Chapter 2

Binomial Model

In this chapter we consider a simple discrete (primary) financial market model called the binomial or Cox-Ross-Rubinstein (CRR) [12] model. In the context of this model, we derive the unique arbitrage free prices for a European and an American contingent claim.

For this chapter and the next, we shall need various notions from probability and the theory of discrete time stochastic processes, including conditional expectations, martingales, supermartingales and stopping times. For the convenience of the reader, a summary of some relevant concepts and results is provided in Appendices A and B. For further details, the reader is encouraged to consult Chung [9] or D. Williams [38].

Regarding general notation used throughout this book, we note that if $X$ is a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub-$\sigma$-algebra of $\mathcal{F}$, we shall sometimes use the notation $X \in \mathcal{G}$ to indicate that $X$ is $\mathcal{G}$-measurable. Also, two random variables will be considered equal if they are equal almost surely, and two stochastic processes will be considered equal if they are indistinguishable (see Appendices B and C). In this chapter and the next, we restrict attention to finite sample spaces and consider only probability measures that give positive probability to each individual outcome. Consequently, in these chapters, equality of random variables and indistinguishability of discrete time stochastic processes actually entail equality surely.

2.1. Binomial or CRR Model

The CRR model is a simple discrete time model for a financial market. There are finitely many times $t = 0, 1, 2, \ldots, T$ (where $T < \infty$ is a positive
integer and successive times are successive integers). At each of these times the values of two assets can be observed. One asset is a risky security called a stock, and the other is a riskless security called a bond.

The bond is assumed to yield a constant rate of return \( r \geq 0 \) over each time period \((t - 1, t]\); and so assuming the bond is valued at $1 at time zero, the value of the bond at time \( t \) is given by

\[
B_t = (1 + r)^t, \quad t = 0, 1, \ldots, T. \tag{2.1}
\]

We measure holdings in the bond in units, where the value of one unit at time \( t \) is \( B_t \).

The stock price process is modeled as an exponential random walk such that \( S_0 \) is a strictly positive constant and

\[
S_t = S_{t-1} \xi_t, \quad t = 1, 2, \ldots, T, \tag{2.2}
\]

where \( \{ \xi_t, t = 1, 2, \ldots, T \} \) is a sequence of independent and identically distributed random variables with

\[
P(\xi_t = u) = p = 1 - P(\xi_t = d), \tag{2.3}
\]

where \( p \in (0, 1) \) and \( 0 < d < 1 + r < u \). The last two conditions are assumed to avoid arbitrage opportunities in the primary market model and to ensure stock prices are random and strictly positive. Note that

\[
S_t = S_0 \prod_{i=1}^{t} \xi_i, \quad t = 0, 1, \ldots, T. \tag{2.4}
\]

(Here we assume that an empty product is defined to take the value 1.) One may represent the possible paths that \( S_t \) follows using a binary tree (see Figure 1). Note that there are only three distinct values for \( S_2 \); i.e., the two middle dots in Figure 1 have the same value for \( S_2 \). The points have been drawn as two distinct points to emphasize the fact that they may be reached by different paths, that is, through different values for the sequence \( S_0, S_1, S_2 \). For pricing and hedging of some contingent claims under the binomial model (e.g., a European contingent claim whose payoff depends only on the terminal value of the stock price or an American contingent claim whose payoff at time \( t \) depends only on the stock price at that time), one may use a so-called recombining tree in which only the distinct values of \( S_t \) are indicated (in particular, the two middle dots associated with values of \( S_2 \) in Figure 1 are identified). However, in general one needs the full path structure of the stock price process to price and hedge contingent claims; cf. Exercise 3 at the end of this chapter.

For concreteness, and without loss of generality, we assume that the probability space \((\Omega, F, P)\) on which our random variables are defined is such that \( \Omega \) is the finite set of \( 2^T \) possible outcomes for the values of the
2.1. Binomial or CRR Model

Figure 1. Binary tree for \( T = 2 \)

\begin{align*}
\text{stock price } (T+1)-\text{tuple, } (S_0, S_1, S_2, \ldots, S_T); \ F \text{ is the } \sigma-\text{algebra consisting of all possible subsets of } \Omega; \text{ and } P \text{ is the probability measure on } (\Omega, \mathcal{F}) \text{ associated with the Bernoulli probability } p. \text{ Then, for example,} \\
P((S_0, S_1, \ldots, S_T) = (S_0, S_0u, S_0u^2, \ldots, S_0u^T)) = p_T. \ (2.5)
\end{align*}

Indeed, each of the \(2^T\) possible outcomes in \(\Omega\) has strictly positive probability under \(P\). The expectation operator under \(P\) will be denoted by \(E[\cdot]\).

To describe the information available to the investor at time \(t\), we introduce the \(\sigma\)-algebra generated by the stock prices up to and including time \(t\); i.e., let

\[ \mathcal{F}_t = \sigma\{S_0, S_1, \ldots, S_t\}, \ t = 0, 1, \ldots, T. \quad (2.6) \]

In particular, with our concrete probability space, \(\mathcal{F}_T = \mathcal{F}\). The family \(\{\mathcal{F}_t, t = 0, 1, \ldots, T\}\) is called a filtration. We will often write it simply as \(\{\mathcal{F}_t\}\).

A trading strategy (in the primary market) is a collection of pairs of random variables

\[ \phi = \{ (\alpha_t, \beta_t), t = 1, 2, \ldots, T \} \quad (2.7) \]

where the random variable \(\alpha_t\) represents the number of shares of stock to be held over the time interval \((t-1, t]\) and the random variable \(\beta_t\) represents
the number of units of the bond to be held over the time interval \((t-1, t]\). For simplicity, we allow \(\alpha_t, \beta_t\) to take any values in \(\mathbb{R}\). In particular, we do not restrict to integer numbers of shares of stock or units of the bond, and we allow short selling of shares (\(\alpha_t < 0\)) and borrowing (\(\beta_t < 0\)). We think of trading occurring at time \(t-1\) to determine the portfolio holdings \((\alpha_t, \beta_t)\) until the next trading time \(t\). To avoid strategies that anticipate the future, it is assumed that \(\alpha_t, \beta_t\) are \(\mathcal{F}_{t-1}\)-measurable random variables for \(t = 1, 2, \ldots, T\). (In this discrete model setting, this simply means that \(\alpha_t\) and \(\beta_t\) can be expressed as real-valued functions of \((S_0, S_1, \ldots, S_{t-1})\). In fact, the dependence on \(S_0\) is trivial since \(S_0\) is assumed to be a constant.) Thus, the holdings in stock and bond over the time period \((t-1, t]\) can depend only on the stock prices observed up to and including time \(t-1\).

