their constructions by paying attention to the algebraic properties of the functors and of the resulting Fock spaces. Our setting is closer to that of Bożejko and Gută [64, Section 2], but we work with arbitrary vector spaces rather than Hilbert spaces.

We point out that the entire theory applies to $V = \mathbb{k}$. In this case, there are canonical choices for $v$ and $f$. This yields results for undecorated Fock functors which go beyond those discussed in Chapter 15. We do not make them explicit.

<table>
<thead>
<tr>
<th>Categories</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$\mathsf{Sp}^u$</td>
<td>Species with up operators</td>
</tr>
<tr>
<td>$\mathsf{Sp}_d$</td>
<td>Species with down operators</td>
</tr>
<tr>
<td>$\mathsf{gVec}^c$</td>
<td>Graded vector spaces with creation operators</td>
</tr>
<tr>
<td>$\mathsf{gVec}_a$</td>
<td>Graded vector spaces with annihilation operators</td>
</tr>
</tbody>
</table>
We will freely use the set up of Sections 2.8, 8.11 and 8.12. The notations for the categories are reviewed in Table 19.3. Throughout this section, $V$ is a fixed vector space, $v \in V$ is a vector, and $f \in V^*$ is a functional.

19.4.1. Constructions of Gută and Maassen. We extend the decorated full Fock functor

$$K_V : \text{Sp} \to \text{gVec}$$

to the category of species with up operators, and its companion

$$K^\vee_V : \text{Sp} \to \text{gVec}$$

to the category of species with down operators. The choices of up operators for $K$ and down operators for $K^\vee$ are not arbitrary; they are justified below (see also Section 19.3).

**Definition 19.11.** We define a functor

$$K_{V,v} : \text{Sp}^u \to \text{gVec}^c$$

by

$$K_{V,v}(p, u) := (K_V(p), c(v)),$$

where $K_V : \text{Sp} \to \text{gVec}$ is the decorated full Fock functor, and the homogeneous map

$$c(v) : K_V(p) \to K_V(p)$$

doing of degree 1 has components

$$p[n] \otimes V^\otimes n \to p[n+1] \otimes V^\otimes (n+1)$$

defined by

$$c(v)(x_0 \otimes 1) := u(x_0) \otimes v, \quad c(v)(x_n \otimes v_1 \cdots v_n) := u(x_n) \otimes vv_1 \cdots v_n,$$

for $x_n \in p[n], v_i \in V$.

Here we make use of Convention 8.49 in order to have $u : p[n] \to p'[n] = p[n+1]$. Note that the dependence of $c(v)$ on $u$ is not manifest in the notation.

A morphism in $\text{Sp}^u$ intertwines the up operators and hence its image under $K_V$ intertwines the creation operators. Thus, $K_{V,v}$ is a functor as stated.

**Definition 19.12.** We define a functor

$$K^\vee_{V,f} : \text{Sp}^d \to \text{gVec}_a$$

by

$$K^\vee_{V,f}(p, d) := (K^\vee_V(p), a(f)),$$

where $K^\vee_V : \text{Sp} \to \text{gVec}$ is the decorated full Fock functor, and the homogeneous map

$$a(f) : K^\vee_V(p) \to K^\vee_V(p)$$

done of degree $-1$ has components

$$p[n] \otimes V^\otimes n \to p[n-1] \otimes V^\otimes (n-1)$$

defined by

$$a(f)(x_0 \otimes 1) := 0, \quad a(f)(x_n \otimes v_1 \cdots v_n) := d(x_n) \otimes f(v_1)v_2 \cdots v_n,$$

for $x_n \in p[n], v_i \in V$. 


19.4.2. Duality between creation and annihilation. Let $V$ be a finite-dimensional vector space and $V^*$ be its dual. Recall that the full Fock functors $K_V$ and $K_V^\vee$ are related by

$$K_V^\vee(p^*_*, u^*) \cong K_V(p)^*,$$

for any species $p$ (Section 19.1.4). The functors $K_{V,v}$ and $K_{V,f}^\vee$ considered above are related in a similar manner. The precise result is given below. The proof is straightforward.

**Proposition 19.13.** Let $(p, u)$ be a species with up operators and $(q, d)$ be a species with down operators. Let $(p^*, u^*)$ and $(q^*, d^*)$ be the dual species with down and up operators, respectively. Then, the following diagrams commute.

$$K_{V,v}(p^*, u^*)_∗ \xrightarrow{a(v)_*} K_{V,v}(p^*, u^*)_∗ \xrightarrow{c(v)_*} K_{V,v}(p, u)^* \xrightarrow{c(v)} K_{V,v}(p, u)^*$$

$$K_{V,f}(q^*, d^*)_∗ \xrightarrow{a(f)_*} K_{V,f}(q^*, d^*)_∗ \xrightarrow{c(f)} K_{V,f}(q, d)^*$$

In other words, on finite-dimensional species, the functors $K_{V,v}$ and $K_{V,f}$ are contragredient to the functors $K_{V,v}^\vee$ and $K_{V,f}^\vee$, respectively.

19.4.3. From up-down to creation-annihilation.

**Notation 19.14.** Recall from Convention 8.49 that we view $[n]^+ = [1 + n]$ with $1$ being the distinguished element of $[n]^+$. We now extend this convention to sets of positive integers. Thus, if $S = \{i_1 < \cdots < i_s\}$ is such a set, we let

$$\{1 + i_1 < \cdots < 1 + i_s\} \quad \text{and} \quad S^+ = \{1 < 1 + i_1 < \cdots < 1 + i_s\}.$$ 

Note that if $[n] = S \sqcup T$, then

$$[n]^+ = S^+ \sqcup (1 + T) = (1 + S) \sqcup T^+.$$ 

Here is a first result that explains why creation goes with $K_V$ and annihilation with $K_V^\vee$. Another reason is given later, in Proposition 19.19.

**Proposition 19.15.** Let $(p, u)$ and $(q, w)$ be species with up operators. The creation operator $c(v)$ and the colax structure of $K_V$ are related by the following commutative diagrams.

$$K_{V,v}(p, u) \cdot (q, w) \xrightarrow{c(v)^{-1}} K_{V,v}(p, u) \cdot K_{V,v}(q, w) \xrightarrow{c(v)} K_{V,v}(p, u) \cdot K_{V,v}(q, w)$$
Let \((p, d)\) and \((q, e)\) be species with down operators. The annihilation operator \(a(f)\) and the lax structure of \(\mathcal{K}_V^\psi\) are related by the following commutative diagrams.

\[
\begin{align*}
\mathcal{K}_{V,f}^\psi(p, d) & \cdot \mathcal{K}_{V,f}^\psi(q, e) \xrightarrow{\psi} \mathcal{K}_{V,f}^\psi((p, d) \cdot (q, e)) & \mathcal{K}_{V,f}^\psi(1) \\
\downarrow a(f) \cdot \text{id} + \text{id} \cdot a(f) & \downarrow a(f) & \downarrow a(f) \\
\mathcal{K}_{V,f}^\psi(p, d) \cdot \mathcal{K}_{V,f}^\psi(q, e) \xrightarrow{\psi} \mathcal{K}_{V,f}^\psi((p, d) \cdot (q, e)) & \mathcal{K}_{V,f}^\psi(1)
\end{align*}
\]

(19.10)

In the finite-dimensional setting, the two statements are duals of each other. The same remark applies to the subsequent corollaries.

**Proof.** Consider the first diagram in (19.9). Start from an element

\[
x \otimes y \otimes v_1 \cdots v_n \in p[S] \otimes q[T] \otimes V^{\otimes n}
\]

in the component of degree \(n\) of \(\mathcal{K}_V((p, u) \cdot (q, w))\) where \(S \cup T = [n]\). Applying \(\psi\) takes us to

\[
p[\text{cano}](x) \otimes v_{i_1} \cdots v_{i_s} \otimes q[\text{cano}](y) \otimes v_{j_1} \cdots v_{j_t} \in p[s] \otimes V^{\otimes s} \otimes q[t] \otimes V^{\otimes t},
\]

where \(S = \{i_1 < \cdots < i_s\}\) and \(T = \{j_1 < \cdots < j_t\}\). Applying now \(c(v) \cdot \text{id} + \text{id} \cdot c(v)\) we obtain

\[
u(p[\text{cano}](x)) \otimes v_{i_1} \cdots v_{i_s} \otimes q[\text{cano}](y) \otimes v_{j_1} \cdots v_{j_t} + p[\text{cano}](x) \otimes v_{i_1} \cdots v_{i_s} \otimes w(q[\text{cano}](y)) \otimes v_{j_1} \cdots v_{j_t}.
\]

On the other hand, applying \(c(v)\) to \(x \otimes y \otimes v_1 \cdots v_n\) we get, in view of (8.68) and (19.8),

\[
u(x) \otimes y \otimes v_1 \cdots v_n + x \otimes w(y) \otimes v_1 \cdots v_n
\]

\[
\in \left(p[S^+] \otimes q[1 + T] \otimes V^{\otimes(n+1)}\right) \oplus \left(p[1 + S] \otimes q[T^+] \otimes V^{\otimes(n+1)}\right).
\]

Therefore, applying \(\psi\) we obtain, in view of (19.7),

\[
p[\text{cano}](u(x)) \otimes v_{i_1} \cdots v_{i_s} \otimes q[\text{cano}](y) \otimes v_{j_1} \cdots v_{j_t} + p[\text{cano}](x) \otimes v_{i_1} \cdots v_{i_s} \otimes q[\text{cano}](w(y)) \otimes v_{j_1} \cdots v_{j_t}.
\]

This coincides with the expression obtained above since the up operators \(u\) and \(w\) are morphisms of species. Thus the first diagram in (19.9) commutes.

