
Contents

Preface	xv
Acknowledgments	xvi
Chapter I. Theory of Distributions	1
Introduction to Chapters I, II, III	1
§1. Spaces of infinitely differentiable functions	2
1.1. Properties of the convolution	2
1.2. Approximation by C_0^∞ -functions	3
1.3. Proof of Proposition 1.1	5
1.4. Proof of property b) of the convolution	5
§2. Definition of a distribution	6
2.1. Examples of distributions	6
2.2. Regular functionals	7
2.3. Distributions in a domain	8
§3. Operations with distributions	9
3.1. Derivative of a distribution	9
3.2. Multiplication of a distribution by a C^∞ -function	9
3.3. Change of variables for distributions	10
§4. Convergence of distributions	10
4.1. Delta-like sequences	12
§5. Regularizations of nonintegrable functions	14
5.1. Regularization in \mathbb{R}^1	15
5.2. Regularization in \mathbb{R}^n	17
§6. Supports of distributions	20

6.1.	General form of a distribution with support at 0	20
6.2.	Distributions with compact supports	22
§7.	The convolution of distributions	24
7.1.	Convolution of $f \in \mathcal{D}'$ and $\varphi \in C_0^\infty$	24
7.2.	Convolution of $f \in \mathcal{D}'$ and $g \in \mathcal{E}'$	26
7.3.	Direct product of distributions	27
7.4.	Partial hypoellipticity	28
§8.	Problems	30
Chapter II. Fourier Transforms		33
§9.	Tempered distributions	33
9.1.	General form of a tempered distribution	35
§10.	Fourier transforms of tempered distributions	37
10.1.	Fourier transforms of functions in \mathcal{S}	38
10.2.	Fourier transform of tempered distributions	39
10.3.	Generalization of Liouville's theorem	41
§11.	Fourier transforms of distributions with compact supports	42
§12.	Fourier transforms of convolutions	45
§13.	Sobolev spaces	46
13.1.	Density of $C_0^\infty(\mathbb{R}^n)$ in $H_s(\mathbb{R}^n)$	49
13.2.	Multiplication by $a(x) \in \mathcal{S}$	50
13.3.	Sobolev's embedding theorem	51
13.4.	An equivalent norm for noninteger	52
13.5.	Restrictions to hyperplanes (traces)	53
13.6.	Duality of Sobolev spaces	54
13.7.	Invariance of $H_s(\mathbb{R}^n)$ under changes of variables	55
§14.	Singular supports and wave front sets of distributions	57
14.1.	Products of distributions	61
14.2.	Restrictions of distributions to a surface	63
§15.	Problems	65
Chapter III. Applications of Distributions to Partial Differential Equations		69
§16.	Partial differential equations with constant coefficients	69
16.1.	The heat equation	70
16.2.	The Schrödinger equation	72
16.3.	The wave equation	73
16.4.	Fundamental solutions for the wave equations	74
16.5.	The Laplace equation	78

16.6.	The reduced wave equation	81
16.7.	Faddeev's fundamental solutions for $(-\Delta - k^2)$	84
§17.	Existence of a fundamental solution	85
§18.	Hypoelliptic equations	87
18.1.	Characterization of hypoelliptic polynomials	89
18.2.	Examples of hypoelliptic operators	90
§19.	The radiation conditions	91
19.1.	The Helmholtz equation in \mathbb{R}^3	91
19.2.	Radiation conditions	93
19.3.	The stationary phase lemma	95
19.4.	Radiation conditions for $n \geq 2$	98
19.5.	The limiting amplitude principle	101
§20.	Single and double layer potentials	102
20.1.	Limiting values of double layers potentials	102
20.2.	Limiting values of normal derivatives of single layer potentials	106
§21.	Problems	107
Chapter IV.	Second Order Elliptic Equations in Bounded Domains	111
	Introduction to Chapter IV	111
§22.	Sobolev spaces in domains with smooth boundaries	112
22.1.	The spaces $\overset{\circ}{H}_s(\Omega)$ and $H_s(\Omega)$	112
22.2.	Equivalent norm in $H_m(\Omega)$	113
22.3.	The density of C_0^∞ in $\overset{\circ}{H}_s(\Omega)$	114
22.4.	Restrictions to $\partial\Omega$	115
22.5.	Duality of Sobolev spaces in Ω	116
§23.	Dirichlet problem for second order elliptic PDEs	117
23.1.	The main inequality	118
23.2.	Uniqueness and existence theorem in $\overset{\circ}{H}_1(\Omega)$	120
23.3.	Nonhomogeneous Dirichlet problem	121
§24.	Regularity of solutions for elliptic equations	122
24.1.	Interior regularity	123
24.2.	Boundary regularity	124
§25.	Variational approach. The Neumann problem	125
25.1.	Weak solution of the Neumann problem	127
25.2.	Regularity of weak solution of the Neumann problem	128
§26.	Boundary value problems with distribution boundary data	129