We will restrict attention here to self-financing trading strategies, namely, those trading strategies \(\phi\) such that
\[
\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t, \quad t = 1, 2, \ldots, T-1, \tag{2.8}
\]
and the investor’s initial wealth is equal to
\[
\alpha_1 S_0 + \beta_1 B_0. \tag{2.9}
\]

We will simply refer to these as trading strategies rather than using the longer term “self-financing trading strategies”. We say that a trading strategy \(\phi\) represents a portfolio whose value at time \(t\) is given by \(V_t(\phi)\), where
\[
V_0(\phi) = \alpha_1 S_0 + \beta_1 B_0, \tag{2.10}
\]
\[
V_t(\phi) = \alpha_t S_t + \beta_t B_t, \quad t = 1, 2, \ldots, T. \tag{2.11}
\]

An arbitrage opportunity (in the primary market) is a trading strategy \(\phi\) such that \(V_0(\phi) = 0, V_T(\phi) \geq 0\) and \(E[V_T(\phi)] > 0\). We note that, in the presence of the preceding conditions, the last condition is equivalent to \(P(V_T(\phi) > 0) > 0\).

### 2.2. Pricing a European Contingent Claim

A European contingent claim (ECC) is represented by an \(\mathcal{F}_T\)-measurable random variable \(X\). This is interpreted to mean that the value of the contingent claim at the expiration time \(T\) is given by \(X\). For example, a European call option with strike price \(K \in (0, \infty)\) and expiration date \(T\) is represented by \(X = (S_T - K)^+ \equiv \max\{0, S_T - K\}\). The meaning of \(X\) in this case is as follows. If \(S_T > K\), the holder of the European call option will exercise it at \(T\) and, after selling a share of stock just purchased for \(K\), will make a profit of \(S_T - K\) (ignoring whatever was paid for the option in the first place). Thus, in this case, the value of the option at \(T\) is \(S_T - K\). On the other hand, if \(S_T \leq K\), the holder of the option will not exercise it at \(T\) and
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its value is 0 at \( T \). Similarly, a European put option with the same strike price and expiration date is represented by \( X = (K - S_T)^+ \).

A replicating (or hedging) strategy for a European contingent claim \( X \) is a trading strategy \( \phi \) such that \( V_T(\phi) = X \). If there exists such a replicating strategy, the contingent claim is said to be attainable (or redundant).

In this section, we derive the (initial) arbitrage free price for a European contingent claim. Existence of such a price depends on the existence of a so-called risk neutral probability, and uniqueness depends on there being a replicating strategy for the contingent claim.

2.2.1. Single Period Case. We begin by examining the single period case where \( T = 1 \). For this case, we first show that there is a replicating strategy for any European contingent claim \( X \).

Given \( X \), we seek a trading strategy \( \phi = (\alpha_1, \beta_1) \), where \( \alpha_1 \) and \( \beta_1 \) are constants such that

\[
V_1(\phi) \equiv \alpha_1 S_1 + \beta_1 B_1 = X. \tag{2.12}
\]

Now \( S_1 \) has two possible values, \( S_0u, S_0d \), and \( X \) is a function of \( S_1 \), since it is \( \mathcal{F}_1 = \sigma\{S_0, S_1\} \)-measurable. Let \( X^u \) denote the value of \( X \) when \( S_1 = S_0u \) and \( X^d \) denote the value of \( X \) when \( S_1 = S_0d \). Then considering these two possible outcomes, (2.12) yields two equations for the two deterministic unknowns \( \alpha_1, \beta_1 \):

\[
\alpha_1 S_0u + \beta_1 (1 + r) = X^u \quad \tag{2.13}
\]
\[
\alpha_1 S_0d + \beta_1 (1 + r) = X^d. \quad \tag{2.14}
\]

Solving for \( \alpha_1, \beta_1 \) yields

\[
\alpha_1 = \frac{X^u - X^d}{(u - d)S_0}, \quad \tag{2.15}
\]
\[
\beta_1 = \frac{1}{1 + r} \left( \frac{u X^d - d X^u}{u - d} \right). \tag{2.16}
\]

The right member of (2.15) is sometimes written informally as \( \frac{dX}{S_0} \). The initial wealth needed to finance this strategy (sometimes called the manufacturing cost of the contingent claim) is

\[
V_0 = \alpha_1 S_0 + \beta_1 B_0
= \frac{1}{(1 + r)(u - d)} \left( (1 + r - d)X^u + (u - (1 + r))X^d \right)
= \frac{1}{1 + r} \left( p^* X^u + (1 - p^*)X^d \right)
= E^{p^*}[X^*], \quad \tag{2.17}
\]
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where \( X^* = X/(1 + r) \), \( p^* = \frac{1+r-d}{u-d} \), and \( E^{p^*}[\cdot] \) denotes the expectation operator with \( p \) replaced by \( p^* \). (For brevity, in this chapter, when probabilities or (conditional) expectations are those obtained by using \( p^* \) in place of \( p \), we shall simply say that the probability or expectation is computed under \( p^* \).)

Note that \( p^* \in (0, 1) \) and so, just as for \( p \), each of the two possible outcomes for \((S_0, S_1)\) has positive probability under \( p^* \). Furthermore, \( E^{p^*}[S_1] = (1 + r)S_0 \), and so the discounted stock price process \( \{S_0, \frac{1}{1+r}S_1\} \) is a martingale under \( p^* \) (relative to the filtration \( \{F_t\} \)). Thus, under \( p^* \), the average rate of return of the risky asset is the same as that of the riskless asset. For this reason \( p^* \) is called the risk neutral probability. (A person is said to be “risk averse” if the person prefers the expected value of a payoff to the random payoff itself. A person is said to be “risk preferring” if the person prefers the random payoff to the expected value of the payoff. A person is “risk neutral” if the person is neither risk averse nor risk preferring; in other words the person is indifferent, having no preference for the expected payoff versus the random payoff. The probability \( p^* \) is called a risk neutral probability because under \( p^* \), a risk neutral investor would be indifferent to the choice at time zero of investing \( S_0 \) in stock or bond, since both investments have the same expected payoff at time 1 under \( p^* \).)