The commutativity of the second diagram in (19.9) follows from that of the first plus unitality of the lax structure (3.6) (or can be easily checked directly).

The proofs for (19.10) are similar. \(\square\)

Recalling the definition of the monoidal structure on \(g\text{Vec}^\mathcal{C}\) (2.75) we see that diagrams (19.9) say that \(\psi\) and \(\psi_0\) are morphisms in \(g\text{Vec}^\mathcal{C}\). A similar remark applies to (19.10). In conjunction with Theorem 19.2 which says that \(\mathcal{K}_V\) and \(\mathcal{K}_V^\psi\) are bilax monoidal functors, we deduce at once the following result.

**Proposition 19.16.** The functor

\[\mathcal{K}_{V,f}^\psi(p, d) : (Sp^u, \cdot) \rightarrow (g\text{Vec}^\mathcal{C}, \cdot)\]

is colax monoidal. The functor

\[\mathcal{K}_{V,f}^\psi(p, d) : (Sp^u, \cdot) \rightarrow (g\text{Vec}^\mathcal{C}, \cdot)\]
is lax monoidal.

Recall that (co)lax monoidal functors preserve (co)monoids (Proposition 3.29). Further, recall that (co)monoids in categories with up, down, creation or annihilation operators are usual (co)monoids equipped with (co)derivations (Sections 8.12.5 and 2.8.3). This yields the following.

**Corollary 19.17.** The functor $K_{V,v}$ preserves up coderivations while the functor $K_{V,f}^\vee$ preserves down derivations. Explicitly:

Let $p$ be a comonoid in species equipped with an up coderivation $u$. Then the creation operator

$$c(v): K_{V,v}(p, u) \to K_{V,v}(p, u)$$

is a coderivation of the coalgebra $K_{V}(p)$ of degree $+1$.

Similarly, if $p$ is a monoid in species equipped with a down derivation $d$, then the annihilation operator

$$a(f): K_{V,f}^\vee(p, d) \to K_{V,f}^\vee(p, d)$$

is a derivation of the algebra $K_{V}(p)$ of degree $-1$.

**Example 19.18.** Recall from Example 8.55 that the exponential species is equipped with a down derivation and an up coderivation. We have $K_{V}(E) = T(V)$ and $K_{V}^\vee(E) = T^\vee(V)$, the tensor and shuffle algebras, respectively. The underlying space is classical full Fock space, in both cases. The creation-annihilation operators of Definitions 19.11 and 19.12 coincide with the classical creation-annihilation operators of Section 19.3. Corollary 19.17 recovers the facts, noted in Section 19.3, that the classical creation operator is a coderivation for the coproduct of the tensor algebra (deshuffle) and the classical annihilation operator is a derivation for the shuffle product.

More examples are discussed in Section 19.6.

### 19.4.4. Creation-annihilation on bosonic Fock functors.

Let $V$ be a vector space. Recall (Theorem 19.2) that the full Fock functors are related to the bosonic Fock functors by means of transformations (morphisms of bilax functors)

$$K_{V} \Rightarrow \overline{K}_{V}, \quad K_{V}^\vee \Rightarrow \overline{K}_{V}^\vee.$$ 

The creation-annihilation operators induce homogeneous maps on the bosonic Fock functors as follows.

**Proposition 19.19.** For any species with up operators $(p, u)$ and $v \in V$, the creation operator $c(v)$ descends to coinvariants

$$K_{V}(p, u) \xrightarrow{c(v)} K_{V}(p, u) \quad \overline{K}_{V}(p, u) \xrightarrow{\bar{c}(v)} \overline{K}_{V}(p, u)$$
yielding a homogeneous map \( \bar{c}(v) : \bar{\mathcal{K}}_V(p, u) \to \bar{\mathcal{K}}_V(p, u) \) of degree +1. Dually, for any species with down operators \((p, d)\) and \(f \in V^*\), the annihilation operator \(\bar{a}(f)\) restricts to invariants

\[
K_\nu^\vee(p, d) - \bar{a}(f) \to \bar{K}_V^\vee(p, d)
\]

yielding a homogeneous map \( \bar{a}(f) : \bar{K}_V^\vee(p, d) \to \bar{K}_V^\vee(p, d) \) of degree −1.

**Proof.** We check the first assertion. Let \( \sigma \in S_n \) and consider the elements

\[
x \otimes v_1 \cdots v_n \quad \text{and} \quad p[\sigma](x) \otimes v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}
\]

in \( p[n] \otimes V^\otimes n \), the component of degree \( n \) of \( K_V(p, u) \). Note that the second is obtained from the first by acting by \( \sigma \). Now applying \( c(v) \) to both of them yields

\[
u(x) \otimes vv_1 \cdots v_n
\]

and

\[
u(p[\sigma](x)) \otimes vv_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)} = p[\sigma^+](u(x)) \otimes vv_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)}.
\]

The equality holds by naturality of \( u \) and the definition of \( \sigma^+ \) (8.65). Observe that acting by \( \sigma^+ \) on the first element gives the second element above; so they yield the same element in the space of coinvariants. \( \square \)

Proposition 19.19 allows us to give the following definition.

**Definition 19.20.** We define functors

\[
\bar{K}_V^\nu : \text{Sp}^u \to \text{gVec}^c \quad \text{and} \quad \bar{K}_V^\vee : \text{Sp}^d \to \text{gVec}^a
\]

by

\[
\bar{K}_V^\nu(p, u) := (\bar{K}_V(p), \bar{c}(v)) \quad \text{and} \quad \bar{K}_V^\vee(p, d) := (\bar{K}_V(p), \bar{a}(f))
\]

where \( \bar{c}(v) \) and \( \bar{a}(f) \) are the maps of Proposition 19.19.

Theorem 19.2 and Proposition 19.16 yield:

**Proposition 19.21.** The functor

\[
(\bar{K}_V^\nu, \bar{\psi}) : (\text{Sp}^u, \cdot) \to (\text{gVec}^c, \cdot)
\]

is bistrong monoidal and \( K_{\nu, v} \Rightarrow \bar{K}_V^\nu \) is a morphism of colax monoidal functors. The functor

\[
(\bar{K}_V^\vee, \bar{\psi}) : (\text{Sp}^d, \cdot) \to (\text{gVec}^a, \cdot)
\]

is bistrong monoidal and \( K_{\vee, f} \Rightarrow \bar{K}_V^\vee \) is a morphism of lax monoidal functors.

By arguing as for Corollary 19.17, we obtain the following consequence.

**Corollary 19.22.** The functor \( \bar{K}_V^\nu \) preserves up derivations and up coderivations while the functor \( \bar{K}_V^\vee \) preserves down derivations and down coderivations.
In view of the above observations, it is natural to consider the functor $\triangledown_V$ of Section 19.2.1. Indeed, there is an induced functor

$$\triangledown_{V,v,f} : \text{Sp}_d^u \to \text{gVec}_a^c$$

from species with up-down operators to graded vector spaces with creation-annihilation operators. This point of view will be taken up later in Section 19.8.4.

19.5. Creation-annihilation on generalized bosonic Fock spaces

Classically, creation and annihilation are viewed as operators on the same (Fock) space, and therefore can be composed. Our presentation leads to operators acting on different spaces ($\mathcal{K}_V^c$ and $\mathcal{K}_V^\vee$ for creation, $\mathcal{K}_V^\vee$ and $\mathcal{K}_V^c$ for annihilation). However, over a field of characteristic 0, the bosonic functors $\mathcal{K}_V^c$ and $\mathcal{K}_V^\vee$ are naturally isomorphic (Proposition 19.8). This identification allows us to compose creation and annihilation operators at the bosonic level.

These operators do not commute. A result of Gut¸˘a and Maassen [158, Lemmas 6 and 7] describes the situation explicitly. This is recalled in Proposition 19.25. Since our setting is slightly more general and the notation is different from theirs, we provide a proof. The main result of this section is Proposition 19.24 which is a variant of this result. It is in fact easier to derive and more useful for later purposes.

We assume throughout this section that $k$ is a field of characteristic 0. We also continue to assume that $V$ is a fixed vector space, $v \in V$ is a vector, and $f \in V^*$ is a functional.

