Chapter 1

Background Material on Rings and Modules

This chapter contains a review of some basic topics from general algebra. Proofs are frequently omitted. The reader should consider making a quick scan of this chapter upon first reading, and when necessary, refer back to the appropriate sections later. To avoid listing many special cases, a theorem will frequently be stated in a form that is more general than any given application may require.

1. Rings and Modules

We first establish some conventions and fundamental definitions. A *ring* is always associative and contains an identity element denoted by 1. In a ring which has at least two elements we require $0 \neq 1$. If $R \neq (0)$ and R has no zero divisors, then we say R is a *domain*. A commutative domain is called an *integral domain*. A domain in which every nonzero element is invertible is called a *division ring*. A commutative division ring is called a *field*. The set of all invertible elements in a ring R is a group which is denoted $\text{Units}(R)$ or R^* and is called *the group of units in R*. The ring of integers is denoted $\mathbb{Z}$, the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$, and the field of complex numbers $\mathbb{C}$. If R is any ring, then the ring of n-by-n matrices over R is denoted by $M_n(R)$. If R is a commutative ring and G a group, then the *group ring* or *group algebra* is denoted $R(G)$. If R is any ring, the *opposite ring of R* is denoted R^o. As an additive abelian group, R^o and R are equal. However, the multiplication of R^o is reversed from that of R.

Example 1.1.1. If k is a field, the ring of *quaternions* over k is the four-dimensional k-vector space with basis $1, i, j, k$, and with multiplication defined by extending the relations

$$i^2 = j^2 = k^2 = -1$$
$$ij = -ji = k$$

by associativity and distributivity. The reader should verify the following.

(1) If k is equal to either $\mathbb{Q}$ or $\mathbb{R}$, then the ring of quaternions is a division ring. Over $\mathbb{R}$, the ring of quaternions is denoted $\mathbb{H}$.
(2) If k is equal to $\mathbb{C}$, then the ring of quaternions is isomorphic to $M_2(k)$.
(3) If k is equal to $\mathbb{Z}/(2)$, the ring of quaternions is isomorphic to the group ring over the elementary 2-group of order 4, hence is commutative.

If A is a ring and B is a subset of A, then we say B is a *subring* of A if B contains both 0 and 1 and B is a ring under the addition and multiplication rules of A. The *center* of A is the set $Z(A) = \{x \in A \mid xy = yx \text{ for all } y \in A\}$. Then $Z(A)$ is a commutative ring which is a subring of A. If $x \in Z(R)$, then we say x is *central.* A homomorphism from the ring A to the ring B maps the identity of A to the identity of B. All modules are required to be unitary, and unless otherwise specified, a module will be a left module. An R-module M is said to be *finitely generated* in case there exists a finite subset $X = \{x_1, \dots, x_n\}$ of M and M is the the smallest submodule of M containing X.

Let R be a commutative ring. An *R-algebra* is a ring A together with a homomorphism of rings $\theta : R \to Z(A)$ mapping R into the center of A. We call θ the *structure homomorphism* of A. Then A is an R-module with action $ra = \theta(r)a$. We write $R \cdot 1$ for the image of θ. If B is a subring of A containing $R \cdot 1$, then we say B is an *R-subalgebra* of A. We say A is a *finitely generated* R-algebra in case there exists a finite subset $X = \{x_1, \dots, x_n\}$ of A and A is the the smallest subalgebra of A containing X and $R \cdot 1$.

1.1. Categories and Functors. A *category* consists of a collection of *objects* and a collection of *morphisms* between pairs of those objects. The composition of morphisms is defined and is again a morphism. For our purposes, a category will always be one of the following:

(1) The category whose objects are modules over a ring R and whose morphisms are homomorphisms of modules. By ${}_R\mathfrak{M}$ we denote the

category of all left R-modules together with R-module homomorphisms. By $\mathfrak{M}_R$ we denote the category of all right R-modules together with R-module homomorphisms. If A and B are R-modules, the set of all R-module homomorphisms from A to B is denoted $\operatorname{Hom}_R(A, B)$.

(2) The category whose objects are rings and whose morphisms are homomorphisms of rings. A subcategory would be the category whose objects are commutative rings.

(3) The category whose objects are finitely generated algebras over a fixed commutative ring R and whose morphisms are R-algebra homomorphisms.

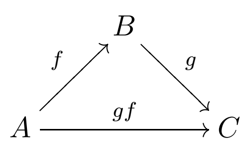
(4) The category whose objects are sets and whose morphisms are functions.

(5) The category of pointed sets. A *pointed set* is a pair (X, x) where X is a nonempty set and x is a distinguished element of X called the *base point*. A morphism from a pointed set (X, x) to a pointed set (Y, y) is a function $f : X \to Y$ such that $f(x) = y$.

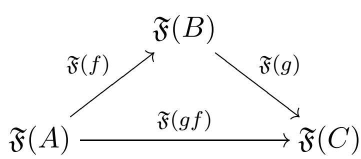
A *covariant functor* from a category $\mathfrak{C}$ to a category $\mathfrak{D}$ is a correspondence $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ which is a function on objects $A \mapsto \mathfrak{F}(A)$ and for any pair of objects $A, B \in \mathfrak{C}$, each f in $\operatorname{Hom}_{\mathfrak{C}}(A, B)$ is mapped to a function $\mathfrak{F}(f)$ in $\operatorname{Hom}_{\mathfrak{D}}(\mathfrak{F}(A), \mathfrak{F}(B))$ such that the following are satisfied

(1) If $1 : A \to A$ is the identity map, then $\mathfrak{F}(1) : \mathfrak{F}(A) \to \mathfrak{F}(A)$ is the identity map.

(2) Given a commutative triangle



in $\mathfrak{C}$, the triangle



commutes in $\mathfrak{D}$.

The definition of a *contravariant functor* is similar, except the arrows get reversed. That is, if $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ is a contravariant functor and f is an element of $\operatorname{Hom}_{\mathfrak{C}}(A, B)$, then $\mathfrak{F}(f)$ is in $\operatorname{Hom}_{\mathfrak{D}}(\mathfrak{F}(B), \mathfrak{F}(A))$.

If $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ is a covariant functor between categories of modules, then $\mathfrak{F}$ is *left exact* if for every short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1.1}$$

in $\mathfrak{C}$, the corresponding sequence

$$0 \to \mathfrak{F}(A) \xrightarrow{\mathfrak{F}(\alpha)} \mathfrak{F}(B) \xrightarrow{\mathfrak{F}(\beta)} \mathfrak{F}(C)$$

is exact in $\mathfrak{D}$. We say $\mathfrak{F}$ is *right exact* if for every short exact sequence (1.1) in $\mathfrak{C}$, the sequence

$$\mathfrak{F}(A) \xrightarrow{\mathfrak{F}(\alpha)} \mathfrak{F}(B) \xrightarrow{\mathfrak{F}(\beta)} \mathfrak{F}(C) \to 0$$

is exact in $\mathfrak{D}$.

If $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ is a contravariant functor between categories of modules, then $\mathfrak{F}$ is *left exact* if for every short exact sequence (1.1) in $\mathfrak{C}$, the sequence

$$0 \to \mathfrak{F}(C) \xrightarrow{\mathfrak{F}(\beta)} \mathfrak{F}(B) \xrightarrow{\mathfrak{F}(\alpha)} \mathfrak{F}(A)$$

is exact in $\mathfrak{D}$. We say the contravariant functor $\mathfrak{F}$ is *right exact* if for every short exact sequence (1.1) in $\mathfrak{C}$, the sequence

$$\mathfrak{F}(C) \xrightarrow{\mathfrak{F}(\beta)} \mathfrak{F}(B) \xrightarrow{\mathfrak{F}(\alpha)} \mathfrak{F}(A) \to 0$$

is exact in $\mathfrak{D}$.

If $\mathfrak{F} : \mathfrak{A} \to \mathfrak{C}$ is a covariant functor, we say that $\mathfrak{F}$ is *fully faithful* if

$$\operatorname{Hom}_{\mathfrak{A}}(A, B) \to \operatorname{Hom}_{\mathfrak{C}}(\mathfrak{F}(B), \mathfrak{F}(A))$$

is a one-to-one correspondence. We say that $\mathfrak{F}$ is *essentially surjective* if for every object C in $\mathfrak{C}$, there exists A in $\mathfrak{A}$ such that C is isomorphic to $\mathfrak{F}(A)$. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories of modules and suppose we have two covariant functors $\mathfrak{F}$ and $\mathfrak{F}'$ from $\mathfrak{C}$ to $\mathfrak{D}$. We say that $\mathfrak{F}$ and $\mathfrak{F}'$ are *naturally equivalent* if for every module M in $\mathfrak{C}$ there is an isomorphism φ_M in $\operatorname{Hom}_{\mathfrak{D}}\big(\mathfrak{F}(M), \mathfrak{F}'(M)\big)$ such that, for every pair of modules M and N in $\mathfrak{C}$ and any $f \in \operatorname{Hom}_{\mathfrak{C}}(M, N)$, the diagram

$$\begin{array}{ccc} \mathfrak{F}(M) & \xrightarrow{\mathfrak{F}(f)} & \mathfrak{F}(N) \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_N} \\ \mathfrak{F}'(M) & \xrightarrow{\mathfrak{F}'(f)} & \mathfrak{F}'(N) \end{array}$$

commutes. We denote by $I_{\mathfrak{C}}$ the identity functor on the category $\mathfrak{C}$ defined by $I_{\mathfrak{C}}(M) = M$ and $I_{\mathfrak{C}}(f) = f$, for modules M and maps f. Then we say two categories $\mathfrak{C}$ and $\mathfrak{D}$ are *equivalent* if there is a functor $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ and a functor $\mathfrak{G} : \mathfrak{D} \to \mathfrak{C}$ such that $\mathfrak{F} \circ \mathfrak{G}$ is naturally equivalent to $I_{\mathfrak{D}}$ and $\mathfrak{G} \circ \mathfrak{F}$ is naturally equivalent to $I_{\mathfrak{C}}$. The functors $\mathfrak{F}$ and $\mathfrak{G}$ are then referred to as *inverse equivalences*. For a proof of the next proposition, the reader is referred to a book on Category Theory. For example, see [**Bas68**, Proposition (1.1), p. 4].

Proposition 1.1.2. *A functor $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ establishes an equivalence of categories if and only if $\mathfrak{F}$ is fully faithful and essentially surjective.*

Example 1.1.3. Let R be a ring and R^o the opposite ring. Any $M \in {}_R\mathfrak{M}$ can be made into a right R^o-module by defining $m * r = rm$. The reader should verify that this defines a covariant functor ${}_R\mathfrak{M} \to \mathfrak{M}_{R^o}$ and that the category of left R-modules is equivalent to the category of right R^o-modules.

1.2. Progenerator Modules. As references for this section, we recommend [**DF04**] and [**Rot79**]. First we recall some of the basic properties of free modules. Let R be a ring and I any index set. Denote by R^I the R-module direct sum $\bigoplus_{i \in I} R$. If $I = \{1, 2, \dots, n\}$, then we write $R^{(n)}$ for R^I. If M is an R-module, then we say M is *free on the index set I* if M is isomorphic to R^I for some index set I. The Kronecker *delta function* is the function $\delta : I \times I \to \{0, 1\}$ defined by:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The *standard basis* for R^I is $\{e_i \in R^I \mid i \in I\}$ where $e_i(j) = \delta_{ij}$. An R-module M is free if and only if there exists a subset $\{b_i \mid i \in I\}$ of M satisfying the following.

(1) Each $x \in M$ can be written $x = \sum_{i \in I'} x_i b_i$, where $I' \subseteq I$ is finite and $x_i \in R$.

(2) If $I' \subseteq I$ is finite and $\sum_{i \in I'} r_i b_i = 0$, then each $r_i = 0$.

The set $\{b_i \mid i \in I\}$ is called a *basis* for M. If R is a commutative ring and M a free R-module with a finite basis $\{b_1, \dots, b_n\}$, then any other basis of M has n elements. In this case, we call n the *rank* of M and write $\mathrm{Rank}_R(M) = n$.

Lemma 1.1.4. *Let R be a ring and M an R-module.*

(1) *There exists a free R-module F and a surjective homomorphism $F \to M$.*

(2) *M is finitely generated if and only if M is the homomorphic image of a free R-module $R^{(n)}$ for some n.*

Proposition 1.1.5. *If R is a ring and M an R-module, then the following are equivalent.*

(1) *M is isomorphic as an R-module to a direct summand of a free R-module.*

(2) *Every short exact sequence of R-modules*

$$0 \to A \to B \xrightarrow{\beta} M \to 0$$

is split exact.

(3) *For any diagram of R-modules*

$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle \exists\psi} \swarrow & \downarrow {\scriptstyle \phi} & & \\ A & \xrightarrow{\alpha} & B & \longrightarrow & 0 \end{array}$$

with the bottom row exact, there exists an R-module homomorphism $\psi : M \to A$ such that $\alpha\psi = \phi$.

(4) *M has a dual basis $\{(m_i, f_i) \mid i \in I\}$ for some index set I consisting of $m_i \in M$, $f_i \in \operatorname{Hom}_R(M, R)$ satisfying:*
   (a) *for each $m \in M$, $f_i(m) = 0$ for all but finitely many $i \in I$, and*
   (b) *for all $m \in M$, $m = \sum_{i \in I} f_i(m) m_i$.*

If M satisfies any of the equivalent conditions of Proposition 1.1.5, then we say M is a *projective* R-module. Proposition 1.1.5 (1) is trivially satisfied by a free R-module. Hence a free module is projective.

Example 1.1.6. Let D be a division ring and $R = M_2(D)$ the ring of two-by-two matrices over D. If we identify D with the scalar matrices, then R is a left vector space of dimension 4 over D. Let

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The reader should verify the following facts.

(1) e_1 and e_2 are idempotents and $e_1 e_2 = e_2 e_1 = 0$. We say e_1 and e_2 are *orthogonal idempotents*.

(2) Re_1 is the set of all matrices with second column consisting of zeros.

(3) Re_2 is the set of all matrices with first column consisting of zeros.

(4) $\dim_D(Re_1) = \dim_D(Re_2) = 2$.

(5) $R = Re_1 \oplus Re_2$ as R-modules.

By (5), Re_1 and Re_2 are projective R-modules. It follows from Exercise 1.1.9 below that Re_1 and Re_2 are not free R-modules. For any $n \geq 2$, we will see in Example 3.2.4 below that $M_n(D)$ is a simple artinian ring. By Theorem 3.2.2, proved below, every module over $M_n(D)$ is projective. The method used above can be modified to show that $M_n(D)$ contains left ideals that are projective but not free.

Let R be a ring and M an R-module. We say that M is *of finite presentation* if there exists an exact sequence

$$R^{(m)} \to R^{(n)} \to M \to 0$$

for some m and n. The proof of Corollary 1.1.7 is left to the reader.

Corollary 1.1.7. *Let R be a ring and M a finitely generated projective R-module. Then M is of finite presentation over R. There exists a finitely generated projective R-module N such that $M \oplus N$ is a finitely generated free R-module.*

Let R be a ring and M an R-module. The reader should verify that $\operatorname{Hom}_R(M,R)$ is a right R-module by the action $(fr)(m) = f(m)r$ (see Lemma 1.3.17 below). Using this, the reader should verify that the set

$$\mathfrak{T}_R(M) = \left\{ \sum_{i=1}^{n} f_i(m_i) \mid n \geq 1, f_i \in \operatorname{Hom}_R(M,R), m_i \in M \right\} \tag{1.2}$$

is a two-sided ideal in R, which is called the *trace ideal of M in R*. We say that M is a *generator* over R in case $\mathfrak{T}_R(M) = R$. We say that an R-module M is a *progenerator over R* in case M is finitely generated, projective and a generator over R.

Let $\theta : R \to S$ be a homomorphism of rings and M a left S-module. Then R acts on M from the left by the rule $rx = \theta(r)x$. Using θ, an S-module M can be viewed as an R-module. Since S is a left S-module, θ makes S into a left R-module. In a similar way, θ can be used to make a right S-module into a right R-module.

Proposition 1.1.8. *Let $\theta : R \to S$ be a homomorphism of rings and M an S-module.*

(1) *(Projective over Projective is Projective) If S is projective as an R-module and M is a projective S-module, then M is projective as an R-module. Moreover, if M is finitely generated over S and S is finitely generated as an R-module, then M is finitely generated as an R-module.*

(2) *(A Generator over a Generator is a Generator) If S is a generator when viewed as an R-module and M is an S-generator, then M is an R-generator.*

(3) *(A Progenerator over a Progenerator is a Progenerator) If S is an R-progenerator and M is an S-progenerator, then M is an R-progenerator.*

Proof. (1): Let $\{(m_i, f_i) \mid i \in I\}$ be a dual basis for M over S and $\{(s_j, g_j) \mid j \in J\}$ a dual basis for S over R. The reader should verify that the functions

$g_j f_i$ in $\operatorname{Hom}_R(M, R)$ and the products $s_j m_i$ in M make up a dual basis for M over R.

(2): For some $m > 0$ there exist $\{f_1, \dots, f_m\}$ in $\operatorname{Hom}_S(M, S)$ and $\{x_1, \dots, x_m\}$ in M such that we can write $1 \in S$ as $1 = \sum_{i=1}^m f_i(x_i)$. For some n there exist $\{g_1, \dots, g_n\}$ in $\operatorname{Hom}_R(S, R)$ and $\{s_1, \dots, s_n\}$ in S such that we can write $1 \in R$ as $1 = \sum_{j=1}^n g_j(s_j)$. For each (i, j), the composite function $g_j f_i$ is in $\operatorname{Hom}_R(M, R)$ and the product $s_j m_i$ is in M. Then

$$\sum_{i=1}^m \sum_{j=1}^n g_j f_i(s_j m_i) = \sum_{i=1}^m \sum_{j=1}^n g_j\big(s_j f_i(m_i)\big) = \sum_{j=1}^n g_j\Big(s_j \sum_{i=1}^m f_i(m_i)\Big) = \sum_{j=1}^n g_j(s_j) = 1$$

shows that the trace ideal is equal to R.

Part (3) follows immediately from Parts (1) and (2). □

1.3. Exercises.

Exercise 1.1.9. (Free over Free is Free) Let $\theta : R \to S$ be a homomorphism of rings such that S is free as an R-module. Let M be a free S-module. Prove:

(1) M is a free R module.

(2) If M has a finite basis over S, and S has a finite basis over R, then M has a finite basis over R. In this case, if R and S are both commutative, then $\operatorname{Rank}_R(M) = \operatorname{Rank}_S(M) \operatorname{Rank}_R(S)$.

(3) If R and S are fields, then $\dim_R(S)$ and $\dim_S(M)$ are both finite if and only if $\dim_R(M)$ is finite.

Exercise 1.1.10. Let R be a ring and $\{M_i \mid i \in I\}$ a family of R-modules. Prove that the direct sum $\bigoplus_{i \in I} M_i$ is projective over R if and only if each M_i is projective over R.

Exercise 1.1.11. Proving that a module is a generator can be a difficult task. This exercise contains conditions on a module that are equivalent to being a generator. For another characterization of generator modules, see Exercise 1.3.30. Let R be a ring and M a left R-module. Prove that the following are equivalent.

(1) M is an R-generator.

(2) The R-module R is the homomorphic image of a direct sum $M^{(n)}$ of finitely many copies of M.

(3) The R-module R is the homomorphic image of a direct sum M^I of copies of M over some index set I.

(4) Every left R-module A is the homomorphic image of a direct sum M^I of copies of M over some index set I.

Exercise 1.1.12. Let k be a field, x an indeterminate, and $n > 1$ an integer. Let $T = k[x]$, $S = k[x^n, x^{n+1}]$, and $R = k[x^n]$. For the tower of subrings $R \subseteq S \subseteq T$, prove:

(1) T is free over R of rank n.

(2) S is free over R of rank n.

(3) T is not free over S.

1.4. Nakayama's Lemma. In this section we prove Nakayama's Lemma for a finitely generated module over a commutative ring. Throughout this book we will encounter various forms of Nakayama's Lemma. The version in Lemma 1.1.13, as well as its proof are from [**DI71**]. Corollary 1.1.17 is a second version, and Theorem 3.1.2 below contains a form that is valid for noncommutative rings.

Let R be a ring, $A \subseteq R$ a left ideal of R, and M an R-module. Denote by AM the R-submodule of M generated by all elements of the form am, where $a \in A$ and $m \in M$. The *annihilator* of M in R is $\operatorname{annih}_R M = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$. The reader should verify that $\operatorname{annih}_R M$ is a two-sided ideal in R. If $\operatorname{annih}_R M = (0)$, then we say M is *faithful*.

Lemma 1.1.13 (Nakayama's Lemma)**.** *Let R be a commutative ring and M a finitely generated R-module. An ideal A of R has the property that $AM = M$ if and only if $A + \operatorname{annih}_R(M) = R$.*

Proof. Assume $A + \operatorname{annih}_R(M) = R$. Write $1 = \alpha + \beta$ for some $\alpha \in A$ and $\beta \in \operatorname{annih}_R(M)$. Given m in M, $m = 1m = (\alpha + \beta)m = \alpha m + \beta m = \alpha m$. Therefore $AM = M$.

Conversely, suppose $AM = M$. Choose a generating set $\{m_1, \ldots, m_n\}$ for M over R. Define

$$\begin{aligned} M = M_1 &= Rm_1 + \cdots + Rm_n \\ M_2 &= Rm_2 + \cdots + Rm_n \\ &\vdots \\ M_n &= Rm_n \\ M_{n+1} &= 0. \end{aligned}$$

We prove that for every $i = 1, 2, \ldots, n+1$, there exists α_i in A such that $(1 - \alpha_i)M \subseteq M_i$. Since $(1 - 0)M = M \subseteq M_1$, take $\alpha_1 = 0$. Proceed inductively. Suppose $i \geq 1$, $\alpha_i \in A$, and $(1-\alpha_i)M \subseteq M_i$. Then

$$\begin{aligned}(1-\alpha_i)M &= (1-\alpha_i)AM \\ &= A(1-\alpha_i)M \\ &\subseteq AM_i.\end{aligned}$$

In particular, $(1-\alpha_i)m_i \in AM_i = Am_i + Am_{i+1} + \cdots + Am_n$. So there exist elements $\alpha_{ii}, \ldots, \alpha_{im}$ in A such that

$$(1-\alpha_i)m_i = \sum_{j=i}^{n} \alpha_{ij} m_j.$$

Subtracting shows that

$$(1-\alpha_i-\alpha_{ii})m_i = \sum_{j=i+1}^{n} \alpha_{ij} m_j$$

is in M_{i+1}. Consider

$$\begin{aligned}(1-\alpha_i)(1-\alpha_i-\alpha_{ii})M &= (1-\alpha_i-\alpha_{ii})\big((1-\alpha_i)M\big) \\ &\subseteq (1-\alpha_i-\alpha_{ii})M_i \\ &\subseteq M_{i+1}.\end{aligned}$$

Set $\alpha_{i+1} = -(-\alpha_i - \alpha_{ii} - \alpha_i + \alpha_i^2 + \alpha_i\alpha_{ii})$. Then $\alpha_{i+1} \in A$ and $(1-\alpha_{i+1})M \subseteq M_{i+1}$. By finite induction, $(1-\alpha_{n+1})M = 0$. Hence $1-\alpha_{n+1} \in \operatorname{annih}_R(M)$ and $1 \in A + \operatorname{annih}_R(M)$. □

Corollary 1.1.14. *Let R be a commutative ring and M a finitely generated R-module. If $\mathfrak{m}M = M$ for every maximal ideal $\mathfrak{m}$ of R, then $M = 0$.*

Proof. If $M \neq 0$, then $1 \notin \operatorname{annih}_R(M)$. Some maximal ideal $\mathfrak{m}$ contains $\operatorname{annih}_R(M)$. So $\mathfrak{m} + \operatorname{annih}_R(M) = \mathfrak{m} \neq R$. By Lemma 1.1.13, $\mathfrak{m}M \neq M$. □

Proposition 1.1.15. *Let R be a commutative ring and M a finitely generated and projective R-module. Then $\mathfrak{T}_R(M) \oplus \operatorname{annih}_R(M) = R$.*

Proof. There exists a dual basis $\{(m_i, f_i) \mid 1 \leq i \leq n\}$ for M. For each $m \in M$, we see that $m = f_1(m)m_1 + \cdots + f_n(m)m_n$ is in $\mathfrak{T}_R(M)M$. Then $\mathfrak{T}_R(M)M = M$. By Lemma 1.1.13, $\mathfrak{T}_R(M) + \operatorname{annih}_R(M) = R$. Therefore, $\mathfrak{T}_R(M)\operatorname{annih}_R(M) = \mathfrak{T}_R(M) \cap \operatorname{annih}_R(M)$. It suffices to show that $\mathfrak{T}_R(M)\operatorname{annih}_R(M) = 0$. A typical generator for $\mathfrak{T}_R(M)$ is $f(m)$ for some $m \in M$ and $f \in \operatorname{Hom}_R(M, R)$. Given $\alpha \in \operatorname{annih}_R(M)$, we see that $\alpha f(m) = f(\alpha m) = f(0) = 0$. □

For a commutative ring, Proposition 1.1.15 supplies us with a powerful criterion for showing that a finitely generated projective module is a generator. As a consequence, we have the following two useful descriptions for progenerator modules.

Corollary 1.1.16. *Let R be a commutative ring and M an R-module. Then the following are true.*

(1) *M is an R-progenerator if and only if M is finitely generated projective and faithful.*

(2) *Assume R has no idempotents except 0 and 1. Then M is an R-progenerator if and only if M is finitely generated, projective, and $M \neq (0)$.*

We end this section with the following useful variation of Nakayama's Lemma.

Corollary 1.1.17. *Let R be a commutative ring. Suppose I is an ideal in R, M is an R-module, and there exist submodules A and B of M such that $M = A + IB$. If*

(1) *I is nilpotent (that is, $I^n = 0$ for some $n > 0$), or*

(2) *I is contained in every maximal ideal of R and M is finitely generated,*

then $M = A$.

Proof. Notice that

$$M/A = \frac{A + IB}{A} \subseteq \frac{A + IM}{A} \subseteq I(M/A) \subseteq M/A.$$

Assuming (1) we get $M/A = I(M/A) = \cdots = I^n(M/A) = 0$. Now assume (2) and let $\mathfrak{m}$ be an arbitrary maximal ideal of R. Then $M/A = I(M/A) \subseteq \mathfrak{m}(M/A)$. By Corollary 1.1.14, $M/A = 0$. □

1.5. Exercise.

Exercise 1.1.18. A *local ring* is a commutative ring R such that R has exactly one maximal ideal. If R is a local ring with maximal ideal $\mathfrak{m}$, then $R/\mathfrak{m}$ is called the *residue field* of R. If $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local rings and $\phi : R \to S$ is a homomorphism of rings, then we say ϕ is a *local homomorphism of local rings* in case $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$. Let $\phi : R \to S$ be a local homomorphism of local rings. Assume S is a finitely generated R-module and $\mathfrak{m}$ is the maximal ideal of R. Show that if the homomorphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ induced by ϕ is an isomorphism, then ϕ is onto. (Hint: S is generated by $\phi(R)$ and $\mathfrak{m}S$.)

1.6. Module Direct Summands of Rings. If a ring R decomposes into a direct sum of finitely many two-sided ideals, then the direct summands are generated by central idempotents. The main result of this section, Lemma 1.1.20, considers the more general situation of when a left ideal of R is an R-module direct summand of R. This important utility lemma has many applications in the rest of the book. In fact, the connection between module direct summands of R and idempotents will play an essential role in the definition of separability given in Chapter 4.

An idempotent $e \in R$ is said to be *primitive* if e cannot be written as a sum of two nonzero orthogonal idempotents. Let I be a nonzero left ideal in R. Then I is a *minimal left ideal of* R if whenever J is a left ideal of R and $J \subseteq I$, then either $J = 0$, or $J = I$.

Example 1.1.19. Let F be a field and $R = M_2(F)$ the ring of two-by-two matrices over F. Let

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We saw in Example 1.1.6 that e_1 and e_2 are orthogonal idempotents and $R = Re_1 \oplus Re_2$ as R-modules. The reader should verify that Re_1 is a minimal left ideal and e_1 is a primitive idempotent.

Lemma 1.1.20. *Let R be a ring and I a left ideal of R.*

(1) *I is an R-module direct summand of R if and only if $I = Re$ for some idempotent e.*

(2) *Let e be an idempotent in R. Then e is primitive if and only if Re cannot be written as an R-module direct sum of proper left ideals of R.*

(3) *If I is a minimal left ideal, then I is an R-module direct summand of R if and only if $I^2 \neq 0$.*

(4) *Suppose $R = I \oplus J$ where I and J are two-sided ideals. Then $I = Re$ for some central idempotent e, I is a ring, and e is the multiplicative identity for I.*

Proof. (1): Assume $R = I \oplus L$. Write $1 = e + f$ where $e \in I$ and $f \in L$. Then $e = e^2 + ef$. Now $ef = e - e^2 \in I \cap L = 0$. Likewise $fe = 0$. Also $e + f = 1 = 1^2 = (e + f)^2 = e^2 + f^2$. In the direct sum the representation of 1 is unique, so $e = e^2$ and $f = f^2$. Let $x \in I$. Then $x = x \cdot 1 = xe + xf$. But $xf = x - xe \in I \cap L = 0$. So $Re = I$. Conversely, assuming $e^2 = e$, we prove that Re is a direct summand of R. Then $0 = e - e^2 = e(1-e) = (1-e)e$. Also $(1-e)^2 = 1-e-e+e^2 = 1-e$. This shows $e, 1-e$ are orthogonal idempotents. Since $1 = e + (1 - e)$ we have $R = Re + R(1 - e)$. Let $x \in Re \cap R(1 - e)$. Then $x = ae = b(1 - e)$ for some $a, b \in R$. Then $xe = ae^2 = ae = x$ and again $xe = b(1 - e)e = 0$. Therefore $R = Re \oplus R(1 - e)$.

(2): Using the same ideas as in (1), we see that e is a sum of nonzero orthogonal idempotents if and only if Re decomposes into a direct sum of proper left ideals of R.

(3): Assume I is a minimal left ideal of R. Suppose $R = I \oplus L$ for some left ideal L of R. By (1), $I = Re$ for some idempotent e. Then $e = e^2 \in I^2$ so $I^2 \neq 0$. Conversely assume $I^2 \neq 0$. There is some $x \in I$ such that $Ix \neq 0$. But Ix is a left ideal of R and since I is minimal, we have $Ix = I$. For some $e \in I$, we have $ex = x$. Let $L = \operatorname{annih}_R(x) = \{r \in R \mid rx = 0\}$. Then L is a left ideal of R. Since $(1-e)x = x - ex = x - x = 0$ it follows that $1 - e \in L$. Therefore $1 = e + (1-e) \in I + L$ so $R = I + L$. Also, $e \in I$ and $ex = x \neq 0$ shows that $e \notin L$. Now $I \cap L$ is a left ideal in R and is contained in the minimal left ideal I. Since $I \cap L \neq I$, it follows that $I \cap L = 0$ which proves that $R = I \oplus L$ as R-modules.

(4): This part is left to the reader. □

Our first application of Lemma 1.1.20 is to prove the following unique decomposition theorem for commutative rings.