It is important to realize that computing expectations under \( p^* \) is a mathematical device. We are not assuming that the stock price actually moves according to this probability. That is, \( p^* \) may be unrelated to the subjective probability \( p \) that we associate with the binomial model for movements in the stock price.

For the next theorem, we need the notion of an arbitrage opportunity in the market consisting of the stock, bond and contingent claim. For this, we suppose that the price of the contingent claim at time zero is \( C_0 \), a constant. A trading strategy in stock, bond and the contingent claim is a triple \( \psi = (\alpha_1, \beta_1, \gamma_1) \) of \( \mathcal{F}_0 \)-measurable random variables (these will actually be constants), where \( \alpha_1 \) represents the number of shares of stock held over \((0, 1]\), \( \beta_1 \) represents the number of units of the bond held over \((0, 1]\), and \( \gamma_1 \) represents the number of units of the contingent claim held over \((0, 1]\). The initial value of the portfolio associated with \( \psi \) is \( V_0(\psi) = \alpha_1S_0 + \beta_1B_0 + \gamma_1C_0 \).

The value of this portfolio at time one is \( V_1(\psi) = \alpha_1S_1 + \beta_1B_1 + \gamma_1X \). An arbitrage opportunity in the stock-bond-contingent claim market is a trading strategy \( \psi = (\alpha_1, \beta_1, \gamma_1) \) such that \( V_0(\psi) = 0 \), \( V_1(\psi) \geq 0 \) and \( E[V_1(\psi)] > 0 \).

Theorem 2.2.1. \( V_0 = E^{p^*}[X^*] \) is the unique arbitrage free initial price for the European contingent claim \( X \).
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Proof. Let \( \phi^* = (\alpha_1^*, \beta_1^*) \) denote the replicating strategy (in stock and bond) for the contingent claim \( X \); cf. (2.15)–(2.16).

First we show that if the initial price \( C_0 \) of the contingent claim is anything other than \( V_0 \), then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose \( C_0 > V_0 \). Then an investor starting with zero initial wealth could sell one contingent claim \( \gamma_1 = -1 \) for \( C_0 \), invest \( V_0 \) in the replicating strategy \( \phi^* = (\alpha_1^*, \beta_1^*) \) and invest the remainder, \( C_0 - V_0 \), in bond. Thus, the trading strategy in stock, bond and contingent claim would be \( (\alpha_1^*, \beta_1^* + C_0 - V_0, -1) \). This has an initial value of zero, and its value at time one is

\[
\alpha_1^* S_1 + \beta_1^* B_1 + (C_0 - V_0) B_1 - X. \tag{2.18}
\]

But the strategy \( (\alpha_1^*, \beta_1^*) \) was chosen so that

\[
\alpha_1^* S_1 + \beta_1^* B_1 = X, \tag{2.19}
\]

and so it follows that the value at time one of the stock-bond-contingent claim portfolio is

\[
(C_0 - V_0) B_1 > 0. \tag{2.20}
\]

Thus, this represents an arbitrage opportunity. Similarly, if \( C_0 < V_0 \), then the investor can use the strategy \( (-\alpha_1^*, -\beta_1^* + V_0 - C_0, 1) \) to create an arbitrage opportunity.

Now we show that if \( C_0 = V_0 \), then there is no arbitrage opportunity in the stock-bond-contingent claim market. Suppose that \( \psi = (\alpha_1, \beta_1, \gamma_1) \) is a trading strategy in stock, bond and the contingent claim, with an initial value \( V_0(\psi) = \alpha_1 S_0 + \beta_1 + \gamma_1 C_0 \) of zero and non-negative value \( V_1(\psi) \) at time one. The value of the portfolio at time one is

\[
V_1(\psi) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X \tag{2.21}
\]

and so

\[
E^p [V_1(\psi)] = \alpha_1 E^p [S_1] + \beta_1 (1 + r) + \gamma_1 E^p [X]
\]

\[
= \alpha_1 (1 + r) S_0 + \beta_1 (1 + r) + \gamma_1 (1 + r) C_0
\]

\[
= (1 + r) V_0(\psi)
\]

\[
= 0,
\]

where we have used the fact that \( \alpha_1, \beta_1, \gamma_1 \) are constants (as they are \( F_0 \)-measurable), plus the martingale property of the discounted stock price process under \( p^* \) and the assumption that \( C_0 = V_0 = E^{p^*} [X] / (1 + r) \).

Now, since \( V_1(\psi) \geq 0 \) and the probability associated with \( p^* \) gives positive probability to all possible outcomes, it follows from the equality above that \( V_1(\psi) = 0 \), and hence \( E[V_1(\psi)] = 0 \). Thus, there cannot be an arbitrage opportunity. \( \square \)
Examination of the proof of Theorem 2.2.1 reveals that existence of an arbitrage free price for the contingent claim depends on the martingale property of \( \{S_0, \frac{1}{1+r}S_1\} \) under \( p^* \) and uniqueness of the price depends on the existence of a replicating strategy for the contingent claim. These aspects will hold true more generally.

2.2.2. Multi-Period Case. We now consider the general binomial model where \( T \) is any fixed positive integer. In this subsection \( p^* \) has the same value as in the single period case, namely, \( p^* = \frac{1+r-d}{u-d} \). Just as in the single period case, under the probability associated with \( p \) or \( p^* \), all the \( 2^T \) possible outcomes for the \((T+1)\)-tuple \((S_0, S_1, S_2, \ldots, S_T)\) have positive probability.

We first show that there is a replicating strategy for a European contingent claim \( X \). For this, given \( X \), we seek a trading strategy \( \phi = \{ (\alpha_t, \beta_t), t = 1, \ldots, T \} \) such that

\[
V_T(\phi) \equiv \alpha_T S_T + \beta_T B_T = X. \tag{2.22}
\]

This is developed by working backwards through the binary tree.