19.5.1. The commutation setup. Let $(p, u, d)$ be a species with up-down operators. Consider the following (noncommutative) diagram

$$\begin{array}{ccc}
\mathcal{K}_{V,v}(p, u) & \xrightarrow{c(v)} & \mathcal{K}_{V,v}(p, u) \\
\downarrow \kappa & & \downarrow \kappa \\
\mathcal{K}_V^\vee(p, d) & \xrightarrow{a(f)} & \mathcal{K}_V^\vee(p, d) \\
\downarrow \kappa & & \downarrow \kappa \\
\mathcal{K}_V^c(p, d) & \xrightarrow{c(v)} & \mathcal{K}_V^c(p, u)
\end{array}$$

in which $\kappa$ is the decorated norm transformation (Definition 19.6). Using the invertibility of $\kappa$ at the bosonic level, this diagram yields two new noncommutative
diagrams as follows.

\[
\begin{array}{ccc}
\mathcal{K}_{V,f}^\vee(p, d) & \xrightarrow{\tilde{c}(v)} & \mathcal{K}_{V,f}^\vee(p, d) \\
\downarrow & & \downarrow \\
\tilde{K}_{V,f}^\vee(p, d) & \xrightarrow{\tilde{c}(v)} & \tilde{K}_{V,f}^\vee(p, d)
\end{array}
\]

Here \(\tilde{c}(v)\) denotes the conjugate of \(\tilde{c}(v)\) by \(\tilde{\kappa}\),

\[
(19.13) \quad \tilde{K}_{V,f}^\vee(p, d) \xrightarrow{\tilde{\kappa}^{-1}} \mathcal{K}_{V,v}^\vee(p, u) \xrightarrow{\tilde{\kappa}} \mathcal{K}_{V,v}^\vee(p, u) \xrightarrow{\tilde{\kappa}} \mathcal{K}_{V,f}^\vee(p, u),
\]

and \(\tilde{a}(f)\) denotes the conjugate of \(\tilde{a}(f)\) by \(\tilde{\kappa}^{-1}\),

\[
(19.14) \quad \mathcal{K}_{V,v}^\vee(p, u) \xrightarrow{\tilde{\kappa}} \tilde{K}_{V,f}^\vee(p, d) \xrightarrow{\tilde{\kappa}} \tilde{K}_{V,f}^\vee(p, d) \xrightarrow{\tilde{\kappa}^{-1}} \mathcal{K}_{V,v}^\vee(p, u).
\]

The lack of commutativity of diagrams \((19.12)\) is of interest; it is systematically studied in the rest of this section. We will use the first diagram in \((19.12)\) for working purposes.

**Notation 19.23.** For each \(1 \leq i, j \leq n\), we let \((i, j) \in S_n\) denote the transposition that switches \(i\) with \(j\). For simplicity, we use \(\sigma \cdot x\) instead of \(p[\sigma](x)\) to denote the action of \(\sigma \in S_n\) on \(x \in p[n]\).

We let \((k, \ldots, 1)\) be the permutation which sends \(k\) to \(k - 1\), \(k - 1\) to \(k - 2\), and so on, and finally \(1\) to \(k\), while fixing the elements greater than \(k\).

**19.5.2. Composition formulas: first version.** We begin by deriving an explicit formula for the creation operator \(\tilde{c}(v)\). We have emphasized that creation goes with \(\mathcal{K}_V\); however there is a sensible creation operator on \(\mathcal{K}_V^\vee\) given by

\[
(19.15) \quad c(v) : x \otimes v_1 \cdots v_n \mapsto \sum_{k=0}^{n} (k + 1, \ldots, 1) \cdot u(x) \otimes v_1 \cdots v_k v \cdots v_n.
\]

With this definition, the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{K}_V(p, u) & \xrightarrow{c(v)} & \mathcal{K}_V(p, u) \\
\downarrow & & \downarrow \\
\mathcal{K}_V^\vee(p, u) & \xrightarrow{c(v)} & \mathcal{K}_V^\vee(p, u)
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{K}_{V,v}(p, u) & \xrightarrow{\tilde{c}(v)} & \mathcal{K}_{V,v}(p, u) \\
\downarrow & & \downarrow \\
\mathcal{K}_{V,v}(p, u) & \xrightarrow{\tilde{c}(v)} & \mathcal{K}_{V,v}(p, u)
\end{array}
\]
To see this, note that
\[
\kappa(c(v)(x \otimes v_1 \cdots v_n)) = \sum_{\tau \in S_{n+1}} \tau \cdot (u(x)) \otimes \tau \cdot (vv_1 \cdots v_n)
\]
\[
= \sum_{k=0}^{n} \sum_{\sigma \in S_n} (k + 1, \ldots, 1) \sigma^+ \cdot u(x) \otimes (v_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)} v \cdots v_{\sigma^{-1}(n)})
\]
\[
= \sum_{k=0}^{n} \sum_{\sigma \in S_n} (k + 1, \ldots, 1) \cdot u(\sigma \cdot x) \otimes (k + 1, \ldots, 1) \cdot (vv_{\sigma^{-1}(1)} \cdots v_{\sigma^{-1}(n)})
\]
\[
= c(v)(\kappa(x \otimes v_1 \cdots v_n)).
\]

Here \(\sigma^+\) is as in (8.65). For the equalities, note that any permutation \(\tau\) of \(vv_1 \cdots v_n\) is the composite of a permutation \(\sigma^+\) of \(vv_1 \cdots v_n\) which fixes \(v\) in the first position followed by a permutation \((k + 1, \ldots, 1)\) which inserts \(v\) in the \((k + 1)\)-st position.

Since \(\mathcal{K}^\vee_V(p)\) is a subspace of \(\mathcal{K}^\vee_V(p)\), it follows that the creation operator \(\check{c}(v)\) is given by the same formula (19.15). Using this, we obtain:

**Proposition 19.24.** Let \((p, u, d)\) be a species with up-down operators and

\[
\sum_{i} x_i \otimes v_i^1 \cdots v_i^n \in p[n] \otimes S_n V^\otimes n
\]

an element of degree \(n\) in \(\mathcal{K}^\vee_V(p, u, d)\). We have

\[
\text{(19.16)} \quad \check{a}(f)\check{c}(v)\left(\sum_{i} x_i \otimes v_i^1 \cdots v_i^n\right) = \sum_{i} d(u(x_i) \otimes f(v_i^1) v_2^i \cdots v_k^i v \cdots v_n^i)
\]

and

\[
\text{(19.17)} \quad \check{c}(v)\check{a}(f)\left(\sum_{i} x_i \otimes v_i^1 \cdots v_i^n\right) = \sum_{i} \sum_{k=1}^{n} (k, \ldots, 1) \cdot ud(x_i) \otimes f(v_i^1) v_2^i \cdots v_k^i v \cdots v_n^i.
\]

A \(q\)-deformation of this result is given later in Proposition 19.40.

**19.5.3. Composition formulas: second version.** We now present variants of the above formulas, mainly for completeness. We begin with a slight variant of (19.15) which is as follows.

\[
\text{(19.18)} \quad \check{c}(v)\left(\sum_{i} x_i \otimes v_i^1 \cdots v_i^n\right) = \sum_{i} u(x_i) \otimes vv_i^1 \cdots v_i^n
\]

\[
+ \sum_{i} \sum_{k=1}^{n} (k + 1, 1) \cdot u(x_i) \otimes v_k^i v_2^i \cdots v_k^i v \cdots v_n^i,
\]

where \(v\) appears at position \(k + 1\).
Proof. To prove this, we note that by $S_n$-invariance, for $1 \leq k \leq n$,
\[
\sum_i x_i \otimes v^i_1 \cdots v^i_n = \sum_i (k, \ldots, 1)^{-1} \cdot x_i \otimes v^i_k v^i_1 \cdots v^i_{k-1} v^i_{k+1} \cdots v^i_n.
\]
Now apply the operator which acts by the up operator $u$ followed by the permutation $(k + 1, \ldots, 1)$ on the first factor and inserts $v$ in position $k + 1$ in the second factor. Now summing over $1 \leq k \leq n$ and adding the term
\[
\sum_i u(x_i) \otimes vv^i_1 \cdots v^i_n
\]
to both sides, one obtains (19.18). Here we made use of (19.15), the observation that
\[
(k + 1, \ldots, 1) = (k + 1, 1)(k + 1, \ldots, 2)
\]
for $1 \leq k \leq n$, and the $S_n$-invariance of $u$. \hfill \square

Proposition 19.25 (Guţă and Maassen). Let $(p, u, d)$ be a species with up-down operators and
\[
\sum_i x_i \otimes v^i_1 \cdots v^i_n \in p[n] \otimes^{S_n} V^\otimes n
\]
an element of degree $n$ in $\bar{K}^\vee_V(p, u, d)$. We have
\[
(19.19) \quad \bar{a}(f)\bar{c}(v) \left( \sum_i x_i \otimes v^i_1 \cdots v^i_n \right)
\]
\[
= \sum_i du(x_i) \otimes f(v) v^i_1 \cdots v^i_n
\]
\[
+ \sum_i \sum_{k=1}^n d((k + 1, 1) \cdot u(x_i)) \otimes f(v^i_k) v^i_1 \cdots v^i_{k-1} v^i_{k+1} \cdots v^i_n
\]
and
\[
(19.20) \quad \bar{c}(v)\bar{a}(f) \left( \sum_i x_i \otimes v^i_1 \cdots v^i_n \right)
\]
\[
= \sum_i \sum_{k=1}^n (k, 1) \cdot ud((k, 1) \cdot x_i) \otimes f(v^i_k) v^i_1 \cdots v^i_{k-1} v^i_{k+1} \cdots v^i_n.
\]
In both summations over $k$, $v$ appears at position $k$.

Proof. The first part follows directly from (19.18). For the second part, we apply the transposition $(k, 1)$ to the $S_n$-invariant element and then apply $\bar{a}(f)$ to obtain
\[
\bar{a}(f) \left( \sum_i x_i \otimes v^i_1 \cdots v^i_n \right) = \sum_i (k, 1) \cdot x_i \otimes f(v^i_k) v^i_1 \cdots v^i_{k-1} v^i_{k+1} \cdots v^i_n,
\]
where $v^i_1$ is in position $k$. This holds for $1 \leq k \leq n$ and further we have the same $S_{n-1}$-invariant element written in $n$ different ways. Note that (19.18) expresses $\bar{c}(v)$ acting on the degree $n$ part as a sum of $n + 1$ operators. Here $\bar{c}(v)$ is acting on the degree $n - 1$ part since we are first applying the annihilation operator. So it is a sum of $n$ operators. By letting the $k$-th operator act on the above formula, and summing over all $1 \leq k \leq n$, the result follows. \hfill \square
For deriving the formulas in the second version, we crucially used the fact that we were dealing with an $S_n$-invariant element. Hence, unlike for the first version, these formulas do not generalize to the $q$-setting. In that scenario, one has to deal with the generalized anyonic Fock spaces $\mathcal{F}_{V,q}(p)$ elements of which cannot be interpreted as invariants.