Theorem 1.1.21. *Let R be a commutative ring and assume R decomposes into an internal direct sum $R = Re_1 \oplus \cdots \oplus Re_n$, where each e_i is a primitive idempotent. Then this decomposition is unique in the sense that, if $R = Rf_1 \oplus \cdots \oplus Rf_p$ is another such decomposition of R, then $n = p$, and after rearranging, $e_1 = f_1, \ldots, e_n = f_n$.*

Proof. Any idempotent of $R = Re_1 \oplus \cdots \oplus Re_n$ is of the form $x_1 + \cdots + x_n$ where x_i is an idempotent in Re_i. By Lemma 1.1.20, the only idempotents of Re_i are 0 and e_i. Hence, R has exactly n primitive idempotents, namely $e_1, \ldots, e_n$. □

1.7. Exercises. All of the exercises below will have significant applications later in the text. The reader is encouraged to prove them before continuing to the next section.

Exercise 1.1.22. Let R be a ring and I a left ideal in R. Prove that the following are equivalent.

(1) R/I is a projective left R-module.

(2) The R-module sequence $0 \to I \to R \to R/I \to 0$ is split-exact.

(3) I is an R-module direct summand of R.

(4) There is an element $e \in R$ such that $1 - e \in I$ and $Ie = (0)$.

(5) There is an idempotent $e \in R$ such that $I = R(1-e)$.

For another condition that is equivalent to (1) – (5), see Exercise 1.3.15.

Exercise 1.1.23. Let A be an R-algebra and e an idempotent in A.

(1) Show that eAe is an R-algebra.
(2) Show that there is an R-module direct sum decomposition:

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e).$$

Exercise 1.1.24. Let R be a commutative ring and J an ideal in R. Prove:

(1) If J is a direct summand of R, then $J^2 = J$.
(2) If J is a finitely generated ideal, and $J^2 = J$, then J is a direct summand of R.

2. Polynomial Functions

This section contains some results of a technical nature. Polynomial functions arise when we consider the coefficients of the characteristic polynomial associated to an element in a finite dimensional algebra over a field. In particular, the norm map and trace map are examples of homogeneous polynomial functions. The resultant operator defines a polynomial function which will be used when we construct a maximal commutative separable subfield of a division algebra. The main theorem on C_1 fields will be applied to prove Tsen's Theorem (Corollary 11.1.9).

2.1. The Ring of Polynomial Functions on a Module. Let R be a commutative ring, M an R-module, and $M^* = \operatorname{Hom}_R(M,R)$ the dual of M. By $\operatorname{Map}(M,R)$ we denote the set of all functions $f : M \to R$. Then $\operatorname{Map}(M,R)$ can be turned into an R-algebra. The addition and multiplication operations are defined pointwise: $(f+g)(x) = f(x)+g(x)$, $(fg)(x) = f(x)g(x)$. An element a in R defines the constant function $a : M \to R$, where $a(x) = a$. We can view M^* as an R-submodule of $\operatorname{Map}(M,R)$. The R-subalgebra of $\operatorname{Map}(M,R)$ generated by the set M^* is denoted $R[M^*]$ and is called the *ring of polynomial functions* on M. If $d \geq 0$, then a polynomial function $f \in R[M^*]$ is said to be *homogeneous of degree* d, if $f(rx) = r^d f(x)$, for all $x \in M$ and $r \in R$. Proposition 1.2.1 shows that the ring $R[M^*]$ is in fact a coordinate-free way to generalize the usual ring of polynomial functions on a vector space.

Proposition 1.2.1. *Let k be an infinite field, and V a finite dimensional k-vector space. If $\dim_k(V) = n$, then $k[V^*] \cong k[x_1,\dots,x_n]$ as k-algebras.*

Proof. Let $\{(v_i, f_i) \mid 1 \leq i \leq n\}$ be a dual basis for V. Define $\theta : k[x_1,\dots,x_n] \to k[V^*]$ by $x_i \mapsto f_i$. The reader should verify that θ is one-to-one and onto. □

Lemma 1.2.2. *Let R be a commutative ring and P a free R-module of finite rank n. Let $\phi \in \operatorname{Hom}_R(P,P)$. If the characteristic polynomial of ϕ is $p(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, then for each $i = 1, \dots, n$, the assignment $\phi \mapsto (-1)^i a_i$ defines a polynomial function $N_i : \operatorname{Hom}_R(P,P) \to R$ which is homogeneous of degree i.*

Proof. Let B be a basis for P. Let $\phi \in \operatorname{Hom}_R(P,P)$, and $(\phi_{ij}) = M(\phi, B)$ the matrix of ϕ with respect to B. By Proposition 1.2.1, a polynomial function on $\operatorname{Hom}_R(P,P)$ corresponds to a polynomial in the n^2 indeterminates $\Phi = \{\phi_{ij} \mid 1 \le i \le n, 1 \le j \le n\}$. The characteristic polynomial of ϕ is given by the combinatorial formula for the determinant

$$\det(xI_n - (\phi_{ij})) = \sum_{\vec{\ell} \in S_n} \operatorname{sign}(\ell) b_{\ell_1,1} \cdots b_{\ell_n,n} \tag{1.3}$$

where $b_{ii} = x - \phi_{ii}$ and $b_{ij} = -\phi_{ij}$ if $i \neq j$. A typical summand in (1.3) can be written in the form

$$\operatorname{sign}(\ell) b_{\ell_1,1} \cdots b_{\ell_n,n} = (x - a_{i_1i_1}) \cdots (x - a_{i_di_d})m$$

where m is a monomial in $k[\Phi]$ of degree $n-d$. Therefore, $b_{\ell_1,1} \cdots b_{\ell_n,n}$ is a polynomial in x of degree d and for $0 \le k < n$, the coefficient of x^k is a homogeneous polynomial of degree $n-k$ in $k[\Phi]$. □

Example 1.2.3. Let k be a field and A a k-algebra. Assume $\dim_k(A) = n$ is finite. Using the left regular representation, we can embed A as a k-subalgebra of $\operatorname{Hom}_k(A,A)$. As in Lemma 1.2.2, let $N_i : \operatorname{Hom}_k(A,A) \to k$ be the homogeneous polynomial function of degree i defined by the coefficient of x^{n-i} in the characteristic polynomial of ϕ. For each i, upon restriction to A, $N_i : A \to k$ defines a homogeneous polynomial function on A of degree i. In particular N_n is the norm $N_k^A : A \to k$ defined in Exercise 1.2.14, and N_1 the trace $T_k^A : A \to k$. Fix a k-basis $\alpha_1, \dots, \alpha_n$ for A. Then this basis can be extended to a basis for $\operatorname{Hom}_k(A,A)$ and N_i can be identified with a homogeneous polynomial in $k[x_1, \dots, x_n]$ of degree i.

2.2. Resultant of Two Polynomials. Assume $m \ge 0$, $n \ge 0$, and $m + n \ge 1$. Let $f = \sum_{i=0}^m f_i x^i$ and $g = \sum_{i=0}^n g_i x^i$ be two polynomials in $k[x]$, where k is a field. So the degree of f is at most m, and the degree of g is at most n. In the general case, m and n are both positive, and the *Sylvester* *matrix* of f and g is the $(m+n)$-by-$(m+n)$ matrix

$$\operatorname{Syl}(f,g) = \begin{bmatrix} f_m & f_{m-1} & f_{m-2} & \cdots & f_0 & & & & \cdots & \\ & f_m & f_{m-1} & \cdots & f_1 & f_0 & & & \cdots & \\ & & f_m & \cdots & f_2 & f_1 & f_0 & & \cdots & \\ & & & \vdots & & & & & & \\ & \cdots & & f_m & f_{m-1} & f_{m-2} & \cdots & f_0 & & \\ & \cdots & & & f_m & f_{m-1} & \cdots & f_1 & f_0 & \\ & \cdots & & & & f_m & \cdots & f_2 & f_1 & f_0 \\ g_n & g_{n-1} & g_{n-2} & \cdots & g_0 & & & & \cdots & \\ & g_n & g_{n-1} & \cdots & g_1 & g_0 & & & \cdots & \\ & & g_n & \cdots & g_2 & g_1 & g_0 & & \cdots & \\ & & & \vdots & & & & & & \\ & \cdots & & g_n & g_{n-1} & f_{m-2} & \cdots & g_0 & & \\ & \cdots & & & g_n & g_{n-1} & \cdots & g_1 & g_0 & \\ & \cdots & & & & g_n & \cdots & g_2 & g_1 & g_0 \end{bmatrix}$$

where blank spaces consist of zeros. The top n rows are constructed from the coefficients of f, shifted and padded with zeros. The bottom m rows are constructed from shifting the coefficients of g, and padding with zeros. In the degenerate case when $m = 0$, $\operatorname{Syl}(f, g)$ is defined to be the n-by-n diagonal matrix $f_0(E_{11} + \cdots + E_{nn})$. In the degenerate case when $n = 0$, $\operatorname{Syl}(f, g)$ is defined to be the m-by-m diagonal matrix $g_0(E_{11} + \cdots + E_{mm})$. The *resultant* of f and g, written $\operatorname{Res}(f, g)$, is the determinant of $\operatorname{Syl}(f, g)$.

Lemma 1.2.4. *In the above context, we view $f_0, \dots, f_m$, $g_0, \dots, g_n$ as variables. Then $\operatorname{Res}(f, g)$ satisfies the following:*

(1) *$\operatorname{Res}(f, g)$ is a polynomial in $\mathbb{Z}[f_0, \dots, f_m, g_0, \dots, g_n]$ which is homogeneous of degree $n + m$.*

(2) *For any constant $c \in \mathbb{Z}$,*

$$\operatorname{Res}(cf, g) = c^n \operatorname{Res}(f, g)$$
$$\operatorname{Res}(f, cg) = c^m \operatorname{Res}(f, g)$$

Thus, $\operatorname{Res}(f, g)$ is homogeneous of degree n in $f_0, \dots, f_m$ and homogeneous of degree m in $g_0, \dots, g_n$.

Proof. The proof is left to the reader. □

Lemma 1.2.5. *In the above context, the following are true.*

(1) *If $\deg(f) < m$ and $\deg(g) < n$, then $\operatorname{Res}(f, g) = 0$.*

(2) *If $m = 0$, then $\operatorname{Res}(f, g) = f_0^n$.*

(3) *If $n = 0$, then $\operatorname{Res}(f, g) = g_0^m$.*

(4) $\operatorname{Res}(f,g) = (-1)^{mn}\operatorname{Res}(g,f)$.

(5) *If* $\deg(f) = m$ *and* $d = \deg(g) < n$, *then* $\operatorname{Res}(f,g) = f_m^{n-d}\operatorname{Res}(f,h)$, *where* $h = g_d x^d + \cdots + g_1 x + g_0$.

Proof. Parts (1) – (4) are left to the reader.

(5): The Sylvester matrix has the form

$$\operatorname{Syl}(f,g) = \begin{bmatrix} T & * \\ 0 & \operatorname{Syl}(f,h) \end{bmatrix}$$

where T is an upper triangular matrix of size $(n-d)$-by-$(n-d)$ with diagonal $(f_m, \dots, f_m)$. □

Lemma 1.2.6. *In the context of Lemma* 1.2.5, *assume* $m \le n$ *and* $\deg(f) = m$. *Let* q *and* r *be the unique polynomials in* $k[x]$ *guaranteed by the Division Algorithm which satisfy:* $q = \sum_{i=0}^{n-m} q_i x^i$, $r = \sum_{i=0}^{m-1} r_i x^i$, *and* $g = qf + r$. *Then* $\operatorname{Res}(f,g) = f_m^{n-m+1}\operatorname{Res}(f,r)$.

Proof. Write $c = -q_{n-m} = -g_n/f_m$, and set $h = g + cx^{n-m}f = \sum_{i=0}^{n-1} h_i x^i$. Let $I_m = E_{11} + \cdots + E_{mm} \in M_m(k)$, $I_n = E_{11} + \cdots + E_{nn} \in M_n(k)$, and $I_{mn} = E_{11} + \cdots + E_{mm} \in M_{mn}(k)$. The product

$$\begin{bmatrix} I_n & 0 \\ cI_{mn} & I_m \end{bmatrix} \operatorname{Syl}(f,g) = \begin{bmatrix} f_m & * \\ & \operatorname{Syl}(f,h) \end{bmatrix}$$

corresponds to elementary row operations. Computing determinants, we find that $\operatorname{Res}(f,g) = f_m \operatorname{Res}(f,h)$. By induction on $n-m$, we are done. □

Theorem 1.2.7. *In the context of Lemma* 1.2.5, *assume* F/k *is an extension of fields such that in the unique factorization domain* $F[x]$ *both polynomials* f *and* g *have no irreducible factor of degree greater than one.*

(1) *If* $m = \deg(f) \ge 1$ *and* $f = f_m(x-\alpha_1)\cdots(x-\alpha_m)$ *is a factorization of* f *into a product of linear polynomials, then*

$$\operatorname{Res}(f,g) = f_m^n \prod_{i=1}^{m} g(\alpha_i).$$

(2) *If* $\deg(g) = n \ge 1$ *and* $g = g_n(x-\beta_1)\cdots(x-\beta_n)$ *is a factorization of* g *into a product of linear polynomials, then*

$$\operatorname{Res}(f,g) = (-1)^{mn} g_n^m \prod_{j=1}^{n} f(\beta_j).$$

(3) *Suppose* $\deg(f) = m \ge 1$ *and* $\deg(g) = n \ge 1$. *If* $f = f_m(x-\alpha_1)\cdots(x-\alpha_m)$ *and* $g = g_n(x-\beta_1)\cdots(x-\beta_n)$ *are factorizations of* f

and g into products of linear polynomials, then

$$\operatorname{Res}(f,g) = f_m^n g_n^m \prod_{i=1}^{m}\prod_{j=1}^{n}(\alpha_i - \beta_j).$$

Proof. We prove (1) and (2) simultaneously. The reader should verify that Part (3) follows from Parts (1) and (2).

The proof is by induction on $m+n$. The basis for the induction, which follows from Lemma 1.2.5, is when $n = 0$ or $m = 0$. Assume from now on that $1 \le m$ and $1 \le n$.

Case 1: $\deg(f) = m \ge 1$, and $\deg(g) = d < n$. If we set $h = \sum_{i=0}^{d} g_i x^i$, then by Lemma 1.2.5 (5), $\operatorname{Res}(f,g) = f_m^{n-d}\operatorname{Res}(f,g)$. By the induction hypothesis, $\operatorname{Res}(f,g) = f_m^{n-d}\operatorname{Res}(f,g) = f_m^{n-d} f_m^d \prod_{i=1}^{m} g(\alpha_i)$, which proves (1) in this case.

Case 2: $\deg(g) = n \ge 1$, and $\deg(f) = d < m$. In this case, Part (2) follows by Case 1 and Lemma 1.2.5 (4).

Case 3: Assume $\deg(f) = m \ge 1$ and $\deg(g) = n \ge 1$, and $m \le n$. As in Lemma 1.2.6, write $g = fq + r$, where $r = \sum_{i=0}^{m-1} r_i x^i$. By Lemma 1.2.6 and the induction hypothesis,

$$\begin{aligned}\operatorname{Res}(f,g) &= f_m^{n-m+1}\operatorname{Res}(f,r)\\ &= f_m^{n-m+1} f_m^{m-1}\prod_{i=1}^{m} r(\alpha_i)\\ &= f_m^n \prod_{i=1}^{m} g(\alpha_i)\end{aligned}$$

where the last equation follows since $r(\alpha_i) = g(\alpha_i) - f(\alpha_i)q(\alpha_i)$. In this case, we have proved Part (1). By

$$\begin{aligned}\operatorname{Res}(f,g) &= f_m^n \prod_{i=1}^{m} g(\alpha_i)\\ &= f_m^n \prod_{i=1}^{m} g_n(\alpha_i - \beta_1)\cdots(\alpha_i - \beta_n)\\ &= g_n^m \prod_{j=1}^{n} f_m(\alpha_1 - \beta_j)\cdots(\alpha_m - \beta_j)\\ &= (-1)^{mn} g_n^m \prod_{j=1}^{n} f(\beta_j)\end{aligned}$$

we see that Part (2) holds in Case 3.

Case 4: Assume $\deg(f) = m \geq 1$ and $\deg(g) = n \geq 1$, and $n \leq m$. Applying Lemma 1.2.5 (4), this reduces to Case 3. □

Corollary 1.2.8. *In the above context,* $\operatorname{Res}(f, g) = 0$ *if and only if one of the following is satisfied:*

(1) $\deg(f) < m$ *and* $\deg(g) < n$.
(2) $(f, g) \neq k[x]$, *or equivalently, f and g have a common irreducible factor, or equivalently, f and g have a common root in some extension field F/k.*

Proof. If (1) is true, then the first column of $\operatorname{Syl}(f, g)$ is made up of zeros, so $\operatorname{Res}(f, g) = 0$. Otherwise, by Lemma 1.2.5, we can reduce to the case where $\deg(f) = m$. By Theorem 1.2.7 (1), $\operatorname{Res}(f, g) = 0$ if and only if f and g have a common root in some extension field F/k, which is equivalent to (2). □

2.3. Polynomial Functions on an Algebraic Curve. A field K is said to be C_1 if every homogeneous polynomial of degree $d > 0$ in n variables, where $n > d$, has a nontrivial zero. The purpose of this section is to proof that an algebraic function field in one variable over an algebraically closed field is C_1. The proof we give is modeled after that of [**Gre69**, Chapter 3]. Fields which are C_1 are the subject of Corollaries 11.1.8 and 7.9.7.

Example 1.2.9. Let k be a field and A a k-algebra. Assume $\dim_k(A) = n$ is finite. Using the left regular representation, we can embed A as a k-subalgebra of $\operatorname{Hom}_k(A, A)$. As in Example 1.2.3, the norm $N_k^A : A \to k$ is a homogeneous polynomial function on A of degree n and the trace $T_k^A : A \to k$ is a homogeneous linear polynomial function on A. Fix a k-basis $\alpha_1, \dots, \alpha_n$ for A. With respect to this basis, we identify A with affine n-space over k (see, for example, [**DF04**, Section 15.2]). That is, an element $a_1\alpha_1 + \cdots + a_n\alpha_n \in A$ corresponds to the point $(a_1, \dots, a_n) \in \mathbb{A}_k^n$. With this identification, the norm $N_k^A : A \to k$ corresponds to a homogeneous polynomial in $k[x_1, \dots, x_n]$ of degree n. Using Exercise 1.2.14 we see that an element α in A is invertible if and only if $N_k^A(\alpha) \neq 0$. If A is a division algebra over k, then the norm defines a homogeneous polynomial in $k[x_1, \dots, x_n]$ of degree n with no nontrivial zeros. We should advise the reader that the norm used in this example is not the norm defined specifically for an Azumaya algebra (or a central simple algebra) in Section 11.1.1.

Proposition 1.2.10. *Let k be an algebraically closed field and $K = k(x)$ the field of rational functions over k in the single variable x. Then K is C_1.*

Proof. Let f be a homogeneous polynomial of degree d in $K[x_1, \dots, x_n]$, where $n > d$. If necessary, we can multiply by a polynomial in $k[x]$, and assume f is in $k[x][x_1,\dots,x_n]$. Expand $f = t_1m_1 + \cdots + t_pm_p$ as a sum where each t_i is a polynomial in $k[x]$ and each m_i is a monomial of degree d in $k[x_1,\dots,x_n]$ with coefficient 1. Let q be the maximum of $\{\deg(t_1),\dots,\deg(t_p)\}$. Pick m such that $m(n-d) > q-n+1$. This is possible, since $n > d$. Let $Y = \{y_{ij} \mid 1 \le i \le n, 0 \le j \le m\}$ be a set of indeterminates. For $1 \le i \le n$, consider the polynomials $p_i = \sum_{j=0}^{n} y_{ij}x^j$ in $k[Y][x]$. For any $e \ge 0$, $(p_i)^e$ is a polynomial in $k[Y][x]$ and in $(p_i)^e$ the coefficient of each power x^j is a homogeneous polynomial in $k[Y]$ of degree e. For each of the monomials m_r we have: $m_r(p_1,\dots,p_n)$ is a polynomial in $k[Y][x]$, the coefficient of each power x^j is a homogeneous polynomial in $k[Y]$ of degree d, and the highest power of x that occurs has degree md. Therefore, $f(p_1,\dots,p_n)$ is a polynomial in $k[Y][x]$ and the coefficient of each power x^j is a homogeneous polynomial in $k[Y]$ of degree d. Because q is the highest degree of all of the polynomials $t_1,\dots,t_p$, the highest power of x appearing in $f(p_1,\dots,p_n)$ has degree $q+md$ or less. Expand $f(p_1,\dots,p_n)$ as a polynomial in x with coefficients in $k[Y]$:

$$f(p_1,\dots,p_n) = \sum_{j=0}^{q+md} f_j(Y)x^j.$$

Consider the ideal $I = (f_0(Y),\dots,f_{q+md}(Y))$ in $k[Y]$. Because the number of indeterminates, $(m+1)n$, is greater than the number of generators, $q+md+1$, by Exercise 1.2.22, $Z(I)$ is an infinite subset of $\mathbb{A}_k^{(m+1)n}$. That is, there are elements a_{ij} in k, not all zero, defining $p_i = \sum_j a_{ij}x^j$ in $k[x]$ such that $f(p_1,\dots,p_n) = 0$. □

Lemma 1.2.11. *Let k be a field which is not algebraically closed. For any positive integer N, there exists $n \ge N$ and a homogeneous polynomial $f \in k[x_1,\dots,x_n]$ of degree n such that $Z(f) = \{(0,\dots,0)\}$.*

Proof. Let K/k be an extension of fields and assume $\dim_k(K) = d > 1$. As in Example 1.2.9, the norm $N_k^K : K \to k$ can be used to define a homogeneous polynomial $\phi \in k[x_1,\dots,x_d]$ of degree d which has only the trivial zero. Let $X = \{x_{ij} \mid 1 \le i \le d, 1 \le j \le d\}$ be a set of d^2 indeterminates. The notation $\phi^{(2)}$ denotes the polynomial

$$\phi^{(2)} = \phi(\phi \mid \phi \mid \cdots \mid \phi) = \phi(\phi(x_{11},\dots,x_{1d}),\dots,\phi(x_{d1},\dots,x_{dd}))$$

in $k[X]$ obtained by substituting ϕ into itself. Then $\phi^{(2)}$ is homogeneous of degree d^2 in d^2 variables. Since ϕ has only the trivial zero, we see that $\phi^{(2)}$ also has only the trivial zero. Using the same notation,

$$\phi^{(3)} = \phi^{(2)}(\phi \mid \phi \mid \cdots \mid \phi)$$

is the polynomial obtained by substituting ϕ into $\phi^{(2)}$. Then $\phi^{(3)}$ is a homogeneous polynomial of degree d^3 in d^3 variables and has only the trivial zero. Recursively, for any $m > 1$ we construct a homogeneous polynomial $\phi^{(m)}$ of degree d^m in d^m variables which has only the trivial zero. Take f to be $\phi^{(m)}$, for any m such that $d^m \geq N$. □

Lemma 1.2.12. *Assume K is a C_1 field and d, m and n are positive integers. Let $f_1, \dots, f_m$ be homogeneous polynomials of degree d in $k[x_1, \dots, x_n]$. If $n > md$, then the zero set $Z(f_1, \dots, f_m)$ is not equal to the singleton set $\{(0, \dots, 0)\}$.*

Proof. By Lemma 1.2.11, there exist an integer $e > m$ and a homogeneous polynomial ϕ in $K[x_1, \dots, x_e]$ of degree e such that ϕ has only the trivial zero. Let $\lfloor e/m \rfloor$ be the greatest integer less than or equal to e/m. In ϕ substitute $\lfloor e/m \rfloor$ groups of $f_1, \dots, f_m$ for $x_1, \dots, x_{n\lfloor e/m \rfloor}$ according to the pattern

$$\psi = \phi(f_1, \dots, f_m \mid f_1, \dots, f_m \mid \cdots \mid f_1, \dots, f_m \mid 0, \dots, 0)$$

where the vertical bar signifies that a different set of n variables are used in the different groups $f_1, \dots, f_m$. So ψ is a polynomial in $n\lfloor e/m \rfloor$ variables with coefficients in K, and is homogeneous of degree de. Since $n - dm > 0$, by taking e larger if necessary, we can pick ϕ and e such that $e > m$ and $(n - dm)\lfloor e/m \rfloor > dm$. Therefore, $n\lfloor e/m \rfloor > dm(1 + \lfloor e/m \rfloor) \geq de$. Since K is C_1, ψ has a nontrivial zero. By the choice of ϕ, this implies there exists a nontrivial simultaneous zero for $f_1, \dots, f_m$. □

Theorem 1.2.13. *Let F be a field which is C_1. If K is an algebraic extension field of F, then K is C_1.*

Proof. Let f be a homogeneous polynomial of degree d in $K[x_1, \dots, x_n]$, with $d < n$. The coefficients of f generate a finite dimensional extension of F in K. For this reason, it suffices to prove the theorem under the assumption that $\dim_F(K) = m$ is finite. Let $\{\kappa_1, \dots, \kappa_m\}$ be an F-basis for K. Let $Y = \{y_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ be a set of nm new indeterminates. Let $p_i = y_{i1}\kappa_1 + \cdots + y_{im}\kappa_m$, for $1 \leq i \leq n$. Then $p_1, \dots, p_n$ are homogeneous linear polynomials in $K[Y]$. We can write $f(p_1, \dots, p_n) \in K[Y]$ as

$$f(p_1, \dots, p_n) = f_1\kappa_1 + \cdots + f_m\kappa_m$$

where each f_j is a homogeneous polynomial of degree d in $F[Y]$. If follows from Lemma 1.2.12 that $Z(f_1, \dots, f_m) \subseteq \mathbb{A}_F^{mn}$ contains a point (a_{ij}) such that not all of the a_{ij} are zero. Evaluating $p_1, \dots, p_n$ at (a_{ij}) shows that there exists a nontrivial zero for f in $\mathbb{A}_K^n$. □

2.4. Exercises.

Exercise 1.2.14. Let R be a commutative ring and suppose A is an R-algebra which is finitely generated and free of rank n as an R-module. Let

$\theta : A \to \operatorname{Hom}_R(A, A)$ be the left regular representation of A in $\operatorname{Hom}_R(A, A)$ which is defined by $\alpha \mapsto \ell_\alpha$. Define $T_R^A : A \to R$ by the assignment $\alpha \mapsto \operatorname{trace}(\ell_\alpha)$. We call T_R^A the *trace from A to R*. Define $N_R^A : A \to R$ by the assignment $\alpha \mapsto \det(\ell_\alpha)$. We call N_R^A the *norm from A to R*.

(1) Show that $T_R^A(r\alpha + s\beta) = rT_R^A(\alpha) + sT_R^A(\beta)$, if $r, s \in R$ and $\alpha, \beta \in A$.

(2) Show that $N_R^A(\alpha\beta) = N_R^A(\alpha)N_R^A(\beta)$ and $N_R^A(r\alpha) = r^n N_R^A(\alpha)$, if $r \in R$ and $\alpha, \beta \in A$.

Exercise 1.2.15. Let R be a commutative ring and M a finitely generated free R-module of rank n. Let $\phi \in \operatorname{Hom}_R(M, M)$. Show that if $\operatorname{char.poly}_R(\phi) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\operatorname{trace}(\phi) = -a_{n-1}$ and $\det(\phi) = (-1)^n a_0$.

Exercise 1.2.16. Let k be a field, V a finitely generated vector space over k, and $\phi \in \operatorname{Hom}_k(V, V)$. Suppose $q = \operatorname{min.poly}_k(\phi) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ is irreducible in $k[x]$. Prove the following.

(1) $\operatorname{char.poly}_k(\phi) = q^r$ for some integer r.

(2) $\det(\phi) = (-1)^{mr} a_0^r$.

(3) $\operatorname{trace}(\phi) = -ra_{m-1}$.

Exercise 1.2.17. Let $\theta : R \to S$ be a homomorphism of commutative rings.

(1) Show that θ induces a homomorphism of rings $\theta : M_n(R) \to M_n(S)$.

(2) Show that $\theta(\det(M)) = \det(\theta(M))$, for every M in $M_n(R)$.

(3) Show that under the homomorphism of rings $R[x] \to S[x]$, the characteristic polynomial of M maps to the characteristic polynomial of $\theta(M)$. That is, $\theta(\operatorname{char.poly}_R(M)) = \operatorname{char.poly}_S(\theta(M))$.

Exercise 1.2.18. Let R be a commutative ring and $n \geq 1$. If $A \in M_n(R)$, show that the trace of A satisfies:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} A e_{ji} = \operatorname{trace}(A) I_n$$

where e_{ij} denotes the elementary matrix and $I_n = e_{11} + \cdots + e_{nn}$ the identity matrix.

Exercise 1.2.19. Let R be a commutative ring and $A = M_n(R)$ the ring of n-by-n matrices over R. The so-called *trace pairing* $\tau : A \times A \to R$ is defined by $\tau(\alpha, \beta) = \operatorname{trace}(\alpha\beta)$. Show that τ satisfies these properties:

(1) $\tau(\alpha, \beta) = \tau(\beta, \alpha)$.

(2) $\tau(a_1\alpha_1 + a_2\alpha_2, \beta) = a_1\tau(\alpha_1, \beta) + a_2\tau(\alpha_2, \beta)$ for $a_1, a_2 \in R$.

(3) $\tau(\alpha, b_1\beta_1 + b_2\beta_2) = b_1\tau(\alpha, \beta_1) + b_2\tau(\alpha, \beta_2)$ for $b_1, b_2 \in R$.

(4) If $\alpha \neq 0$ is fixed, then $\tau(\alpha, \ ) : A \to R$ is nonzero. That is, there exists β such that $\tau(\alpha, \beta) \neq 0$.

We say that τ is a *symmetric nondegenerate bilinear form.*

Exercise 1.2.20. Let R be an integral domain.

(1) A polynomial f in $R[x]$ defines a function $f : R \to R$. If R is infinite, show that f is the zero function (that is, $f(a) = 0$ for all $a \in R$) if and only if f is the zero polynomial.
(2) A polynomial f in $R[x_1, \dots, x_r]$ defines a function $f : R^r \to R$. If R is infinite, use induction on r to show f is the zero function if and only if f is the zero polynomial.

Exercise 1.2.21. Let k be a field. Show that for any $n \geq 1$ there exists a polynomial $f \in k[x]$ of degree n such that f has no repeated roots.

Exercise 1.2.22. Let k be an algebraically closed field, $n \geq 1$, and $S = k[x_0, \dots, x_n]$. Then $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ is a graded ring. Let $S_+ = (x_0, x_1, \dots, x_n)$ be the maximal ideal generated by the monomials of degree one. Prove:

(1) Let $f \in S_d$ be a homogeneous polynomial of degree $d > 0$. If $(a_0, \dots, a_n)$ is in $Z(f)$, and $\alpha \in K$, then $(\alpha a_0, \dots, \alpha a_n) \in Z(f)$.
(2) Let $I = (f_1, \dots, f_m)$ be an ideal in S, where each f_i is homogeneous and has degree greater than or equal to one. If $\operatorname{Rad} I$, the nil radical of I, is not equal to S_+, then $Z(I)$ is infinite. (See Section 2.2 for the definition of $\operatorname{Rad} I$.)