Let \( V_T = X \). Since \( X \) is an \( \mathcal{F}_T \)-measurable random variable and the \((T+1)\)-tuple \((S_0, S_1, \ldots, S_T)\) can take only finitely many possible values, it follows that

\[
V_T = f(S_0, S_1, \ldots, S_T)
\]

for some real-valued function \( f \) of \( T+1 \) variables. Firstly, suppose we condition on knowing the value of \( S_0, S_1, \ldots, S_{T-1} \). Then the cost and associated trading strategy for manufacturing the contingent claim over the time period \((T-1, T]\) can be computed in a very similar manner to that for the single period model. Given \( S_0, S_1, \ldots, S_{T-1} \), there are two possible values for \( V_T \) at time \( T \), depending on whether \( S_T = S_{T-1}u \) or \( S_T = S_{T-1}d \). Denote these two values by \( V^u_T \) and \( V^d_T \). In fact, \( V^u_T = f(S_0, S_1, \ldots, S_{T-1}, S_{T-1}u) \) and \( V^d_T = f(S_0, S_1, \ldots, S_{T-1}, S_{T-1}d) \). Note that these are \( \mathcal{F}_{T-1} \)-measurable random variables (the notation \( V^u_T, V^d_T \) hides this fact but has the advantage that it makes the formulas for the replicating strategy appear simpler). If the contingent claim is a European call option with strike price \( K \) and expiration date \( T \), then \( X = (S_T - K)^+ \) and \( V^u_T = (S_{T-1}u - K)^+ \), \( V^d_T = (S_{T-1}d - K)^+ \).

Now, for any European contingent claim \( X \), by similar analysis to that for the single period case, to ensure that \( V_T(\phi) = X \) we obtain the following allocations for the time period \((T-1, T]\):

\[
\alpha_T = \frac{V^u_T - V^d_T}{(u-d)S_{T-1}} \tag{2.23}
\]

\[
\beta_T = \frac{1}{(1+r)^T} \left( \frac{uV^d_T - dV^u_T}{u-d} \right) \tag{2.24}
\]
and the capital required at time $T - 1$ to finance these allocations in a self-financing manner is

$$V_{T-1} = \frac{1}{1 + r} \left( p^* V^u_T + (1 - p^*) V^d_T \right)$$

$$= \frac{1}{1 + r} E^{p^*}[V_T | \mathcal{F}_{T-1}],$$

$$= \frac{1}{1 + r} E^{p^*}[X | \mathcal{F}_{T-1}],$$

(2.25)

where $p^* = \frac{1 + r - d}{u - d}$ and $E^{p^*}[\cdot | \mathcal{F}_{T-1}]$ denotes the conditional expectation, given $\mathcal{F}_{T-1}$, under $p^*$.

We can find a trading strategy $\phi = \{ (\alpha_t, \beta_t), t = 1, 2, \ldots, T \}$ with associated value process $\{ V_t(\phi), t = 0, 1, \ldots, T \}$ such that $V_T = X$ by proceeding inductively backwards through the binary tree as follows. For the induction step, fix $t \in \{ 1, \ldots, T - 1 \}$ and assume that (self-financing) allocations $(\alpha_{t+1}, \beta_{t+1}), \ldots, (\alpha_T, \beta_T)$ have been determined for the time periods $(t, t+1], \ldots, (T - 1, T]$ with values $V_t, \ldots, V_{T-1}$, at times $t, t+1, \ldots, T - 1$, respectively, such that $V_s = \frac{1}{(1+r)^{T-s}} E^{p^*}[X | \mathcal{F}_s]$ for $s = t, t+1, \ldots, T - 1$, and the value of $(\alpha_T, \beta_T)$ at time $T$ is $V_T = X$. Given $S_0, S_1, \ldots, S_{t-1}$, the holdings $\alpha_t$ and $\beta_t$ for the time period $(t-1, t]$ are chosen so that the value associated with these holdings at time $t$ is the same as that of the random variable $V_t$; i.e., letting $V^u_t$ and $V^d_t$ denote the two possible values of $V_t$ given $S_0, S_1, \ldots, S_{t-1}$, define

$$\alpha_t = \frac{V^u_t - V^d_t}{(u - d) S_{t-1}},$$

(2.26)

$$\beta_t = \frac{1}{(1 + r)^t} \left( \frac{u V^d_t - d V^u_t}{u - d} \right).$$

(2.27)

One can readily check, using the induction hypothesis, that the capital needed at time $t - 1$ to finance these holdings in a self-financing manner is

$$V_{t-1} = \frac{1}{1 + r} E^{p^*}[V_t | \mathcal{F}_{t-1}]$$

$$= \frac{1}{(1 + r)^{T-t+1}} E^{p^*}[E^{p^*}[X | \mathcal{F}_t] | \mathcal{F}_{t-1}]$$

$$= \frac{1}{(1 + r)^{T-t+1}} E^{p^*}[X | \mathcal{F}_{t-1}].$$

Here we have used the tower property of conditional expectations; cf. Appendix A.

This completes the induction step and so it follows that the above procedure constructs a (self-financing) trading strategy, $\phi = \{ (\alpha_t, \beta_t), t =
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Lemma 2.2.2. Let \( \{ S_t, F_t, t = 0, 1, \ldots, T \} \) be a trading strategy in stock and bond with value process \( \{ S_t^*, t = 0, 1, \ldots, T \} \) called the discounted stock price process.