### 19.6. Species with balanced operators

So far, we have dealt with species with up-down operators but never specified any relations between these operators. The discussion in Section 19.5 motivates the following definition.

**Definition 19.26.** A species with balanced operators is a species $(p,u,d)$ with up-down operators such that all relations (19.21)–(19.23c) below hold. We use Notation 19.23.

**The up-up and down-down relations.** For $n = 0, 1, 2, \ldots$, 

\[
(1,2) \cdot u^2(-) = u^2(-), 
\]

\[
d^2((1,2) \cdot (-)) = d^2(-) 
\]

as maps 

\[
p[n] \to p[n+2] \quad \text{and} \quad p[n+2] \to p[n] 
\]

respectively.

**The up-down relations.** For $n = 0, 1, 2, \ldots$ and $1 \leq k \leq n$,

\[
du = \lambda_n \id, 
\]

\[
d((k+1,\ldots,1) \cdot u(-)) = (k,\ldots,1) \cdot u(d(-)), 
\]

\[
d((k+1,1) \cdot u(-)) = (k,1) \cdot u(d((k,1) \cdot (-))), 
\]

where $\lambda_n$ is an arbitrary scalar. In the left-hand sides above,

\[
u : p[n] \to p[n+1] \quad \text{and} \quad d : p[n+1] \to p[n], 
\]

while in the right-hand sides,

\[
u : p[n-1] \to p[n] \quad \text{and} \quad d : p[n] \to p[n-1]. 
\]

By using invariance under the appropriate symmetric groups, one sees that relations (19.23b) and (19.23c) imply each other; in other words, they are equivalent.

**Proposition 19.27.** Let $(p,u,d)$ be a species with up-down operators. Let $v,w \in V$ and $f,g \in V^*$, and let $\bar{c}(v)$ and $\bar{a}(f)$ be as in (19.12).

(i) If (19.21) holds, then $\bar{c}(w)\bar{c}(v) = \bar{c}(v)\bar{c}(w)$.

(ii) If (19.22) holds, then $\bar{a}(g)\bar{a}(f) = \bar{a}(f)\bar{a}(g)$.

(iii) If (19.23) holds, then $\bar{a}(f)\bar{c}(v) - \bar{c}(v)\bar{a}(f) = \lambda_n f(v) \id$.

In the third statement, (19.23) refers to all three relations (19.23a)–(19.23c).

In particular, the generalized bosonic Fock space of a species with balanced operators satisfies the usual bosonic commutation relations (19.4). Conjugating by $\bar{\kappa}$ or its inverse and using (19.13) and (19.14), the above result is equivalent to:

(i) If (19.21) holds, then $\bar{c}(w)\bar{c}(v) = \bar{c}(v)\bar{c}(w)$.

(ii) If (19.22) holds, then $\bar{a}(g)\bar{a}(f) = \bar{a}(f)\bar{a}(g)$.

(iii) If (19.23) holds, then $\bar{a}(f)\bar{c}(v) - \bar{c}(v)\bar{a}(f) = \lambda_n f(v) \id$. 


Proof. Consider (i), the case of up operators. It is convenient here to work with \( \bar{c} \) rather than \( \tilde{c} \). Applying the operators \( c(w)c(v) \) and \( c(v)c(w) \) to \( x \otimes v_1 v_2 \cdots v_n \) yields
\[
u^2(x) \otimes wvv_1 v_2 \cdots v_n \quad \text{and} \quad u^2(x) \otimes vv_1 v_2 \cdots v_n
\]
respectively. If (19.21) holds, then applying the transposition \((1, 2)\) to one yields the other. So they represent the same element in the space of coinvariants, thus \( \bar{c}(w)\bar{c}(v) = \bar{c}(v)\bar{c}(w) \), proving (i).

Case (ii) is similar and omitted. Case (iii) follows by applying either Propositions 19.24 or 19.25.

We illustrate this result with some interesting examples. Examples 19.31 and 19.32 are due to Gut\'a and Maassen [158, Section 4.1].

Example 19.28. Let \((E, u, d)\) be the exponential species with up-down operators, as in Example 8.55. In this case \( u \) and \( d \) are inverse and the symmetric group action is trivial. Thus \((E, u, d)\) is a species with balanced operators with \( \lambda_n = 1 \). By applying Proposition 19.27, one recovers the bosonic commutation relations (19.4).

Example 19.29. Let \( e \) be the species of elements defined in Section 8.13.7. Thus, \( e[I] = kI \), the vector space with basis the elements of \( I \). We first note that \( e \) carries up-down operators, the up operator being the natural inclusion and the down operator being given by
\[
\begin{align*}
e[I^+] & \to e[I] \\
i \mapsto & \begin{cases} i & i \in I, \\
0 & i = \ast_I. \end{cases}
\end{align*}
\]

It is straightforward to check that \((e, u, d)\) is a species with balanced operators with \( \lambda_n = 1 \). By applying Proposition 19.27, one sees that the same commutation relations (19.4) hold on the generalized bosonic space of the species of elements.

This can be understood more explicitly as follows. The generalized bosonic space of the species of elements can be identified with
\[
K^\infty(V) \cong k \oplus V \otimes S(V), \quad v_0 \otimes v_1 \cdots v_n \mapsto v_0 \otimes v_1 \cdots v_{i-1} v_i v_{i+1} \cdots v_n.
\]

Under this identification, the creation operator \( \bar{c}(v) \) and annihilation operator \( \tilde{a}(f) \) send \( v_0 \otimes v_1 \cdots v_n \) to
\[
\begin{align*}
v_0 \otimes vv_1 \cdots v_n \quad \text{and} \quad \sum_{i=1}^{n} v_0 \otimes v_1 \cdots f(v_i) \cdots v_n
\end{align*}
\]
respectively. With these descriptions, the commutation relations may also be checked directly.

Example 19.30. Let \( E^2 = E \cdot E \) be the species of subsets (Example 8.17). We may use the up-down operators of \( E \) on either factor to define
\[
u_1, u_2 : E^2[I] \to E^2[I^+] \quad \text{and} \quad d_1, d_2 : E^2[I^+] \to E^2[I]
\]
by
\[
u_1(S) = S, \quad u_2(S) = S \cup \{\ast_I\},
\]
and
\[
\begin{align*}
d_1(S) & = \begin{cases} S & \ast_I \notin S, \\
0 & \ast_I \in S, \end{cases} \\
d_2(S) & = \begin{cases} 0 & \ast_I \notin S, \\
S \setminus \{\ast_I\} & \ast_I \in S. \end{cases}
\end{align*}
\]
It is straightforward to check that \((E_2, u_1, d_1)\) and \((E_2, u_2, d_2)\) are species with balanced operators with \(\lambda_0 = 1\), while \((E_2, u_1, d_2)\) and \((E_2, u_2, d_1)\) are species with balanced operators with \(\lambda_n = 0\).

Proceeding more directly, generalized bosonic space of the subset species can be identified with
\[
\mathcal{K}_V(E_2) \xrightarrow{\cong} S(V) \otimes S(V), \quad T \otimes v_1 \cdots v_n \mapsto v_1 \cdots v_i \otimes v_{j_1} \cdots v_{j_s},
\]
where \(T = \{i_1, \ldots, i_t\}\) and its complement in \([n]\) is \(\{j_1, \ldots, j_s\}\). Under this identification, the creation operators \(\bar{c}_1(v)\) and \(\bar{c}_2(v)\) send \(u_1 \cdots u_m \otimes v_1 \cdots v_n\) to
\[
v u_1 \cdots u_m \otimes v_1 \cdots v_n \quad \text{and} \quad u_1 \cdots u_m \otimes v v_1 \cdots v_n
\]
respectively, and the annihilation operators \(\bar{a}_1(f)\) and \(\bar{a}_2(f)\) send it to
\[
\sum_{i=1}^m u_1 \cdots f(u_i) \cdots u_m \otimes v_1 \cdots v_n \quad \text{and} \quad \sum_{i=1}^n u_1 \cdots u_m \otimes v_1 \cdots f(v_i) \cdots v_n
\]
respectively. With these descriptions, the various commutation relations asserted above can be checked directly.

**Example 19.31.** Let \((L, u, d)\) be the species of linear orders with up-down operators, as in Example 8.56. Interestingly, \((L, u, d)\) is not a species with balanced operators; among the required relations only \((19.23a)\) holds. So Proposition 19.27 does not apply.

Let us write down the creation and annihilation operators explicitly. First observe that
\[
\mathcal{K}_V(L) \xrightarrow{\cong} T(V), \quad \bar{1} \cdots \bar{n} \otimes v_1 \cdots v_n \mapsto v_1 \cdots v_n,
\]
which is classical full Fock space. Under this identification, the creation operator \(\bar{c}(v)\) and the annihilation operator \(\bar{a}(f)\) send \(v_1 \cdots v_n\) to
\[
v v_1 \cdots v_n \quad \text{and} \quad f(v_1)v_2 \cdots v_n
\]
respectively. These are the classical creation and annihilation operators on full Fock space of Section 19.3. It follows that
\[
\bar{a}(f)\bar{c}(v) = \bar{a}(f)\bar{c}(v) = f(v) \text{id}.
\]
This can also be verified from equation (19.16) as follows.