26.1.	Partial hypoellipticity property of elliptic equations	129
26.2.	Applications to nonhomogeneous Dirichlet and Neumann problems	132
§27.	Variational inequalities	134
27.1.	Minimization of a quadratic functional on a convex set.	134
27.2.	Characterization of the minimum point	135
§28.	Problems	137
Chapter V. Scattering Theory		141
Introduction to Chapter V		141
§29.	Agmon's estimates	142
§30.	Nonhomogeneous Schrödinger equation	148
30.1.	The case of $q(x) = O\left(\frac{1}{(1+ x)^{\frac{n+1}{2}+\alpha+\varepsilon}}\right)$	148
30.2.	Asymptotic behavior of outgoing solutions (the case of $q(x) = O\left(\frac{1}{(1+ x)^{\frac{n+1}{2}+\alpha+\varepsilon}}\right), \alpha > 0$)	149
30.3.	The case of $q(x) = O\left(\frac{1}{(1+ x)^{1+\varepsilon}}\right)$	149
§31.	The uniqueness of outgoing solutions	151
31.1.	Absence of discrete spectrum for $k^2 > 0$	155
31.2.	Existence of outgoing fundamental solution (the case of $q(x) = O\left(\frac{1}{(1+ x)^{\frac{n+1}{2}+\delta}}\right)$)	156
§32.	The limiting absorption principle	157
§33.	The scattering problem	160
33.1.	The scattering problem (the case of $q(x) = O\left(\frac{1}{(1+ x)^{n+\alpha}}\right)$)	160
33.2.	Inverse scattering problem (the case of $q(x) = O\left(\frac{1}{(1+ x)^{n+\alpha}}\right)$)	162
33.3.	The scattering problem (the case of $q(x) = O\left(\frac{1}{(1+ x)^{1+\varepsilon}}\right)$)	163
33.4.	Generalized distorted plane waves	164
33.5.	Generalized scattering amplitude	164
§34.	Inverse boundary value problem	168
34.1.	Electrical impedance tomography	171
§35.	Equivalence of inverse BVP and inverse scattering	172
§36.	Scattering by obstacles	175
36.1.	The case of the Neumann conditions	179
36.2.	Inverse obstacle problem	179
§37.	Inverse scattering at a fixed energy	181
37.1.	Relation between the scattering amplitude and the Faddeev's scattering amplitudes	181

37.2.	Analytic continuation of T_r	184
37.3.	The limiting values of T_r and Faddeev's scattering amplitude	187
37.4.	Final step: The recovery of $q(x)$	190
§38.	Inverse backscattering	191
38.1.	The case of real-valued potentials	192
§39.	Problems	193
Chapter VI.	Pseudodifferential Operators	197
	Introduction to Chapter VI	197
§40.	Boundedness and composition of ψ do's	198
40.1.	The boundedness theorem	198
40.2.	Composition of ψ do's	199
§41.	Elliptic operators and parametrices	204
41.1.	Parametrix for a strongly elliptic operator	204
41.2.	The existence and uniqueness theorem	206
41.3.	Elliptic regularity	206
§42.	Compactness and the Fredholm property	207
42.1.	Compact operators	207
42.2.	Fredholm operators	208
42.3.	Fredholm elliptic operators in \mathbb{R}^n	211
§43.	The adjoint of a pseudodifferential operator	211
43.1.	A general form of ψ do's	211
43.2.	The adjoint operator	214
43.3.	Weyl's ψ do's	215
§44.	Pseudolocal property and microlocal regularity	215
44.1.	The Schwartz kernel	215
44.2.	Pseudolocal property of ψ do's	217
44.3.	Microlocal regularity	218
§45.	Change-of-variables formula for ψ do's	221
§46.	The Cauchy problem for parabolic equations	223
46.1.	Parabolic ψ do's	223
46.2.	The Cauchy problem with zero initial conditions	225
46.3.	The Cauchy problem with nonzero initial conditions	226
§47.	The heat kernel	228
47.1.	Solving the Cauchy problem by Fourier-Laplace transform	228
47.2.	Asymptotics of the heat kernel as $t \rightarrow +0$.	230
§48.	The Cauchy problem for strictly hyperbolic equations	231
48.1.	The main estimate	233