Proof. Clearly \( S_t^* \) is \( F_t \)-measurable, and \( S_t^* \) has finite mean for each \( t \), since \( S_t^* \) takes only finitely many values. To verify the conditional expectation property, fix \( t \in \{ 1, \ldots, T \} \). Then, using the fact that \( F_{t-1} \) is generated by \( S_0, S_1, \ldots, S_{t-1} \) and \( \xi_t \) is independent of this \( \sigma \)-algebra, we have

\[
E^{p^*}[S_t^* | F_{t-1}] = \frac{1}{(1+r)^t} E^{p^*}[S_{t-1}\xi_t | F_{t-1}]
\]

\[
= \frac{1}{(1+r)^t} S_{t-1} E^{p^*}[\xi_t]
\]

\[
= \frac{S_{t-1}}{1+r} (p^* u + (1 - p^*) d)
\]

\[
= S_{t-1}^*, \tag{2.30}
\]

where we have used the definition of \( p^* = \frac{1+r-d}{u-d} \) to obtain the last line. \( \square \)

Lemma 2.2.3. Let \( \phi \) be a trading strategy in stock and bond with value process \( \{ V_t(\phi), t = 0, 1, \ldots, T \} \). Consider the discounted value process \( \{ V_t^*(\phi) = V_t(\phi)/(1+r)^t, t = 0, 1, \ldots, T \} \). Then \( \{ V_t^*(\phi), F_t, t = 0, 1, \ldots, T \} \) is a martingale under \( p^* \).

Proof. Let \( \phi = \{ (\alpha_t, \beta_t), t = 1, \ldots, T \} \). Note that \( V_t^*(\phi) = \alpha_t S_t^* + \beta_t \) and \( \alpha_t, \beta_t \in F_{t-1} \) for \( t = 1, \ldots, T \), and \( V_0^*(\phi) = \alpha_0 S_0^* + \beta_1 \). It is straightforward to show that \( V_t^*(\phi) \) is \( F_t \)-measurable and integrable for \( t = 0, 1, \ldots, T \). Recall from Lemma 2.2.2 that \( \{ S_t^*, F_t, t = 0, 1, \ldots, T \} \) is a martingale under \( p^* \). Therefore, for \( t = 1, \ldots, T \),

\[
E^{p^*}[V_t^*(\phi) | F_{t-1}] = \alpha_t E^{p^*}[S_t^* | F_{t-1}] + \beta_t
\]

\[
= \alpha_t S_{t-1}^* + \beta_t. \tag{2.31}
\]
By factoring out $1/(1+r)^{t-1}$ and using the self-financing property of $\phi$, it follows that for $t = 2,\ldots,T$

$$E^p[V^*_t(\phi) | \mathcal{F}_{t-1}] = \frac{1}{(1+r)^{t-1}} (\alpha_t S_{t-1} + \beta_t B_{t-1})$$

$$= \frac{1}{(1+r)^{t-1}} (\alpha_{t-1} S_{t-1} + \beta_{t-1} B_{t-1})$$

$$= V^*_{t-1}(\phi).$$

For $t = 1$,

$$E^p[V^*_1(\phi) | \mathcal{F}_0] = \alpha_1 S_0 + \beta_1 B_0$$

$$= V_0(\phi) = V^*_0(\phi),$$

by definition. Hence the desired martingale property holds. $\square$

The following theorem is the multi-period analogue of the single period Theorem 2.2.1. Before proceeding to this, we must specify the notion of arbitrage in stock, bond and contingent claim to be used in the multi-period context. Let $C_0$ be the price charged for the contingent claim at time zero. Since we are specifying a price only for the contingent claim at time zero, trading in the contingent claim will be allowed only initially, whereas changes in the stock and bond holdings can occur at each of the times $t = 0, 1,\ldots,T-1$.

A trading strategy in stock, bond and the contingent claim is a collection $\psi = \{(\alpha_t, \beta_t), t = 1, 2,\ldots,T; \gamma_1\}$ where for $t = 1, 2,\ldots,T$, $\alpha_t, \beta_t$ are $\mathcal{F}_{t-1}$-measurable random variables representing the holdings in stock and bond, respectively, to be held over the time interval $(t-1, t]$, and $\gamma_1$ is an $\mathcal{F}_0$-measurable random variable (actually a constant) representing the number of units of the contingent claim to be held over the time interval $(0, T]$. The trading strategy must be self-financing; i.e., its initial value is

$$V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0,$$  \hspace{1cm} (2.32)

and at each time $t = 1,\ldots,T-1$,

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t.$$  \hspace{1cm} (2.33)

The last equation does not involve the contingent claim since this is not traded after time zero. The value of the portfolio at time $T$ is

$$V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X.$$  \hspace{1cm} (2.34)

An arbitrage opportunity in the stock-bond-contingent claim market is a trading strategy $\psi$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$.

**Theorem 2.2.4.** Let $X^* = X/(1+r)^T$. Then $V_0 = E^p[X^*]$ is the unique arbitrage free initial price for the European contingent claim $X$. 

Proof. The proof is very similar to that of Theorem 2.2.1.

Let \( \phi^* = \{(\alpha^*_t, \beta^*_t), t = 1, \ldots, T\} \) denote the replicating strategy for the contingent claim \( X \) obtained by applying the inductive procedure described above (cf. (2.26)–(2.27)). Then the value process \( V = \{V_t(\phi^*), t = 0, 1, \ldots, T\} \) for \( \phi^* \) satisfies (2.28) for \( t = 0, 1, 2, \ldots, T \). Note that, by (2.29), \( V_0 \) is the initial value of this replicating strategy.

We use the existence of the replicating strategy to show that if the initial price \( C_0 \) of the contingent claim is not \( V_0 \), then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose \( C_0 > V_0 \). Then an investor could sell one contingent claim initially, use \( V_0 \) of the proceeds to invest in the stock-bond replicating strategy \( \phi^* \), and buy \( C_0 - V_0 \) additional units of bond at time zero and hold them over the entire period \((0, T]\). Thus, the trading strategy is \( \psi = \{(\alpha^*_t, \beta^*_t + C_0 - V_0), t = 1, 2, \ldots, T; \gamma_1 = -1\} \). This has initial value \( V_0(\psi) = \alpha^*_1 S_0 + \beta^*_1 + C_0 - V_0 - C_0 = 0 \), since \( B_0 = 1 \) and \( V_0 = V_0(\phi^*) = \alpha^*_1 S_0 + \beta^*_1 \). The value of this portfolio at time \( T \) is

\[
V_T(\psi) = \alpha^*_T S_T + \beta^*_T B_T + (C_0 - V_0) B_T - X \]

\[
= X + (C_0 - V_0) B_T - X = (C_0 - V_0) B_T > 0, \tag{2.35}
\]

where we have used the fact that \( \{(\alpha^*_t, \beta^*_t), t = 1, 2, \ldots, T\} \) is a replicating strategy for the contingent claim \( X \) and so has value \( X \) at time \( T \). Thus, \( \psi \) is an arbitrage opportunity. Similarly, if \( C_0 < V_0 \), the strategy \(-\psi\) is an arbitrage opportunity.