For any \(l \in L[n]\) we have
\[
du(l) = l \quad \text{and} \quad d((k + 1, \ldots, 1) \cdot u(l)) = 0 \quad \text{for every} \quad k = 1, \ldots, n,
\]
since the minimum element of \((k + 1, \ldots, 1) \cdot u(l)\) is not 1.

The relation noted above does not look like a commutation relation. However, it is indeed the case \(q = 0\) of a \(q\)-commutation relation. This point will be clarified later in Example 19.43.

**Example 19.32.** Consider the species \(a\) of rooted trees (Section 13.3.1): \(a[I]\) is the space with basis consisting of all rooted trees with vertex set \(I\). Given a rooted tree \(t \in a[I]\), a vertex \(i \in I\), and a new element \(j \notin I\), let
\[
t_i^j \in a[I \cup \{j\}]
\]
be the rooted tree obtained by attaching a new leaf with label \(j\) to vertex \(i\). In addition, given a leaf \(k\) of \(t\), let
\[
t \setminus k \in a[I \setminus \{k\}]
\]
be the rooted tree obtained by removing leaf $k$ from $t$. These constructions are illustrated below, for $I = \{i, k, y, z\}$.

The maps

$$a[I] \xrightarrow{\text{u}} a[I^+]$$

given by

$$u(t) := \sum_{i \in I} t_{i}^* \quad \text{for} \; t \in a[I]$$

and

$$d(t) := \begin{cases} t \setminus *_{I} & \text{if} \; *_{I} \text{is a leaf of} \; t, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for} \; t \in a[I^+]$$

turn $a$ into a species with up-down operators. However, $(a, u, d)$ is not a species with balanced operators; relations (19.21) and (19.22) do not hold. However, relation (19.23) holds with $\lambda_n = n$. Details are as follows. For any $t \in a[n]$ we have

$$du(t) = nt$$

and for any $k = 1, \ldots, n$

$$d((k + 1, 1) \cdot u(t)) = \begin{cases} \sum_{i \in [n] \setminus \{k\}} (t \setminus k)_i^k & \text{if} \; k \text{is a leaf of} \; t, \\ 0 & \text{otherwise} \end{cases} = (k, 1) \cdot ud((k, 1) \cdot t).$$

The summation above consists of all trees obtained from $t$ by removing leaf $k$ and reattaching it to a vertex of the remaining tree. An example follows.

It follows from Proposition 19.27 that

$$(\tilde{a}(f)\tilde{c}(v) - \tilde{c}(v)\tilde{a}(f))(x) = nf(v)x$$

for any unlabeled, $V$-decorated rooted tree $x$ with $n$ vertices.

Proceeding more directly, the generalized bosonic Fock space of rooted trees can be identified with the space of unlabeled rooted trees with $n$ vertices, each
vertex decorated by an element of $V$. For instance, the unlabeled $V$-decorated tree below

![Tree](image)

if viewed as a coinvariant, is the image of

$$\otimes v_1v_2v_3 \quad \text{or, say,} \quad \otimes v_2v_1v_3$$

and, if viewed as an invariant, is the following element of $a[3] \otimes_{S_3} V \otimes 3$.

$$\otimes (v_1v_2v_3 + v_3v_2v_1) + \otimes (v_2v_1v_3 + v_2v_3v_1) + \otimes (v_1v_3v_2 + v_3v_1v_2)$$

In keeping with the previous notation, we make the following definitions. For an unlabeled $V$-decorated tree $t$, vertex $n$ and $v \in V$, we let $t^v_n$ be the tree obtained by attaching a new leaf with label $v$ to vertex $n$. In addition, for a leaf $l$, we let $t \setminus l$ be the tree obtained by removing leaf $l$ from $t$.

Under the above identification (say with invariants), the creation and annihilation operators are:

$$\tilde{c}(v)(t) = \sum_{n: \text{vertex of } t} t^v_n \quad \text{and} \quad \bar{a}(f)(t) = \sum_{l: \text{leaf of } t} f(v_l) t \setminus l,$$

where $v_l \in V$ is the label of the leaf $l$.

The creation operator $\tilde{c}(v)$ applied to the above example yields

![Tree](image)

while the annihilation operator $\bar{a}(f)$ yields

![Tree](image)

One can now check directly that (19.24) holds. Further, one sees that neither the creation operators nor the annihilation operators commute. This is consistent with the conclusion drawn from Proposition 19.27.
19.7. Deformations of decorated Fock functors

The theory of undecorated Fock functors developed in Chapter 15 admits several generalizations, one of them being the one-parameter deformations of Chapter 16. In this section we briefly consider this type of deformation in the decorated case.

19.7.1. The decorated fermionic Fock functor. It is natural to ask for a decorated version of the fermionic Fock functor \( \overline{K}_{-1} \). Recall that the bosonic and fermionic Fock functors determine each other by precomposing with the signature functor. Following this idea, we define the decorated fermionic Fock functor, denoted \( \overline{K}_{V,-1} \), as the composite:

\[
(\text{Sp}, \cdot, \beta_p) \xrightarrow{(-) -} (\text{Sp}, \cdot, \beta_{-p}) \xrightarrow{\overline{K}_V} (\text{gVec}, \cdot, \beta_{-p}),
\]

where we recall that \((-) - \) is the signature functor (9.10). As a functor, \( \overline{K}_{V,-1}(p) \) is given by the same formula as given for \( \overline{K}_V(p) \) in Definition 19.1, with the understanding that the coinvariants are taken with respect to the signed action of \( S_n \) on \( V \otimes \bigwedge^n \) as in (19.5). Since the composite of bilax functors is bilax, the above formulation defines \( \overline{K}_{V,-1} \) not just as a functor but as a bilax functor. Since \( \overline{K}_V \) itself can be viewed as a composite (Proposition 19.3), there are alternative ways of viewing \( \overline{K}_{V,-1} \) as a composite. For example, the composites

\[
(\text{Sp}, \cdot, \beta_p) \xrightarrow{(-) \times (E_V)} (\text{Sp}, \cdot, \beta_{-p}) \xrightarrow{\overline{K}_V} (\text{gVec}, \cdot, \beta_{-p}),
\]

both yield \( \overline{K}_{V,-1} \). Here \( (E_V) - \) is the signed partner of \( E_V \) which is the same as the signature functor applied to \( E_V \).

The functor \( \overline{K}_{V,-1} \) can be defined similarly as the composite:

\[
(\text{Sp}, \cdot, \beta_p) \xrightarrow{(-) \times \overline{E}_V} (\text{Sp}, \cdot, \beta_{-p}) \xrightarrow{\overline{K}_V} (\text{gVec}, \cdot, \beta_{-p}),
\]

and described in alternative ways by using \( \overline{K}^\vee \) and \( \overline{K}_{V,-1} \) in the above discussion. It follows from Proposition 19.5 that

\[
\overline{K}_{V,-1}^\vee \cong (\overline{K}_{V,-1})^\vee.
\]

19.7.2. One-parameter deformations. Recall that the full Fock functors can be deformed to yield functors \( K_q \) and \( K^\vee_q \) which depend on a scalar \( q \). This was explained in Section 16.1. In a similar manner, the decorated full Fock functors can be deformed to yield functors \( K_{V,q} \) and \( K^\vee_{V,q} \). We call them the deformed decorated full Fock functors. We now comment on their bilax structures.

Recall the structure maps \( \varphi \) and \( \psi \) of \( K_V \) from Section 19.1.3. For the functor \( K_{V,q} \), the lax structure \( \varphi \) remains unchanged while the colax structure \( \psi \) is deformed to \( \psi_q \) by the coefficient \( q^{\text{sch}_n(S)} \), the exponent being the Schubert statistic. The structure maps of \( K^\vee_{V,q} \), denoted \( \varphi^\vee_q \) and \( \psi^\vee_q \), can be made explicit in the same manner.

**Theorem 19.33.** The functors

\[
(K_{V,q}, \varphi, \psi_q), (K^\vee_{V,q}, \psi_q^\vee, \varphi^\vee_q) : (\text{Sp}, \cdot, \beta_p) \to (\text{gVec}, \cdot, \beta_{pq}).
\]

are bilax monoidal.
Proposition 19.34. The functors $\mathcal{K}_{V,q}$ and $\mathcal{K}_{V,q}^\vee$ are the following composites of bilax functors

\[
\begin{align*}
(\mathbf{Sp}, \cdot, \beta_p) \xrightarrow{(-) \times \mathbf{E}_V} (\mathbf{Sp}, \cdot, \beta_p) & \xrightarrow{\mathcal{K}_q} (\mathbf{gVec}, \cdot, \beta_{pq}) \\
(\mathbf{Sp}, \cdot, \beta_p) \xrightarrow{(-) \times \mathbf{E}_V} (\mathbf{Sp}, \cdot, \beta_p) & \xrightarrow{\mathcal{K}_{V,q}^\vee} (\mathbf{gVec}, \cdot, \beta_{pq})
\end{align*}
\]

respectively.

Decorated versions of results in Chapter 16 can be obtained as a consequence of this result. We state some of them below.