Exercise 1.2.23. Let k be a field and F an extension field of k. Suppose α and β are elements of F that are algebraic over k. Using resultants, show that $\alpha + \beta$ and $\alpha\beta$ are algebraic over k. Show how to find the minimal polynomials for $\alpha + \beta$ and $\alpha\beta$.

3. Hom and Tensor

We assume the reader has some familiarity with the tensor product of modules as well as Hom groups. We give quick reviews of the definitions and list the fundamental properties. As basic references we recommend [**DF04**] and [**Hun80**].

3.1. Tensor Product. Let S and R be rings. If $M \in \mathfrak{M}_R$ and $M \in {}_S\mathfrak{M}$, then M is said to be a *left S right R bimodule* if $s(mr) = (sm)r$ for all possible $s \in S$, $m \in M$ and $r \in R$. Denote by ${}_S\mathfrak{M}_R$ the category of all left S right R bimodules. We say that M is a *left R left S bimodule* if M is both a left R-module and a left S-module and $r(sm) = s(rm)$ for all possible

$r \in R$, $m \in M$ and $s \in S$. For example, if R is a ring and I is an ideal in R, then the associative law for multiplication in R shows that I is a left R right R bimodule. If R is a commutative ring, any left R-module M can be made into a left R right R bimodule by defining mr to be rm. If M is in ${}_R\mathfrak{M}$, N is in $\mathfrak{M}_R$, and C is a $\mathbb{Z}$-module, then a function $f : M \times N \to C$ is said to be an *R-balanced map* if it satisfies

(1) $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$,
(2) $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$, and
(3) $f(mr, n) = f(m, rn)$,

for all possible $m_i \in M$, $n_i \in N$, $r \in R$. The tensor product of M and N over R consists of an abelian group denoted $M \otimes_R N$, and an R-balanced map $\tau : M \times N \to M \otimes_R N$ satisfying the following universal mapping property. If C is an abelian group and $f : M \times N \to C$ is R-balanced, then there exists a unique homomorphism $\phi : M \otimes_R N \to C$ such that $\phi\tau = f$. The reader should verify that the tensor product exists, is unique up to isomorphism, and satisfies the properties enumerated in the following lemma.

Lemma 1.3.1. *Let R and S be rings.*

(1) *If M and M' are in ${}_S\mathfrak{M}_R$, and N and N' are in ${}_R\mathfrak{M}$, then the following are true.*
   (a) *If $s \in S$, $m \in M$ and $n \in N$, the multiplication rule $s(m \otimes n) = sm \otimes n$ turns $M \otimes_R N$ into a left S-module.*
   (b) *Let $f : M \to M'$ be a homomorphism of left S right R bimodules and $g : N \to N'$ a homomorphism of left R-modules. Then $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ is a homomorphism of left S-modules.*

(2) *Since R is a left R right R bimodule, given $M \in {}_R\mathfrak{M}$ we view $R \otimes_R M$ as a left R-module. In this case, $R \otimes_R M \cong M$ as left R-modules under the map $x \otimes y \mapsto xy$.*

(3) *Assume $L \in \mathfrak{M}_R$, $M \in {}_R\mathfrak{M}_S$ and $N \in {}_S\mathfrak{M}$. Then $(L \otimes_R M) \otimes_S N$ is isomorphic as an abelian group to $L \otimes_R (M \otimes_S N)$ under the map which sends $(x \otimes y) \otimes z$ to $x \otimes (y \otimes z)$.*

(4) *Let M and $\{M_i\}_{i \in I}$ be in $\mathfrak{M}_R$. Let N and $\{N_j\}_{j \in J}$ be in ${}_R\mathfrak{M}$. There are isomorphisms of abelian groups*

$$M \otimes_R \Big(\bigoplus_{j \in J} N_j \Big) \cong \bigoplus_{j \in J} (M \otimes_R N_j)$$

and

$$\Big(\bigoplus_{i \in I} M_i \Big) \otimes_R N \cong \bigoplus_{i \in I} (M_i \otimes_R N) .$$

(5) *Let R be a ring, $M \in \mathfrak{M}_R$, and $N \in {}_R\mathfrak{M}$. Then $M \otimes_R N \cong N \otimes_{R^o} M$ under the map $x \otimes y \mapsto y \otimes x$.*

(6) *Let R be a ring and M a right R-module. Given a short exact sequence*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

in ${}_R\mathfrak{M}$, the sequence

$$M \otimes_R A \xrightarrow{1\otimes\alpha} M \otimes_R B \xrightarrow{1\otimes\beta} M \otimes_R C \to 0$$

is an exact sequence of $\mathbb{Z}$-modules.

If S is a ring and M is a left S right R bimodule, then by Lemma 1.3.1 it follows that $M \otimes_R (\cdot)$ defines a right exact covariant functor from ${}_R\mathfrak{M}$ to ${}_S\mathfrak{M}$. In case $M \otimes_R (\cdot)$ is also left exact, then we say M is a *flat* R-module. Let $\phi : R \to S$ be a homomorphism of rings. Then R acts on S from both the left and right by the rules $rx = \phi(r)x$ and $xr = x\phi(r)$. Using the ring structure of S, the reader should verify that S is a left S right R bimodule. Theorem 1.3.2 below describes a general technique, called *change of base*. The proof is left to the reader.

Theorem 1.3.2. *Let $\phi : R \to S$ be a homomorphism of rings. As mentioned above, ϕ makes S into a left S right R bimodule.*

(1) *The assignment $M \mapsto S \otimes_R M$ defines a right exact covariant functor ${}_R\mathfrak{M} \to {}_S\mathfrak{M}$ which satisfies:*
  (a) *R is mapped to S.*
  (b) *$\bigoplus_{i\in I} M_i \mapsto \bigoplus_{i\in I} (S \otimes_R M_i)$. That is, a direct sum is mapped to a direct sum.*
  (c) *The free module R^I is mapped to the free S-module S^I.*

(2) *If M is R-projective, then $S \otimes_R M$ is S-projective.*

(3) *If M is an R-generator, then $S \otimes_R M$ is a S-generator.*

(4) *If M is finitely generated over R, then $S \otimes_R M$ is finitely generated over S.*

(5) *If M is a flat R-module, then $S \otimes_R M$ is a flat S-module.*

For a commutative ring R, Proposition 1.3.3 states that tensor product defines a product on the category of R-progenerator modules. The proof is left to the reader.

Proposition 1.3.3. *Let R be a commutative ring and let M and N be two R-modules.*

(1) *If M and N are finitely generated over R, then so is $M \otimes_R N$.*

(2) *If M and N are projective over R, then so is $M \otimes_R N$.*

(3) *If M and N are generators over R, then so is $M \otimes_R N$.*

(4) *If M and N are progenerators over R, then so is $M \otimes_R N$.*

Proposition 1.3.4 provides partial converses to the four parts of Proposition 1.3.3.

Proposition 1.3.4. *Let R be a ring. Let M and N be left R right R-bimodules. Assume $M \otimes_R N$ is a left R-generator module. Then the following are true.*

(1) *M and N are both left R-generator modules.*

(2) *If $M \otimes_R N$ is projective as a left R-module, then M and N are both projective as left R-modules.*

(3) *If $M \otimes_R N$ is finitely generated as a left R-module, then M and N are both finitely generated as a left R-modules.*

(4) *If $M \otimes_R N$ is a left progenerator over R, then M and N are both left progenerators over R.*

If $M \otimes_R N$ is a right R-generator module, then right hand versions of (1) – (4) hold for M and N.

Proof. (1): By Exercise 1.1.11 there is a free R-module F_1 of finite rank and a homomorphism f_1 of left R-modules such that $f_1 : F_1 \otimes_R (M \otimes_R N) \to R$ is onto. By Lemma 1.1.4 there is a free R-module F_2 and a left R-module homomorphism f_2 such that $f_2 : F_2 \to M$ is onto. By Lemma 1.3.1,

$$F_2 \otimes_R N \xrightarrow{f_2 \otimes 1} M \otimes_R N \to 0$$

is exact. For the same reason,

$$F_1 \otimes_R (F_2 \otimes_R N) \xrightarrow{1 \otimes f_2 \otimes 1} F_1 \otimes_R (M \otimes_R N) \to 0$$

is exact. Since $F_1 \otimes_R F_2$ is a free R-module, Lemma 1.3.1 shows that $F_1 \otimes_R (F_2 \otimes_R N)$ is a direct sum of copies of N. Then $f_1 \circ (1 \otimes f_2 \otimes 1)$ maps a direct sum of copies of N onto R. By another application of Exercise 1.1.11, this shows N is a left R-module generator. The other case is left to the reader.

(2) and (3): By Part (1) and Exercise 1.1.11 there exist a free R-module F of finite rank and a left R-module epimorphism $f : N \otimes_R F \to R$. But f is split since R is projective over R. By Exercise 1.3.7,

$$M \otimes_R N \otimes_R F \xrightarrow{f \otimes 1} M \to 0$$

is split exact. If $M \otimes_R N$ is projective, then by Lemma 1.3.1 and Exercise 1.1.10, M is projective. If $M \otimes_R N$ is finitely generated, then so is M. The other cases are left to the reader. □

In proving Exercise 1.3.15, Corollary 1.3.5 will be helpful. In Corollary 2.6.6 it will be combined with a test for locally free modules over an integral domain to give us a useful flatness criterion. For a proof, see [**Rot79**, Corollary 3.58], for example.

Corollary 1.3.5. *Let R be any ring and M a finitely generated left R-module. Then M is projective if and only if M is of finite presentation and flat.*

3.2. Exercises.

Exercise 1.3.6. Let R be a ring and let R^I and R^J be free R-modules. Prove:

(1) $R^I \otimes_R R^J$ is a free R-module.

(2) Assume R is commutative. If A is a free R-module of rank m and B is a free R-module of rank n, then $A \otimes_R B$ is free of rank mn.

Exercise 1.3.7. Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1.4}$$

be a short exact sequence of left R-modules. Given a right R-module M, consider the sequence

$$0 \to M \otimes_R A \xrightarrow{1\otimes\alpha} M \otimes_R B \xrightarrow{1\otimes\beta} M \otimes_R C \to 0. \tag{1.5}$$

Prove:

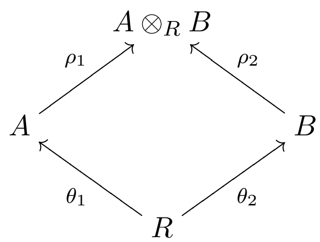
(1) If (1.4) is split exact, then (1.5) is split exact.

(2) If M is a free right R-module, then (1.5) is exact, hence M is flat.

(3) If M is a projective right R-module, then (1.5) is exact, hence M is flat.

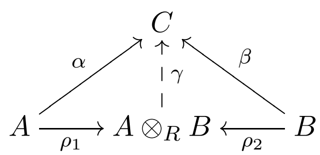
Exercise 1.3.8. Let R be a commutative ring. This exercise outlines a proof that the tensor product is a coproduct in the category of commutative R-algebras. Suppose A and B are R-algebras. Then A and B come with homomorphisms $\theta_1 : R \to A$ and $\theta_2 : R \to B$ satisfying $\operatorname{im}(\theta_1) \subseteq Z(A)$ and $\operatorname{im}(\theta_2) \subseteq Z(B)$. Begin by proving that $A \otimes_R B$ is an R-algebra with multiplication induced by $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes y_1y_2$.

(1) Show that there exist R-algebra homomorphisms $\rho_1 : A \to A \otimes_R B$ and $\rho_2 : B \to A \otimes_R B$ such that the diagram



commutes. Show that elements in $\operatorname{im}(\rho_1)$ commute with elements in $\operatorname{im}(\rho_2)$. That is, $\rho_1(x)\rho_2(y) = \rho_2(y)\rho_1(x)$ for all $x \in A$, $y \in B$.

(2) Suppose there exist R-algebra homomorphisms $\alpha : A \to C$ and $\beta : B \to C$ such that $\operatorname{im}(\alpha)$ commutes with $\operatorname{im}(\beta)$. Show that there exists a unique R-algebra homomorphism $\gamma : A \otimes_R B \to C$ such that the diagram



commutes.

(3) Show that if there exists an R-algebra homomorphism $\gamma : A \otimes_R B \to C$, then there exist R-algebra homomorphisms $\alpha : A \to C$ and $\beta : B \to C$ such that the image of α commutes with the image of β and the diagram of Part (2) commutes.

Exercise 1.3.9. Let S be a commutative R-algebra. Show that there is a well defined homomorphism of R-algebras $\mu : S \otimes_R S \to S$ which maps a typical element $\sum x_i \otimes y_i$ in the tensor algebra to $\sum x_i y_i$ in S.

Exercise 1.3.10. Let R be a commutative ring and let A and B be R-algebras. Prove that $A \otimes_R B \cong B \otimes_R A$ as R-algebras.

Exercise 1.3.11. Prove the following, if S and T are commutative R-algebras.

(1) If S and T are both finitely generated R-algebras, then $S \otimes_R T$ is a finitely generated R-algebra.

(2) If T is a finitely generated R-algebra, then $S \otimes_R T$ is a finitely generated S-algebra.

Exercise 1.3.12. Let $\theta : R \to S$ be a homomorphism of rings. Let $M \in \mathfrak{M}_S$ and $N \in {}_S\mathfrak{M}$. Via θ, M can be viewed as a right R-module and N as a left R-module. Show that θ induces a well defined $\mathbb{Z}$-module epimorphism

$M \otimes_R N \to M \otimes_S N$. (Note: The dual result, how a Hom group behaves when the ring in the middle is changed, is studied in Exercise 1.3.35.)

Exercise 1.3.13. Let $\theta : R \to S$ be a homomorphism of rings. Let $M \in \mathfrak{M}_R$ and $N \in {}_R\mathfrak{M}$, $M' \in \mathfrak{M}_S$ and $N' \in {}_S\mathfrak{M}$. Via θ, M' and N' are viewed as R-modules. In this context, let $f : M \to M'$ be a right R-module homomorphism and $g : N \to N'$ a left R-module homomorphism. Show that there is a well defined $\mathbb{Z}$-module homomorphism $M \otimes_R N \to M' \otimes_S N'$ which satisfies $x \otimes y \mapsto f(x) \otimes g(y)$.

Exercise 1.3.14. Let R be a commutative ring and S a commutative R-algebra. Let A be an S-algebra. Using Exercise 1.3.12, show that there is a well defined epimorphism of rings $A \otimes_R A \to A \otimes_S A$.

Exercise 1.3.15. Let R be a ring and I a left ideal in R. Prove that R/I is a projective left R-module if and only if I is finitely generated and R/I is a flat left R-module. This adds one more equivalent condition to the list in Exercise 1.1.22.

Exercise 1.3.16. Let k be a field and $n > 1$ an integer. Let $T = k[x, y]$, $S = k[x^n, xy, y^n]$, and $R = k[x^n, y^n]$. For the tower of subrings $R \subseteq S \subseteq T$, prove:

(1) T is free over R of rank n^2.
(2) S is free over R of rank n.
(3) T is not free over S. (Hint: Consider the quotient rings

$$S/(x^n, xy, y^n) \text{ and } T/(x^n, xy, y^n).)$$

For more properties of the ring $k[x^n, xy, y^n]$, see Exercise 6.2.9.

3.3. Hom Groups. If R is a ring and M and N are R-modules, then $\operatorname{Hom}_R(M, N)$ is the set of R-module homomorphisms from M to N. Then $\operatorname{Hom}_R(M, N)$ is an additive group under point-wise addition:

$$(f + g)(x) = f(x) + g(x).$$

If R is commutative, then $\operatorname{Hom}_R(M, N)$ can be turned into a left R-module by defining $(rf)(x) = rf(x)$. If R is noncommutative, then $\operatorname{Hom}_R(M, N)$ cannot be turned into an R-module per se. If S is another ring and M or N is a bimodule over R and S, then we can turn $\operatorname{Hom}_R(M, N)$ into an S-module. Lemma 1.3.17 lists four such possibilities. The proof is left to the reader.

Lemma 1.3.17. *Let R and S be rings.*

(1) *If M is a left R right S bimodule and N is a left R-module, then $\operatorname{Hom}_R(M, N)$ is a left S-module, with the action of S given by $(sf)(m) = f(ms)$.*

(2) *If M is a left R-module and N is a left R right S bimodule, then* $\mathrm{Hom}_R(M, N)$ *is a right S-module, with the action of S given by* $(fs)(m) = (f(m))s$.

(3) *If M is a left R left S bimodule and N is a left R-module, then* $\mathrm{Hom}_R(M, N)$ *is a right S-module, with the action of S given by* $(fs)(m) = f(sm)$.

(4) *If M is a left R-module and N is a left R left S bimodule, then* $\mathrm{Hom}_R(M, N)$ *is a left S-module, with the action of S given by* $(sf)(m) = s(f(m))$.

Let R be a ring and M a left R-module. Then $\mathrm{Hom}_R(M, M)$ is a ring where multiplication is composition of functions:

$$(fg)(x) = f(g(x)).$$

The ring $\mathrm{Hom}_R(M, M)$ acts from the left as a ring of functions on M. This multiplication makes M into a left $\mathrm{Hom}_R(M, M)$-module. If R is commutative, then $\mathrm{Hom}_R(M, M)$ is an R-algebra. The next two results are corollaries to Nakayama's Lemma (Lemma 1.1.13).

Corollary 1.3.18. *Let R be a commutative ring and M a finitely generated R-module. Let $f : M \to M$ be an R-module homomorphism such that f is onto. Then f is one-to-one.*

Proof. Let $R[x]$ be the polynomial ring in one variable over R. We turn M into an $R[x]$-module using f. Given $m \in M$ and $p(x) \in R[x]$, define

$$p(x) \cdot m = p(f)(m).$$

Since M is finitely generated over R, M is finitely generated over $R[x]$. Let I be the ideal in $R[x]$ generated by x. Then $IM = M$ because f is onto. By Lemma 1.1.13, $I + \mathrm{annih}_{R[x]} M = R[x]$. For some $p(x)x \in I$, $1 + p(x)x \in \mathrm{annih}_{R[x]} M$. Then $(1 - p(x)x)M = 0$ which says for each $m \in M$, $m = (p(f)f)(m)$. Then $p(f)f$ is the identity function, so f is one-to-one. □

Corollary 1.3.19. *Let R be a commutative ring, M an R-module, N a finitely generated R-module, and $f \in \mathrm{Hom}_R(M, N)$. Then f is onto if and only if for each maximal ideal $\mathfrak{m}$ in R, the induced map $\bar{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is onto.*

Proof. Let C denote the cokernel of f and let $\mathfrak{m}$ be an arbitrary maximal ideal of R. Since N is finitely generated, so is C. Tensor the exact sequence

$$M \xrightarrow{f} N \to C \to 0$$

with $(\cdot) \otimes_R R/\mathfrak{m}$ to get

$$M/\mathfrak{m}M \xrightarrow{\bar{f}} N/\mathfrak{m}N \to C/\mathfrak{m}C \to 0$$

which is exact since tensoring is right exact. If f is onto, then $C = 0$ so $\bar{f}$ is onto. Conversely if $\mathfrak{m}C = C$ for every $\mathfrak{m}$, then Corollary 1.1.14 implies $C = 0$. □

Given a fixed left R-module M, the assignment $N \mapsto \operatorname{Hom}_R(M, N)$ defines a covariant functor from R-modules to abelian groups. Proposition 1.3.20 summarizes some of the first properties of this functor. If M is in the second factor, then the assignment $N \mapsto \operatorname{Hom}_R(N, M)$ defines a contravariant functor, and in Proposition 1.3.21 we enumerate some of its basic properties. The proofs are left to the reader.

Proposition 1.3.20. *For a ring R and a left R-module M, the following are true.*

(1) $\operatorname{Hom}_R(M, \cdot)$ *is a left exact covariant functor from ${}_R\mathfrak{M}$ to ${}_\mathbb{Z}\mathfrak{M}$ which sends a left R module N to the abelian group* $\operatorname{Hom}_R(M, N)$. *Given any R-module homomorphism $f : A \to B$, the homomorphism of groups*

$$\operatorname{Hom}_R(M, A) \xrightarrow{\mathrm{H}_f} \operatorname{Hom}_R(M, B)$$

is defined by the assignment $g \mapsto fg$.

(2) *M is projective if and only if* $\operatorname{Hom}_R(M, \cdot)$ *is an exact functor.*

Proposition 1.3.21. *Given a ring R and a left R-module M,* $\operatorname{Hom}_R(\cdot, M)$ *defines a left exact contravariant functor from ${}_R\mathfrak{M}$ to ${}_\mathbb{Z}\mathfrak{M}$ which sends a left R module N to the abelian group* $\operatorname{Hom}_R(N, M)$. *Given any R-module homomorphism $f : A \to B$, the homomorphism of groups*

$$\operatorname{Hom}_R(B, M) \xrightarrow{\mathrm{H}_f} \operatorname{Hom}_R(A, M)$$

is defined by the assignment $g \mapsto gf$.

Lemma 1.3.22 is a partial converse to the left exactness property of the functor $\operatorname{Hom}_R(\cdot, M)$.

Lemma 1.3.22. *Let R be a ring. The sequence of R-modules*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact, if for all R-modules M

$$\operatorname{Hom}_R(C, M) \xrightarrow{\mathrm{H}_\beta} \operatorname{Hom}_R(B, M) \xrightarrow{\mathrm{H}_\alpha} \operatorname{Hom}_R(A, M)$$

is an exact sequence of $\mathbb{Z}$-modules.

Proof. First we show $\operatorname{im}\alpha \subseteq \ker\beta$. Suppose there exists $a \in A$ such that $\beta\alpha a \neq 0$. We take M to be the nonzero module C. By assumption,

$$\operatorname{Hom}_R(C, C) \xrightarrow{\mathrm{H}_\beta} \operatorname{Hom}_R(B, C) \xrightarrow{\mathrm{H}_\alpha} \operatorname{Hom}_R(A, C)$$

is an exact sequence of $\mathbb{Z}$-modules. Let 1 denote the identity element in the ring $\operatorname{Hom}_R(C, C)$. By evaluating at the element a, we see that $\mathrm{H}_\alpha \mathrm{H}_\beta(1) \neq 0$, a contradiction. Now we show $\operatorname{im}\alpha \supseteq \ker\beta$. Suppose there exists $b \in B$ such that $\beta b = 0$ and $b \notin \operatorname{im}\alpha$. By Proposition 1.3.21, the exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\pi} B/\operatorname{im}\alpha \to 0$$

gives rise to the exact sequence

$$0 \to \operatorname{Hom}_R(B/\operatorname{im}\alpha, B/\operatorname{im}\alpha) \xrightarrow{\mathrm{H}_\pi} \operatorname{Hom}_R(B, B/\operatorname{im}\alpha) \xrightarrow{\mathrm{H}_\alpha} \operatorname{Hom}_R(A, B/\operatorname{im}\alpha).$$

The identity map $1 \in \operatorname{Hom}_R(B/\operatorname{im}\alpha, B/\operatorname{im}\alpha)$ maps to the nonzero map $\pi = \mathrm{H}_\pi(1)$. Since $\mathrm{H}_\alpha(\pi) = \pi\alpha = 0$, we see that $\pi \in \ker \mathrm{H}_\alpha$. If we take M to be the nonzero module $B/\operatorname{im}\alpha$, then by assumption,

$$\operatorname{Hom}_R(C, B/\operatorname{im}\alpha) \xrightarrow{\mathrm{H}_\beta} \operatorname{Hom}_R(B, B/\operatorname{im}\alpha) \xrightarrow{\mathrm{H}_\alpha} \operatorname{Hom}_R(A, B/\operatorname{im}\alpha)$$

is an exact sequence of $\mathbb{Z}$-modules. So $\pi \in \operatorname{im}\mathrm{H}_\beta$. Therefore, there exists g in $\operatorname{Hom}_R(C, B/\operatorname{im}\alpha)$ such that $g\beta = \pi$. On the one hand we have $g\beta(b) = 0$. On the other hand we have $\pi(b) \neq 0$, a contradiction. □

Part (2) of Proposition 1.3.23 describes how the Hom functor commutes with a direct product in the second variable. Note however that a direct sum in the first variable is transformed into a direct product. As an important special case, it follows that Hom commutes with a finite direct sum (or finite direct product) in either variable. The proof is left to the reader.

Proposition 1.3.23. *Let R be a ring. Let M, N, $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ be R-modules.*

(1) *The map $f \mapsto f(1)$ defines an isomorphism $\operatorname{Hom}_R(R, M) \cong M$ of R-modules.*

(2) *There are isomorphisms*

(a) $\operatorname{Hom}_R\Big(\bigoplus_{i\in I} M_i, N\Big) \cong \prod_{i\in I} \operatorname{Hom}_R(M_i, N)$ *and*

(b) $\operatorname{Hom}_R\Big(M, \prod_{j\in J} N_j\Big) \cong \prod_{j\in J} \operatorname{Hom}_R(M, N_j)$

of $\mathbb{Z}$-modules.

3.4. Hom Tensor Relations. In this section we prove several identities involving Hom groups and the tensor product. Following the convention of [**DI71**], we refer to these as "Hom Tensor Relations".

Theorem 1.3.24 (Adjoint Isomorphism)**.** *Let R and S be rings.*

(1) *If $A \in {}_R\mathfrak{M}$, $B \in {}_S\mathfrak{M}_R$ and $C \in {}_S\mathfrak{M}$, then there is an isomorphism of $\mathbb{Z}$-modules*

$$\operatorname{Hom}_S(B \otimes_R A, C) \xrightarrow{\psi} \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C))$$

defined by $\psi(f)(a) = f(\cdot \otimes a)$.

(2) *If $A \in \mathfrak{M}_R$, $B \in {}_R\mathfrak{M}_S$ and $C \in \mathfrak{M}_S$, then there is an isomorphism of $\mathbb{Z}$-modules*

$$\operatorname{Hom}_S(A \otimes_R B, C) \xrightarrow{\phi} \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C))$$

defined by $\phi(f)(a) = f(a \otimes \cdot)$.

In both cases, the isomorphism is natural in both variables A and C. The "Tensor-Hom" pair, $(B \otimes_R (\cdot), \operatorname{Hom}_S(B, \cdot))$, is an adjoint pair.

Proof. See, for example, [**Rot79**, Theorem 2.11]. □

Lemma 1.3.25. *Let R and S be rings. Let $A \in {}_R\mathfrak{M}$ be finitely generated and projective. For any $B \in {}_R\mathfrak{M}_S$ and $C \in \mathfrak{M}_S$ there is a natural isomorphism*

$$\operatorname{Hom}_S(B, C) \otimes_R A \xrightarrow{\alpha} \operatorname{Hom}_S(\operatorname{Hom}_R(A, B), C)$$

of abelian groups. On generators, the map is defined by $\alpha(f \otimes a)(g) = f(g(a))$.

Proof. Note that $\operatorname{Hom}_S(B, C)$ is a right R-module by the action $(fr)(b) = f(rb)$ and $\operatorname{Hom}_R(A, B)$ is a right S-module by the action $(gs)(a) = g(a)s$. Given any (f, a) in $\operatorname{Hom}_S(B, C) \times A$, define $\phi(f, a) \in \operatorname{Hom}_S(\operatorname{Hom}_R(A, B), C)$ by $\phi(f, a)(g) = f(g(a))$. The reader should verify that ϕ is a well defined balanced map. Therefore α is a well defined group homomorphism. Also note that if $\psi : A \to A'$ is an R-module homomorphism, then the diagram

$$\begin{array}{ccc} \operatorname{Hom}_S(B, C) \otimes_R A & \xrightarrow{\alpha} & \operatorname{Hom}_S(\operatorname{Hom}_R(A, B), C) \\ \downarrow{\scriptstyle 1 \otimes \psi} & & \downarrow{\scriptstyle \mathrm{H}(\mathrm{H}(\psi))} \\ \operatorname{Hom}_S(B, C) \otimes_R A' & \xrightarrow{\alpha} & \operatorname{Hom}_S(\operatorname{Hom}_R(A', B), C) \end{array}$$

commutes. If $A = R$, then by Proposition 1.3.23 (1) we see that α is an isomorphism. If $A = R^n$ is finitely generated and free, then using Proposition 1.3.23 we see that α is an isomorphism. If A is a direct summand of a free R-module of finite rank, then the proof follows by combining the above results. □

Theorem 1.3.26. *Let R be a commutative ring and let A and B be R-algebras. Let M be a finitely generated projective A-module and N a finitely generated projective B-module. Then for any A-module M' and any B-module N', the mapping*

$$\operatorname{Hom}_A(M, M') \otimes_R \operatorname{Hom}_B(N, N') \xrightarrow{\psi} \operatorname{Hom}_{A\otimes_R B}(M \otimes_R N, M' \otimes_R N')$$

induced by $\psi(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$ is an R-module isomorphism. If $M = M'$ and $N = N'$, then ψ is also a homomorphism of rings.

Proof. Define a function

$$\operatorname{Hom}_A(M, M') \times \operatorname{Hom}_B(N, N') \xrightarrow{\rho} \operatorname{Hom}_{A\otimes_R B}(M \otimes_R N, M' \otimes_R N')$$

by $\rho(f, g)(x \otimes y) = f(x) \otimes g(y)$. The equations

$$\begin{aligned}
\rho(f_1 + f_2, g)(x \otimes y) &= (f_1 + f_2)(x) \otimes g(y)\\
&= (f_1(x) + f_2(x)) \otimes g(y)\\
&= f_1(x) \otimes g(y) + f_2(x) \otimes g(y)\\
&= \rho(f_1, g)(x \otimes y) + \rho(f_2, g)(x \otimes y)\\
&= \big(\rho(f_1, g) + \rho(f_2, g)\big)(x \otimes y)
\end{aligned}$$

and

$$\begin{aligned}
\rho(fr, g)(x \otimes y) &= (fr)(x) \otimes g(y)\\
&= f(x)r \otimes g(y)\\
&= f(x) \otimes rg(y)\\
&= f(x) \otimes (rg)(y)\\
&= \rho(f, rg)(x \otimes y)
\end{aligned}$$

show that ρ is R-balanced. Therefore ψ is well defined. Now we show that ψ is an isomorphism. The method of proof is to reduce to the case where M and N are free modules.

Case 1: Show that ψ is an isomorphism if $M = A$ and $N = B$. By Proposition 1.3.23 (1), both sides are naturally isomorphic to $M' \otimes_R N'$.

Case 2: Show that ψ is an isomorphism if M is free of finite rank m over A and N is free of finite rank n over B. By Proposition 1.3.23, Exercise 1.3.6, and Case 1, both sides are naturally isomorphic to $(M' \otimes_R N')^{(mn)}$.