Now suppose \( C_0 = V_0 \). We show that there is no arbitrage opportunity in stock, bond and contingent claim trading. Let \( \psi = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T; \gamma_1 \} \) be a trading strategy in stock, bond and contingent claim with an initial value of zero and final value \( V_T(\psi) \) that is a non-negative random variable. Then,

\[
0 = V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \tag{2.36}
\]

\[
0 \leq V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \tag{2.37}
\]

Note that \( \phi = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T\} \) is a trading strategy in stock and bond. Using the martingale property of \( \{V^*_t(\phi), \mathcal{F}_t, t = 0, 1, \ldots, T\} \) under \( p^* \), on setting \( V^*_t(\psi) = V_t(\psi)/(1 + r)^t \) for \( t = 0, 1, \ldots, T \), we have

\[
\frac{1}{(1 + r)^T} E^{p^*}[V_T(\psi)] = E^{p^*}[V^*_T(\psi)] = E^{p^*}[V^*_T(\phi)] + E^{p^*}[\gamma_1 X^*]
\]

\[
= E^{p^*}[V^*_0(\phi)] + \gamma_1 E^{p^*}[X^*]
\]

\[
= \alpha_1 S_0^* + \beta_1 B_0^* + \gamma_1 C_0
\]

\[
= V_0(\psi) = 0
\]
2.3. Pricing an American Contingent Claim

An American contingent claim (ACC) is represented by a (finite) sequence \( Y = \{Y_t, t = 0, 1, \ldots, T\} \) of real-valued random variables such that \( Y_t \) is \( \mathcal{F}_t \)-measurable for \( t = 0, 1, 2, \ldots, T \). The random variable \( Y_t, t = 0, 1, 2, \ldots, T \), is interpreted as the payoff for the claim if the owner cashes it in at time \( t \). The time at which the owner cashes in the claim is required to be a stopping time taking values in \( \{0, 1, \ldots, T\} \); i.e., a random variable \( \tau : \Omega \rightarrow \{0, 1, \ldots, T\} \) such that \( \{\tau = t\} \equiv \{\omega \in \Omega : \tau(\omega) = t\} \in \mathcal{F}_t, t = 0, 1, 2, \ldots, T \). For \( s, t \in \{0, 1, \ldots, T\} \) such that \( s \leq t \), let \( \mathcal{T}_{[s,t]} \) denote the set of integer-valued stopping times that take values in the interval \([s, t]\). An example of an American contingent claim is an American call option with strike price \( K \) which has payoff \( Y_t = (S_t - K)^+ \) at time \( t, t = 0, 1, 2, \ldots, T \). Note that if \( S_t \leq K \), cashing in the contingent claim at time \( t \) has an equivalent payoff to that obtained by not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

An important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them — the buyer may cash in the claim at any stopping time \( \tau \in \mathcal{T}_{[0,T]} \), whereas the seller seeks protection from the risk associated with all possible choices of the stopping time \( \tau \) by the buyer. As with the pricing of European contingent claims, for the pricing of American contingent claims, an essential role will be played by a trading strategy that hedges the risk for the seller of an American contingent claim. However, unlike the European contingent claim setting, the seller will not always be able to exactly replicate the payoff of the American contingent claim at all times \( t \). Instead, the seller of an American contingent claim seeks a superhedging strategy which is a (self-financing) trading strategy \( \phi \) whose value is at least as great as the payoff of the American contingent claim at each time \( t \).

More precisely, let \( Y = \{Y_t, t = 0, 1, \ldots, T\} \) be the payoff sequence for an American contingent claim (abbreviated as ACC). For \( t = 0, 1, \ldots, T \), let \( U_t \) denote the minimum amount of wealth that the seller of the ACC must have at time \( t \) in order to ensure that the seller has enough to cover the payoff if the buyer cashes in the claim at some stopping time \( \tau \in \mathcal{T}_{[t,T]} \). A superhedging strategy for the seller is a (self-financing) trading strategy

where we have used the facts that \( \gamma_1 \in \mathcal{F}_0, C_0 = E^{p^*}[X^*], S_0 = S_0, B_0^* = B_0, \) and \( V_0(\psi) = 0 \). Since it was assumed that \( V_T(\psi) \geq 0 \) and the probability associated with \( p^* \) gives positive probability to all possible values of \( V_T(\psi) \), it follows that \( V_T(\psi) = 0 \) and hence \( E[V_T(\psi)] = 0 \). Thus, there cannot be any arbitrage opportunity when \( C_0 = V_0 \). \( \square \)
Chapter 3

Finite Market Model

The binomial model considered in the previous chapter is an example of a finite market model. In that example, we saw that the existence of both a risk neutral probability and a replicating strategy played a key role in justifying the unique arbitrage free price for any European contingent claim. In this chapter, we extend that idea to the pricing of European contingent claims in a general finite market model. We characterize those finite market models in which there is a risk neutral probability and in which all European contingent claims can be replicated. Indeed, we will prove the first fundamental theorem of asset pricing, which shows the equivalence of the absence of arbitrage in a finite market model to the existence of a risk neutral probability. We will also prove the second fundamental theorem of asset pricing, which shows that all European contingent claims in a finite market model without arbitrage can be replicated if and only if there is a unique risk neutral probability. Then, assuming there is such a unique risk neutral probability and allowing trading of the European contingent claim at all times, we show that there is a unique arbitrage free price process for each European contingent claim. These results have their origins in the 1979 seminal paper of Harrison and Kreps [20]. In the binomial model, there is a unique risk neutral probability and hence there is a unique arbitrage free price process for every European contingent claim (cf. Exercise 4 of Chapter 2). We round out this chapter with a discussion of (single period) markets in which there is more than one risk neutral probability. Although such markets have no arbitrage opportunities, there are some European contingent claims that are not replicable; these are called incomplete markets.
3. Finite Market Model

3.1. Definition of the Finite Market Model

Throughout this chapter we will be working within the framework of the following discrete time, finite state market model. For short we will call this a finite market model.