Proposition 19.35. There are isomorphisms of bilax functors

\[\mathcal{K}_{V,q}(-) \cong \mathcal{K}_q(L_q \times (-)) \quad \text{and} \quad \mathcal{K}_{V,q}^\vee(-) \cong \mathcal{K}_q^\vee(L_q^* \times (-))\]

from $(\mathbf{Sp}, \cdot, \beta_p)$ to $(\mathbf{gVec}, \cdot, \beta_{pq})$.

**Proof.** We give the argument for the first part.

\[
\mathcal{K}_{V,q}(-) \cong \mathcal{K}_q((-) \times \mathbf{E}_V) \cong \mathcal{K}_q^\vee((-) \times \mathbf{E}_V) \cong \mathcal{K}_q^\vee(L_q \times (-)).
\]

The first isomorphism follows from Proposition 19.34, the second follows from Proposition 16.6, and the third follows from Proposition 19.3. \(\square\)

We now discuss a deformation of the decorated norm transformation of Definition 19.6. Let

\[
\kappa_q : \mathcal{K}_{V,q} \Rightarrow \mathcal{K}_{V,q}^\vee
\]

be defined as follows. For any species $\mathbf{p}$, let

\[
(k_q)_\mathbf{p}(x \otimes v_1 \cdots v_n) := \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} \sigma \cdot (x \otimes v_1 \cdots v_n),
\]

for any $x \in \mathbf{p}[n], v_i \in V$. This is the decorated $q$-norm. The dependence of $\kappa_q$ on $V$ is not manifest in the notation.

It follows directly from the definition that the decorated $q$-norm is the result of precomposing the undecorated $q$-norm $\kappa_q$ of Definition 16.13 with the functor $(-) \times \mathbf{E}_V$. It then follows from Proposition 16.15 that the decorated $q$-norm is a morphism of bilax functors. Similarly, one can deduce from Proposition 16.14 that the contragredient for the decorated $q$-norm for $V$ is the decorated $q$-norm for $V^*$.

Let $\mathcal{Z}_{V,q}$ be the image of the decorated $q$-norm. We call it the **decorated anyonic Fock functor**. It fits into the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{K}_{V,q} & \xrightarrow{\kappa_q} & \mathcal{K}_{V,q}^\vee \\
\downarrow & & \downarrow \\
\mathcal{Z}_{V,q} & & \\
\end{array}
\]

On finite-dimensional species, the dual of this diagram is the same diagram with $V$ replaced by $V^*$.

It follows that $\mathcal{Z}_{V,q}$ is the composite:

\[
(\mathbf{Sp}, \cdot, \beta_p) \xrightarrow{(-) \times \mathbf{E}_V} (\mathbf{Sp}, \cdot, \beta_p) \xrightarrow{\mathcal{Z}_q} (\mathbf{gVec}, \cdot, \beta_{pq}),
\]
where $\mathcal{Fq}$ is the anyonic Fock functor (Section 16.3.5). We define anyonic Fock space to be the value of the decorated anyonic Fock functor on the exponential species. We note that

$$\mathcal{F} = \mathcal{G} = \mathcal{K},$$

the functor considered in Section 19.2.1. This is the case $q = 1$. For $q = -1$, in characteristic 0, we have

$$\mathcal{F}_{V,1} \cong \mathcal{K}_{V,-1} \cong \mathcal{K}_{V,-1},$$

the decorated fermionic Fock functors. For $q = 0$, the decorated 0-norm is the identity, so

$$\mathcal{F}_{V,0} = \mathcal{K}_{V,0} = \mathcal{K}_{V,0} = \mathcal{K}_{V,0}.$$

We call this the decorated free Fock functor. In this situation, the result of Proposition 19.35 says that

$$\mathcal{F}_{V,0}(-) \cong \mathcal{K}_V \left( \mathcal{L}_0 \times (\cdot) \right).$$

This is an isomorphism of bilax functors from $(\mathcal{S}p, \cdot, \beta_p)$ to $(\mathcal{G}Vec, \cdot, \beta_0)$.

19.8. Deformations related to up-down and creation-annihilation

In the previous section, we looked at deformations of decorated Fock functors. We now look at the behavior of these functors on species with up-down operators. The main result is Proposition 19.41 which says that if the up-down operators are balanced (in a weaker sense) then the resulting creation-annihilation operators satisfy a $q$-commutation relation.

19.8.1. The Hadamard and signature functors. We know from Proposition 8.58 that the Hadamard functor $(\times, \varphi, \psi)$ on species is bilax with respect to the Cauchy product. We would like to upgrade this result to species with up or down operators. This is not possible. The best one can say is the following.

**Proposition 19.36.** The functor

$$(\times, \psi): (\mathcal{S}p^u \times \mathcal{S}p^u, \cdot_p \times \cdot_q) \to (\mathcal{S}p^u, \cdot_{pq})$$

is colax. Dually, the functor

$$(\times, \varphi): (\mathcal{S}p^d \times \mathcal{S}p^d, \cdot_p \times \cdot_q) \to (\mathcal{S}p^d, \cdot_{pq})$$

is lax.

**Proof.** We give the argument for the first part. First, define the Hadamard product of two species with up operators $(p, u)$ and $(q, v)$ to be $(p \cdot q, w)$ where

$$(u_1 \cdot id + \tau_p \cdot u_2) \times (u_2 \cdot id + \tau_q \cdot v_2)$$

Let $(p_1, u_1)$, $(p_2, u_2)$, $(q_1, v_1)$ and $(q_2, v_2)$ be species with up operators. We need to check that the following diagram commutes, with $\tau_q$ as in (8.69).

$$
\begin{array}{ccc}
(p_1 \cdot q_1) \times (p_2 \cdot q_2) & \xrightarrow{\psi} & (p_1 \times p_2) \cdot (q_1 \times q_2) \\
\downarrow & & \downarrow
\end{array}
$$

$$
(u_1 \times u_2 \cdot id + \tau_{pq} \cdot (v_1 \times v_2))
$$

$$
\begin{array}{ccc}
(p_1 \cdot q_1)' \times (p_2 \cdot q_2)' & \xrightarrow{\psi} & ((p_1 \times p_2) \cdot (q_1 \times q_2))'
\end{array}
$$

where $\mathcal{Fq}$ is the anyonic Fock functor (Section 16.3.5). We define anyonic Fock space to be the value of the decorated anyonic Fock functor on the exponential species. We note that

$$\mathcal{F}_{V,1} = \mathcal{G},$$

the function considered in Section 19.2.1. This is the case $q = 1$. For $q = -1$, in characteristic 0, we have

$$\mathcal{F}_{V,-1} \cong \mathcal{K}_{V,-1} \cong \mathcal{K}_{V,-1},$$

the decorated fermionic Fock functors. For $q = 0$, the decorated 0-norm is the identity, so

$$\mathcal{F}_{V,0} = \mathcal{F}_{V,0} = \mathcal{F}_{V,0} = \mathcal{F}_{V,0}.$$

We call this the decorated free Fock functor. In this situation, the result of Proposition 19.35 says that

$$\mathcal{F}_{V,0}(-) \cong \mathcal{K}_V \left( \mathcal{L}_0 \times (\cdot) \right).$$

This is an isomorphism of bilax functors from $(\mathcal{S}p, \cdot, \beta_p)$ to $(\mathcal{G}Vec, \cdot, \beta_0)$. 
One verifies this on each component. The first observation is that the diagram commutes trivially unless one starts in a component of the form
\[ (p_1[S] \otimes q_1[T]) \otimes (p_2[S] \otimes q_2[T]). \]
On this component, the check is straightforward. One needs to use \( \tau_p \tau_q = \tau_{pq} \).

Recall the signature functor \((-)^-\) which sends a species \( p \) to its signed partner \( p^- = p \times E^- \). The above result along with the observation of Section 9.3 that \( E^- \) is a comonoid in \( (\text{Sp}^u, \cdot, -1) \) and a monoid in \( (\text{Sp}_d, \cdot, -1) \) implies the following.

**Proposition 19.37.** The functor
\[ (-)^-: (\text{Sp}^u, \cdot, q) \to (\text{Sp}^u, \cdot, -q) \]
is colax. Dually, the functor
\[ (-)^-: (\text{Sp}_d, \cdot, q) \to (\text{Sp}_d, \cdot, -q) \]
is lax.

**19.8.2. The up-down properties of the deformed Fock functors.** Let \( v \in V \) and \( f \in V^* \) be fixed. Consider the functors
\[ \mathcal{K}_{V, q, v}^\lor: \text{Sp}^u \to g\text{Vec}^c \quad \text{and} \quad \mathcal{K}_{V, q, f}^\lor: \text{Sp}_d \to g\text{Vec}_a, \]
with the creation operator on the former and the annihilation operator on the latter defined in the same way as before. So far, there is no dependence on \( q \). The dependence comes when one considers the monoidal properties of these functors. Keeping in mind the undeformed case, we do not expect these functors to be bilax; rather, we expect the former to be colax, and dually the latter to be lax. Accordingly:

**Proposition 19.38.** The functor
\[ (\mathcal{K}_{V, q, v}^\lor, \psi_q): (\text{Sp}^u, \cdot, p) \to (g\text{Vec}^c, \cdot, pq) \]
is colax monoidal. The functor
\[ (\mathcal{K}_{V, q, f}^\lor, \psi_q^\lor): (\text{Sp}_d, \cdot, p) \to (g\text{Vec}_a, \cdot, pq) \]
is lax monoidal.

The proof is straightforward and omitted.