Case 3: The general case. By Proposition 1.1.5 (1), we can write $M \oplus L \cong F$ where F is a free A module of finite rank and $N \oplus K \cong G$ where G is a free B module of finite. Using Proposition 1.3.23 and Lemma 1.3.1

(1.6)
$$\operatorname{Hom}_A(F, M') \otimes_R \operatorname{Hom}_B(G, N') = \big(\operatorname{Hom}_A(M, M') \otimes_R \operatorname{Hom}_B(N, N')\big) \oplus H$$

is an internal direct sum of the left hand side for some submodule H. Likewise,

(1.7) $\operatorname{Hom}_{A\otimes_R B}(F\otimes_R G, M'\otimes_R N') = \operatorname{Hom}_{A\otimes_R B}(M\otimes_R N, M'\otimes_R N')\oplus H'$

is an internal direct sum of the right hand side, for some submodule H'. By Case 2, the natural map Ψ is an isomorphism between the left hand sides of (1.6) and (1.7). The restriction of Ψ gives the desired isomorphism ψ. □

Corollary 1.3.27 allows us to change the base ring for the Hom group from R to A, if the module N is finitely generated and projective. When A is a flat commutative R-algebra, see Proposition 2.4.10 for another important theorem on changing the base ring.

Corollary 1.3.27. *Let R be a commutative ring and N a finitely generated projective R-module. Let A be an R-algebra. Then*

$$A\otimes_R \operatorname{Hom}_R(N,N') \xrightarrow{\psi} \operatorname{Hom}_A(A\otimes_R N, A\otimes_R N')$$

is an R-module isomorphism for any R-module N'.

Proof. In Theorem 1.3.26, take $B=R$, $M=M'=A$. □

Corollary 1.3.28. *If R is commutative and M and N are finitely generated projective R-modules, then*

$$\operatorname{Hom}_R(M,M)\otimes_R \operatorname{Hom}_R(N,N) \xrightarrow{\psi} \operatorname{Hom}_R(M\otimes_R N, M\otimes_R N)$$

is an R-algebra isomorphism.

Proof. In Theorem 1.3.26, take $A=B=R$, $M=M'$ and $N=N'$. □

Theorem 1.3.29. *Let A and B be rings. Let L be a finitely generated and projective left A-module. Let M be a left A right B bimodule. Let N be a left B-module. Then*

$$\operatorname{Hom}_A(L,M)\otimes_B N \xrightarrow{\psi} \operatorname{Hom}_A(L, M\otimes_B N)$$

is a $\mathbb{Z}$-module isomorphism, where $\psi(f\otimes y)(x)=f(x)\otimes y$ for all $y\in N$ and $x\in L$.

Proof. By Lemma 1.3.17, $\operatorname{Hom}_A(L,M)$ is a right B-module by the action $(fb)(x)=f(x)b$. The reader should verify that ψ is balanced, hence well defined.

Case 1: Show that ψ is an isomorphism if $L=A$. This follows because both sides are naturally isomorphic to $M\otimes_B N$, by Proposition 1.3.23 (1).

Case 2: Show that ψ is an isomorphism if L is free of rank n over A. By Proposition 1.3.23 (2), Lemma 1.3.1 and Case 1, both sides are naturally isomorphic to $(M\otimes_R N)^{(n)}$.

Case 3: The general case. By Proposition 1.1.5 (1), we can write $L \oplus K \cong F$ where F is a free A module of rank n. Using Proposition 1.3.23 (2) and Lemma 1.3.1

$$\operatorname{Hom}_A(F, M) \otimes_B N = \operatorname{Hom}_A(L, M) \otimes_R N \oplus H \tag{1.8}$$

is an internal direct sum of the left hand side for some submodule H. Likewise,

$$\operatorname{Hom}_A(F, M \otimes_B N) = \operatorname{Hom}_A(L, M \otimes_R N) \oplus H' \tag{1.9}$$

is an internal direct sum of the right hand side, for some submodule H'. By Case 2, the natural map Ψ is an isomorphism between the left hand sides of (1.8) and (1.9). The restriction of Ψ gives the desired isomorphism ψ. □

3.5. Exercises.

Exercise 1.3.30. This exercise is based on [**Bas68**, Proposition 1.1(a), p. 52]. Let R be a ring and M a left R-module. The functor $\operatorname{Hom}_R(M, \cdot)$ from the category of left R-modules to the category of $\mathbb{Z}$-modules is said to be *faithful* in case for every R-module homomorphism $\beta : A \to B$, if $\beta \neq 0$, then there exists $h \in \operatorname{Hom}_R(M, A)$ such that $\beta h \neq 0$. This exercise outlines a proof that M is an R-generator if and only if the functor $\operatorname{Hom}_R(M, \cdot)$ is faithful.

(1) For any left R-module A, set $H = \operatorname{Hom}_R(M, A)$. Let M^H denote the direct sum of copies of M over the index set H. Show that there is an R-module homomorphism
$$\alpha : M^H \to A$$
defined by $\alpha(f) = \sum_{h \in H} h(f(h))$.

(2) Show that if $\operatorname{Hom}_R(M, \cdot)$ is faithful, then for any left R-module A, the map α defined in Part (1) is surjective. Conclude that M is an R-generator. (Hint: Let $\beta : A \to B$ be the cokernel of α. Show that the composition $M \xrightarrow{h} A \xrightarrow{\beta} B$ is the zero map for all $h \in H$.)

(3) Prove that if M is an R-generator, then $\operatorname{Hom}_R(M, \cdot)$ is faithful. (Hint: Use Exercise 1.1.11.)

Exercise 1.3.31. Let R be any ring and $\phi : A \to B$ a homomorphism of left R-modules. Prove that the following are equivalent.

(1) ϕ has a left inverse. That is, there exists an R-module homomorphism $\psi : B \to A$ such that $\psi\phi = 1_A$.

(2) The sequence $\operatorname{Hom}_R(B, M) \xrightarrow{\mathrm{H}_\phi} \operatorname{Hom}_R(A, M) \to 0$ is exact, for every left R-module M.

(3) The sequence $\operatorname{Hom}_R(B, A) \xrightarrow{\mathrm{H}_\phi} \operatorname{Hom}_R(A, A) \to 0$ is exact.

Exercise 1.3.32. Let R be a ring. View R as a left R-module. Show that there is an isomorphism of rings $\operatorname{Hom}_R(R,R) \cong R^o$, where R^o denotes the opposite ring.

Exercise 1.3.33. Let R be a ring, M a left R-module, and N a right R-module. Prove the following.

(1) $M^* = \operatorname{Hom}_R(M,R)$ is a right R-module by the multiplication rule given in Lemma 1.3.17 (2).

(2) $N^* = \operatorname{Hom}_R(N,R)$ is a left R-module by the rule $(rf)(x) = rf(x)$.

(3) Let $M^{**} = \operatorname{Hom}_R(M^*,R)$ be the double dual of M. For $m \in M$, let $\varphi_m : M^* \to R$ be the "evaluation at m" map. That is, if $f \in M^*$, then $\varphi_m(f) = f(m)$. Prove that $\varphi_m \in M^{**}$, and that the assignment $m \mapsto \varphi_m$ defines a homomorphism of left R-modules $M \to M^{**}$.

Exercise 1.3.34. Let R be a ring. We say a left R-module M is *reflexive* in case the homomorphism $M \to M^{**}$ of Exercise 1.3.33 is an isomorphism. Prove the following.

(1) If $M_1, \dots, M_n$ are left R-modules, then the direct sum $\bigoplus_{i=1}^n M_i$ is reflexive if and only if each M_i is reflexive.

(2) A finitely generated free R-module is reflexive.

(3) A finitely generated projective R-module is reflexive.

(4) If P is a finitely generated projective R-module and M is a reflexive R-module, then $P \otimes_R M$ is reflexive.

(5) If R is a PID and M is a nonzero finitely generated torsion R-module, then M is not a reflexive R-module.

Exercise 1.3.35. Exercise 1.3.12 shows how the tensor group behaves when the ring in the middle is changed. The dual result for Hom groups is the object of this exercise. Let $\theta : R \to S$ be a homomorphism of rings. Let M and N be S-modules. Via θ, M and N can be viewed as R-modules. Show that θ induces a well defined $\mathbb{Z}$-module monomorphism $\operatorname{Hom}_S(M,N) \to \operatorname{Hom}_R(M,N)$.

4. Direct Limit and Inverse Limit

In this section we give the definitions and state (usually without proofs) the fundamental properties of the direct limit and the inverse limit. As references for this section, we recommend [**DF04**] and [**AM69**]. Perhaps the most fundamental tool in homological algebra is the so-called Snake Lemma which is stated below. The proof is left to the reader.

Theorem 1.4.1. *(The Snake Lemma) Let R be any ring and*

$$\begin{array}{ccccccccc} & & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & & \end{array}$$

a commutative diagram of R-modules with exact rows. Then there is an exact sequence

$$\ker\alpha \xrightarrow{f_1^*} \ker\beta \xrightarrow{f_2^*} \ker\gamma \xrightarrow{\partial} \operatorname{coker}\alpha \xrightarrow{g_1^*} \operatorname{coker}\beta \xrightarrow{g_2^*} \operatorname{coker}\gamma.$$

If f_1 is one-to-one, then f_1^* is one-to-one. If g_2 is onto, then g_2^* is onto.

4.1. The Direct Limit. An index set I is called a *directed set* in case there is a reflexive transitive binary relation on I denoted $\leq$ such that for any two elements $i, j \in I$, there is an element $k \in I$ with $i \leq k$ and $j \leq k$. Let I be a directed set and $\mathfrak{C}$ a category. Usually $\mathfrak{C}$ will be a category of R-modules for some ring R. At other times $\mathfrak{C}$ will be a category of R-algebras for some commutative ring R. Suppose that for each $i \in I$ there is an object $A_i \in \mathfrak{C}$ and for each pair $i, j \in I$ such that $i \leq j$ there is a $\mathfrak{C}$-morphism $\phi_j^i : A_i \to A_j$ such that the following are satisfied.

(1) For each $i \in I$, $\phi_i^i : A_i \to A_i$ is the identity on A_i, and
(2) for all $i, j, k \in I$ with $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\phi_k^i} & A_k \\ & {\scriptstyle\phi_j^i}\searrow \quad \nearrow{\scriptstyle\phi_k^j} & \\ & A_j & \end{array}$$

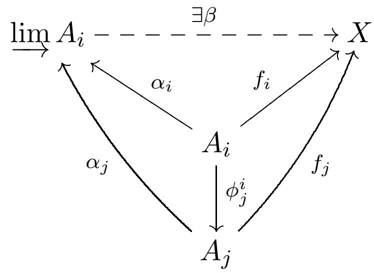
commutes.

Then the collection of objects and morphisms $\{A_i, \phi_j^i\}$ is called a *directed system* in $\mathfrak{C}$ with index set I.

Let $\{A_i, \phi_j^i\}$ be a directed system in $\mathfrak{C}$ for a directed index set I. The *direct limit* of this system, denoted $\varinjlim A_i$, is an object in $\mathfrak{C}$ together with a set of morphisms $\alpha_i : A_i \to \varinjlim A_i$ indexed by I such that the following are satisfied.

(1) For all $i \leq j$, $\alpha_i = \alpha_j \phi_j^i$, and
(2) $\varinjlim A_i$ satisfies the universal mapping property. Namely, if X is an object in $\mathfrak{C}$ and $f_i : A_i \to X$ is a set of morphisms indexed by I such that for all $i \leq j$, $f_i = f_j \phi_j^i$, then there exists a unique morphism

$\beta : \varinjlim A_i \to X$ such that the diagram



commutes for all $i \leq j$ in I.

Proposition 1.4.2 guarantees that for the categories in which we are interested, direct limits exist and are unique.

Proposition 1.4.2. *Let R be a ring.*

(1) *If $\{A_i, \phi^i_j\}$ is a directed system of R-modules for a directed index set I, then the direct limit $\varinjlim A_i$ exists. The direct limit is unique up to isomorphism.*

(2) *If R is a commutative ring and $\{A_i, \phi^i_j\}$ is a directed system of R-algebras for a directed index set I, then the direct limit $\varinjlim A_i$ exists and is unique up to isomorphism.*

Let R be a ring and I a directed index set. Suppose $\{A_i, \phi^i_j\}$ and $\{B_i, \psi^i_j\}$ are two directed systems of R-modules. A *morphism* from $\{A_i, \phi^i_j\}$ to $\{B_i, \psi^i_j\}$ is a set of R-module homomorphisms $\alpha = \{\alpha_i : A_i \to B_j\}_{i \in I}$ indexed by I such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & B_i \\ {\scriptstyle \phi^i_j}\downarrow & & \downarrow{\scriptstyle \psi^i_j} \\ A_j & \xrightarrow{\alpha_j} & B_j \end{array}$$

commutes whenever $i \leq j$. Define $f_i : A_i \to \varinjlim B_i$ by composing α_i with the structure map $B_i \to \varinjlim B_i$. The universal mapping property guarantees a unique R-module homomorphism $\vec{\alpha} : \varinjlim A_i \to \varinjlim B_i$. According to the next theorem, the direct limit defines an exact functor on the category of directed systems of R-modules.

Theorem 1.4.3. *Let R be a ring, I a directed index set, and*

$$\{A_i, \phi^i_j\} \xrightarrow{\alpha} \{B_i, \psi^i_j\} \xrightarrow{\beta} \{C_i, \rho^i_j\}$$

a sequence of morphisms of directed systems of R-modules such that

$$0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$$

is exact for every $i \in I$. Then

$$0 \to \varinjlim A_i \xrightarrow{\vec{\alpha}} \varinjlim B_i \xrightarrow{\vec{\beta}} \varinjlim C_i \to 0$$

is an exact sequence of R-modules.

Let $\{R_i, \theta_j^i\}$ be a directed system of rings for a directed index set I. Each R_i can be viewed as a $\mathbb{Z}$-algebra, hence the direct limit $R = \varinjlim R_i$ exists, by Proposition 1.4.2. For the same index set I, let $\{M_i, \phi_j^i\}$ and $\{N_i, \psi_j^i\}$ be directed systems of $\mathbb{Z}$-modules such that each M_i is a right R_i-module and each N_i is a left R_i-module. For each $i \le j$, M_j and N_j are R_i-modules via $\theta_j^i : R_i \to R_j$. In this context, we also assume that the transition homomorphisms ϕ_j^i and ψ_j^i are R_i-linear:

$$\phi_j^i(ax) = \theta_j^i(a)\phi_j^i(x)$$
$$\psi_j^i(ax) = \theta_j^i(a)\phi_j^i(x)$$

for all $a \in R_i$, $x \in M_i$ and $y \in N_i$. By Exercise 1.3.13 there are $\mathbb{Z}$-module homomorphisms

$$\tau_j^i : M_i \otimes_{R_i} N_i \to M_j \otimes_{R_j} N_j$$

such that $\{M_i \otimes_{R_i} N_i, \tau_j^i\}$ is a directed system for I. Let $M = \varinjlim M_i$, $N = \varinjlim N_i$.

Proposition 1.4.4. *In the above context, $\varinjlim M_i \otimes_{R_i} N_i = M \otimes_R N$.*

Proof. By Exercise 1.3.13 there are $\mathbb{Z}$-module homomorphisms

$$\alpha_i : M_i \otimes_{R_i} N_i \to M \otimes_R N$$

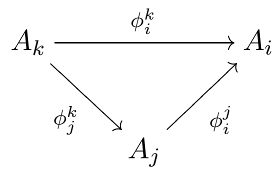
such that $\alpha_i = \alpha_j \tau_j^i$. We show that $M \otimes_R N$ satisfies the universal mapping property. Suppose we are given $\mathbb{Z}$-module homomorphisms

$$f_i : M_i \otimes_{R_i} N_i \to X$$

such that $f_i = f_j \tau_j^i$. Suppose $(x, y) \in M \times N$. Then for some $i \in I$, (x, y) comes from $M_i \times N_i$. The reader should verify that $(x, y) \mapsto f_i(x \otimes y)$ defines an R-balanced map $M \times N \to X$. This induces $\beta : M \otimes_R N \to X$. By the universal mapping property for tensor products, β is unique and satisfies $\beta\alpha_i = f_i$. □

4.2. The Inverse Limit. Let $\mathfrak{C}$ be a category. Usually $\mathfrak{C}$ will be a category of modules or a category of algebras over a commutative ring. At other times $\mathfrak{C}$ will be a category of topological groups. Let I be an index set with a reflexive transitive binary relation denoted $\le$. (Do not assume I is a directed set.) Suppose that for each $i \in I$ there is an object $A_i \in \mathfrak{C}$ and for each pair $i, j \in I$ such that $i \le j$ there is a $\mathfrak{C}$-morphism $\phi_i^j : A_j \to A_i$ such that the following are satisfied.

(1) For each $i \in I$, $\phi_i^i : A_i \to A_i$ is the identity on A_i, and
(2) for all $i, j, k \in I$ with $i \le j \le k$, the diagram

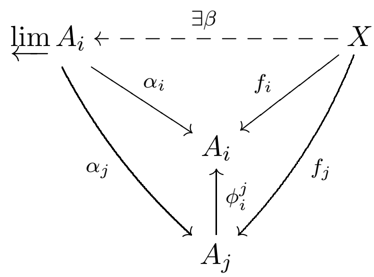


commutes.

Then the collection of objects and morphisms $\{A_i, \phi_i^j\}$ is called an *inverse system* in $\mathfrak{C}$ with index set I.

Let $\{A_i, \phi_i^j\}$ be an inverse system in $\mathfrak{C}$ for an index set I. The *inverse limit* of this system, denoted $\varprojlim A_i$, is an object in $\mathfrak{C}$ together with a set of morphisms $\alpha_i : \varprojlim A_i \to A_i$ indexed by I such that the following are satisfied.

(1) For all $i \le j$, $\alpha_i = \phi_i^j \alpha_j$, and
(2) $\varprojlim A_i$ satisfies the universal mapping property. Namely, if X is an object in $\mathfrak{C}$ and $f_i : X \to A_i$ is a set of morphisms indexed by I such that for all $i \le j$, $f_i = \phi_i^j f_j$, then there exists a unique morphism $\beta : X \to \varprojlim A_i$ such that the diagram



commutes for all $i \le j$ in I.

For the category of modules over a ring, or the category of algebras over a commutative ring, inverse limits exist and are unique.

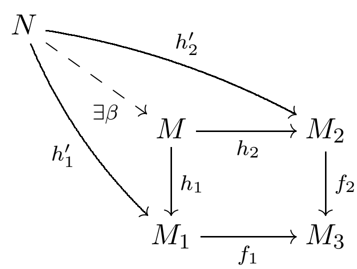
Proposition 1.4.5. *Let R be a ring.*

(1) *If $\{A_i, \phi_i^j\}$ is an inverse system of R-modules for an index set I, then the inverse limit $\varprojlim A_i$ exists. The inverse limit is unique up to isomorphism.*
(2) *If R is a commutative ring and $\{A_i, \phi_i^j\}$ is an inverse system of R-algebras for an index set I, then the inverse limit $\varprojlim A_i$ exists and is unique up to isomorphism.*

Example 1.4.6. Let A be a ring. Suppose $f_1 : M_1 \to M_3$ and $f_2 : M_2 \to M_3$ are homomorphisms of left A-modules. Then the *pullback* (or *fiber product*) is defined to be $M = \{(x_1, x_2) \in M_1 \oplus M_2 \mid f_1(x_1) = f_2(x_2)\}$. Notice that M is the kernel of the A-module homomorphism $M_1 \oplus M_2 \to M_3$ defined by $(x_1, x_2) \mapsto f_1(x_1) - f_2(x_2)$, hence M is a left A-module. If h_1 and h_2 are induced by the coordinate projections, then

$$\begin{array}{ccc} M & \xrightarrow{h_2} & M_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & M_3 \end{array} \tag{1.10}$$

is a commutative diagram of A-modules. An important feature of the pullback is that it can be interpreted as an inverse limit. For the index set, take $I = \{1, 2, 3\}$ with the ordering $1 < 3$, $2 < 3$. The reader should verify that if f_1, f_2 are the transition homomorphisms, then $\{M_1, M_2, M_3\}$ is an inverse system and the inverse limit $\varprojlim M_i$ is isomorphic to the pullback M of (1.10). In particular, the pullback M satisfies the universal mapping property. That is, if N is an R-module and there exist h'_1 and h'_2 such that $f_1 h'_1 = f_2 h'_2$, then there exists a unique morphism $\beta : N \to M$ such that the diagram



commutes. A commutative square of R-modules such as (1.10) is called a *cartesian square* (or *fiber product diagram*, or *pullback diagram*), if M is isomorphic to the pullback $\varprojlim M_i$. Let A_1, A_2, A_3 be rings. If $f_1 : A_1 \to A_3$ and $f_2 : A_2 \to A_3$ are homomorphisms, then the inverse limit $A = \varprojlim A_i$ with respect to the index set $I = \{1, 2, 3\}$ is a ring. As above, A can be identified with the pullback $A = \{(x_1, x_2) \in A_1 \oplus A_2 \mid f_1(x_1) = f_2(x_2)\}$.

4.3. Inverse Systems Indexed by Nonnegative Integers. For the index set $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$, the notation for an inverse system simplifies. Let R be any ring and $\{A_i, \phi_i^j\}$ an inverse system of R-modules for the index set $\{0, 1, 2, \dots\}$. Simply write ϕ_{i+1} for ϕ_i^{i+1}. Then for any $j > i$ we can multiply to get $\phi_i^j = \phi_{i+1}\phi_{i+2}\cdots\phi_j$. Using this notation, and Proposition 1.4.5, the inverse limit $\varprojlim A_i$ can be identified with the set of all sequences $(x_0, x_1, x_2, \dots)$ in $\prod_{n=0}^{\infty} A_n$ such that $x_n = \phi_{n+1} x_{n+1}$ for all $n \geq 0$.

Let R be a ring and suppose $\{A_i, \phi_{i+1}\}$ and $\{B_i, \psi_{i+1}\}$ are two inverse systems of R-modules indexed by $I = \{0, 1, 2, 3, \dots\}$. A *morphism* from $\{A_i, \phi_{i+1}\}$ to $\{B_i, \psi_{i+1}\}$ is a sequence of R-module homomorphisms $\alpha = \{\alpha_i : A_i \to B_j\}_{i\geq 0}$ such that the diagram

$$\begin{array}{ccc} A_{i+1} & \xrightarrow{\alpha_{i+1}} & B_{i+1} \\ {\scriptstyle \phi_{i+1}}\downarrow & & \downarrow{\scriptstyle \psi_{i+1}} \\ A_i & \xrightarrow{\alpha_i} & B_i \end{array}$$

commutes whenever $i \geq 0$. Define $f_i : \varprojlim A_i \to B_i$ by composing the structure map $\varprojlim A_i \to A_i$ with α_i. The universal mapping property guarantees a unique R-module homomorphism $\overleftarrow{\alpha} : \varprojlim A_i \to \varprojlim B_i$. For the category of inverse systems of R-modules, sufficient conditions for an inverse limit to be exact are described below in Proposition 1.4.8. To simplify its proof, we need another definition and a lemma. Define

$$d : \prod_{n=0}^{\infty} A_n \longrightarrow \prod_{n=0}^{\infty} A_n$$

by the formula $d(x_0, x_1, x_2, \dots) = (x_0 - \phi_1 x_1,\ x_1 - \phi_2 x_2,\ x_2 - \phi_3 x_3,\ \dots,\ x_n - \phi_{n+1} x_{n+1},\ \dots)$.

Lemma 1.4.7. *Let R be any ring and $\{A_i, \phi_{i+1}\}$ an inverse system of R-modules for the index set $\{0, 1, 2, \dots\}$. If $\phi_{n+1} : A_{n+1} \to A_n$ is onto for each $n \geq 0$, then there is an exact sequence*

$$0 \to \varprojlim A_n \to \prod_{n=0}^{\infty} A_n \xrightarrow{d} \prod_{n=0}^{\infty} A_n \to 0$$

where d is defined in the previous paragraph.

Proof. It follows at once that $\ker d = \varprojlim A_n$. Let $(y_0, y_1, y_2, \dots) \in \prod A_n$. To show that d is surjective, it is enough to solve the equations

$$\begin{aligned} x_0 - \phi_1 x_1 &= y_0 \\ x_1 - \phi_2 x_2 &= y_1 \\ &\vdots \\ x_n - \phi_{n+1} x_{n+1} &= y_n \end{aligned}$$

for $(x_0, x_1, x_2, \dots)$. This is possible because each ϕ_{n+1} is surjective. Simply take $x_0 = 0$, $x_1 = (\phi_1)^{-1}(-y_0)$, and recursively, $x_{n+1} = (\phi_{n+1})^{-1}(x_n - y_n)$. $\square$

Proposition 1.4.8. *Let R be a ring, and*

$$\{A_i, \phi_{i+1}\} \xrightarrow{\alpha} \{B_i, \psi_{i+1}\} \xrightarrow{\beta} \{C_i, \rho_{i+1}\}$$

a sequence of morphisms of inverse systems of R-modules indexed by the nonnegative integers $\{0, 1, 2, 3, \dots\}$ such that

(1) $0 \to A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \to 0$ *is exact for every* $i \ge 0$, *and*

(2) $\phi_{i+1} : A_{i+1} \to A_i$ *is onto for every* $i \ge 0$.

Then

$$0 \to \varprojlim A_i \xrightarrow{\overleftarrow{\alpha}} \varprojlim B_i \xrightarrow{\overleftarrow{\beta}} \varprojlim C_i \to 0$$

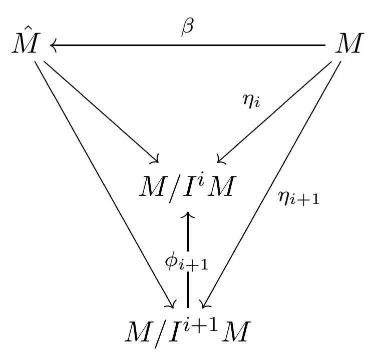
is an exact sequence of R-modules.

Proof. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod A_n & \xrightarrow{\alpha} & \prod B_n & \xrightarrow{\beta} & \prod C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & \prod A_n & \xrightarrow{\alpha} & \prod B_n & \xrightarrow{\beta} & \prod C_n & \longrightarrow & 0 \end{array}$$

commutes and the rows are exact. By Lemma 1.4.7, the leftmost vertical map is onto. The rest of the proof follows from Theorem 1.4.1 and Lemma 1.4.7. □

Let R be a commutative ring, I an ideal in R and M an R-module. Then for all integers $n \ge 1$, I^n denotes the ideal generated by all products of the form $x_1 x_2 \cdots x_n$ where each x_i is in I. The chain of ideals $R \supseteq I^1 \supseteq I^2 \supseteq I^3 \supseteq \dots$ gives rise to the chain of submodules $M \supseteq I^1 M \supseteq I^2 M \supseteq I^3 M \supseteq \dots$. Then $I^{i+1}M \subseteq I^i M$ so there is a natural projection $\phi_{i+1} : M/I^{i+1}M \to M/I^i M$. The set of R-modules and homomorphisms $\{M/I^i M, \phi_{i+1}\}$ is an inverse system indexed by $\{1, 2, 3, 4, \dots\}$. The inverse limit of this system $\hat{M} = \varprojlim M/I^i M$, is called the *$I$-adic completion* of M. For each i, let $\eta_i : M \to M/I^i M$ be the natural projection. Clearly $\eta_i = \phi_{i+1}\eta_{i+1}$ so by the universal mapping property, there is a unique $\beta : M \to \hat{M}$ such that the diagram



commutes. The proof of the next proposition is left to the reader.

Proposition 1.4.9. *Let I be an ideal in the commutative ring R and $\hat{R}$ the I-adic completion of R. Let M be an R-module and $\hat{M}$ the I-adic completion of M. Then the following are true.*

(1) *$\hat{M}$ is an $\hat{R}$-module.*

(2) *The natural map $\beta : M \to \hat{M}$ is one-to-one if and only if $\cap I^n M = 0$.*

4.4. Exercises.

Exercise 1.4.10. Let R be an arbitrary ring. Let I be an index set, $X = \{x_i\}_{i \in I}$ a set of indeterminates indexed by I. Let J be the set of all finite subsets of I, ordered by set inclusion. For each $\alpha \in J$, let $X_\alpha = \{x_j \mid j \in \alpha\}$. Show how to make the set of polynomial rings $\{R[X_\alpha]\}_{\alpha \in J}$ into a directed system of rings. Define $R[X] = \varinjlim R[X_\alpha]$ as the direct limit.

Exercise 1.4.11. Let A be an R-module.

(1) Suppose $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ is a chain of submodules of the R-module A. Show how to make $\{A_i\}$ into a directed system and prove that $\varinjlim A_i = \bigcup_i A_i$.

(2) Let $\{A_\alpha\}$ be the set of all finitely generated R-submodules of A. Show how to make $\{A_\alpha\}$ into a directed system and prove that $A = \varinjlim A_\alpha$.

Exercise 1.4.12. Let R be a commutative ring and A an R-algebra.

(1) Show that $A = \varinjlim A_\alpha$ where A_α runs over the set of all finitely generated R-subalgebras of A.

(2) Assume A is integral over R (see Section 3.3). Show that $A = \varinjlim A_\alpha$ where A_α runs over the set of all R-subalgebras of A such that A_α is finitely generated as an R-module.

(3) If $f \in A$, show that $A = \varinjlim A_\alpha$ where A_α runs over all finitely generated R-subalgebras of A such that $R[f] \subseteq A_\alpha \subseteq A$.

Exercise 1.4.13. Let A be a commutative ring and $R = A[x]$ the polynomial ring in one variable with coefficients in A. Let $I = Rx$ be the ideal in R generated by x. Show that the I-adic completion of R is isomorphic to the power series ring $A[[x]]$ in one variable over A.

Exercise 1.4.14. Let R be any ring and $\{A_i, \phi^i_j\}$ a directed system of flat R-modules for a directed index set I. Prove that the direct limit $\varinjlim A_i$ is a flat R-module.

Exercise 1.4.15. Let $\{R_i, \theta^i_j\}$ be a directed system of rings for a directed index set I. Let $R = \varinjlim R_i$ be the direct limit. As in Proposition 1.4.4, let $\{M_i, \phi^i_j\}$ a directed system of $\mathbb{Z}$-modules for the same index set I such that each M_i is a left R_i-module and the transition homomorphisms ϕ^i_j are R_i-module homomorphisms. If each M_i is a flat R_i-module, prove that $M = \varinjlim M_i$ is a flat R-module. (Hint: $\{R \otimes_{R_i} M_i, 1 \otimes \phi^i_j\}$ is a directed system of flat R-modules.)

Exercise 1.4.16. Let R be a ring and $\{A_i, \phi^i_j\}$ a directed system of R-modules for a directed index set I. Let P be a finitely generated projective R-module.

(1) Show that $\operatorname{Hom}_R(P, \varinjlim A_i) \cong \varinjlim \operatorname{Hom}_R(P, A_i)$. (Hint: Following the proof of Theorem 1.3.26, reduce to the case where P is free.)

(2) Show that $\operatorname{Hom}_R(P, \bigoplus_i A_i) \cong \bigoplus_i \operatorname{Hom}_R(P, A_i)$.

Exercise 1.4.17. Let R be a commutative ring and $\{A_i, \phi^i_j\}$ a directed system of R-algebras for a directed index set I. Show that an idempotent in $\varinjlim A_i$ comes from an idempotent in A_i, for some $i \in I$. In other words, if $e \in \varinjlim A_i$ and $e^2 = e$, then for some $i \in I$, there exists $e_i \in A_i$ such that $e_i^2 = e_i$ and if $\alpha_i : A_i \to \varinjlim A_i$ is the natural map, then $\alpha_i(e_i) = e$.