Let \((\Omega, \mathcal{F}, P)\) be a probability space where \(\Omega\) is a finite set of possible outcomes, \(\mathcal{F}\) is the \(\sigma\)-algebra consisting of all subsets of \(\Omega\) and \(P\) is a probability measure on \((\Omega, \mathcal{F})\) such that \(P(\{\omega\}) > 0\) for all \(\omega \in \Omega\). Expectations under \(P\) will be written simply as \(E[\cdot]\). Whenever another probability is to be used, this will be explicitly indicated in the notation.

We assume that there are finitely many times \(t = 0, 1, 2, \ldots, T\) (where \(T < \infty\) is a positive integer and successive times are successive integers). At each of these times the values of \(d + 1\) assets can be observed. Here \(d\) is a strictly positive integer. One asset is a riskless security called a bond, and the other \(d\) assets are risky securities called stocks.

A \(\sigma\)-algebra \(\mathcal{F}_t \subset \mathcal{F}\) describes the information available to an investor at time \(t\). It is assumed that \(\mathcal{F}_0 = \{\emptyset, \Omega\}\), \(\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T\) and \(\mathcal{F}_T = \mathcal{F}\). The collection \(\{\mathcal{F}_t, t = 0, 1, \ldots, T\}\) is called a filtration.

The bond (asset labelled 0) is assumed to have price process \(S^0 = \{S^0_t, t = 0, 1, \ldots, T\}\), where \(S^0_t\) denotes the price of the bond at time \(t\). We assume that for each \(t\), \(S^0_t > 0\) and \(S^0_t\) is deterministic (i.e., \(S^0_t \in \mathcal{F}_0\)). For example, if the bond has an interest rate of \(r \geq 0\) per unit of time, then \(S^0_t = (1 + r)^t\) for all \(t\). The bond is considered to be a “numeraire”; i.e., it tells us what a dollar at time 0 is worth (due, amongst other things, to the effects of inflation) at time \(t\).

The \(d\) stocks are assumed to have price processes \(S^1, \ldots, S^d\), where \(S^i_t\) is the price of the \(i^{th}\) stock at time \(t\). It is assumed that \(S^i_t\) is an \(\mathcal{F}_t\)-measurable random variable for \(i = 1, \ldots, d\) and \(t = 0, 1, \ldots, T\). Note that since \(\Omega\) is finite, \(S_t = (S^0_t, S^1_t, \ldots, S^d_t), t = 0, 1, \ldots, T\), can take on at most finitely many values. It follows that in the development below, all of the expectations we write will be automatically finite.

A trading strategy (in the finite market model) is a collection of \((d + 1)\)-dimensional vectors indexed by \(t = 1, \ldots, T\):

\[
\phi = \{\phi_t, t = 1, \ldots, T\},
\]

where for each \(t \in \{1, \ldots, T\}\), \(\phi_t = (\phi^0_t, \phi^1_t, \ldots, \phi^d_t)\) is such that \(\phi^i_t\) is a real-valued \(\mathcal{F}_{t-1}\)-measurable random variable for \(i = 0, 1, \ldots, d\). We regard \(\phi^i_t\) as representing the number of “shares” of asset \(i\) to be held over the time interval \((t-1, t]\). In particular, \(\phi^0_t\) denotes the number of units of the bond to be held over this interval, and \(\phi^i_t\) denotes the number of shares of stock \(i\) to be held over the interval for \(i = 1, \ldots, d\). A positive value for \(\phi^i_t\) indicates
3.1. Definition of the Finite Market Model

that one buys that number of shares of asset $i$, at a price of $S_{t-1}^i$ per share, and holds them over the interval $[t-1, t]$. A negative value for $\phi_t^i$ indicates that asset $i$ will be “sold short”. For example, if $\phi_t^i = -1$, one is effectively borrowing the value $S_{t-1}^i$ of asset $i$ at time $t-1$ with the understanding that the cost to repay this loan at time $t$ is the value of one share of asset $i$ at time $t$; i.e., $S_t^i$. We will restrict attention to self-financing trading strategies, namely, those trading strategies $\phi$ such that the investor’s initial wealth is given by

$$\phi_1 \cdot S_0,$$

and

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \quad t = 1, \ldots, T - 1,$$  \hspace{1cm} (3.3)

where $\cdot$ denotes the dot product in $\mathbb{R}^{d+1}$. In this chapter, the term “trading strategy” will always mean self-financing trading strategy.

The initial value of a trading strategy $\phi$ is $V_0(\phi) = \phi_1 \cdot S_0$ and its value at time $t \in \{1, \ldots, T\}$ is

$$V_t(\phi) \equiv \phi_t \cdot S_t = \sum_{i=0}^{d} \phi_t^i S_t^i.$$  \hspace{1cm} (3.4)

Using the self-financing property we also have that

$$V_t(\phi) = \phi_{t+1} \cdot S_t, \quad t = 0, 1, \ldots, T - 1.$$  \hspace{1cm} (3.5)

The gains process associated with a trading strategy $\phi$ is defined by

$$G_t(\phi) = V_t(\phi) - V_0(\phi), \quad t = 0, 1, \ldots, T.$$  \hspace{1cm} (3.6)

Using the equivalent forms for $V_s(\phi)$ that come from the self-financing property of $\phi$, we can rewrite this process as follows for $t = 1, \ldots, T$:

$$G_t(\phi) = \sum_{s=1}^{t} (V_s(\phi) - V_{s-1}(\phi))$$

$$= \sum_{s=1}^{t} (\phi_s \cdot S_s - \phi_s \cdot S_{s-1})$$

$$= \sum_{s=1}^{t} \phi_s \cdot (S_s - S_{s-1})$$

$$= \sum_{s=1}^{t} \phi_s \cdot \Delta S_s,$$  \hspace{1cm} (3.7)

where $\Delta S_s \equiv S_s - S_{s-1}$. In fact, the last expression is a discrete time stochastic integral (recall that $\phi_s$ is $\mathcal{F}_{s-1}$-measurable).
An arbitrage opportunity (in the finite market model) is a trading strategy \( \phi \) such that
\[
V_0(\phi) = 0, \quad V_T(\phi) \geq 0, \quad E[V_T(\phi)] > 0.
\]
The finite market model is said to be viable if it has no arbitrage opportunities.