**19.8.3. Creation-annihilation on generalized fermionic Fock spaces.** The entire discussion in Section 19.5 can be carried out for the decorated fermionic Fock functors of Section 19.7.1. The starting point is to take diagram (19.11) and replace \( \mathcal{K}_V, \mathcal{K}_V^\lor \) and \( \kappa \) with \( \mathcal{K}_{V, -1}, \mathcal{K}_{V, -1}^\lor \) and \( \kappa^{-1} \) respectively. Up to isomorphism, this is equivalent to precomposing (19.11) with the signature functor. Using the invertibility of \( \kappa^{-1} \) at the fermionic level, this yields two new noncommutative diagrams as follows.

\[
\begin{align*}
\mathcal{K}_{V, -1, f}^\lor(p, d) \xrightarrow{a(f)} \mathcal{K}_{V, -1, f}(p, d) \\
\mathcal{K}_{V, -1, f}^\lor(p, d) \xrightarrow{\bar{a}(f)} \mathcal{K}_{V, -1, f}(p, d)
\end{align*}
\]

\[
\begin{align*}
\mathcal{K}_{V, -1, v}^\lor(p, u) \xrightarrow{\bar{a}(f)} \mathcal{K}_{V, -1, v}(p, u) \\
\mathcal{K}_{V, -1, v}^\lor(p, u) \xrightarrow{\bar{a}(f)} \mathcal{K}_{V, -1, v}(p, u)
\end{align*}
\]
Formulas (19.16) and (19.17) hold with the coefficients $(-1)^k$ and $(-1)^{k-1}$ respectively inserted inside the double summations. More details are given in the $q$-version below. The same is true for the formulas (19.19) and (19.20). This leads to the following fermionic version of Proposition 19.27.

**Proposition 19.39.** Let $(p, u, d)$ be a species with up-down operators. Let $v, w \in V$ and $f, g \in V^*$, and let $\tilde{c}(v)$ and $\tilde{a}(f)$ be as in (19.31).

(i) If (19.21) holds, then $\tilde{c}(w)\tilde{c}(v) = -\tilde{c}(v)\tilde{c}(w)$.

(ii) If (19.22) holds, then $\tilde{a}(g)\tilde{a}(f) = -\tilde{a}(f)\tilde{a}(g)$.

(iii) If (19.23) holds, then $\tilde{a}(f)\tilde{c}(v) + \tilde{c}(v)\tilde{a}(f) = \lambda_n f(v) \text{id}$.

In the third statement, (19.23) refers to all three relations (19.23a)–(19.23c).

In particular, the generalized fermionic Fock space of a species with balanced operators satisfies the usual fermionic commutation relations (19.6). Conjugating by $\kappa_{-1}$ or its inverse, the same relations hold with $\tilde{c}$ and $\tilde{a}$ replaced by $\bar{c}$ and $\bar{a}$.

In the examples of Section 19.6, we had derived various bosonic commutation relations. In light of the above result, we see that they have corresponding fermionic analogues. We content ourselves by mentioning that for the exponential species, the above result recovers the commutation relations on fermionic Fock space (19.6).


We would like to unify the bosonic and fermionic settings. For that purpose, we consider the decorated anyonic Fock functor $\mathcal{I}_{V,q}$. It turns out that there is an induced functor $\mathcal{I}_{V,q,v,f} : \mathcal{S}p_{u,d}^{u} \rightarrow g\mathcal{V}ec_{c,a}$, from species with up-down operators to graded vector spaces with creation-annihilation operators. This means that $K_{V,q,v} \Rightarrow \mathcal{I}_{V,q,v,f}$ and $\mathcal{I}_{V,q,v,f} \Rightarrow K_{V,q,f}$ are natural transformations, the former for up and creation, and the latter for down and annihilation.

From now on, we simply write $\mathcal{I}_{V,q}$, suppressing the dependence on $v$ and $f$. The value of this functor on a species with up-down operators is a generalized anyonic Fock space. It carries both creation and annihilation operators, which we denote simply by $c(v)$ and $a(f)$ without bar or tilde, keeping in view their unbiased nature.

Explicitly, the creation operator is given by

\[(19.32) \quad c(v) : x \otimes v_1 \cdots v_n \mapsto \sum_{k=0}^{n} q^k (k + 1, \ldots, 1) \cdot u(x) \otimes v_1 \cdots v_k v \cdots v_n.\]

Note that this is (19.15) with the coefficient $q^k$ inserted. This formula can be derived in the same manner using the decorated $q$-norm (19.26). One sees that the exponent of $q$ must be the number of inversions of $(k + 1, \ldots, 1)$ which is $k$.

The annihilation operator is given by

\[(19.33) \quad a(f) : x \otimes v_1 v_2 \cdots v_n \mapsto d(x) \otimes f(v_1) v_2 \cdots v_n,\]

as before with no dependence on $q$.

These formulas imply the following $q$-analogue of Proposition 19.24:
Proposition 19.40. Let \((p, u, d)\) be a species with up-down operators and
\[
\sum_i x_i \otimes v_1^i \cdots v_n^i
\]
an element of degree \(n\) in \(\mathcal{V}_{q,r}(p)\). Then (19.16) and (19.17) hold with coefficients \(q^k\) and \(q^{k-1}\) respectively inserted inside the double summations.

We point out that there is no similar statement for (19.19) and (19.20). As a further implication, we obtain the following \(q\)-analogue which simultaneously generalizes part (iii) of Propositions 19.27 and 19.39. Note that no claim is being made about the commutativity of creation operators or of annihilation operators.

Proposition 19.41. Let \((p, u, d)\) be a species with up-down operators. If (19.23) holds, then
\[
a(f)c(v) - q c(v)a(f) = \lambda_n f(v) \text{id}
\]
where the operators are acting on the degree \(n\) component of \(\mathcal{V}_{q,r}(p)\).

These type of deformed creation-annihilation operators and \(q\)-commutation relations for anyonic Fock space \(\mathcal{V}_{q,r}(E)\) have been considered in the literature, starting with the work of Bożejko and Speicher \[65, Section 2\]; for additional work and more recent references see also Anshelevich \[26\].

19.8.5. Relating the decorated anyonic Fock functors. Consider the decorated free Fock functor
\[
\mathcal{V}_{V,0} : \mathcal{S}p_u^d \to \mathcal{G}Vec_a^c.
\]
One can see from (19.32) and (19.33), or using (19.28) that creation and annihilation on generalized free Fock space are given by
\[
c(v) : x \otimes v_1 \cdots v_n \mapsto u(x) \otimes vv_1 \cdots v_n,
\]
\[
a(f) : x \otimes v_1 \cdots v_n \mapsto d(x) \otimes f(v_1) v_2 \cdots v_n.
\]
They verify Proposition 19.41 for \(q = 0\).

Now recall from Proposition 19.36 and its proof that if \(p\) and \(q\) are species with up-down operators, then so is their Hadamard product \(p \times q\). Also let \(L\) be the linear order species with up-down operators as defined in Example 8.56.

Proposition 19.42. The following is an isomorphism of functors
\[
(19.34) \quad \mathcal{V}_{V,0}(-) \cong \mathcal{V}(L \times (-))
\]
from \(\mathcal{S}p_u^d\) to \(\mathcal{G}Vec_a^c\).

This is a straightforward check which we omit.

It is worth comparing the claim made above with (19.29). We point out that in the present situation we are not making any claims about the monoidal properties of the functors; so it does not matter whether we write \(L\) or \(L_0\).

Example 19.43. Applying (19.34) to the exponential species, we obtain
\[
\mathcal{V}_{V,0}(E) \cong \mathcal{V}(L).
\]

The first space is the free Fock space. Creation-annihilation operators on this space satisfy the 0-commutation relation, that is, the relation of Proposition 19.41 with \(q = 0\). The second space is the generalized bosonic Fock space of the linear order species. However, \(L\) is not a species with balanced operators; so we do not expect the bosonic commutation relations to hold on this space. Rather, we are seeing
that the 0-commutation relation should hold which is exactly what was noted in Example 19.31.

**Example 19.44.** Applying (19.34) to the species of elements (Example 19.29), we obtain

\[ \mathcal{Z}_{V,0}(e) \cong \mathcal{Z}_V(L \times e). \]

The first space is the generalized free Fock space of the species of elements. Since \( e \) is a species with balanced operators, the creation-annihilation operators acting on it satisfy the 0-commutation relation.

The pointing of the species of linear orders \( L^\bullet = L \times e \) carries up-down operators; however, they do not turn \( L^\bullet \) into a species with balanced operators. Hence we do not expect the creation-annihilation operators acting on generalized bosonic Fock space of \( L^\bullet \) to satisfy the usual commutation relation. Instead, the above isomorphism shows that they satisfy the 0-commutation relation.

### 19.9. Yang–Baxter deformations of decorated Fock functors

In Section 19.7, we discussed one-parameter deformations of the decorated Fock functors. We now sketch a more general framework for deformations in the decorated setting. To express the monoidal properties of these deformed functors, one needs to generalize the notion of a bilax functor to the context where the source category is braided but the target category is only partially braided. We explain the main idea behind this notion, and then discuss the examples of interest to us.


Let \((D, \bullet)\) be a monoidal category. Recall that a Yang–Baxter operator on a functor \( F : C \to D \) consists of a natural isomorphism

\[ \nu : F(A) \bullet F(B) \to F(B) \bullet F(A) \]

satisfying the dodecagon axiom [184, Definition 2.4]. If \( C \) has only one arrow, then we recover the more common notion of a Yang–Baxter operator [191, Definition XIII.3.1].