Exercise 1.4.18. In the context of the pullback diagram (1.10), prove the following:

(1) $\ker h_1 \cong \ker f_2$ and $\ker h_2 \cong \ker f_1$.

(2) If f_2 is onto, then h_1 is onto. If f_1 is onto, then h_2 is onto.

Exercise 1.4.19. Let A be a ring and let I and J be two-sided ideals in A. Show that

$$\begin{array}{ccc} \frac{A}{I\cap J} & \xrightarrow{h_2} & \frac{A}{J} \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \frac{A}{I} & \xrightarrow{f_1} & \frac{A}{I+J} \end{array}$$

is a cartesian square of rings, where all of the homomorphisms are the natural maps.

Exercise 1.4.20. Let B be a ring and I a two-sided ideal of B. Assume $A \subseteq B$ is a subring such that $I \subseteq A$. Show that

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \frac{A}{I} & \xrightarrow{f_1} & \frac{B}{I} \end{array}$$

is a cartesian square of rings, where all of the homomorphisms are the natural maps.

5. The Morita Theorems

In the literature, the Morita Theorems are stated and proved by various authors in a number of ways and in different levels of generality. The approach we choose to follow is that used by [**DI71**]. Our functors are defined only on categories of modules. To define the functors, we will start with a ring R and a left R-module M. There are occasions when it is preferable to define the functor starting with a right R-module M. For example, when we apply the theorems in Section 7.8, it will be convenient to use a right handed Morita equivalence. Rather than state in this section more than one version of the theorems, we will leave it to the reader to make the necessary straightforward translations.

We begin by establishing some notation that will be in effect throughout this section. For any ring R and any left R-module M, set

$$M^* = \operatorname{Hom}_R(M, R)$$

and

$$S = \operatorname{Hom}_R(M, M).$$

Since R is a left R right R bimodule, by Lemma 1.3.17 (2), M^* is a right R-module under the operation $(fr)(m) = f(m)r$. Since S is a ring of R-module endomorphisms of M, M is a left S-module by $sm = s(m)$ and under this operation M is a left R left S bimodule. By Lemma 1.3.17 (3), we make M^* a right S-module by $(fs)(m) = f\big(s(m)\big)$, which is just composition of functions. The reader should verify that M^* is in fact a right R right S bimodule. It follows that we can form $M^* \otimes_R M$ and $M^* \otimes_S M$. Moreover $M^* \otimes_R M$ is a left S right S bimodule by virtue of M being a left R left S bimodule and M^* being a right R right S bimodule. Similarly $M^* \otimes_S M$ is a left R right R bimodule. Define

$$\theta_R : M^* \otimes_R M \to S = \operatorname{Hom}_R(M, M)$$

by the rule: $\theta_R(f \otimes m)(x) = f(x)m$. The reader should check that θ_R is both a left and a right S-module homomorphism. Define

$$\theta_S : M^* \otimes_S M \to R$$

by the rule: $\theta_S(f \otimes m) = f(m)$. The reader should verify that θ_S is a right and left R-module homomorphism whose image is the trace ideal $\mathfrak{T}_R(M)$ (see Eq.(1.2)).

Lemma 1.5.1. *In the above context,*

(1) *θ_R is onto if and only if M is finitely generated and projective. If θ_R is onto, it is one-to-one.*

(2) *θ_S is onto if and only if M is a generator. If θ_S is onto, it is one-to-one.*

Proof. (1): If θ_R is onto, then there exist $f_i \in M^*$ and $m_i \in M$ such that the identity map $1 : M \to M$ is equal to $\theta_R(\sum_{i=1}^n f_i \otimes m_i)$. That is, for every $x \in M$, $x = \sum_{i=1}^n f_i(x)m_i$. Then $\{(f_i, m_i)\}$ is a finite dual basis for M. By Proposition 1.1.5, M is finitely generated and projective. Conversely, if a finite dual basis exists, then the identity map $1 : M \to M$ is in the image of θ_R. Since θ_R is an S-module homomorphism, θ_R is onto.

To prove the second statement in (1), assume θ_R is onto. Let $f_1, \dots, f_n \in M^*$, $m_1, \dots, m_n \in M$ be a finite dual basis for M. Let $\alpha = \sum_j h_j \otimes n_j \in M^* \otimes_R M$ and $\theta_R(\alpha) = 0$. That is, $\sum_j h_j(x)n_j = 0$ for every x in M. In particular, for each f_i, we have

$$\Big[\sum_j h_j \cdot f_i(n_j)\Big](x) = \sum_j h_j(x)f_i(n_j) = \\ \sum_j f_i\big(h_j(x)n_j\big) = f_i\big(\sum_j h_j(x)n_j\big) = f_i(0) = 0.$$

From this we get

$$\begin{aligned}\alpha = \sum_j h_j \otimes n_j &= \sum_j \big[h_j \otimes \big(\sum_i f_i(n_j)m_i\big)\big] = \sum_{i,j} h_j \otimes f_i(n_j)m_i \\ &= \sum_{i,j} \big(h_j \cdot f_i(n_j)\big) \otimes m_i = \sum_i \big[\big(\sum_j h_j \cdot f_i(n_j)\big) \otimes m_i\big] = \sum_i 0 \otimes m_i = 0.\end{aligned}$$

Hence, θ_R is one-to-one.

(2): Because the image of θ_S equals $\mathfrak{T}_R(M)$, the trace ideal of M, it is clear that θ_S is onto if and only if M is an R-generator.

To prove the second statement in (2), assume θ_S is onto. Let $\sum_j h_j \otimes n_j \in \ker \theta_S$. That is, $\sum_j h_j(n_j) = 0$. Since θ_S is onto, there exist $f_1, \dots, f_n$ in M^*, $m_1, \dots, m_n$ in M with $\sum_i f_i(m_i) = 1 \in R$. Notice that for each f_i and every $x \in M$,

$$\sum_j h_j \cdot \theta_R(f_i \otimes n_j)(x) = \sum_j h_j\big(f_i(x)n_j\big) = f_i(x) \sum_j h_j(n_j) = 0.$$

From this we get

$$\begin{aligned}\sum_j h_j \otimes n_j &= \sum_j h_j \otimes \big(\sum_i f_i(m_i)\big)n_j = \sum_j h_j \otimes \big(\sum_i f_i(m_i)n_j\big) \\ &= \sum_j h_j \otimes \big(\sum_i \theta_R(f_i \otimes n_j)(m_i)\big) = \sum_{i,j} h_j \otimes \theta_R(f_i \otimes n_j)(m_i) \\ &= \sum_i \big(\sum_j h_j \cdot \theta_R(f_i \otimes n_j)\big) \otimes (m_i) = \sum_i 0 \otimes m_i = 0.\end{aligned}$$

Therefore, θ_S is one-to-one. □

The main theorem of this section, Theorem 1.5.2, shows that if M is an R-progenerator, then the functors $(\,) \otimes_R M : \mathfrak{M}_R \to {}_S\mathfrak{M}$ and $M^* \otimes_S (\,) : {}_S\mathfrak{M} \to \mathfrak{M}_R$ are inverse equivalences. In this case, we say that the categories $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$ are *Morita equivalent.*

Theorem 1.5.2. *Let R be any ring and let M be a left R-progenerator. Set $S = \mathrm{Hom}_R(M, M)$ and $M^* = \mathrm{Hom}_R(M, R)$. Then*

$$(\,) \otimes_R M : \mathfrak{M}_R \to {}_S\mathfrak{M}$$

and

$$M^* \otimes_S (\,) : {}_S\mathfrak{M} \to \mathfrak{M}_R$$

are inverse equivalences, establishing $\mathfrak{M}_R \sim {}_S\mathfrak{M}$.

Proof. Let L be any right R-module. Then, by the basic properties of the tensor product and Lemma 1.5.1 (2), we have

$$\begin{aligned} M^* \otimes_S (L \otimes_R M) &\cong M^* \otimes_S (M \otimes_{R^o} L) \\ &\cong (M^* \otimes_S M) \otimes_{R^o} L \cong R \otimes_{R^o} L \cong L \otimes_R R \cong L \end{aligned}$$

where the composite isomorphism is given by $f \otimes (l \otimes m) \mapsto l \cdot \theta_S(f \otimes m) = l \cdot f(m)$. This isomorphism allows one to verify that $(\,) \otimes_R M$ followed by $M^* \otimes_S (\,)$ is naturally equivalent to the identity functor on $\mathfrak{M}_R$. Likewise, given any left S-module N, the isomorphism of Lemma 1.5.1 (1) shows that

$$\begin{aligned} (M^* \otimes_S N) \otimes_R M &\cong (N \otimes_{S^o} M^*) \otimes_R M \\ &\cong N \otimes_{S^o} (M^* \otimes_R M) \cong N \otimes_{S^o} S \cong S \otimes_S N \cong N \end{aligned}$$

under the map $(f \otimes n) \otimes m \mapsto \theta_R(f \otimes m) \cdot n$. Again this gives us that $M^* \otimes_S (\,)$ followed by $(\,) \otimes_R M$ is naturally equivalent to the identity on ${}_S\mathfrak{M}$. ☐

Corollary 1.5.3. *In the context of Theorem* 1.5.2*, the following are true.*

(1) *$R \cong \mathrm{Hom}_S(M, M)$ (as rings) where r in R maps to "left multiplication by r".*
(2) *$M^* \cong \mathrm{Hom}_S(M, S)$ (as right S-modules) where f in M^* maps to the homomorphism $\theta_R\big(f \otimes (\,)\big)$.*
(3) *$M \cong \mathrm{Hom}_R(M^*, R) = M^{**}$ (as left R-modules) where m in M maps to the element in M^{**} which is "evaluation at m".*
(4) *$S^o \cong \mathrm{Hom}_R(M^*, M^*)$ (as rings) where s in S^o maps to "right multiplication by s".*
(5) *M is an S-progenerator.*
(6) *M^* is an R-progenerator.*
(7) *M^* is an S-progenerator.*

Proof. The fully faithful part of Proposition 1.1.2 applied to the functor $(\ )\otimes_R M$ says that for any two right R-modules A and B, the assignment

$$\operatorname{Hom}_R(A,B)\to\operatorname{Hom}_S(A\otimes_R M, B\otimes_R M) \tag{1.11}$$

is a one-to-one correspondence. Under this equivalence, the right R-module R corresponds to the left S-module $R\otimes_R M\cong M$ and the right R-module M^* corresponds to the left S-module $M^*\otimes_R M\cong S$. For (1), use (1.11) with $A=B=R$. For (2), use (1.11) with $A=R$ and $B=M^*$. In each case, the reader should verify that the composite isomorphisms are the correct maps.

The fully faithful part of Proposition 1.1.2 applied to the functor $M^*\otimes_S (\ ) : {}_S\mathfrak{M}\to\mathfrak{M}_R$ says that for any two left S-modules C and D, the assignment

$$\operatorname{Hom}_S(C,D)\to\operatorname{Hom}_R(M^*\otimes_S C, M^*\otimes_S D) \tag{1.12}$$

is a one-to-one correspondence. By Proposition 1.3.23 (1), M is isomorphic to $\operatorname{Hom}_S(S,M)$. By (1.12) with $C=S$ and $D=M$, we get $\operatorname{Hom}_S(S,M)\cong\operatorname{Hom}_R(M^*,R)=M^{**}$, which is (3). For (4), use (1.12) with $C=D=S$. Since $M^*\otimes_S S\cong M^*$, we get the isomorphism of rings $\operatorname{Hom}_S(S,S)\cong\operatorname{Hom}_R(M^*,M^*)$. By Exercise 1.3.32, $S^o\cong\operatorname{Hom}_S(S,S)$ as rings. In each case, the reader should verify that the composite isomorphisms are the correct maps.

(5): Because M is an R-progenerator, we have $\theta_S : M^*\otimes_S M\cong R$ and $\theta_R : M^*\otimes_R M\cong S$. By (1) and (2) above, this gives rise to isomorphisms

$$\theta_S : \operatorname{Hom}_S(M,S)\otimes_S M\cong\operatorname{Hom}_S(M,M)$$

and

$$\theta_R : \operatorname{Hom}_S(M,S)\otimes_{\operatorname{Hom}_S(M,M)} M\cong S.$$

By Lemma 1.5.1 with R and S interchanged, it follows that M is an S-progenerator.

(6): Using $M^*\otimes_S M\cong R$ and substituting (3) and (4), we obtain

$$\begin{aligned} R &\cong M^*\otimes_S M\\ &\cong M^*\otimes_S \operatorname{Hom}_R(M^*,R)\\ &\cong \operatorname{Hom}_R(M^*,R)\otimes_{S^o} M^*\\ &\cong \operatorname{Hom}_R(M^*,R)\otimes_{\operatorname{Hom}_R(M^*,M^*)} M^*. \end{aligned} \tag{1.13}$$

We also have

$$\begin{aligned} \operatorname{Hom}_{R^o}(M^*,R^o)\otimes_{R^o} M^* &\cong M^*\otimes_R \operatorname{Hom}_R(M^*,R)\\ &\cong M^*\otimes_R M\\ &\cong S\\ &\cong \operatorname{Hom}_R(M^*,M^*)\\ &\cong \operatorname{Hom}_{R^o}(M^*,M^*) \end{aligned} \tag{1.14}$$

where the last isomorphism in (1.14) is set identity and M^* is considered as a left R^o-module since it is a right R-module. By Lemma 1.5.1 with M^* in place of M, we see that M^* is an R-generator by (1.13) and a finitely generated and projective left R^o-module by (1.14). This implies that M^* is a right R-progenerator.

(7): By (5), M is an S-progenerator. Applying (6) to the S-module M proves that $\operatorname{Hom}_S(M, S)$ is an S-progenerator. By Part (2), $\operatorname{Hom}_S(M, S) \cong M^*$. □

Corollary 1.5.4. *Let R, M and S be as in Theorem 1.5.2.*

(1) *For any two-sided ideal $\mathfrak{a}$ of R, $M^* \otimes_R (\mathfrak{a} \otimes_R M)$ is naturally isomorphic to the two-sided ideal of S consisting of all elements of the form $\sum_i \theta_R(f_i \otimes \alpha_i m_i)$, where $f_i \in M^*$, $\alpha_i \in \mathfrak{a}$, and $m_i \in M$.*

(2) *For any two-sided ideal $\mathfrak{b}$ of S, $M^* \otimes_S (\mathfrak{b} \otimes_S M)$ is naturally isomorphic to the two-sided ideal of R consisting of all elements of the form $\sum_i \theta_S\big(f_i \otimes \beta_i(n_i)\big) = \sum_i f_i\big(\beta_i(n_i)\big)$, where $f_i \in M^*$, $\beta_i \in \mathfrak{b}$, and $n_i \in M$.*

The correspondences (1) and (2) are inverses of each other and establish a one-to-one, order preserving correspondence between the two-sided ideals of R and the two-sided ideals of S.

Proof. Since M and M^* are both R-projective, they are flat. The exact sequence $0 \to \mathfrak{a} \to R$ yields the exact sequence

$$0 \to M^* \otimes_R (\mathfrak{a} \otimes_R M) \to M^* \otimes_R (R \otimes_R M) \cong M^* \otimes_R M \cong S.$$

We consider $M^* \otimes_R (\mathfrak{a} \otimes_R M)$ as a subset of $M^* \otimes_R (R \otimes_R M)$. By θ_R, $M^* \otimes_R (R \otimes_R M)$ is isomorphic to S. This maps this submodule $M^* \otimes_R (\mathfrak{a} \otimes_R M)$ onto the ideal of S made up of elements of the form $\sum_i \theta_R(f_i \otimes \alpha_i m_i)$.

Likewise, M and M^* are S-projective. The exact sequence $0 \to \mathfrak{b} \to S$ yields the exact sequence

$$0 \to M^* \otimes_S (\mathfrak{b} \otimes_S M) \to M^* \otimes_S M \cong R.$$

We view $M^* \otimes_S (\mathfrak{b} \otimes_S M)$ as the ideal of R made up of elements looking like $\sum_i f_i\big(\beta_i(n_i)\big)$. The reader should verify that the correspondences are inverses of each other. □

Corollary 1.5.5. *In the context of Theorem 1.5.2, let L be a right R-module and $L \otimes_R M$ its corresponding left S-module.*

(1) *L is finitely generated over R if and only if $L \otimes_R M$ is finitely generated over S.*

(2) *L is R-projective if and only if $L \otimes_R M$ is S-projective.*

(3) *L is an R-generator if and only if $L \otimes_R M$ is an S-generator.*

Proof. We apply Lemma 1.1.4 to write L as the homomorphic image of a free R-module. Suppose $R^I \to L \to 0$ is exact, for some index set, I. After tensoring with $(\cdot) \otimes_R M$, we get an exact sequence

$$M^I \to L \otimes_R M \to 0 \tag{1.15}$$

of S-modules. By Corollary 1.5.3 (5), M is finitely generated and projective as an S-module. For each biconditional, we prove only one direction. Each converse follows by categorical equivalence.

(1): If L is finitely generated over R, we may assume I is a finite set. In (1.15), $M^I = \bigoplus_{i \in I} M$ is a finite sum of finitely generated modules and is finitely generated. So $L \otimes_R M$ is finitely generated.

(2): If L is projective, then by Proposition 1.1.5 the exact sequence $R^I \to L \to 0$ splits. It follows that (1.15) also splits. By Exercise 1.1.10, the S-modules M^I and $L \otimes_R M$ are projective.

(3): Let L be an R-generator. Let $\delta : C \to D$ be a nonzero homomorphism of left S-modules. By Exercise 1.3.30 (3), to show that $L \otimes_R M$ is an S-generator it suffices to show that there exists an S-module homomorphism $f : L \otimes_R M \to C$ such that $\delta \circ f$ is nonzero. By the fully faithful part of Proposition 1.1.2, $1 \otimes \delta : M^* \otimes_S C \to M^* \otimes_S D$ is a nonzero homomorphism of right R-modules. Since L is an R-generator, by Exercise 1.3.30 (4), there exists an R-module homomorphism $\alpha : L \to M^* \otimes_S C$ such that $(1 \otimes \delta) \circ \alpha$ is nonzero. Again by Proposition 1.1.2, $\delta \circ (\alpha \otimes 1)$ is nonzero. □

5.1. Exercises.

Exercise 1.5.6. Let R be any ring and let M be a left R-progenerator. Set $S = \operatorname{Hom}_R(M, M)$. Prove that $(\ ) \otimes_R M : \mathfrak{M}_R \to {}_S\mathfrak{M}$ and $\operatorname{Hom}_S(M, \ ) : {}_S\mathfrak{M} \to \mathfrak{M}_R$ are inverse equivalences, establishing $\mathfrak{M}_R \sim {}_S\mathfrak{M}$. (Hint: Corollary 1.5.3 (2) and Theorem 1.3.29.)

Exercise 1.5.7. Let R be any ring. A left R-module M is said to be *faithfully flat* if M is flat and M has the property that $N \otimes_R M = 0$ implies $N = 0$. Show that a left R-progenerator is faithfully flat.

Chapter 2

Modules over Commutative Rings

This chapter, like Chapter 1, contains material of a preliminary nature that will be applied in subsequent chapters. Proofs that are readily available in the standard references are usually omitted. To avoid listing many special cases, a theorem will frequently be stated in a form that is more general than any given application may require. Some readers will probably prefer to make a quick scan through this chapter on the first pass, and return later as needed.

1. Localization of Modules and Rings

As a basic reference for this section, we recommend [**DF04**]. Throughout this section, R denotes a commutative ring. First we make a quick review of the localization construction and state some of the fundamental properties. Then in Section 2.1.1 we prove some local to global lemmas.

A *multiplicative subset* of R is a subset W that is closed under multiplication and such that $1 \in W$. Let M be an R-module and W a multiplicative subset of R. Define a relation on $M \times W$ by $(m_1, w_1) \sim (m_2, w_2)$ if and only if there exists $w \in W$ such that $w(w_2 m_1 - w_1 m_2) = 0$. The reader should verify that $\sim$ is an equivalence relation on $R \times W$. The set of equivalence classes is denoted $W^{-1}M$ and the equivalence class containing (m, w) is denoted by the fraction m/w. We call $W^{-1}M$ the *localization of* M *at* W. The assignment $m \mapsto m/1$ defines an R-module homomorphism $\sigma : M \to W^{-1}M$. The kernel of σ is equal to the the set of all $m \in M$ such that $wm = 0$ for some w in W. The multiplication rule $(r/w_1)(m/w_2) = (rm)/(w_1 w_2)$ makes

$W^{-1}M$ into a $W^{-1}R$-module. If M is an R-algebra, the multiplication rule $(m_1/w_1)(m_2/w_2) = (m_1m_2)/(w_1w_2)$ makes $W^{-1}M$ into an R-algebra. The assignment $\phi(m/w) = 1/w \otimes m$ defines a $W^{-1}R$-module isomorphism

$$W^{-1}M \xrightarrow{\phi} W^{-1}R \otimes_R M.$$

The inverse of ϕ is given by $a \otimes b \mapsto ab$. If $0 \in W$, then $W^{-1}M = (0)$.

Given a prime ideal P in R, $W = R - P = \{x \in R \mid x \notin P\}$ is a multiplicative set. The R-algebra $W^{-1}R$ is usually written R_P. The ideal generated by P in R_P is $PR_P = \{x/y \in R_P \mid x \in P, y \notin P\}$. If $x/y \notin PR_P$, then $x \notin P$ so $y/x \in R_P$ is the multiplicative inverse of x/y. Since the complement of PR_P consists of units, the ideal PR_P contains every noninvertible element. So PR_P is the unique maximal ideal of R_P. A local ring is a commutative ring that has a unique maximal ideal (Exercise 1.1.18). Hence R_P is a local ring with maximal ideal PR_P, which is called the *local ring of R at P*. The factor ring R_P/PR_P is a field and is also called the residue field of R_P. The factor ring R/P is an integral domain and by Exercise 2.1.9, R_P/PR_P is isomorphic to the quotient field of R/P.

Lemma 2.1.1 shows that a localization of a commutative ring R is a flat R-module. In general, a localization $W^{-1}R$ is not projective. For an example, see Exercise 2.1.14.

Lemma 2.1.1. *If R is a commutative ring and W a multiplicative set in R, then $W^{-1}R$ is a flat R-module.*

1.1. Local to Global Lemmas. Our first local to global result is Proposition 2.1.2. It simply says that if a module is locally trivial (for every maximal ideal), then it is trivial (globally). The proof can be found in [**DF04**].

Proposition 2.1.2. *Let R be a commutative ring and M an R-module. If $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ of R, then $M = 0$.*

Lemma 2.1.3 is a technical lemma whose proof is left to the reader.

Lemma 2.1.3. *Let R be a commutative ring, M a finitely generated R-module, and $W \subseteq R$ a multiplicative subset. Then $W^{-1}M = 0$ if and only if there exists $w \in W$ such that $wM = 0$.*

In the following, we write M_α instead of $M[\alpha^{-1}]$ for the localization of an R-module M with respect to the multiplicative set $\{1, \alpha, \alpha^2, \dots\}$. Lemmas 2.1.4 and 2.1.5 are called local to global lemmas because if $W \subseteq R$ is a multiplicative set and $\alpha \in W$, then $W^{-1}M$ is a localization of M_α. Lemma 2.1.4 provides sufficient conditions for which a local isomorphism of R-modules can be globalized.

Lemma 2.1.4. *Let R be a commutative ring and $\varphi : M \to N$ a homomorphism of R-modules. Let $W \subseteq R$ be a multiplicative subset and $\varphi_W : M \otimes_R W^{-1}R \to N \otimes_R W^{-1}R$.*

(1) *If φ_W is one-to-one and $\ker \varphi$ is a finitely generated R-module, then there exists $\alpha \in W$ such that $\varphi_\alpha : M_\alpha \to N_\alpha$ is one-to-one.*

(2) *If φ_W is onto and $\operatorname{coker} \varphi$ is a finitely generated R-module, then there exists $\beta \in W$ such that $\varphi_\beta : M_\beta \to N_\beta$ is onto.*

(3) *If φ_W is an isomorphism and both $\ker \varphi$ and $\operatorname{coker} \varphi$ are finitely generated R-modules, then there exists $w \in W$ such that $\varphi_w : M_w \to N_w$ is an isomorphism.*

Proof. Start with the exact sequence of R-modules

$$0 \to \ker(\varphi) \to M \xrightarrow{\varphi} N \to \operatorname{coker}(\varphi) \to 0. \tag{2.1}$$

Tensoring (2.1) with $(\cdot) \otimes_R R[W^{-1}]$ we get

$$0 \to W^{-1} \ker(\varphi) \to W^{-1}M \xrightarrow{\varphi_W} W^{-1}N \to W^{-1} \operatorname{coker}(\varphi) \to 0 \tag{2.2}$$

which is exact, by Lemma 2.1.1.

(1): If φ_W is one-to-one, then by Lemma 2.1.3 there is $\alpha \in W$ such that $\alpha(\ker(\varphi)) = 0$. Therefore, $\ker(\varphi) \otimes_R R[\alpha^{-1}] = 0$, and φ_α is one-to-one.

(2): If φ_W is onto, then by Lemma 2.1.3 there is an element β in W such that $\beta(\operatorname{coker}(\varphi)) = 0$. Therefore, $\operatorname{coker}(\varphi) \otimes_R R[\beta^{-1}] = 0$, and φ_β is onto.

(3): Let α be as in (1) and β as in (2). If we set $w = \alpha\beta$, then φ_w is an isomorphism of R_w-modules. □

Lemma 2.1.5 provides sufficient conditions for which a local isomorphism of R-algebras can be globalized.

Lemma 2.1.5. *Let R be a commutative ring. Let A and B be commutative R-algebras and $\varphi : A \to B$ an R-algebra homomorphism. Assume $\ker \varphi$ is a finitely generated ideal of A, and B is a finitely generated A-algebra. If $W \subseteq R$ is a multiplicative subset and $\varphi \otimes 1 : A \otimes_R W^{-1}R \to B \otimes_R W^{-1}R$ is an isomorphism of $W^{-1}R$-algebras, then there exists $w \in W$ such that $\varphi_w : A_w \to B_w$ is an isomorphism of R_w-algebras.*

Proof. Suppose $\ker \varphi = Ax_1 + \cdots + Ax_n$. By Lemma 2.1.3 there is $\alpha \in W$ such that $\alpha(Rx_1 + \cdots + Rx_n) = 0$. Therefore, $\alpha \ker \varphi = 0$. Suppose the A-algebra B is generated by $y_1, \ldots, y_m$. By Lemma 2.1.3 there is $\beta \in W$ such that $\beta(Ry_1 + \cdots + Ry_m) \subseteq \varphi(A)$. If we set $w = \alpha\beta$, then $\varphi_w : A_w \to B_w$ is an isomorphism of R_w-algebras. □

Lemma 2.1.6 provides sufficient conditions for which an R-module that is free over the local ring at a prime ideal can be globalized to a free module.

Lemma 2.1.6. *Let R be a commutative ring and M an R-module of finite presentation. Let $\mathfrak{p} \in \operatorname{Spec} R$ and assume $M_{\mathfrak{p}} = M \otimes_R R_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. Then there exists $\alpha \in R - \mathfrak{p}$ such that M_α is a free R_α-module.*

Proof. Since M is finitely generated, we know that $M_{\mathfrak{p}}$ is free of finite rank. Pick a basis $\{m_1/\alpha_1, \dots, m_n/\alpha_n\}$ for $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. Since $1/\alpha_1, \dots, 1/\alpha_n$ are units in $R_{\mathfrak{p}}$, it follows that $\{m_1/1, \dots, m_n/1\}$ is a basis for $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. Define $\varphi : R^n \to M$ by $(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i m_i$, and consider the exact sequence of R-modules

$$0 \to \ker \varphi \to R^n \xrightarrow{\varphi} M \to \operatorname{coker} \varphi \to 0. \tag{2.3}$$

Tensoring (2.3) with $(\cdot) \otimes_R R_{\mathfrak{p}}$, we get

$$0 \to (\ker \varphi)_{\mathfrak{p}} \to R^n_{\mathfrak{p}} \xrightarrow{\varphi_{\mathfrak{p}}} M_{\mathfrak{p}} \to (\operatorname{coker} \varphi)_{\mathfrak{p}} \to 0 \tag{2.4}$$

which is exact, by Lemma 2.1.1. As mentioned above, $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ with basis $\{m_1/1, \dots, m_n/1\}$ and $\varphi_{\mathfrak{p}}$ maps the standard basis to this basis. That is, $\varphi_{\mathfrak{p}}$ is an isomorphism. So $0 = (\ker \varphi)_{\mathfrak{p}} = (\operatorname{coker} \varphi)_{\mathfrak{p}}$. Since M is finitely generated over R so is $\operatorname{coker} \varphi$. By Lemma 2.1.3 there exists $\beta \in R - \mathfrak{p}$ such that $\beta \cdot \operatorname{coker} \varphi = 0$. Then $(\operatorname{coker} \varphi)_\beta = 0$. Tensoring (2.3) with $(\ ) \otimes_R R_\beta$ we get the exact sequence

$$0 \to (\ker \varphi)_\beta \to R^n_\beta \xrightarrow{\varphi_\beta} M_\beta \to 0. \tag{2.5}$$

Since M is a finitely presented R-module, M_β is a finitely presented R_β-module. By Lemma 2.1.7 which is proved below, $(\ker \varphi)_\beta$ is a finitely generated R_β-module. Since $\beta \in R - \mathfrak{p}$, there exists a homomorphism of rings $R_\beta \to R_{\mathfrak{p}}$ so we can tensor (2.5) with $(\cdot) \otimes_{R_\beta} R_{\mathfrak{p}}$ to get (2.4) again. That is, $(\ker \varphi)_\beta \otimes_{R_\beta} R_{\mathfrak{p}} \cong (\ker \phi)_{\mathfrak{p}} = 0$. Lemma 2.1.3 says there exists μ/β^k in $R_\beta - \mathfrak{p} R_\beta$ such that $(\mu/\beta^k)\,(\ker \phi)_\beta = 0$. But β is a unit in R_β so this is equivalent to $\mu\,(\ker \phi)_\beta = 0$. It is easy to check that $R_{\mu\beta} = R[(\mu\beta)^{-1}] = (R_\beta)_\mu$. This means $0 = \big((\ker \phi)_\beta\big)_\mu = (\ker \phi)_{\beta\mu}$. We also have $(\operatorname{coker} \phi)_{\beta\mu} = 0$. Tensoring (2.3) with $R_{\mu\beta}$ results in $R^{(n)}_{\mu\beta} \cong M_{\mu\beta}$. □

Lemma 2.1.7, which was used in the proof of Lemma 2.1.6, provides useful criteria for proving that a submodule of a finitely generated module is finitely generated. Notice that the lemma is valid when the ring R is noncommutative.

Lemma 2.1.7. *Let R be any ring and*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

an exact sequence of R-modules.

(1) *If B is finitely generated, then C is finitely generated.*

(2) *If A and C are finitely generated, then B is finitely generated.*

(3) *If B is finitely generated and C is of finite presentation, then A is finitely generated.*

Proof. Parts (1) and (2) are left to the reader.

