Preface

This book is an introduction to *braid foliation techniques*, which is a theory developed to study knots and links and related surfaces in 3-manifolds, and which from its inception has been intimately related to contact topology. The original idea of braid foliation techniques is due to Daniel Bennequin, who in the early 1980s first used a preliminary version of them to study transverse links and contact structures on $\mathbb{R}^3$, and established the existence of non-contactomorphic contact structures on $\mathbb{R}^3$. In the 1990s Joan Birman and William Menasco developed and systematized the theory of braid foliations, and in a series of papers spanning over ten years they used these techniques in $\mathbb{R}^3$ and $S^3$ to probe the landscape of closed braids representing topological link types, with their work culminating in the Markov Theorem without Stabilization and accompanying applications to the study of link types whose transverse classification is non-trivial. A number of researchers have since applied braid foliation techniques in new ways to solve foundational problems in braid theory and contact topology, most notably in Ivan Dynnikov and Maxim Prasolov’s proof of the Legendrian grid number conjecture and the generalized Jones conjecture. Along the way, Tetsuya Ito and Malyutin-Netsvetaev discovered interesting interplay between braid foliations and Dehornoy’s ordering of braids; furthermore, Ito and Keiko Kawamuro have recently extended the bulk of braid foliation techniques to arbitrary closed 3-manifolds supported by open book decompositions, terming this new generalized theory that of *open book foliations*.

We therefore believe that braid foliation techniques can be a highly useful implement in the toolbox of low-dimensional geometric topologists, and we have endeavored to present an accessible and detailed introduction to the theory in this book, including all of the above applications of braid foliation
techniques. The primary audience that we have in mind is graduate students in geometric topology, but we hope that this work will also prove useful to the more experienced researcher as well. Rather than present all the details of braid foliation techniques at the outset, and overwhelm the reader with meaningless detail, we have constructed the book so that each chapter centers around a key theorem or collection of theorems, and the particular braid foliation techniques needed to prove that theorem are introduced in parallel, so that the reader has an immediate “take-home” for the techniques involved. The book does not need to be read entirely linearly, but we do recommend that the reader new to the subject read Chapters 2 and 3 in detail, as these two chapters form the core of braid foliation techniques. Following these chapters, the reader interested in transverse links in the standard contact structure may turn to Chapters 4, 5, 6 and 7; the reader interested in Legendrian links, including the work of Dynnikov and Prasolov and relations of braid foliations to convex surface theory, can skip ahead to Chapters 8, 9, and 12; those interested in studying braids algebraically can turn to Chapter 10; and the reader interested in Ito and Kawamuro’s theory of open book foliations can proceed to Chapter 11.

An exercise section has been included at the end of each chapter, and we encourage the student to take time to work through these exercises before proceeding to the next chapter. Braid foliation techniques are highly visual, and we have therefore freely included figures throughout the book that will hopefully help the reader gain his or her own insight into the theory. We have also tried to point the reader to other powerful tools which can be used to solve similar problems to those addressed here, most notably, characteristic foliation and convex surface theory techniques of Emmanuel Giroux and Ko Honda, knot Floer homology theories of Peter Ozsváth and Zoltan Szabó, and knot contact homology techniques of Lenny Ng. In fact there are still open questions of how best to understand braid foliation techniques in the context of these other theories, which we encourage the interested reader and researcher to pursue.

One final comment: We emphasize that throughout the book our links will be oriented and smooth, and our ambient isotopies will be smooth. This will allow us to have well-defined notions of transversality, and utilize standard applications of Sard’s theorem involving regular projections and general position. At times, however, we will find it very useful to employ various piecewise-linear approximations of our smooth links to organize combinatorial arguments. At the outset, therefore, we note that these piecewise-linear approximations will always have a smooth link close by, to which we are actually performing smooth isotopies.
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