48.2.	Uniqueness and parabolic regularization	235
48.3.	The Cauchy problem on a finite time interval	237
48.4.	Strictly hyperbolic equations of second order	240
§49.	Domain of dependence	243
§50.	Propagation of singularities	247
50.1.	The null-bicharacteristics	247
50.2.	Operators of real principal type	247
50.3.	Propagation of singularities for operators of real principal type	249
50.4.	Propagation of singularities in the case of a hyperbolic Cauchy problem	255
§51.	Problems	258
Chapter VII. Elliptic Boundary Value Problems and Parametrices		263
Introduction to Chapter VII		263
§52.	Pseudodifferential operators on a manifold	264
52.1.	Manifolds and vector bundles	264
52.2.	Definition of a pseudodifferential operator on a manifold	265
52.3.	Elliptic ψ do's on a manifold	266
§53.	Boundary value problems in the half-space	266
53.1.	Factorization of an elliptic symbol	266
53.2.	Explicit solution of the boundary value problem	268
§54.	Elliptic boundary value problems in a bounded domain	270
54.1.	The method of "freezing" coefficients	270
54.2.	The Fredholm property	273
54.3.	Invariant form of the ellipticity of boundary conditions	276
54.4.	Boundary value problems for elliptic systems of differential equations	276
§55.	Parametrices for elliptic boundary value problems	278
55.1.	Plus-operators and minus-operators	278
55.2.	Construction of the parametrix in the half-space	281
55.3.	Parametrix in a bounded domain	284
§56.	The heat trace asymptotics	285
56.1.	The existence and the estimates of the resolvent	285
56.2.	The parametrix construction	286
56.3.	The heat trace for the Dirichlet Laplacian	288
56.4.	The heat trace for the Neumann Laplacian	293
56.5.	The heat trace for the elliptic operator of an arbitrary order	294
§57.	Parametrix for the Dirichlet-to-Neumann operator	296

57.1.	Construction of the parametrix	296
57.2.	Determination of the metric on the boundary	300
§58.	Spectral theory of elliptic operators	301
58.1.	The nonselfadjoint case	301
58.2.	Trace class operators	302
58.3.	The selfadjoint case	305
58.4.	The case of a compact manifold	309
§59.	The index of elliptic operators in \mathbb{R}^n	311
59.1.	Properties of Fredholm operators	311
59.2.	Index of an elliptic ψ do	313
59.3.	Fredholm elliptic ψ do's in \mathbb{R}^n	316
59.4.	Elements of K -theory	317
59.5.	Proof of the index theorem	321
§60.	Problems	324
Chapter VIII.	Fourier Integral Operators	329
	Introduction to Chapter VIII	329
§61.	Boundedness of Fourier integral operators (FIO's)	330
61.1.	The definition of a FIO	330
61.2.	The boundedness of FIO's	331
61.3.	Canonical transformations	333
§62.	Operations with Fourier integral operators	334
62.1.	The stationary phase lemma	334
62.2.	Composition of a ψ do and a FIO	335
62.3.	Elliptic FIO's	337
62.4.	Egorov's theorem	338
§63.	The wave front set of Fourier integral operators	340
§64.	Parametrix for the hyperbolic Cauchy problem	342
64.1.	Asymptotic expansion	342
64.2.	Solution of the eikonal equation	344
64.3.	Solution of the transport equation	346
64.4.	Propagation of singularities	348
§65.	Global Fourier integral operators	349
65.1.	Lagrangian manifolds	349
65.2.	FIO's with nondegenerate phase functions	350
65.3.	Local coordinates for a graph of a canonical transformation	353
65.4.	Definition of a global FIO	358
65.5.	Construction of a global FIO given a global canonical transformation	360

65.6.	Composition of global FIO's	365
65.7.	Conjugation by a global FIO and the boundedness theorem	369
§66.	Geometric optics at large	370
66.1.	Generating functions and the Legendre transforms	370
66.2.	Asymptotic solutions	374
66.3.	The Maslov index	377
§67.	Oblique derivative problem	381
67.1.	Reduction to the boundary	381
67.2.	Formulation of the oblique derivative problem	382
67.3.	Model problem	384
67.4.	First order differential equations with symbols depending on x'	387
67.5.	The boundary value problem on $\partial\Omega$	394
§68.	Problems	399
	Bibliography	403
	Index	407