(3): Consider the commutative diagram

$$
\begin{array}{ccccccccc}
 & & R^{(n)} & \xrightarrow{\phi} & R^{(n)} & \xrightarrow{\psi} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \exists\rho} & & \downarrow{\scriptstyle \exists\eta} & & \downarrow{=} & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0
\end{array}
\tag{2.6}
$$

where the top row exists because C is of finite presentation. The homomorphism η exists by Proposition 1.1.5 (3) because $R^{(n)}$ is projective. Now $\beta\eta\phi = \psi\phi = 0$ so $\operatorname{im}\eta\phi \subseteq \ker\beta = \operatorname{im}\alpha$. Again, since $R^{(n)}$ is projective there exists ρ making the diagram commute. Since B is finitely generated, so is $\operatorname{coker}\eta$ by Part (1). The Snake Lemma 1.4.1 applied to (2.6) says that $\operatorname{coker}\rho \cong \operatorname{coker}\eta$ so $\operatorname{coker}\rho$ is finitely generated. Because $\operatorname{im}\rho$ is finitely generated, the exact sequence

$$0 \to \operatorname{im}\rho \to A \to \operatorname{coker}\rho \to 0$$

and Part (2) show that A is finitely generated. □

1.2. Exercises.

Exercise 2.1.8. Suppose R is a commutative ring, $R = R_1 \oplus \cdots \oplus R_n$ is a direct sum, and $\pi_i : R \to R_i$ is the projection. Assume each R_i is a local ring with maximal ideal $\mathfrak{n}_i$. Let $\mathfrak{m}_i = \pi_i^{-1}(\mathfrak{n}_i)$. Prove the following.

(1) $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ is the complete list of maximal ideals of R.

(2) π_i induces an isomorphism on local rings $R_{\mathfrak{m}_i} \cong R_i$.

(3) The natural homomorphism $R \to R_{\mathfrak{m}_1} \oplus \cdots \oplus R_{\mathfrak{m}_n}$ is an isomorphism.

Exercise 2.1.9. Let R be a commutative ring and P a prime ideal in R. Prove that $R_P/(PR_P)$ is isomorphic to the quotient field of R/P.

Exercise 2.1.10. Let $f : R \to S$ be a homomorphism of commutative rings and W a multiplicative subset of R. Prove the following.

(1) $f(W) \subseteq S$ is a multiplicative subset of S.

(2) If $Z = f(W)$ is the image of W, then $Z^{-1}S \cong W^{-1}S = S \otimes_R W^{-1}R$.

(3) If I is an ideal in R, then there are isomorphisms

$$W^{-1}(R/I) \cong (R/I) \otimes_R W^{-1}R \cong (W^{-1}R)/(I(W^{-1}R)).$$

Exercise 2.1.11. Let R be a commutative ring, P a prime ideal in R, and $\alpha \in R - P$. Prove that $R_P \cong (R_\alpha)_{PR_\alpha} \cong R_\alpha \otimes_R R_P$.

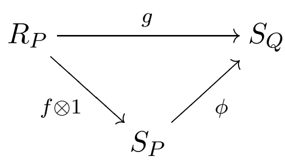
Exercise 2.1.12. Let $f : R \to S$ be a homomorphism of commutative rings. Let Q be a prime ideal in S and $P = f^{-1}(Q)$. Let $Q_P = Q \otimes_R R_P$ and $S_P = S \otimes_R R_P$. Prove the following.

(1) f induces a local homomorphism of local rings $g : R_P \to S_Q$. For the definition of local homomorphism of local rings, see Exercise 1.1.18.

(2) Q_P is a prime ideal of S_P.

(3) S_Q is isomorphic to the local ring of S_P at Q_P.

(4) The diagram



commutes, where ϕ is the localization map.

Exercise 2.1.13. (Local to Global Property for Idempotents) Let R be a commutative ring and $\mathfrak{p} \in \operatorname{Spec} R$. Let A be an R-algebra and e an idempotent in $A_\mathfrak{p}$. Show that there exists $\alpha \in R - \mathfrak{p}$ and an idempotent e_0 in $A_\alpha = A \otimes_R R[\alpha^{-1}]$ such that e is equal to the image of e_0 under the natural map $A_\alpha \to A_\mathfrak{p}$.

Exercise 2.1.14. Let R be a UFD which is not a field and α a nonzero element of R which is not invertible. Prove that the localization $W^{-1}R$ is not projective. (Hint: Show that $\operatorname{Hom}_R\big(R[\alpha^{-1}], R\big) = (0)$.)

2. The Prime Spectrum of a Commutative Ring

This section contains a brief review of the fundamental properties of the prime ideal spectrum of a commutative ring which is furnished with the Zariski topology. The notation and results established here play a central role throughout the rest of the book. We assume the reader has some familiarity with the prime spectrum, as well as a basic working knowledge of point set topology. As a basic reference, we suggest [**DF04**].

Let R be a commutative ring. The *prime ideal spectrum* of R, denoted $\operatorname{Spec} R$, is the set of all prime ideals in R. The *maximal ideal spectrum* of R, denoted $\operatorname{Max} R$, is the set of all maximal ideals in R. Given a subset $L \subseteq R$, let

$$V(L) = \{P \in \operatorname{Spec} R \mid P \supseteq L\}.$$

Given any collection $\{L_i\}$ of subsets of R, the reader should verify that

(1) $V(\{1\}) = \emptyset$, $V(\{0\}) = \operatorname{Spec} R$,

(2) $\bigcap_i V(L_i) = V(\bigcup L_i)$, and

(3) $V(L_1) \cup V(L_2) = V\big(\{x_1 x_2 \mid x_1 \in L_1, x_2 \in L_2\}\big)$.

The collection of sets $\{V(L) \mid L \subseteq R\}$ make up the closed sets for a topology on $\operatorname{Spec} R$, called the *Zariski topology*. Given a nonempty subset $Y \subseteq \operatorname{Spec} R$, let

$$I(Y) = \bigcap_{P \in Y} P.$$

Being an intersection of ideals, $I(Y)$ is an ideal. By definition, we take $I(\emptyset)$ to be the unit ideal R.

Lemma 2.2.1. *Let R be a commutative ring and W a multiplicative subset of $R - \{0\}$. Then there exists a prime ideal $P \in \operatorname{Spec} R$ such that $P \cap W = \emptyset$.*

Proof. Let $\mathcal{S} = \{I \subseteq R \mid I \text{ is an ideal and } I \cap W = \emptyset\}$. Then $(0) \in \mathcal{S}$. By Zorn's Lemma, $\mathcal{S}$ has a maximal element, say P. To see that P is a prime ideal, assume $x \notin P$ and $y \notin P$ and show that $xy \notin P$. By maximality of P we know $(Rx + P) \cap W \neq \emptyset$. Hence, there exists $a \in R$ and $u \in W$ such that $ax - u \in P$. Likewise, $(Ry + P) \cap W \neq \emptyset$, so there exists $b \in R$ and $v \in W$ such that $by - v \in P$. Multiplying, we find that $abxy - uv \in P$. Since $uv \in W$ and $P \cap W = \emptyset$, this implies $xy \notin P$. □

If R is a commutative ring and $\mathfrak{a}$ is an ideal in R, then the *nil radical of* $\mathfrak{a}$ is defined to be the set

$$\operatorname{Rad}(\mathfrak{a}) = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n > 0\}.$$

Notice that $\mathfrak{a}$ is always a subset of $\operatorname{Rad}(\mathfrak{a})$ and $\mathfrak{a}$ is called a *radical ideal* if $\mathfrak{a} = \operatorname{Rad}(\mathfrak{a})$. When the ring R is ambiguous, we write $\operatorname{Rad}_R(\mathfrak{a})$ for $\operatorname{Rad}(\mathfrak{a})$. The nil radical of the zero ideal, $\operatorname{Rad}_R(0)$, is the set of all nilpotent elements in R. The next lemma draws the connection between the prime ideals containing $\mathfrak{a}$ and the nil radical of $\mathfrak{a}$.

Lemma 2.2.2. *Let R be a commutative ring and $\mathfrak{a}$ an ideal in R. Then $\operatorname{Rad}(\mathfrak{a})$ is an ideal in R which contains $\mathfrak{a}$ and*

$$\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}_R(\mathfrak{a}) = I(V(\mathfrak{a})) = \bigcap_{P \in V(\mathfrak{a})} P.$$

In particular, $\operatorname{Rad}(0) = \operatorname{Rad}_R(0) = \bigcap_{P \in \operatorname{Spec} R} P$.

Proof. We begin by proving the last statement. Pick $x \in \operatorname{Rad}(0)$. Fix $P \in \operatorname{Spec} R$. If $x^n = 0$, then either $x = 0$ or $n \geq 2$. If $n \geq 2$, then $x \cdot x^{n-1} \in P$ so $x \in P$ or $x^{n-1} \in P$. Inductively, $x \in P$ so $\operatorname{Rad}(0) \subseteq P$. If

$x \notin \text{Rad}(0)$, let $W = \{1, x, x^2, \dots\}$. Lemma 2.2.1 shows that there exists $P \in \text{Spec}\, R$ such that $x \notin P$.

Now we prove the first statement. Under the natural map $\eta : R \to R/\mathfrak{a}$ there is a one-to-one correspondence between ideals of R containing $\mathfrak{a}$ and ideals of $R/\mathfrak{a}$. Under this correspondence, prime ideals correspond to prime ideals. To get the nil radical $\text{Rad}_R\, \mathfrak{a}$, we lift the nil radical $\text{Rad}_{R/\mathfrak{a}}(0)$. □

Let X be a topological space and Y a nonempty subset. We say Y is *irreducible* if whenever $Y \subseteq Y_1 \cup Y_2$ and Y_1, Y_2 are closed subsets of X, then $Y \subseteq Y_1$, or $Y \subseteq Y_2$. We say Y is *connected* if whenever $Y \subseteq Y_1 \cup Y_2$ and Y_1, Y_2 are disjoint closed subsets of X, then $Y \subseteq Y_1$, or $Y \subseteq Y_2$. The empty set is not considered to be irreducible or connected. Notice that an irreducible set is connected. If Z is a subset of the topological space X, then the *closure* of Z, denoted $\bar{Z}$, is the smallest closed subset of X that contains Z. Equivalently, $\bar{Z}$ is equal to the intersection of all closed sets that contain Z.

Lemma 2.2.3. *Let X be a topological space.*

(1) *If X is irreducible and $U \subseteq X$ is a nonempty open of X, then U is irreducible and dense.*

(2) *Let Z be a subset of X and denote by $\bar{Z}$ the closure of Z in X. Then Z is irreducible if and only if $\bar{Z}$ is irreducible.*

(3) *If X is irreducible, then X is connected.*

Proof. The proof is left to the reader. □

Lemma 2.2.4. *Let $\mathfrak{a}$ be an ideal in R and Y a subset of* $\text{Spec}\, R$. *Then*

(1) *$V(\mathfrak{a}) = V(\text{Rad}\, \mathfrak{a})$, and*

(2) *$V(I(Y)) = \bar{Y}$, the closure of Y in the Zariski topology.*

Proof. (1): Since $\mathfrak{a} \subseteq \text{Rad}\, \mathfrak{a}$, it follows that $V(\mathfrak{a}) \supseteq V(\text{Rad}\, \mathfrak{a})$. Conversely, if $P \in \text{Spec}\, R$ and $P \supseteq \mathfrak{a}$, then by Lemma 2.2.2, $P \supseteq \text{Rad}\, \mathfrak{a}$. Then $P \in V(\text{Rad}\, \mathfrak{a})$.

(2): Since $V(I(Y))$ is closed we have $V(I(Y)) \supseteq \bar{Y}$. Since $\bar{Y}$ is closed, $\bar{Y} = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$. Since $Y \subseteq \bar{Y}$, $I(Y) \supseteq I(\bar{Y}) = I(V(\mathfrak{a})) = \text{Rad}\, \mathfrak{a} \supseteq \mathfrak{a}$. Thus, $V(I(Y)) \subseteq V(\mathfrak{a}) = \bar{Y}$. □

Corollary 2.2.5. *There is a one-to-one order-reversing correspondence between closed subsets of* $\text{Spec}\, R$ *and radical ideals in R given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto V(\mathfrak{a})$. Under this correspondence, irreducible closed subsets correspond to prime ideals, and closed points correspond to maximal ideals.*

Proof. For the first part, see [**DF04**, Proposition 15.5.54], for example. The last part is proved in Lemma 2.2.6. □

Lemma 2.2.6. *Let R be a commutative ring and Y a subset of* $\operatorname{Spec} R$*. Then Y is irreducible if and only if $P = I(Y)$ is a prime ideal in R. If Z is an irreducible closed subset of* $\operatorname{Spec} R$*, then $P = I(Z)$ is the unique minimal element of Z, and is called the* generic point *of Z.*

Proof. Suppose Y is irreducible. Assume $x, y \in R$ and $xy \in I(Y)$. Notice that $Y \subseteq \bar{Y} = V(I(Y)) \subseteq V(xy) = V(x) \cup V(y)$. Since Y is irreducible, $Y \subseteq V(x)$ or $Y \subseteq V(y)$. Therefore, $x \in I(Y)$, or $y \in I(Y)$. This shows $I(Y)$ is a prime ideal. Conversely, assume $P = I(Y)$ is a prime ideal of R. The singleton set $\{P\}$ is irreducible, and by Lemma 2.2.3 the closure of $\{P\}$ is irreducible. By Lemma 2.2.4, the closure of $\{P\}$ is equal to $V(P)$, which is equal to $\bar{Y}$. By Lemma 2.2.3, Y is irreducible. The rest is left to the reader. □

Let R be a commutative ring. If $\alpha \in R$, the *basic open subset* of $\operatorname{Spec} R$ associated to α is

$$U(\alpha) = \operatorname{Spec} R - V(\alpha) = \{Q \in \operatorname{Spec} R \mid \alpha \notin Q\}.$$

By Lemma 2.2.7 (2) we see that the collection of all basic open sets $\{U(\alpha) \mid \alpha \in R\}$ is a basis for the Zariski topology on $\operatorname{Spec} R$.

Lemma 2.2.7. *Let R be a commutative ring.*

(1) *Let $\alpha, \beta \in R$. The following are equivalent:*
   (a) $V(\alpha) = V(\beta)$.
   (b) $U(\alpha) = U(\beta)$.
   (c) *There exist $a \geq 1$, $b \geq 1$ such that $\alpha^a \in R\beta$ and $\beta^b \in R\alpha$.*

(2) *If I is an ideal in R, then*

$$\operatorname{Spec} R - V(I) = \bigcup_{\alpha \in I} U(\alpha)$$

Every open set can be written as a union of basic open sets.

Proof. (1): By Lemma 2.2.2, $\operatorname{Rad}(R\alpha) = I(V(\alpha))$, and by Lemma 2.2.4, $V(\operatorname{Rad}(R\alpha)) = V(\alpha)$. So $V(\alpha) = V(\beta)$ if and only if $\operatorname{Rad}(R\alpha) = \operatorname{Rad}(R\beta)$ which is true if and only if there exist $a \geq 1$, $b \geq 1$ such that $\alpha^a \in R\beta$ and $\beta^b \in R\alpha$. The rest is left to the reader. □

We end this section with a series of results that describe the fundamental correspondence that exists between idempotents in R, ideals that are direct summands of R, and subsets of $\operatorname{Spec} R$ that are both open and closed. This correlation is the subject of Theorem 2.2.9 and its two corollaries. Playing a central role in this development is Lemma 1.1.20, which we proved in Chapter 1. First we show that the basic open set associated to an idempotent is also a closed set.

Lemma 2.2.8. *Let* R *be a commutative ring and* $\operatorname{idemp}(R) = \{e \in R \mid e = e^2\}$ *the set of all idempotents of* R.

(1) *If* $e \in \operatorname{idemp}(R)$, *then the closed set* $V(1-e)$ *is equal to the open set* $U(e)$.
(2) *Let* $e, f \in \operatorname{idemp}(R)$. *Then* $V(e) = V(f)$ *if and only if* $e = f$.
(3) *Let* $e, f \in \operatorname{idemp}(R)$. *Then* $Re = Rf$ *if and only if* $e = f$.

Proof. (1): Let $P \in \operatorname{Spec} R$. Since $e(1-e) = 0$, either $e \in P$, or $1 - e \in P$. Since $1 = e + (1-e)$, P does not contain both e and $1-e$.

(2): Assume $V(e) = V(f)$. By Lemma 2.2.7, there exist $a \geq 1$, $b \geq 1$ such that $e = e^a \in Rf$ and $f = f^b \in Re$. Write $e = xf$ and $f = ye$ for some $x, y \in R$. Then $e = xf = xf^2 = (xf)f = ef = eye = ye^2 = ye = f$.

(3): $Re = Rf$ implies $V(e) = V(f)$, which by Part (2) implies $e = f$. □

Theorem 2.2.9. *Let* R *be a commutative ring. Define* $\mathcal{C}$ *to be the set of all subsets* $Y \subseteq \operatorname{Spec} R$ *such that* Y *is both open and closed. Define* $\mathcal{D}$ *to be the set of all ideals* A *in* R *such that* A *is an* R*-module direct summand of* R. *Then there are one-to-one correspondences:*

$$\gamma : \operatorname{idemp}(R) \to \mathcal{C},$$

defined by $e \mapsto V(1-e) = U(e)$, *and*

$$\delta : \operatorname{idemp}(R) \to \mathcal{D},$$

defined by $e \mapsto Re$.

Proof. Parts (1) and (2) of Lemma 2.2.8 show that γ is well defined and one-to-one. By Lemma 1.1.20 (1), δ is well defined and onto. By Lemma 2.2.8 (3), δ is one-to-one. It remains to be proven that γ is onto. Assume A_1, A_2 are ideals in R, $X_1 = V(A_1)$, $X_2 = V(A_2)$, $X_1 \cup X_2 = \operatorname{Spec} R$, $X_1 \cap X_2 = \emptyset$. We prove that $X_i = V(e_i)$ for some $e_i \in \operatorname{idemp}(R)$. Since $\emptyset = X_1 \cap X_2 = V(A_1 + A_2)$, we know A_1 and A_2 are comaximal. Therefore, $A_1 A_2 = A_1 \cap A_2$. Since $\operatorname{Spec} R = X_1 \cup X_2 = V(A_1 A_2) = V(A_1 \cap A_2)$, Lemma 2.2.2 implies

$$A_1 \cap A_2 \subseteq \bigcap_{P \in \operatorname{Spec} R} P = \operatorname{Rad}_R(0).$$

That is, $A_1 \cap A_2$ consists of nilpotent elements. Write $1 = \alpha_1 + \alpha_2$, where $\alpha_i \in A_i$. Then $R = R\alpha_1 + R\alpha_2$ so $R\alpha_1$ and $R\alpha_2$ are comaximal. Also $R\alpha_1 \cap R\alpha_2 = R\alpha_1\alpha_2 \subseteq A_1 \cap A_2 \subseteq \operatorname{Rad}_R(0)$. So there exists $m > 0$ such that $(\alpha_1\alpha_2)^m = 0$. Then $R\alpha_1^m$ and $R\alpha_2^m$ are comaximal (Exercise 2.2.18) and

$R\alpha_1^m \cap R\alpha_2^m = (0)$. Therefore R is isomorphic to the internal direct sum $R \cong R\alpha_1^m \oplus R\alpha_2^m$. By Lemma 1.1.20 there are orthogonal idempotents $e_1, e_2 \in R$ such that $1 = e_1 + e_2$ and $Re_i = R\alpha_i^m$. Then $\operatorname{Spec} R = V(e_1) \cup V(e_2)$ and $V(e_1) \cap V(e_2) = \emptyset$. Moreover, $V(e_i) \supseteq V(R\alpha_i^m) \supseteq V(A_i) = X_i$. From this it follows that $X_i = V(e_i)$, hence γ is onto. □

Corollary 2.2.10. *Suppose R is a commutative ring and $\operatorname{Spec} R = X_1 \cup \cdots \cup X_r$, where each X_i is a nonempty closed subset and $X_i \cap X_j = \emptyset$ whenever $i \neq j$. Then there are idempotents $e_1, \dots, e_r$ in R such that $X_i = U(e_i) = V(1 - e_i)$ is homeomorphic to $\operatorname{Spec} Re_i$, and $R = Re_1 \oplus \cdots \oplus Re_r$. In particular, the topological space $\operatorname{Spec} R$ is connected if and only if 0 and 1 are the only idempotents of R.*

Proof. By Theorem 2.2.9 there are unique idempotents $e_1, \dots, e_r$ in R such that $X_i = U(e_i) = V(1-e_i)$. Since $R = Re_i \oplus R(1-e_i)$, the map $\pi_i : R \to Re_i$ defined by $x \mapsto xe_i$ is a homomorphism of rings with kernel $R(1 - e_i)$. By Exercise 2.2.14, π_i induces a homeomorphism $\operatorname{Spec} Re_i \to X_i$. If $i \neq j$, then $V(1-e_i) \cap V(1-e_j) = X_i \cap X_j = \emptyset$. It follows that the ideals $R(1-e_i)$ are pairwise relatively prime. By the Chinese Remainder Theorem, the direct sum map

$$R \xrightarrow{\phi} Re_1 \oplus \cdots \oplus Re_r$$

is onto. By Exercise 2.2.17, the kernel of ϕ is the principal ideal generated by the product $(1 - e_1) \cdots (1 - e_r)$. But $X = X_1 \cup \cdots \cup X_r = V((1 - e_1) \cdots (1 - e_r))$. Therefore, $(1 - e_1) \cdots (1 - e_r) \in \operatorname{Rad}_R(0)$. Since the only nilpotent idempotent is 0, ϕ is an isomorphism. □

Corollary 2.2.11. *Let e be an idempotent of R. The following are equivalent:*

(1) *e is a primitive idempotent.*

(2) *$V(1 - e) = U(e)$ is a connected component of $\operatorname{Spec} R$.*

(3) *0 and 1 are the only idempotents of the ring Re.*

Proof. (1) is equivalent to (3): This follows from Lemma 1.1.20 (2).

(2) is equivalent to (3): Since $R = Re \oplus R(1 - e)$, it follows from Exercise 2.2.14 that $V(1-e)$ is homeomorphic to $\operatorname{Spec} Re$. The rest follows from Corollary 2.2.10. □

We say a commutative ring R is *connected* if $\operatorname{Spec} R$ is connected. By the above, R is connected if and only if R has only two idempotents.

2.1. Exercises. In the following, R always denotes a commutative ring.

Exercise 2.2.12. If $\theta : R \to S$ is a homomorphism of commutative rings, show that $P \mapsto \theta^{-1}(P)$ induces a function $\theta^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ which is continuous for the Zariski topology. Show that if $\sigma : S \to T$ is another homomorphism, then $(\sigma\theta)^\sharp = \theta^\sharp \sigma^\sharp$.

Exercise 2.2.13. For the following, let I and J be ideals in R. Prove that the nil radical satisfies the following properties.

(1) $I \subseteq \operatorname{Rad} I$
(2) $\operatorname{Rad}(\operatorname{Rad} I) = \operatorname{Rad} I$
(3) $\operatorname{Rad}(IJ) = \operatorname{Rad}(I \cap J) = \operatorname{Rad} I \cap \operatorname{Rad} J$
(4) $\operatorname{Rad} I = R$ if and only if $I = R$
(5) $\operatorname{Rad}(I + J) = \operatorname{Rad}(\operatorname{Rad} I + \operatorname{Rad} J)$
(6) If $P \in \operatorname{Spec} R$, then for all $n > 0$, $P = \operatorname{Rad}(P^n)$.
(7) $I + J = R$ if and only if $\operatorname{Rad} I + \operatorname{Rad} J = R$.

Exercise 2.2.14. Let $I \subsetneq R$ be an ideal in R, $\eta : R \to R/I$ the natural map, and $\eta^\sharp : \operatorname{Spec}(R/I) \to \operatorname{Spec} R$ the continuous map of Exercise 2.2.12. Prove the following.

(1) $\eta^\sharp$ is a one-to-one order-preserving correspondence between the prime ideals of R/I and $V(I)$.
(2) There is a one-to-one correspondence between radical ideals in R/I and radical ideals in R containing I.
(3) Under $\eta^\sharp$ the image of a closed set is a closed set.
(4) $\eta^\sharp : \operatorname{Spec}(R/I) \to V(I)$ is a homeomorphism.
(5) If $I \subseteq \operatorname{Rad}_R(0)$, then $\eta^\sharp : \operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ is a homeomorphism.

Exercise 2.2.15. Let W be a multiplicative set in R and $\theta : R \to W^{-1}R$ the localization. For any subset $S \subseteq W^{-1}R$, use the intersection notation $S \cap R = \theta^{-1}(S)$ for the preimage. Prove the following.

(1) If J is an ideal in $W^{-1}R$, then $J = W^{-1}(J \cap R)$.
(2) The continuous map $\theta^\sharp : \operatorname{Spec}(W^{-1}R) \to \operatorname{Spec}(R)$ is one-to-one.
(3) If $P \in \operatorname{Spec} R$ and $P \cap W = \emptyset$, then $W^{-1}P$ is a prime ideal in $W^{-1}R$.
(4) The image of $\theta^\sharp : \operatorname{Spec}(W^{-1}R) \to \operatorname{Spec}(R)$ consists of those prime ideals in R that are disjoint from W.
(5) If $P \in \operatorname{Spec} R$, there is a one-to-one correspondence between prime ideals in R_P and prime ideals of R contained in P.

Exercise 2.2.16. Show that $\operatorname{Spec} R$ is quasi-compact. That is, every open cover of $\operatorname{Spec} R$ has a finite subcover.

Exercise 2.2.17. Let $I_1, I_2, \dots, I_n$ be pairwise comaximal ideals in R. Show that $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n$.

Exercise 2.2.18. Show that if I and J are comaximal ideals in R, then for every $m \geq 1$ and $n \geq 1$, I^m and J^n are comaximal. In this case $I^m J^n = I^m \cap J^n$.

3. Finitely Generated Projective Modules

Throughout this section, R is a commutative ring. Our goal in this section is to generalize the notion of rank of a free module to projective modules. We show that if R is connected, then there is a well defined notion of rank associated with any finitely generated projective R-module. For a free module, this rank agrees with the rank we already defined in Section 1.1.2.

Our first main result in this section is a proof that every finitely generated projective module over a local ring is free. Hence for local rings, there is a well defined notion of rank for finitely generated projective modules. Notice that the next two proofs are both applications of Nakayama's Lemma.

Lemma 2.3.1. *Let R be a commutative ring and I an ideal in R. Let M be an R-module. If*

(1) *I is nilpotent, or*
(2) *I is contained in every maximal ideal of R and M is finitely generated,*

then a subset $X \subseteq M$ generates M as an R-module if and only if the image of X generates M/IM as an R/I-module.

Proof. Let $\eta : M \to M/IM$. Suppose $X \subseteq M$ and let T be the R-submodule of M spanned by X. Then $\eta(T) = (T + IM)/IM$ is spanned by $\eta(X)$. If $T = M$, then $\eta(T) = M/IM$. Conversely, $\eta(T) = M/IM$ implies $M = T + IM$. By Corollary 1.1.17, this implies $M = T$. □

Proposition 2.3.2. *Let R be a commutative local ring. If P is a finitely generated projective R-module, then P is free of finite rank. If $\mathfrak{m}$ is the maximal ideal of R and $\{x_i + \mathfrak{m}P \mid 1 \leq i \leq n\}$ is a basis for the vector space $P/\mathfrak{m}P$ over the residue field $R/\mathfrak{m}$, then $\{x_1, \dots, x_n\}$ is a basis for P over R. It follows that* $\operatorname{Rank}_R(P) = \dim_{R/\mathfrak{m}}(P/\mathfrak{m}P)$.

Proof. Define $\phi : R^{(n)} \to P$ by $\phi(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^n \alpha_i x_i$. The goal is to show that ϕ is onto and one-to-one, in that order. Denote by T the image of ϕ. Then $T = Rx_1 + \cdots + Rx_n$ which is the submodule of P generated by $\{x_1, \ldots, x_n\}$. It follows from Lemma 2.3.1 that ϕ is onto. To show that ϕ is one-to-one we prove that $\ker \phi = 0$. Since P is R-projective, the sequence

$$0 \to \ker \phi \to R^{(n)} \xrightarrow{\phi} P \to 0$$

is split exact. Therefore, $\ker \phi$ is a finitely generated projective R-module. Upon tensoring with $(\,) \otimes_R R/\mathfrak{m}$, ϕ becomes the isomorphism $(R/\mathfrak{m})^{(n)} \cong P/\mathfrak{m}P$. By Exercise 1.3.7,

$$0 \to \ker \phi \otimes_R R/\mathfrak{m} \to (R/\mathfrak{m})^{(n)} \xrightarrow{\phi} P/\mathfrak{m}P \to 0$$

is split exact. Therefore, $\ker \phi \otimes_R R/\mathfrak{m} = 0$, or in other words $\mathfrak{m}(\ker \phi) = \ker \phi$. By Corollary 1.1.14, $\ker \phi = (0)$. □

Before continuing our discussion of ranks, we state and prove a corollary of Lemma 2.3.1 and Proposition 2.3.2.

Corollary 2.3.3. *Let R be a commutative local ring with residue field k. Let $\psi : M \to N$ be a homomorphism of R-modules, where M is finitely generated and N is finitely generated and free. Then*

$$0 \to M \xrightarrow{\psi} N$$

is split exact if and only if $\psi \otimes 1 : M \otimes_R k \to N \otimes_R k$ is one-to-one.

Proof. Assume $\psi \otimes 1$ is one-to-one. By Proposition 2.3.2 we can pick a generating set $\{x_1, \ldots, x_n\}$ for the R-module M such that $\{x_1 \otimes 1, \ldots, x_n \otimes 1\}$ is a basis for the k-vector space $M \otimes_R k$. Define $\pi : R^{(n)} \to M$ by mapping the ith standard basis vector to x_i. Then $\pi \otimes 1 : k^{(n)} \to M \otimes_R k$ is an isomorphism. The diagram

$$\begin{array}{ccccc} R^{(n)} & \xrightarrow{\pi} & M & \xrightarrow{\psi} & N \\ \downarrow & & \downarrow & & \downarrow \\ k^{(n)} & \xrightarrow{\pi \otimes 1} & M \otimes_R k & \xrightarrow{\psi \otimes 1} & N \otimes_R k \end{array}$$

commutes. The composite map $\psi\pi \otimes 1$ is one-to-one. By Exercise 2.3.11, there is an R-module homomorphism $\tau : N \to R^{(n)}$ which is a left inverse for $\psi\pi$. Since π is onto, it follows that $\pi\tau$ is a left inverse for ψ.

Conversely, if ψ has a left inverse, then clearly $\psi \otimes 1$ is one-to-one. □

Let M be a finitely generated projective module over a commutative ring R. For any prime ideal P of R, the localization M_P is a finitely generated projective R_P-module (Theorem 1.3.2). Therefore M_P is a finitely generated free R_P-module (Proposition 2.3.2), and M_P has a well defined rank. If there is an integer $n \geq 0$ such that $n = \operatorname{Rank}_{R_P}(M_P)$ for all $P \in \operatorname{Spec} R$, then we say M has *constant rank* and write $\operatorname{Rank}_R(M) = n$. We now prove that the constant rank property is preserved by change of base.