Now suppose that \( C \) is a braided monoidal category and let \((F, \nu)\) be as above. Note that we do not require \( D \) to be braided. Even then we can make sense of when \((F, \nu)\) is bilax: in the braiding axiom (3.11) use \( \nu \) instead of \( \beta \). This idea can be used to give an abstract definition of a bilax functor in this setting. In the recent paper [266], McCurdy and Street have discussed this notion (independently from our work). However, we point out that in addition to the usual axioms one would also need compatibilities of \( \nu \) with the lax and colax structures of \( F \). In the usual setting, these compatibilities follow from properties of the braiding. One reason for this can be seen from the requirement: If \( C \) has only one object, then a bilax functor should specialize to a braided bialgebra [356, Definition 5.1].

The results of Takeuchi [356, Section 5] relating braided bialgebras to bialgebras in certain braided monoidal categories can be extended to results relating bilax monoidal functors in the context of a Yang–Baxter operator to bilax monoidal functors between braided monoidal categories.
Yang–Baxter deformations of decorated full Fock functors. Start with a Yang–Baxter operator \( R \) on \( V \). This implies that the tensor power \( V^\otimes n \) carries an action of the braid group \( B_n \). Recall that there is a canonical section

\[
s : S_n \to B_n
\]

which sends generators to generators in the usual presentations of these groups [246, Section 2.1.2]. This section is not a group homomorphism.

For any decomposition \( S \sqcup T = [n] \), consider the permutation \( \zeta : [n] \to [n] \) whose restrictions to \( S \) and \( T \) are the order-preserving maps \( \text{cano} : S \to [s] \) and \( \text{cano} : T \to [s + 1, s + t] \), where \( s = |S| \) and \( t = |T| \). Then there is an induced map

\[
s(\zeta) : V^\otimes n \to V^\otimes s \otimes V^\otimes t
\]

given by the action of the element \( s(\zeta) \in B_n \). More explicitly, we repeatedly apply the Yang–Baxter operator so that the \( V \)'s which lie in the positions specified by \( S \) move to the first \( s \) positions.

Now for \( s + t = n \), consider the permutation \( \zeta : [n] \to [n] \) whose restrictions to \( [s] \) and \( [s + 1, s + t] \) are the order-preserving maps \( \text{cano} : [s] \to [t + 1, s + t] \) and \( \text{cano} : [s + 1, s + t] \to [t] \). Then there is an induced map

\[
V^\otimes s \otimes V^\otimes t \to V^\otimes t \otimes V^\otimes s
\]

given by the action of the element \( s(\zeta) \in B_n \).

We now explain how these ideas can be used to construct a bilax functor

\[
K_{V,R} : (Sp, \cdot, \beta) \to (gVec, \cdot).
\]

To start with, the functor is defined by:

\[
K_{V,R}(p) := p[n] \otimes V^\otimes n.
\]

The lax structure is the same as for \( K_V \) while the colax structure is defined using (19.37). The structure map \( \nu \) of (19.35) is defined using (19.38). This turns \( K_{V,R} \) into a bilax functor.

The functor \( K_{V,R} \) is constructed along similar lines.

For any species \( p \), let \( \kappa_p : K_{V,R}(p) \to K_{V,R}^\vee(p) \) be the map of graded vector spaces given by

\[
\kappa_p(x \otimes v_1 \cdots v_n) := \sum_{\sigma \in S_n} \sigma \cdot x \otimes s(\sigma) \cdot (v_1 \cdots v_n),
\]

for any \( x \in p[n], v_i \in V \). This defines the norm transformation

\[
\kappa : K_{V,R} \Rightarrow K_{V,R}^\vee.
\]

It is a morphism of bilax functors. The image gives rise to the bilax functor \( \Im_{V,R} \).

Example 19.45. Consider the flip operator on \( V \) which interchanges the two tensor factors of \( V \otimes V \). This, as well as any scalar multiple \( q \) of it, is a Yang–Baxter operator \( R_q \) on \( V \). This gives the representation of \( B_n \) in which the action of the standard generators is by multiplication by \( q \). In this case, the Yang–Baxter operator \( \nu \) on \( K_{V,R_q} \) extends in fact to the braiding \( \beta_q \) on \( gVec \). The functors we obtain in this situation are

\[
K_{V,R_q} = K_{V,q}, \quad K_{V,R_q}^\vee = K_{V,q}^\vee, \quad \Im_{V,R_q} = \Im_{V,q},
\]

the functors of diagram (19.27). These are bilax in the usual sense.
Example 19.46. We now generalize the previous example. Let $Q$ be a square matrix of size $r$, where $r$ is the dimension of $V$. Fix a basis $x_1, x_2, \ldots, x_r$ of $V$, and consider the Yang–Baxter $R_Q$ operator on $V$:

$$V \otimes V \to V \otimes V, \quad x_i \otimes x_j \mapsto q_{ji} x_j \otimes x_i$$

where $i$ and $j$ vary between 1 and $r$, and $q_{ji}$ denotes the entries of the matrix $Q$. (We recover the previous example if all entries are equal.) The operator $R_Q$ is an involution precisely if $Q$ is log-antisymmetric. Let us denote the resulting functors by

$$K_{V,R_Q}, \ K_{V,R_Q}^\vee, \ \mathcal{S}_{V,R_Q}.$$ They are not bilax in the usual sense in general.

In Chapter 20, we construct these functors using colored species and multi-graded vector spaces. We highlight an important result. If the field characteristic is 0 and $Q$ is such that no monomial in the $q_{ij}$’s equals 1 then the norm transformation is an isomorphism and the three functors above are isomorphic (Theorem 20.11).

Example 19.47. We continue the discussion in the preceding example. Applying the above deformed functors to a bimonoid in species yields a braided bialgebra rather than a usual bialgebra.

Recall the bimonoid $E$ associated to the exponential species. The object $K_{V,R_Q}(E)$ is the free algebra on $r$ generators $k\langle x_1, \ldots, x_r \rangle$ of Example 2.14. The object $K_{V,R_Q}^\vee(E)$ is the quantum shuffle algebra as defined by Green [152] and Rosso [316, Proposition 9]. It has the same underlying space as the free algebra but the structure maps are different: the product is a deformation of the shuffle product and the coproduct is deconcatenation. The object $\mathcal{S}_{V,R_Q}(E)$ is Rosso’s quantum symmetric algebra associated to the matrix $Q$. This is also called the Nichols algebra of diagonal type associated to $Q$ [23, Proposition 2.11]. These objects appear in the classification of pointed Hopf algebras with abelian coradical [20, 22, 24, 25]. Sections 3.2 and 4 of the survey by Andruskiewitsch and Schneider [23] contain results on Nichols algebras of diagonal type. More information can be found in the lecture notes by Heckenberger [165].

A more detailed discussion of this example is given later in Example 20.21.

Other Yang–Baxter operators would lead to more general deformations of the decorated Fock functors. The resulting braided Hopf algebras after applying the functor $\mathcal{S}_{V,R}$ would include, for the special case of the exponential species, the Nichols algebra (also called quantum symmetric algebra) associated to the Yang–Baxter operator $R$. For information on Nichols algebras, see [23] and [165]. They are named after Warren Nichols who considered them in [284].

19.9.3. Up-down and creation-annihilation. The construction in Section 19.8 of creation-annihilation operators from species with up-down operators can also be extended to the setting of Yang–Baxter operators. The creation-annihilation operators act on usual anyonic Fock space but the action is deformed by the Yang–Baxter operator. We explain this briefly.

Fix $v \in V$ and $f \in V^*$. Then the value of the functor $\mathcal{S}_{V,R}$ on a species with up-down operators carries both creation and annihilation operators which are as follows.
The creation operator is given by

$$c(v) : x \otimes v_1 \cdots v_n \mapsto \sum_{k=0}^n (k+1, \ldots, 1) \cdot u(x) \otimes s(k+1, \ldots, 1) \cdot (v v_1 \cdots v_n),$$

where $s$ is the canonical section of (19.36). The Yang–Baxter operator $R$ appears in this map via the action on the tensors.

The annihilation operator is given by

$$a(f) : x \otimes v_1 v_2 \cdots v_n \mapsto d(x) \otimes f(v_1) v_2 \cdots v_n,$$

as before with no dependence on $R$.

To get the commutation relations, pick a basis $x_1, x_2, \ldots, x_r$ of $V$, and write

$$R(x_a \otimes x_b) = \sum_{c,d} R_{ab}^{cd} x_c \otimes x_d$$

for suitable coefficients $R_{ab}^{cd}$. One may check that:

**Proposition 19.48.** Let $(p, u, d)$ be a species with up-down operators, and let $R$ be a Yang–Baxter operator. If (19.23) holds, then for any $i$ and $j$ between 1 and $r$,

$$a(x_i^*) c(x_j) - \sum_{k,l} R_{ij}^{kl} c(x_k) a(x_l^*) = \lambda_n \delta_{ij} \text{id},$$

where the operators are acting on the degree $n$ component of $\mathcal{F}_{V,q}(p)$.

If $R = R_Q$ as in Example 19.46, then the above commutation relation becomes:

$$a(x_i^*) c(x_j) - q_{ij} c(x_j) a(x_i^*) = \lambda_n \delta_{ij} \text{id}.$$

If all the $q_{ij}$’s are equal, then one recovers the $q$-commutation relation of Proposition 19.41.

The special case of the exponential species recovers the creation-annihilation operators introduced by Bożejko and Speicher [66]. The commutation relation of Proposition 19.48 is given on [66, p. 109].