Preface

This book is based on the lectures on partial differential equations that I have given for many years at UCLA. It does not assume any knowledge of partial differential equations and can be considered as a first graduate course in linear PDE. However, some basic knowledge of the Fourier transform, Lebesgue integrals and elementary functional analysis is required. It is organized as lecture notes with emphasis on clarity and accessibility.

We shall briefly describe the content of the book. The first three chapters are the elementary theory of distributions and Fourier transforms of distributions with applications to partial differential equations with constant coefficients. It is similar to the first chapters of the books by Gelfand and Shilov [**GSh**] and Shilov [**Sh**]. Additional material includes the wave front sets of distributions, Sobolev spaces, the stationary phase lemma, the radiation conditions, and potential theory.

In Chapter IV the Dirichlet and the Neumann boundary value problems are considered for second order elliptic equations in a smooth bounded domain. The existence, uniqueness, and regularity of solutions are proven. A nontraditional topic of this chapter is the proof of the existence and uniqueness of the solutions of the Neumann and Dirichlet problems for homogeneous equations in Sobolev spaces of negative order on the boundary.

Chapter V is devoted to scattering theory including inverse scattering, inverse boundary value problem, and the obstacle problem.

Chapter VI starts with the theory of pseudodifferential operators with classical symbols. It is followed by the theory of parabolic Cauchy problems based on pseudodifferential operators with symbols analytic in the half-plane and heat kernel asymptotics.

The next topic of Chapter VI is the Cauchy problem for hyperbolic equations of order $m \geq 2$, the domains of dependence of solutions to hyperbolic equations, and Hörmander's theory [**H1**] of propagation of singularities for the equations of real principal type with applications to hyperbolic equations.

In Chapter VII the Fredholm property for elliptic boundary value problems and parametrices in smooth domains are studied following the approach of the author's book [**E1**]. The main application of the parametrix is the study of heat trace asymptotics as $t \to 0$. The parametrix construction allows one to compute explicitly two leading terms of the heat trace asymptotics for the cases of Dirichlet and Neumann boundary conditions. Chapter VII concludes with elements of the spectral theory of elliptic operators and the proof of the index theorem for elliptic operators in $\mathbb{R}^n$ following the works of Atiyah-Singer [**AtS1**], [**AtS2**] and Seeley [**Se3**].

The last Chapter VIII is devoted to the theory of Fourier integral operators. Starting with the local theory of FIO, we proceed to the global theory. We consider only a subclass of Hörmander's FIOs (see [**H1**]), assuming that the Lagrangian manifold of the FIO corresponds to the graph of a canonical transformation. In particular, having a global canonical transformation, we construct a global FIO corresponding to this canonical transformation. Next, following Maslov [**M1**], [**M2**], [**MF**], we construct a global geometric optic solution for a second order hyperbolic equation on arbitrary time interval $[0, T]$.

Chapter VIII concludes with a section on the oblique derivative problem. The oblique derivative problem is a good example of nonelliptic boundary value problem, and it attracted the attention of many mathematicians: Egorov-Kondrat'ev [**EgK**], Malutin [**Mal**], Mazya-Paneah [**MaP**], Mazya [**Ma**], and others. The section is based on the author's paper [**E3**], and it uses the FIOs to greatly simplify the problem. Similar results are obtained independently by Sjöstrand [**Sj**] and Duistermaat-Sjöstrand [**DSj**].

At the end of each chapter there is a problem section. Some problems are relatively simple exercises that help to study the material. Others are more difficult problems that cover additional topics not included in the book. In those cases hints or references to the original sources are given.

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