Proposition 2.3.4. *Let R be a commutative ring and S a commutative R-algebra. If M is a finitely generated projective R-module of constant rank $\operatorname{Rank}_R(M) = n$, then $M \otimes_R S$ is a finitely generated projective S-module of constant rank and $\operatorname{Rank}_S(M) = n$.*

Proof. By Theorem 1.3.2, $M \otimes_R S$ is a finitely generated projective S-module. Let $\theta : R \to S$ be the structure map. Let $Q \in \operatorname{Spec} S$ and $P = \theta^{-1}(Q) \in \operatorname{Spec} R$. Then by Exercise 2.1.12, θ extends to a local homomorphism of local rings $\theta : R_P \to S_Q$. The proof follows from the string of S_Q-module isomorphisms

$$\begin{aligned}(M \otimes_R S) \otimes_S S_Q &\cong M \otimes_R (S \otimes_S S_Q) \cong M \otimes_R S_Q \cong M \otimes_R (R_P \otimes_{R_P} S_Q)\\ &\cong (M \otimes_R R_P) \otimes_{R_P} S_Q \cong (R_P)^{(n)} \otimes_{R_P} S_Q \cong (S_Q)^{(n)} .\end{aligned}$$

□

In the following we write R_α for the localization of R at the multiplicative set $\{1, \alpha, \alpha^2, \dots\}$. If M is a finitely generated projective R-module and $P \in \operatorname{Spec} R$, then as an application of the local to global lemmas we show that there is a basic open neighborhood $U(\alpha)$ of P such that M_α has constant rank on $U(\alpha)$.

Theorem 2.3.5. *Let R be a commutative ring and M a finitely generated projective R-module.*

(1) *Given $P \in \operatorname{Spec} R$ there exists $\alpha \in R - P$ such that M_α is a free R_α-module.*
(2) *If α is as in (1), then the values $\operatorname{Rank}_{R_Q}(M_Q)$ are constant for all $Q \in U(\alpha)$.*
(3) *The map*

$$\operatorname{Spec} R \xrightarrow{\phi} \{0, 1, 2, \dots\}$$

defined by $P \mapsto \operatorname{Rank}_{R_P} M_P$ is continuous if $\{0, 1, 2, \dots\}$ is given the discrete topology (that is, the topology where every subset is closed, or equivalently, "points are open").

Proof. (1): By Proposition 2.3.2 we know that M_P is a free module over R_P. By Corollary 1.1.7, M is an R-module of finite presentation. An application of Lemma 2.1.6 completes the proof.

(2): If $Q \in U(\alpha)$, then $\alpha \in R - Q$. By Exercise 2.1.11, $R_Q = (R_\alpha)_{QR_\alpha}$. Since M_α is R_α-free of rank n, it follows from Proposition 2.3.4 that M_Q is R_Q-free of rank n.

(3): We need to prove that for every $n \geq 0$, the preimage $\phi^{-1}(n)$ is open in $\operatorname{Spec} R$. Let $P \in \operatorname{Spec} R$ such that $\operatorname{Rank}_{R_P} M_P = n$. It is enough to find an open neighborhood of P in the preimage of n. By Part (1), there exists $\alpha \in R - P$ such that M_α is free of rank n over R_α. Since $U(\alpha)$ is an open neighborhood of P in $\operatorname{Spec} R$, it is enough to show that $\operatorname{Rank}_{R_Q}(M_Q) = n$ for all $Q \in U(\alpha)$. This shows that (3) follows from (2). □

Let M be a finitely generated projective R-module. Combining Theorem 2.3.5 with the decomposition theorem for $\operatorname{Spec} R$ (Corollary 2.2.10) yields the following unique decomposition theorem for M. In the terminology of [**McD84**, Theorem IV.27], the elements $e_1, \dots, e_t$ that appear in Corollary 2.3.6 are called the *structure idempotents* of M.

Corollary 2.3.6. *Let R be a commutative ring and M a finitely generated projective R-module. Then there are idempotents $e_1, \dots, e_t$ in R satisfying the following.*

(1) $R = Re_1 \oplus \cdots \oplus Re_t$.
(2) $M = Me_1 \oplus \cdots \oplus Me_t$.
(3) *If $R_i = Re_i$ and $M_i = M \otimes_R R_i$, then M_i is a finitely generated projective R_i-module of constant rank.*
(4) *If $\operatorname{Rank}_{R_i}(M_i) = n_i$, then $n_1, \dots, n_t$ are distinct.*
(5) *The integer t and the idempotents $e_1, \dots, e_t$ are uniquely determined by M.*

Proof. The rank function $\phi : \operatorname{Spec} R \to \{0, 1, 2, \dots\}$ defined by $\phi(P) = \operatorname{Rank}_P(M)$ is continuous and locally constant (Theorem 2.3.5). Let $U_n = \phi^{-1}(\{n\})$ for each $n \geq 0$. Then $\{U_n \mid n \geq 0\}$ is a collection of subsets of $\operatorname{Spec} R$ each of which is open and closed. Moreover, the sets U_n are pairwise disjoint. Since $\operatorname{Spec} R$ is quasi-compact (Exercise 2.2.16) the image of ϕ is a finite set, say $\{n_1, \dots, n_t\}$. Let $e_1, \dots, e_t$ be the idempotents in R corresponding to the disjoint union $\operatorname{Spec} R = U_{n_1} \cup \cdots \cup U_{n_t}$ (Corollary 2.2.10). The rest is left to the reader. □

3.1. Exercises. For the following, R always denotes a commutative ring.

Exercise 2.3.7. Let L and M be finitely generated projective R-modules. Assume L and M both have constant rank. Prove:

(1) $L \oplus M$ has constant rank and $\operatorname{Rank}_R(L \oplus M)$ is equal to the sum $\operatorname{Rank}_R(L) + \operatorname{Rank}_R(M)$.
(2) $L \otimes_R M$ has constant rank and $\operatorname{Rank}_R(L \otimes_R M)$ is equal to the product $\operatorname{Rank}_R(L) \operatorname{Rank}_R(M)$.

(3) $\operatorname{Hom}_R(L, M)$ has constant rank and $\operatorname{Rank}_R(\operatorname{Hom}_R(L, M))$ is equal to the product $\operatorname{Rank}_R(L)\operatorname{Rank}_R(M)$.

Exercise 2.3.8. Let $f : R \to S$ be a homomorphism of commutative rings and $P \in \operatorname{Spec} R$. Let $k_P = R_P/PR_P$ be the residue field and $S_P = S \otimes_R R_P$. Let $Q \in \operatorname{Spec} S$ such that $P = f^{-1}(Q)$. Prove:

(1) $S \otimes_R k_P \cong S_P/PS_P$.

(2) $Q \otimes_R k_P$ is a prime ideal of $S \otimes_R k_P$ and QS_P/PS_P is the corresponding prime ideal of S_P/PS_P.

(3) The localization of S_P/PS_P at QS_P/PS_P is S_Q/PS_Q.

(4) The localization of $S \otimes_R k_P$ at the prime ideal $Q \otimes_R k_P$ is $S_Q \otimes_R k_P$.

Exercise 2.3.9. Let $f : R \to S$ be a homomorphism of commutative rings and let $f^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ the continuous map of Exercise 2.2.12. Assume

(a) $f^\sharp$ is one-to-one,

(b) the image of $f^\sharp$ is an open subset of $\operatorname{Spec} R$, and

(c) for every $\mathfrak{q} \in \operatorname{Spec} S$, if $\mathfrak{p} = \mathfrak{q} \cap R$, then the natural map $R_\mathfrak{p} \to S_\mathfrak{q}$ is an isomorphism.

If (a), (b) and (c) are satisfied, then we say $f^\sharp$ is an *open immersion*. Under these hypotheses, prove the following.

(1) For every $\mathfrak{q} \in \operatorname{Spec} S$, if $\mathfrak{p} = \mathfrak{q} \cap R$, then $S \otimes_R R_\mathfrak{p}$ is isomorphic to $S_\mathfrak{q}$.

(2) If $\alpha \in R$ and $U(\alpha)$ is a nonempty basic open subset of the image of $f^\sharp$, then $R[\alpha^{-1}]$ is isomorphic to $S \otimes_R R[\alpha^{-1}]$.

Exercise 2.3.10. Let M and N be finitely generated projective R-modules and $\varphi : M \to N$ an R-module homomorphism. Let $\mathfrak{p} \in \operatorname{Spec} R$ and assume $\varphi \otimes 1 : M \otimes_R R_\mathfrak{p} \to N \otimes_R R_\mathfrak{p}$ is an isomorphism. Show that there exists $\alpha \in R - \mathfrak{p}$ such that $\varphi \otimes 1 : M \otimes_R R_\alpha \to N \otimes_R R_\alpha$ is an isomorphism.

Exercise 2.3.11. Let R be a commutative local ring with residue field k. Let M and N be finitely generated free R-modules and $\psi : M \to N$ a homomorphism of R-modules. Show that if $\psi \otimes 1 : M \otimes_R k \to N \otimes_R k$ is one-to-one, then ψ has a left inverse. That is, there exists an R-module homomorphism $\sigma : N \to M$ such that $\sigma\psi = 1$ is the identity mapping on M.

Exercise 2.3.12. Let S be a commutative R-algebra that as an R-module is a progenerator. Show that if $\operatorname{Spec} R$ is connected, then the number of connected components of $\operatorname{Spec} S$ is bounded by $\operatorname{Rank}_R(S)$, hence is finite.

Exercise 2.3.13. Let R_1 and R_2 be rings (not necessarily commutative) and $S = R_1 \oplus R_2$ the direct sum. Let M be a left S-module. Using the projection maps $\pi_i : S \to R_i$, show that the R_i-modules $M_i = R_i \otimes_S M$ are S-modules. Show that M is isomorphic as an S-module to the direct sum $M_1 \oplus M_2$.

4. Faithfully Flat Modules and Algebras

As general references for this section, we suggest [**Mat80**] and [**Rot79**]. Recall that in Section 1.3.1 we defined a left R-module N to be flat if the functor $(\ ) \otimes_R N$ is both left and right exact. In Exercise 1.5.7 we defined N to be faithfully flat if N is flat, and N has the property that for any right R-module M, $M \otimes_R N = 0$ implies $M = 0$. If R is a commutative ring, then Lemma 2.4.1 below adds two more necessary and sufficient conditions for N to be faithfully flat. In the remainder of the book, we will have many important applications which are based on properties of faithfully flat R-algebras. For example, the theory of faithfully flat descent (see Section 5.3) forms the basis for the formulation of Amitsur cohomology (see Section 5.5).

Lemma 2.4.1. *Let R be a commutative ring and N an R-module. The following are equivalent.*

(1) *A sequence of R-modules $0 \to A \to B \to C \to 0$ is exact if and only if $0 \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ is exact.*
(2) *N is flat and for any R-module M, if $M \otimes_R N = 0$, then $M = 0$.*
(3) *N is flat and for every maximal ideal $\mathfrak{m}$ of R, $N \neq \mathfrak{m}N$.*

Example 2.4.2. Let R be a commutative ring and N an R-module. By the Morita Theorems, (see Exercise 1.5.7), if N is a progenerator, then N is faithfully flat. If N is projective, then N is flat (Exercise 1.3.7) but not necessarily faithfully flat. For example, suppose the ring $R = I \oplus J$ is an internal direct sum of two nonzero ideals I and J. Then $IJ = 0$, $I^2 = I$, $J^2 = J$ and $I + J = R$. The sequence $0 \to I \to 0$ is not exact. Tensor with $(\cdot) \otimes_R J$ and get the exact sequence $0 \to 0 \to 0$. So J is not faithfully flat.

Proposition 2.4.3. *Let R be a commutative ring. The R-module*

$$E = \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} R_{\mathfrak{m}}$$

is faithfully flat.

Proof. Each $R_{\mathfrak{m}}$ is flat by Lemma 2.1.1, so E is flat by Exercise 2.4.15. For every maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}R_{\mathfrak{m}} \neq R_{\mathfrak{m}}$ so $\mathfrak{m}E \neq E$. By Lemma 2.4.1, we are done. □

Lemma 2.4.4. *Let $\theta : R \to S$ be a homomorphism of commutative rings such that S is a faithfully flat R-module. Then the following are true.*

(1) *For any R-module M, the map $M \to M \otimes_R S$ defined by $x \mapsto x \otimes 1$ is one-to-one. In particular, θ is one-to-one, so we can view $R = \theta(R)$ as a subring of S.*

(2) *For any ideal $I \subseteq R$, $IS \cap R = I$.*

(3) *The continuous map $\theta^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ of Exercise 2.2.12 is onto.*

Lemma 2.4.5. *If $\theta : R \to S$ is a homomorphism of commutative rings, then the following are equivalent.*

(1) *S is faithfully flat as an R-module.*

(2) *S is a flat R-module and the continuous map $\theta^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ is onto.*

(3) *S is a flat R-module and for each maximal ideal $\mathfrak{m}$ of R, there is a maximal ideal $\mathfrak{n}$ of S such that $\mathfrak{n} \cap R = \mathfrak{m}$.*

Proposition 2.4.6. *Let R be a commutative ring and $\epsilon : R \to A$ a homomorphism of rings such that ϵ makes A into a progenerator R-module.*

(1) *Under ϵ, R is mapped isomorphically onto an R-module direct summand of A.*

(2) *If B is a subring of A containing the image of ϵ, then the image of ϵ is an R-module direct summand of B.*

(3) *A is faithfully flat as an R-module.*

Proof. (1): By Corollary 1.1.16, A is R-faithful. Therefore the sequence

$$0 \to R \xrightarrow{\epsilon} A$$

is exact. By Exercise 1.3.31, there is a left inverse for ϵ if and only if

$$\operatorname{Hom}_R(A, R) \xrightarrow{\mathrm{H}_\epsilon} \operatorname{Hom}_R(R, R) \to 0 \tag{2.7}$$

is exact. Let $\mathfrak{m}$ be a maximal ideal in R. By Theorem 1.3.2, $A \otimes_R R/\mathfrak{m} = A/\mathfrak{m}A$ is a progenerator over the field $R/\mathfrak{m}$. In other words, $A/\mathfrak{m}A$ is a nonzero finite dimensional vector space over $R/\mathfrak{m}$. The diagram

$$\begin{array}{ccccc} R/\mathfrak{m} \otimes_R \operatorname{Hom}_R(A, R) & \xrightarrow{1 \otimes \mathrm{H}_\epsilon} & R/\mathfrak{m} \otimes_R \operatorname{Hom}_R(R, R) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \\ \operatorname{Hom}_{R/\mathfrak{m}}(A/\mathfrak{m}A, R/\mathfrak{m}) & \xrightarrow{\mathrm{H}_\epsilon} & \operatorname{Hom}_{R/\mathfrak{m}}(R/\mathfrak{m}, R/\mathfrak{m}) & \longrightarrow & 0 \end{array}$$

commutes. The bottom row is exact since $0 \to R/\mathfrak{m} \to A/\mathfrak{m}A$ is split exact over $R/\mathfrak{m}$. The vertical maps are isomorphisms by Corollary 1.3.27. Therefore the top row is exact. Corollary 1.3.19 says that (2.7) is exact. This proves (1).

(2): Assume $R \subseteq B \subseteq A$ is a tower of subrings. If $\sigma : A \to R$ is a left inverse for $\epsilon : R \to A$, then σ restricts to a left inverse for $R \to B$.

(3): This follows from Exercise 1.5.7. □

Proposition 2.4.7 and its corollary show that flatness is a local property.

Proposition 2.4.7. *Let R be a commutative ring and A an R-module. The following are equivalent.*

(1) *A is a flat R-module.*

(2) *A_p is a flat R_p-module, for every $p \in \operatorname{Spec} R$.*

(3) *$A_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module, for every $\mathfrak{m} \in \operatorname{Max} R$.*

Proof. It follows from Theorem 1.3.2 that (1) implies (2). It is immediate that (2) implies (3).

(3) implies (1): Denote by S the exact sequence

$$0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$$

of R-modules. Let $\mathfrak{m} \in \operatorname{Max} R$. Because $R_{\mathfrak{m}}$ is flat over R and $A_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$,

$$(S) \otimes_R R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} A_{\mathfrak{m}} = (S) \otimes_R A_{\mathfrak{m}}$$

is an exact sequence. Take the direct sum over all $\mathfrak{m}$. The reader should verify that

$$(S) \otimes_R \Big(\bigoplus_{\mathfrak{m} \in \operatorname{Max} R} A_{\mathfrak{m}} \Big) = (S) \otimes_R A \otimes_R \Big(\bigoplus_{\mathfrak{m} \in \operatorname{Max} R} R_{\mathfrak{m}} \Big)$$

is exact. By Proposition 2.4.3,

$$E = \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} R_{\mathfrak{m}}$$

is a faithfully flat R-module, so $(S) \otimes_R A$ is exact. □

Proposition 2.4.8. *Let $f : R \to S$ be a homomorphism of commutative rings. The following are equivalent.*

(1) *S is a flat R-algebra.*

(2) *$S_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-algebra, for every $\mathfrak{q} \in \operatorname{Spec} S$, if $f^{-1}(\mathfrak{q}) = \mathfrak{p}$.*

(3) *$S_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-algebra, for every $\mathfrak{m} \in \operatorname{Max} S$, if $f^{-1}(\mathfrak{m}) = \mathfrak{p}$.*

Proof. The proof is left to the reader. □

If R is any ring, M is any left R-module, and I is a right ideal in R, the multiplication map $\mu : I \otimes_R M \to M$ is defined by $r \otimes x \mapsto rx$. The image of μ, which is denoted IM, is a $\mathbb{Z}$-submodule of M. If I is a two-sided ideal, then IM is an R-submodule of M. The following finiteness criterion for flatness will be applied in Chapter 9.

Corollary 2.4.9. *Let R be any ring and M a left R-module. The following are equivalent.*

(1) *M is a flat R-module.*

(2) *For every right ideal I of R, the sequence*

$$0 \to I \otimes_R M \xrightarrow{\mu} M \to M/IM \to 0$$

is an exact sequence of $\mathbb{Z}$-modules.

(3) *For every finitely generated right ideal I of R, the sequence*

$$0 \to I \otimes_R M \xrightarrow{\mu} M \to M/IM \to 0$$

is an exact sequence of $\mathbb{Z}$-modules.

(4) *If there exist elements $a_1, \dots, a_r$ in R and $x_1, \dots, x_r$ in M such that $\sum_i a_i x_i = 0$, then there exist an integer s, elements $\{b_{ij} \in R \mid 1 \le i \le r, 1 \le j \le s\}$ in R, and $y_1, \dots, y_s$ in M satisfying $\sum_i a_i b_{ij} = 0$ for all j and $x_i = \sum_j b_{ij} y_j$ for all i.*

The following proposition specifies sufficient conditions for changing the base ring of the Hom group. For a version of Proposition 2.4.10 that applies when the module M is finitely generated projective and S is noncommutative, see Corollary 1.3.27.

Proposition 2.4.10. *Let S be a flat commutative R-algebra. Let M be a finitely presented R-module and N any R-module. The natural map*

$$S \otimes_R \operatorname{Hom}_R(M, N) \xrightarrow{\alpha} \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$$

is an isomorphism of S-modules.

Proof. If M is free, then this follows from Corollary 1.3.27. There is some n and an exact sequence of R-modules

$$R^{(n)} \to R^{(n)} \to M \to 0. \tag{2.8}$$

The functor $S \otimes_R (\cdot)$ is right exact, so

$$S^{(n)} \to S^{(n)} \to S \otimes_R M \to 0 \tag{2.9}$$

is an exact sequence of S-modules. By Proposition 1.3.21, the contravariant functor $\operatorname{Hom}_R(\cdot, N)$ is left exact. Applying it to (2.8), we get the exact sequence

$$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(R^{(n)}, N) \to \operatorname{Hom}_R(R^{(n)}, N). \tag{2.10}$$

By Proposition 1.3.23 (1), and Proposition 1.3.23, (2.10) can be written as

$$0 \to \operatorname{Hom}_R(M, N) \to N^{(n)} \to N^{(n)}. \tag{2.11}$$

Tensoring (2.11) with the flat module S gives the exact sequence

$$0 \to S \otimes_R \operatorname{Hom}_R(M, N) \to (S \otimes_R N)^{(n)} \to (S \otimes_R N)^{(n)}. \tag{2.12}$$

Now apply the left exact functor $\operatorname{Hom}_S(\cdot, S \otimes_R N)$ to (2.9) and simplify using Proposition 1.3.23. This gives the exact sequence

$$0 \to \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N) \to (S \otimes_R N)^{(n)} \to (S \otimes_R N)^{(n)}. \tag{2.13}$$

Combining (2.12) and (2.13) with the natural maps yields a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & S \otimes_R \operatorname{Hom}_R(M, N) & \xrightarrow{f_1} & (S \otimes_R N)^{(n)} & \xrightarrow{f_2} & (S \otimes_R N)^{(n)} \\
& \downarrow \alpha & & \downarrow \beta \; = & & \downarrow \gamma \; = \\
0 \longrightarrow & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N) & \xrightarrow{g_1} & (S \otimes_R N)^{(n)} & \xrightarrow{g_2} & (S \otimes_R N)^{(n)}
\end{array}$$

with exact rows. Since f_1 and β are one-to-one, α is one-to-one. To see that α is onto, let x be an element of the lower left corner. Set $y = \beta^{-1}(g_1(x))$. Then $\gamma(f_2(y)) = g_2(\beta(y)) = g_2(g_1(x)) = 0$. So $y = f_1(z)$ for some z in the upper left corner. Then $x = \alpha(z)$. Note that this also follows from a slight variation of the Snake Lemma 1.4.1. □

Proposition 2.4.11. *Let S be a flat commutative R-algebra and A an R-algebra. Let M be a finitely presented A-module and N any A-module. The natural map*

$$S \otimes_R \operatorname{Hom}_A(M, N) \xrightarrow{\alpha} \operatorname{Hom}_{S \otimes_R A}(S \otimes_R M, S \otimes_R N)$$

is an isomorphism of S-modules.

Proof. The proof is left to the reader. □

4.1. Exercises.

Exercise 2.4.12. Let R be a commutative ring, let M and N be R-modules, and $f \in \operatorname{Hom}_R(M, N)$. For any prime ideal $P \in \operatorname{Spec} R$ there is the R_P-module homomorphism $f_P : M_P \to N_P$ obtained by "localizing at P". Prove:

(1) f is one-to-one if and only if f_P is one-to-one for all $P \in \operatorname{Max} R$.
(2) f is onto if and only if f_P is onto for all $P \in \operatorname{Max} R$.

Exercise 2.4.13. Let R be a commutative ring. Let M and N be finitely generated and projective R-modules of constant rank such that $\operatorname{Rank}_R(M) = \operatorname{Rank}_R(N)$. Let $f \in \operatorname{Hom}_R(M, N)$. Show that if f is onto, then f is one-to-one.

Exercise 2.4.14. (Faithfully Flat Is Preserved by a Change of Base) If A is a commutative R-algebra and M is a faithfully flat R-module, show that $A \otimes_R M$ is a faithfully flat A-module.

Exercise 2.4.15. Let R be a ring and $\{M_i \mid i \in I\}$ a set of right R-modules. Show that the direct sum $\bigoplus_{i \in I} M_i$ is a flat R-module if and only if each M_i is a flat R-module.

Exercise 2.4.16. (Flat over Flat is Flat) Let $\theta : R \to A$ be a homomorphism of rings and M a left A-module. Using θ, view A as a left R-right A-bimodule and M as a left R-module. Show that if A is a flat R-module, and M is a flat A-module, then M is a flat R-module.

Exercise 2.4.17. (Faithfully Flat over Faithfully Flat is Faithfully Flat) Prove that if A is a commutative faithfully flat R-algebra and M is a faithfully flat A-module, then M is a faithfully flat R-module.

Exercise 2.4.18. Let R be a ring, $M \in {}_R\mathfrak{M}_R$ and $N \in {}_R\mathfrak{M}$.

(1) If M and N are flat left R-modules, show that $M \otimes_R N$ is a flat left R-module.
(2) Assume R is commutative. Show that if M and N are faithfully flat R-modules, then $M \otimes_R N$ is a faithfully flat R-module.

Exercise 2.4.19. Let $\theta : R \to S$ be a local homomorphism of local rings. Show that if S is a flat R-algebra, then S is faithfully flat.

Exercise 2.4.20. Let R be a commutative ring and $\{\alpha_i \mid i \in I\}$ a subset of $R - (0)$. Let $S = \prod_{i \in I} R[\alpha_i^{-1}]$. Then S is an R-algebra, where the structure homomorphism is the unique map $R \to S$ which commutes with each natural map $R \to R[\alpha_i^{-1}]$. Show that the following are equivalent.

(1) S is a faithfully flat R-module.
(2) There exists a finite subset $\{i_1, \dots, i_n\} \subseteq I$ such that the finite direct sum $R[\alpha_{i_1}^{-1}] \oplus \cdots \oplus R[\alpha_{i_n}^{-1}]$ is faithfully flat over R.
(3) There exists a finite subset $\{i_1, \dots, i_n\} \subseteq I$ such that $R = R\alpha_{i_1} + \cdots + R\alpha_{i_n}$.

Exercise 2.4.21. Let R be a commutative ring and I an ideal of R which is contained in the nil radical of R. Show that R/I is a flat R-algebra if and only if $I = (0)$.

5. Chain Conditions

We assemble in this section some background results on rings, modules, and topological spaces satisfying either an ascending chain condition or a descending chain condition. References are provided for those proofs that are omitted.

Let S be a set which is partially ordered by $\leq$. We say that S satisfies the *minimum condition* if every nonempty subset of S contains a minimal element. We say that S satisfies the *maximum condition* if every nonempty subset of S contains a maximal element. We say that S satisfies the *descending chain condition* (DCC) if every chain in S of the form $\{\dots, x_3 \leq x_2 \leq x_1 \leq x_0\}$ is eventually constant. That is, there is a subscript n such that $x_n = x_i$ for all $i \geq n$. We say that S satisfies the *ascending chain condition* (ACC) if every chain in S of the form $\{x_0 \leq x_1 \leq x_2 \leq x_3, \dots\}$ is eventually constant. The reader should verify that S satisfies the descending chain condition (DCC) if and only if S satisfies the minimum condition. Likewise, S satisfies the ascending chain condition (ACC) if and only if S satisfies the maximum condition.

Let R be any ring and M an R-module. We say M is *noetherian* if M satisfies the ACC on submodules (equivalently, M satisfies the maximum condition on submodules). The reader should verify that an R-module M is noetherian if and only if every submodule of M is finitely generated. We say M is *artinian* if M satisfies the DCC on submodules (equivalently, M satisfies the minimum condition on submodules). The ring R is said to be (left) *noetherian* if R is noetherian when viewed as a left R-module. In this case we say R satisfies the ACC on left ideals. The ring R is said to be (left) *artinian* if R is artinian when viewed as a left R-module. In this case we say R satisfies the DCC on left ideals.

Lemma 2.5.1. *Let R be any ring and*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

a short exact sequence of R-modules.

(1) *The following are equivalent.*
   (a) *B satisfies the ACC on submodules.*
   (b) *A and C satisfy the ACC on submodules.*
(2) *The following are equivalent.*
   (a) *B satisfies the DCC on submodules.*
   (b) *A and C satisfy the DCC on submodules.*

Proof. See, for example, [**DF04**] or [**Hun80**]. □

Corollary 2.5.2. *If R is a noetherian ring and M is a finitely generated R-module, then*

(1) *M satisfies the ACC on submodules,*
(2) *M is finitely presented,*

(3) *M satisfies the maximum condition on submodules, and*

(4) *every submodule of M is finitely generated.*

Proof. See, for example, [**DF04**] or [**Hun80**]. □

A topological space X is said to be *noetherian* if X satisfies the ascending chain condition on open sets. Notice that X is noetherian if and only if X satisfies the descending chain condition on closed sets.

Proposition 2.5.3. *Let X be a noetherian topological space and Z a nonempty closed subset of X.*

(1) *There are unique irreducible closed subsets $Z_1, \dots, Z_r$ such that $Z = Z_1 \cup \cdots \cup Z_r$ and $Z_i \not\subseteq Z_j$ for all $i \neq j$. The sets Z_i are called the* irreducible components *of Z.*

(2) *There are unique connected closed subsets $Y_1, \dots, Y_c$ such that $Z = Y_1 \cup \cdots \cup Y_c$ and $Y_i \cap Y_j = \emptyset$ for all $i \neq j$. The sets Y_i are called the* connected components *of Z.*

(3) *The number of connected components is less than or equal to the number of irreducible components.*

Proof. (1): See [**Har77**, Proposition I.1.5], or [**DF04**, Proposition 15.2.17]. The rest is left to the reader. □

Proposition 2.5.4. *Let R be a commutative noetherian ring.*

(1) $\operatorname{Spec} R$ *is a noetherian topological space.*

(2) $\operatorname{Spec} R$ *has a finite number of irreducible components.*

(3) $\operatorname{Spec} R$ *has a finite number of connected components.*

Proof. This follows from Corollary 2.2.5 and Proposition 2.5.3. □

Corollary 2.5.5. *Let R be a commutative noetherian ring and I an ideal of R which is not the unit ideal. There is a one-to-one correspondence between the irreducible components of $V(I)$ and the minimal prime over-ideals of I given by $Z \mapsto I(Z)$.*

Proof. Let $V(I) = Z_1 \cup \cdots \cup Z_r$ be the decomposition into irreducible components, which exists by Propositions 2.5.4 and 2.5.3. For each i, let $P_i = I(Z_i)$. By Lemma 2.2.6, each of the ideals $P_1, \dots, P_r$ is prime. First we show that each P_i is minimal. Assume $I \subseteq Q \subseteq P_i$, for some prime Q. Then $V(I) \supseteq V(Q) \supseteq Z_i$. By Lemma 2.2.6, $V(Q)$ is irreducible. By the uniqueness part of Proposition 2.5.3, $V(Q) = Z_i$. Therefore, $Q = I(V(Q)) = P_i$. Now let P be a minimal prime over-ideal of I. We show that P is equal to one of $P_1, \dots, P_r$. By Lemma 2.2.6, $V(P)$ is an irreducible subset of $V(I)$.

Since $V(P) \subseteq Z_1 \cup \cdots \cup Z_r$, $V(P) \subseteq Z_i$, for some i. Therefore, $I \subseteq P_i \subseteq P$. Since P is minimal, $P = P_i$. □

Theorem 2.5.6. *Let R be a commutative noetherian ring. Then there exist primitive idempotents $e_1, \dots, e_n$ in R such that R is the internal direct sum $R = Re_1 \oplus \cdots \oplus Re_n$. This decomposition is unique in the sense that, if $R = Rf_1 \oplus \cdots \oplus Rf_p$ is another such decomposition of R, then $n = p$, and after rearranging, $e_1 = f_1, \dots, e_n = f_n$.*

Proof. Let $\operatorname{Spec} R = X_1 \cup \cdots \cup X_n$ be the decomposition into connected components, which exists by Propositions 2.5.4 and 2.5.3. By Corollary 2.2.10 there are idempotents $e_1, \dots, e_n$ in R such that $X_i = U(e_i) = V(1 - e_i)$ is homeomorphic to $\operatorname{Spec} Re_i$, and $R = Re_1 \oplus \cdots \oplus Re_n$. Corollary 2.2.11 implies each e_i is a primitive idempotent. The uniqueness claim comes from Theorem 1.1.21. □

Before ending this section, we include for reference some definitions that will be needed for later chapters, as well as in some of the exercises below. Let R be any ring and M an R-module. We say M is *simple* if $M \neq (0)$ and (0) is a maximal submodule of M. So if M is a simple module, then (0) and M are the only submodules. Let R be any ring and M an R-module. Suppose there is a strictly descending finite chain of submodules

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n = 0$$

starting with $M = M_0$ and ending with $M_n = 0$. The *length* of the chain is n. A *composition series* for M is a chain such that M_i/M_{i+1} is simple. If M has no composition series, then we say the length of M is infinite and write $\ell(M) = \infty$. Otherwise, the length of M, denoted $\ell(M)$, is defined to be the minimum of the lengths of all composition series of M. The following proposition will be required in Chapter 3. Its proof can be found, for example, in [**AM69**].

Proposition 2.5.7. *Let R be any ring and M an R-module. The following are equivalent.*

(1) *M has a composition series.*
(2) *M satisfies both the ACC and the DCC on submodules.*

5.1. Exercises.

Exercise 2.5.8. Let $R_1, \dots, R_n$ be rings. Prove that the direct sum $R_1 \oplus \cdots \oplus R_n$ is an artinian ring if and only if each R_i is an artinian ring.

Exercise 2.5.9. Prove the following, if R is an artinian ring.

(1) If M is a finitely generated R-module, then M satisfies the DCC on submodules.

(2) If I is a two-sided ideal in R, then R/I is artinian.

(3) If R is commutative and W is a multiplicative set in R, then $W^{-1}R$ is artinian.

(4) If R is an artinian domain, then R is a division ring.

Exercise 2.5.10. Let $\theta : R \to S$ be a homomorphism of commutative rings such that S is a faithfully flat R algebra. Show that if S is artinian, then R is artinian. Show that if S is noetherian, then R is noetherian.

Exercise 2.5.11. Let R be a commutative ring and M and N two R-modules. Prove:

(1) If R is noetherian, and M and N are finitely generated, then $\operatorname{Hom}_R(M, N)$ is a finitely generated R-module.

(2) If R is an integral domain, and N is torsion free, then $\operatorname{Hom}_R(M, N)$ is torsion free.

Exercise 2.5.12. Let D be a division ring and V a finite dimensional vector space over D. Prove:

(1) V is a simple module if and only if $\dim_D(V) = 1$.

(2) $\dim_D(V) = \ell(V)$.

Exercise 2.5.13. Let D be a division ring and V a vector space over D. Prove that the following are equivalent.

(1) V is finite dimensional over D.

(2) V is a D-module of finite length.

(3) V satisfies the ACC on submodules.

(4) V satisfies the DCC on submodules.

Exercise 2.5.14. Let D be a division ring. Prove:

(1) The ring $M_n(D)$ of all n-by-n matrices over D is both artinian and noetherian.

(2) If M is a finite dimensional D-vector space, then the ring of endomorphisms $\operatorname{Hom}_D(M, M)$ is both artinian and noetherian.

Exercise 2.5.15. Let k be a field and R a k-algebra which is finite dimensional as a k-vector space. Prove that the ring R is both artinian and noetherian.

Exercise 2.5.16. By the Hilbert Basis Theorem, a finitely generated $\mathbb{Z}$-algebra is a noetherian ring (see, for example, [**Hun80**, Theorem VIII.4.9]). Let R be a commutative ring. Viewing R as a $\mathbb{Z}$-algebra, show that $R = \varinjlim R_\alpha$, where $\{R_\alpha\}$ is a directed system of noetherian subrings of R.

Exercise 2.5.17. Let R be a commutative local ring with maximal ideal $\mathfrak{m}$. Show that there is a directed system $\{R_\alpha\}$ of noetherian local subrings of R satisfying the following:

(1) The maximal ideal of R_α is $\mathfrak{m}_\alpha = \mathfrak{m} \cap R_\alpha$.

(2) $R = \varinjlim R_\alpha$.

(3) $\mathfrak{m} = \varinjlim \mathfrak{m}_\alpha$.

(4) $R/\mathfrak{m} = \varinjlim (R_\alpha/\mathfrak{m}_\alpha)$.

6. Faithfully Flat Base Change

This section contains background results on modules over commutative rings. We first prove the fundamental theorem on faithfully flat base change. These results are then applied to show that a module is finitely generated projective if and only if it is locally free of finite rank. Lastly, we show that if R is a commutative ring, then the tensor product operation defines an abelian group on the set of all isomorphism classes of R-progenerator modules with rank one. This group is called the Picard group of R.

6.1. Fundamental Theorem on Faithfully Flat Base Change. Let S be a commutative faithfully flat R-algebra. Suppose A is either an R-module, or R-algebra. The faithfully flat base change results proved in this section are all of the type: "A has property P over R if $S \otimes_R A$ has property P over S". These powerful results allow us to prove that A has certain properties by restricting to a larger base ring S, where presumably the structure of $S \otimes_R A$ is easier to understand.

Theorem 2.6.1. *Let S be a commutative faithfully flat R-algebra and M an R-module.*

(1) *M is finitely generated over R if and only if $S \otimes_R M$ is finitely generated over S.*

(2) *M is of finite presentation over R if and only if $S \otimes_R M$ is of finite presentation over S.*

(3) *M is finitely generated projective over R if and only if $S \otimes_R M$ is finitely generated projective over S.*

(4) *M is flat over R if and only if $S \otimes_R M$ is flat over S.*

(5) *M is faithfully flat over R if and only if $S \otimes_R M$ is faithfully flat over S.*

(6) *M is a generator module over R if and only if $S \otimes_R M$ is a generator over S.*

(7) *M is faithful over R if and only if $S \otimes_R M$ is faithful over S.*

Proof. (1): If M is finitely generated, then Theorem 1.3.2 shows $S \otimes_R M$ is finitely generated. Conversely, choose generators $\{t_1, \dots, t_n\}$ for $S \otimes_R M$. After breaking up summations and factoring out elements of S, we can assume each t_i looks like $1 \otimes x_i$ where $x_i \in M$. Consider the sequence

$$R^{(n)} \to M \to 0 \tag{2.14}$$

which is defined by $(r_1, \dots, r_n) \mapsto \sum r_i x_i$. Tensoring (2.14) with S gives the sequence

$$S^{(n)} \to S \otimes_R M \to 0$$

which is exact. Since S is faithfully flat, (2.14) is exact.

(2): Assume M is finitely presented. Suppose $R^{(n)} \to R^{(n)} \to M \to 0$ is exact. Tensoring is right exact, so $S^{(n)} \to S^{(n)} \to S \otimes_R M \to 0$ is exact. Therefore $S \otimes_R M$ is finitely presented. Conversely assume $S \otimes_R M$ is finitely presented. By Part (1), M is finitely generated over R. Suppose $\phi : R^{(n)} \to M$ is onto. Let $N = \ker \phi$. It is enough to show that N is finitely generated. Since

$$0 \to N \to R^{(n)} \xrightarrow{\phi} M \to 0$$

is exact and S is faithfully flat,

$$0 \to S \otimes_R N \to S^{(n)} \xrightarrow{1 \otimes \phi} S \otimes_R M \to 0$$

is exact. By Lemma 2.1.7 (3), $S \otimes_R N$ is finitely generated over S. Part (1) says that N is finitely generated over R.

(3): If M is finitely generated and projective over R, then Theorem 1.3.2 says the same holds for $S \otimes_R M$ over S. Conversely, suppose $S \otimes_R M$ is finitely generated and projective over S. By Corollary 1.1.7, $S \otimes_R M$ is of finite presentation over S. By Part (2), M is of finite presentation over R. To show that M is R-projective, by Proposition 1.3.20 (2) it is enough to show $\operatorname{Hom}_R(M, \cdot)$ is a right exact functor. Start with an exact sequence

$$A \xrightarrow{\alpha} B \to 0 \tag{2.15}$$

of R-modules. It is enough to show

$$\operatorname{Hom}_R(M, A) \xrightarrow{\mathrm{H}_\alpha} \operatorname{Hom}_R(M, B) \to 0 \tag{2.16}$$

is exact. Since S is faithfully flat over R, it is enough to show

$$S \otimes_R \operatorname{Hom}_R(M, A) \xrightarrow{1 \otimes \mathrm{H}_\alpha} S \otimes_R \operatorname{Hom}_R(M, B) \to 0 \tag{2.17}$$

is exact. Tensoring is right exact, so tensoring (2.15) with $S \otimes_R (\cdot)$ gives the exact sequence

$$S \otimes_R A \xrightarrow{1 \otimes \alpha} S \otimes_R B \to 0. \tag{2.18}$$

Since we are assuming $S \otimes_R M$ is S-projective, by Proposition 1.3.20 (2) we can apply the functor $\operatorname{Hom}_S(S \otimes_R M, \cdot)$ to (2.18) yielding

$$\operatorname{Hom}_S(S \otimes_R M, S \otimes_R A) \xrightarrow{\mathrm{H}_{1\otimes\alpha}} \operatorname{Hom}_S(S \otimes_R M, S \otimes_R B) \to 0 \tag{2.19}$$

which is exact. Combine (2.17) and (2.19) to get the commutative diagram

$$\begin{array}{ccccc}
S \otimes_R \operatorname{Hom}_R(M, A) & \xrightarrow{1\otimes \mathrm{H}_\alpha} & S \otimes_R \operatorname{Hom}_R(M, B) & & \\
\downarrow \cong & & \downarrow \cong & & \\
\operatorname{Hom}_S(S \otimes_R M, S \otimes_R A) & \xrightarrow{\mathrm{H}_{1\otimes\alpha}} & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R B) & \longrightarrow & 0
\end{array}$$

where the vertical maps are the natural maps from Proposition 2.4.10. Since the bottom row is exact and the vertical maps are isomorphisms, it follows that $1 \otimes \mathrm{H}_\alpha$ is onto.

(4): Assume $M \otimes_R S$ is a flat S-module. By Exercise 2.4.16, $M \otimes_R S$ is flat over R. Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of R-modules. Then

$$0 \to A \otimes_R M \otimes_R S \to B \otimes_R M \otimes_R S \to C \otimes_R M \otimes_R S \to 0$$

is an exact sequence of R-modules. Since S is faithfully flat over R,

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is an exact sequence of R-modules.

Parts (5), (6) and (7) are left to the reader. □

Definition 2.6.2. Let R be a commutative ring and A an R-algebra. If the structure homomorphism $R \to Z(A)$ from R to the center of A is an isomorphism, then we say A is a *central* R-algebra.

Proposition 2.6.3. *Let R be a commutative ring and A an R-algebra.*

(1) *If S is a commutative faithfully flat R-algebra and $A \otimes_R S$ is a central S-algebra, then A is a central R-algebra.*

(2) *If $A_\mathfrak{m} = A \otimes_R R_\mathfrak{m}$ is a central $R_\mathfrak{m}$-algebra for every maximal ideal $\mathfrak{m}$ of R, then A is a central R-algebra.*

Proof. (1): Assume $A \otimes_R S$ is a central S-algebra. Since S is flat over R, $Z(A) \otimes_R S \to A \otimes_R S$ is one-to-one. By hypothesis, the composite map

$$R \otimes_R S \to Z(A) \otimes_R S \to Z(A \otimes_R S)$$

is an isomorphism. Since S is faithfully flat over R, $R \to Z(A)$ is an isomorphism.

(2): Let $\mathfrak{m}$ be a maximal ideal of R. Since $R_{\mathfrak{m}}$ is a flat R-module, $Z(A) \otimes_R R_{\mathfrak{m}} \to A_{\mathfrak{m}}$ is one-to-one. Clearly, $Z(A) \otimes_R R_{\mathfrak{m}} \subseteq Z(A_{\mathfrak{m}})$. We are given that the composite map

$$R_{\mathfrak{m}} \to Z(A) \otimes_R R_{\mathfrak{m}} \subseteq Z(A_{\mathfrak{m}})$$

is an isomorphism. Therefore, $R_{\mathfrak{m}} \to Z(A) \otimes_R R_{\mathfrak{m}}$ is an isomorphism. By Exercise 2.4.12, $R \to Z(A)$ is an isomorphism. □

6.2. Locally Free Finite Rank is Finitely Generated Projective. Let R be a commutative ring and M an R-module. Then M is *locally free of finite rank* if there exist elements $f_1, \dots, f_n$ in R such that $R = Rf_1 + \cdots + Rf_n$ and for each i, $M_{f_i} = M \otimes_R R_{f_i}$ is free of finite rank over R_{f_i}.

Proposition 2.6.4. *Let R be a commutative ring and M an R-module. The following are equivalent.*

(1) *M is finitely generated projective.*
(2) *M is locally free of finite rank.*
(3) *M is an R-module of finite presentation and for each $\mathfrak{p} \in \operatorname{Spec} R$, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module.*
(4) *M is an R-module of finite presentation and for each $\mathfrak{m} \in \operatorname{Max} R$, $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module.*

Proof. It follows from Corollary 1.1.7 and Proposition 2.3.2 that (1) implies (3). It is trivial that (3) implies (4).

(4) implies (2): Using Lemma 2.1.6, for each $\mathfrak{m} \in \operatorname{Max} R$, choose $\alpha_{\mathfrak{m}} \in R - \mathfrak{m}$ such that $M_{\alpha_{\mathfrak{m}}} = M \otimes_R R_{\alpha_{\mathfrak{m}}}$ is free of finite rank over $R_{\alpha_{\mathfrak{m}}}$. Let $U(\alpha_{\mathfrak{m}}) = \operatorname{Spec} R - V(\alpha_{\mathfrak{m}})$ be the basic open set associated to $\alpha_{\mathfrak{m}}$. Since $U(\alpha_{\mathfrak{m}})$ is an open neighborhood of $\mathfrak{m}$, we have an open cover $\{U(\alpha_{\mathfrak{m}}) \mid \mathfrak{m} \in \operatorname{Max} R\}$ of $\operatorname{Spec} R$ (Corollary 2.2.5). By Exercise 2.2.16, there is a finite subset of $\{\alpha_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max} R\}$, say $\{\alpha_1, \dots, \alpha_n\}$, such that $\{U(\alpha_1), \dots, U(\alpha_n)\}$ is an open cover of $\operatorname{Spec} R$. For each i, M_{α_i} is free of finite rank over R_{α_i} which proves M is locally free of finite rank.

(2) implies (1): Assume $\{U(f_1), \dots, U(f_n)\}$ is an open cover of $\operatorname{Spec} R$ and that for each i, M_{f_i} is free of rank N_i over R_{f_i}. Define N to be the maximum of $\{N_1, \dots, N_n\}$. Then

$$F_i = M_{f_i} \oplus R_{f_i}^{(N-N_i)}$$

is free of rank N over R_{f_i}. Set $S = \bigoplus_i R_{f_i}$. Then $R \to S$ is faithfully flat (Exercise 2.6.15). Set $F = \bigoplus_i F_i$. Then F is free over S of rank N and $M \otimes_R S = \bigoplus_i M_{f_i}$ is a direct summand of F (Exercise 2.6.13). Now apply Theorem 2.6.1 (3). □

Let R be a commutative ring. For any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$, write $k_{\mathfrak{p}}$ for the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. If M is a finitely generated R-module, then M can be used to define a rank function $\operatorname{Spec} R \to \{0, 1, 2, \dots\}$, where $\mathfrak{p} \mapsto \dim_{k_{\mathfrak{p}}}(M \otimes_R k_{\mathfrak{p}})$. The next two corollaries to Proposition 2.6.4 utilize this rank function to give us a powerful test for locally free modules and for flatness over an integral domain.

Corollary 2.6.5. *Let R be an integral domain with quotient field K. For each maximal ideal $\mathfrak{m} \in \operatorname{Max}(R)$, write $k_{\mathfrak{m}}$ for $R/\mathfrak{m}$. The following are equivalent for any finitely generated R-module M.*

(1) *M is a locally free R-module of constant rank n.*

(2) $\dim_K(M \otimes_R K) = n$ *and for every* $\mathfrak{m} \in \operatorname{Max}(R)$, $\dim_{k_{\mathfrak{m}}}(M/\mathfrak{m}M) = n$.

Proof. (1) implies (2): If $M \cong R^{(n)}$, then $M \otimes_R k_{\mathfrak{m}} \cong k_{\mathfrak{m}}^{(n)}$ and $M \otimes_R K \cong K^{(n)}$.

(2) implies (1): Let $\mathfrak{m}$ be a maximal ideal of R and write $M_{\mathfrak{m}}$ for $M \otimes_R R_{\mathfrak{m}}$. Since $M/\mathfrak{m}M$ is free of dimension n over $k_{\mathfrak{m}}$, there exist $x_1, \dots, x_n$ in $M_{\mathfrak{m}}$ which restrict to a $k_{\mathfrak{m}}$-basis under the natural map $M_{\mathfrak{m}} \to M/\mathfrak{m}M$. For some $\alpha \in R - \mathfrak{m}$, the finite set $x_1, \dots, x_n$ is in the image of the natural map $M_\alpha \to M_{\mathfrak{m}}$. Define $\theta : R_\alpha^{(n)} \to M_\alpha$ by mapping the standard basis vector e_i to x_i. By Lemma 2.3.1, $M_{\mathfrak{m}}$ is generated by $x_1, \dots, x_n$ as an $R_{\mathfrak{m}}$-module. Therefore, upon localizing θ at the maximal ideal $\mathfrak{m}R_\alpha$, it becomes onto. Because the cokernel of θ is a finitely generated R_α-module, by Lemma 2.1.4, there exists $\beta \in R_\alpha - \mathfrak{m}R_\alpha$ such that if we replace α with $\alpha\beta$, then θ is onto. The diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\theta & \longrightarrow & R_\alpha^{(n)} & \xrightarrow{\theta} & M_\alpha & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
0 & \longrightarrow & \ker\theta \otimes_R K & \longrightarrow & K^{(n)} & \xrightarrow{\theta\otimes 1} & M \otimes_R K & \longrightarrow & 0
\end{array}
$$

commutes, where the second row is obtained by tensoring the top row with $(\ ) \otimes_R K$. Since the top row is exact, by Lemma 2.1.1 so is the second row. Since R is an integral domain, $R \to K$ is one-to-one. Therefore β is one-to-one. Since $M \otimes K$ has dimension n and $\theta \otimes 1$ is onto, it follows that $\ker\theta \otimes_R K = 0$. The Snake Lemma implies that $\ker\theta = 0$. We have shown that every maximal ideal $\mathfrak{m} \in \operatorname{Max}(R)$ has a basic open neighborhood $U(\alpha)$ such that M_α is a free R_α-module of rank n. The argument that was used to show (4) implies (2) in Proposition 2.6.4 can now be applied to finish the proof. □

Corollary 2.6.6. *Let R be an integral domain with quotient field K and M a finitely generated R-module. Then the following are equivalent.*

(1) *M is of finite presentation and flat.*

(2) *M is an R-progenerator.*

(3) *There exists $n > 0$ such that $\dim_K(M \otimes_R K) = n$ and for every maximal ideal $\mathfrak{m}$ in $\operatorname{Max}(R)$, $\dim_{k_\mathfrak{m}}(M/\mathfrak{m}M) = n$.*

Proof. By Corollaries 1.3.5 and 1.1.16, (1) and (2) are equivalent. Proposition 2.6.4, Corollary 2.6.5, and Corollary 1.1.16 imply that (2) and (3) are equivalent. □

6.3. Invertible Modules and the Picard Group. In this section we show that to any commutative ring R is associated an abelian group $\operatorname{Pic}(R)$, which is called the Picard group of R. The Picard group of R parametrizes up to isomorphism the R-progenerator modules of constant rank one. The assignment $R \mapsto \operatorname{Pic}(R)$ defines a covariant functor from the category of commutative rings to the category of abelian groups. In Section 6.4 we will see that the Picard group of a noetherian integrally closed integral domain can be identified with a subgroup of the divisor class group.

Lemma 2.6.7. *Let M be a finitely generated projective faithful module over the commutative ring R. Then the following are equivalent:*

(1) $\operatorname{Rank}_R(M) = 1$.

(2) $\operatorname{Rank}_R(M^*) = 1$.

(3) $\operatorname{Hom}_R(M, M) \cong R$.

(4) $M^* \otimes_R M \cong R$.

(5) *For some R-module N, there is an isomorphism $M \otimes_R N \cong R$.*

Proof. The hypotheses on M imply that M is an R-progenerator module.

(1) is equivalent to (2): By Corollary 1.3.27, for each prime ideal $P \in \operatorname{Spec} R$, $M^* \otimes_R R_P = \operatorname{Hom}_R(M, R) \otimes_R R_P \cong \operatorname{Hom}_{R_P}(M_P, R_P)$. Since M is finitely generated projective, M_P is free of finite rank by Proposition 2.3.2. Assume $M_P \cong R_P^{(n)}$. Then $(M_P)^* = \operatorname{Hom}_{R_P}(M_P, R_P) \cong \operatorname{Hom}_{R_P}(R_P^{(n)}, R_P) \cong \operatorname{Hom}_{R_P}(R_P, R_P)^{(n)} \cong R_P^{(n)}$. Therefore $\operatorname{Rank}_R(M) = 1$ if and only if $\operatorname{Rank}_R(M^*) = 1$.

(1) implies (3): Let $\phi : R \to \operatorname{Hom}_R(M, M)$ be the left regular representation of R in $\operatorname{Hom}_R(M, M)$. Then $\phi(r)(x) = rx$. For each prime ideal $P \in \operatorname{Spec} R$, $M_P \cong R_P$. Therefore localizing ϕ at P yields

$$R_P \xrightarrow{\phi_P} \operatorname{Hom}_{R_P}(M_P, M_P) \cong \operatorname{Hom}_{R_P}(R_P, R_P) \cong R_P$$

which is an isomorphism. By Exercise 2.4.12, ϕ is an isomorphism.

(3) is equivalent to (4): It follows from Lemma 1.5.1 (1) that $M^* \otimes_R M$ is isomorphic to $\operatorname{Hom}_R(M, M)$.

(4) implies (5): If we take N to be M^*, then this is immediate.

(5) implies (1): Fix a prime $P \in \operatorname{Spec} R$. Then $M_P \cong R_P^{(n)}$ for some n. Now $R_P \cong R_P \otimes_R (M \otimes_R N) \cong M_P \otimes_R N \cong R_P^{(n)} \otimes_R N \cong N_P^{(n)}$. Therefore, N_P is finitely generated projective over the local ring R_P, hence is free of finite rank over R_P. Then $n = 1$. Since P was arbitrary, we are done. □

Definition 2.6.8. If M is an R-module that satisfies any of the equivalent properties of Lemma 2.6.7, then we say M is *invertible*. Given a commutative ring R let $\operatorname{Pic}(R)$ be the set of isomorphism classes of invertible R-modules. The isomorphism class containing a module M is denoted by $|M|$. As stated in Proposition 2.6.9 below, $\operatorname{Pic}(R)$ is an abelian group, which is called the *Picard group* of R. A generalization of the Picard group for a noncommutative R-algebra A is defined in Section 7.8.1. When R is an integral domain with field of fractions K, a description of $\operatorname{Pic}(R)$ in terms of invertible fractional ideals of R in K is made in Exercise 6.4.15.

Proposition 2.6.9. *Under the binary operation* $|P| \cdot |Q| = |P \otimes_R Q|$, $\operatorname{Pic}(R)$ *is an abelian group. The identity element is the class* $|R|$. *The inverse of* $|M| \in \operatorname{Pic}(R)$ *is* $|M^*|$. *The assignment* $R \mapsto \operatorname{Pic}(R)$ *defines a covariant functor from the category of commutative rings to the category of abelian groups.*

Proof. The proof is left to the reader. □

Example 2.6.10. Let k be any field. Let x and y be indeterminates. Let f be the polynomial $f = y^2 - x(x^2 - 1)$. Let R be the factor ring

$$R = \frac{k[x, y]}{(y^2 - x(x^2 - 1))}.$$

Then R is an integral domain. Let M be the maximal ideal of R generated by x and y. If we invert x^2-1, then $x = y^2(x^2-1)^{-1}$, so M becomes principal. If we invert x, then M becomes the unit ideal, and is principal. Since $R(x^2-1)$ and $R(x)$ are comaximal, there is an open cover $U(x^2-1) \cup U(x) = \operatorname{Spec} R$ on which M is locally free of rank 1. Proposition 2.6.4 shows that $|M| \in \operatorname{Pic} R$. Note that M^2 is generated by x^2, xy, y^2. But an ideal that contains x^2 and y^2 also contains x. We see that M^2 is generated by x, hence is free of rank one. The map

$$M \otimes_R M \to M^2$$
$$a \otimes b \mapsto ab$$

is R-linear. Since this map is onto and both sides are projective of rank one, it is an isomorphism. This proves that $M^* \cong M$ and $|M|^{-1} = |M|$.

Example 2.6.11. If R is a commutative ring with the property that every progenerator module is free, then $\operatorname{Pic}(R)$ contains just one element, namely $|R|$. Using the notation of abelian groups, we usually write $\operatorname{Pic}(R) = (0)$ in this case, despite the obvious inconsistency. For example, $\operatorname{Pic}(R) = (0)$ in each of the following cases.

(1) R is a field.

(2) R is a local ring (Proposition 2.3.2).

(3) R is a principal ideal domain.

We saw in Example 2.6.11 that the Picard group of a principal ideal domain is trivial. More generally, we will see in Chapter 6 that if R is a Dedekind domain with field of fractions K (see Section 6.4), then the Picard group $\operatorname{Pic}(R)$ is isomorphic to the quotient $\operatorname{Frac}(R)/\operatorname{Prin}(R)$, where $\operatorname{Frac}(R)$ denotes the group of all fractional ideals of R in K, and $\operatorname{Prin}(R)$ is the group of all principal R-submodules of K. In [**Cla66**], L. Claborn proved that given any abelian group G, there is a Dedekind domain R such that $\operatorname{Pic}(R) \cong G$ (see also [**Fos73**, Theorem 14.10]). Included throughout the following chapters are more examples of rings for which the Picard group can be completely determined. See Exercise 2.6.18 for one such example.

Before ending this section, we include for reference the definition of an operator that will be needed for later chapters, as well as in Exercise 2.6.17 below. Let R be a ring and M a left R-module. If I and J are submodules of M, then the *module quotient* is

$$I : J = \{r \in R \mid rJ \subseteq I\}.$$

The reader should verify that

$$I : J = \operatorname{annih}_R((I + J)/I) = \operatorname{annih}_R(J/(I \cap J))$$

hence $I : J$ is a two-sided ideal in R. The reader should be advised that the colon notation does not specify the ring R or the module M, hence is ambiguous. For example, if A is a commutative ring and R a subring of A, then the *conductor* from A to R is

$$R : A = \{\alpha \in A \mid \alpha A \subseteq R\}.$$

The reader should verify that $R : A$ is an A-submodule of R, hence it is an ideal of both R and A. The following technical lemma will be required in Chapter 6. Its proof can be found, for example, in [**Mat80**].

Lemma 2.6.12. *Let S be a commutative flat R-algebra. If I and J are ideals in R, then*

(1) $(I \cap J)S = IS \cap JS$.

(2) *If J is finitely generated, then* $(I : J)S = (IS : JS)$.

6.4. Exercises.

Exercise 2.6.13. Let R_1 and R_2 be rings and let $S = R_1 \oplus R_2$ be the direct sum. Let M_1 be an R_1-module and M_2 an R_2-module and let $M = M_1 \oplus M_2$. Prove:

(1) M is an S-module.
(2) If M_i is free of rank N over R_i for each i, then M is free of rank N over S.
(3) If M_i is finitely generated and projective over R_i for each i, then M is finitely generated and projective over S.

Exercise 2.6.14. Let R_1 and R_2 be commutative rings. Show that $\operatorname{Pic}(R_1 \oplus R_2)$ is isomorphic to $\operatorname{Pic}(R_1) \oplus \operatorname{Pic}(R_2)$.

Exercise 2.6.15. Let R be a commutative ring. Assume $f_1, \dots, f_n$ are elements of R that generate the unit ideal. That is, $R = Rf_1 + \cdots + Rf_n$. Let $S = R_{f_1} \oplus \cdots \oplus R_{f_n}$ be the direct sum. Let $\theta : R \to S$ be defined by $\theta(x) = (x/1, \dots, x/1)$. Show that S is a faithfully flat R-module.

Exercise 2.6.16. Let R be a commutative ring and M a finitely generated projective R-module of constant rank n. Show that there exist elements $f_1, \dots, f_m$ of R satisfying the following:

(1) $R = Rf_1 + \cdots + Rf_m$.
(2) If $S = R_{f_1} \oplus \cdots \oplus R_{f_m}$, then $M \otimes_R S$ is a free S-module of rank n.

Exercise 2.6.17. Let k be a field and $A = k[x]$ the polynomial ring over k in one variable. Let $R = k[x^2, x^3]$ be the k-subalgebra of A generated by x^2 and x^3. Show:

(1) R and A have the same quotient field, namely $K = k(x)$.
(2) A is a finitely generated R-module.
(3) The conductor ideal from A to R is $\mathfrak{m} = (x^2, x^3)$ which is a maximal ideal of R.
(4) Use Corollary 2.6.6 to show that A is not flat over R. (Hint: Consider $R/\mathfrak{m}$ and $A/\mathfrak{m}A$.)
(5) The rings $R[x^{-2}]$ and $A[x^{-2}]$ are equal, hence the extension $R \to A$ is flat upon localization to the nonempty basic open set $U(x^2)$.

For more properties of the rings R and A, see Exercises 2.6.18 and 8.1.35.

Exercise 2.6.18. This exercise is a continuation of Exercise 2.6.17. Let k be a field, $A = k[x]$ and $R = k[x^2, x^3]$. Show:

(1) For each $\alpha \in k$, set $P_\alpha = R(1-\alpha x)+\mathfrak{m}$. Then P_α is an R-submodule of A, and P_α is isomorphic to R if and only if $\alpha = 0$.

(2) The R-module homomorphisms $P_\alpha \otimes_R P_\beta \to P_\alpha P_\beta \to P_{\alpha+\beta}$ are isomorphisms. (Hints: $x^4 \in \mathfrak{m}^2$, $x^3 \in P_\alpha \mathfrak{m}$, $x^2 \in P_\alpha \mathfrak{m}$, $1-(\alpha+\beta)x \in P_\alpha P_\beta$.)

(3) $\operatorname{Pic} R$ contains a subgroup isomorphic to the additive group k.

(4) $\operatorname{Pic} R \cong k$. (This is a challenge and may involve tools not yet covered in this text. See Exercise 14.2.19 for a method involving the Mayer-Vietoris sequence.)

(5) A is equal to the integral closure of R in K. (As in (4), this is a challenge. You can attempt to do it now, or come back to it later. See Section 3.3.)

Exercise 2.6.19. Let k be a field, $n > 1$ an integer, $T = k[x,y]$, $S = k[x^n, xy, y^n]$, and $S \to T$ the set containment map. Using Corollary 2.6.6 and Exercise 1.3.16, show that T is not flat over S. See Exercise 4.4.19 for more properties of the extension T/S.