It will simplify computations to use discounted asset price processes, obtained by normalizing so that the value of a dollar at any time \( t \) is the same as it is at time 0. This is often called a change of numeraire. For \( i = 0, 1, \ldots, d \), we define
\[
S^*,i_t = \frac{S_i_t}{S^*_0}, \quad \text{for} \quad t = 0, 1, \ldots, T.
\]
Note that \( S^*,0_t \equiv 1 \) for all \( t \). Then \( S^*_t = (S^*,0_t, S^*,1_t, \ldots, S^*,d_t) \) is the value of the vector of discounted asset prices at time \( t \). We will refer to \( S^* = \{ S^*_t, t = 0, 1, \ldots, T \} \) as the (vector) discounted asset price process. The associated discounted value process for a trading strategy \( \phi \) is defined by
\[
V^*_t(\phi) \equiv \frac{V_t(\phi)}{S^*_0}, \quad t = 0, 1, \ldots, T, \tag{3.8}
\]
and using (3.4) and (3.5) we see that
\[
V^*_t(\phi) = \phi_t \cdot S^*_t, \quad t = 1, \ldots, T, \tag{3.9}
\]
and
\[
V^*_t(\phi) = \phi_{t+1} \cdot S^*_t, \quad t = 0, 1, \ldots, T - 1. \tag{3.10}
\]
The discounted gains process for \( \phi \) is defined by
\[
G^*_t(\phi) = V^*_t(\phi) - V^*_0(\phi), \quad t = 0, 1, \ldots, T, \tag{3.11}
\]
and by very similar manipulations to those used in deriving (3.7), this can be reexpressed as \( G^*_0 = 0 \) and
\[
G^*_t(\phi) = \sum_{s=1}^t \phi_s \cdot \Delta S^*_s, \quad t = 1, \ldots, T, \tag{3.12}
\]
where \( \Delta S^*_s = S^*_s - S^*_{s-1} \). An advantage of this last expression is that it involves only the risky assets, since \( \Delta S^*,0 = 0 \) for \( s = 1, \ldots, T \).

3.2. First Fundamental Theorem of Asset Pricing

The following definitions will be needed to state the first fundamental theorem of asset pricing, which characterizes viable finite market models.
Definition 3.2.1. Two probability measures \( Q \) and \( Q' \) on \((\Omega, \mathcal{F})\) are equivalent (or mutually absolutely continuous) provided for each \( A \in \mathcal{F} \),

\[
Q(A) = 0 \quad \text{if and only if} \quad Q'(A) = 0.
\] (3.13)

Remark. In the finite market model, \( P \) gives positive probability to every \( \omega \in \Omega \), and so for a probability measure \( P^* \) on \((\Omega, \mathcal{F})\), \( P \) is equivalent to \( P^* \) if and only if \( P^*(\{\omega\}) > 0 \) for all \( \omega \in \Omega \).

Definition 3.2.2. An equivalent martingale measure (abbreviated as EMM) is a probability measure \( P^* \) defined on \((\Omega, \mathcal{F})\) such that \( P^* \) is equivalent to \( P \) and \( S^* \) is a martingale under \( P^* \) (relative to the filtration \( \mathcal{F}_t, t = 0, 1, \ldots, T \)); i.e., for each \( t \in \{1, \ldots, T\} \),

\[
E^{P^*}[S^*_t | \mathcal{F}_{t-1}] = S^*_{t-1},
\] (3.14)

where \( E^{P^*}[\cdot] \) denotes expectation under \( P^* \), and the above equality is to be interpreted componentwise.

Remark. An equivalent martingale measure is sometimes also called a risk neutral probability. We will use the terms interchangeably.

Lemma 3.2.3. Suppose that \( P^* \) is an equivalent martingale measure and \( \phi \) is a trading strategy in the finite market model. Then \( \{V^*_t(\phi), \mathcal{F}_t, t = 0, 1, \ldots, T\} \) is a martingale under \( P^* \).

Proof. For \( t = 1, \ldots, T \), \( V^*_t(\phi) = \phi_t \cdot S^*_t \in \mathcal{F}_t \) and for \( t = 0 \), \( V^*_0(\phi) = \phi_0 \cdot S^*_0 \in \mathcal{F}_0 \), since \( \phi_t \in \mathcal{F}_{t-1} \) for \( t = 1, \ldots, T-1 \) and \( S^*_t \in \mathcal{F}_t \) for all \( t \). For each \( t \), \( V^*_t(\phi) \) is integrable because \( \Omega \) is a finite set. Finally, for each \( t \in \{1, 2, \ldots, T\} \),

\[
E^{P^*}[V^*_t(\phi) | \mathcal{F}_{t-1}] = \phi_t \cdot E^{P^*}[S^*_t | \mathcal{F}_{t-1}] = \phi_t \cdot S^*_{t-1} = V^*_t(\phi),
\]

where we have used the fact that \( \phi_t \in \mathcal{F}_{t-1} \), the martingale property (3.14) of \( S^* \) and the self-financing property (3.5) of \( \phi \). \( \square \)

Theorem 3.2.4. (First Fundamental Theorem of Asset Pricing) The finite market model is viable if and only if there exists an equivalent martingale measure \( P^* \).

Proof. We first prove the “if” part of the theorem. Suppose there exists an equivalent martingale measure \( P^* \). For a proof by contradiction, suppose that \( \phi \) is an arbitrage opportunity; that is, \( \phi \) is a trading strategy with initial
value \( V_0(\phi) = 0 \), final value \( V_T(\phi) \geq 0 \), and \( E[V_T(\phi)] > 0 \). It follows that the discounted values satisfy \( V_0^*(\phi) = 0 \), \( V_T^*(\phi) \geq 0 \), and since \( P^* \) is equivalent to \( P \), \( E^{P^*}[V_T^*(\phi)] > 0 \). By Lemma 3.2.3, \( \{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \ldots, T\} \) is a martingale under \( P^* \) and so

\[
E^{P^*}[V_T^*(\phi)] = E^{P^*}[V_0^*(\phi)].
\]

However, the left member above is strictly positive whereas the right member is zero, which yields the desired contradiction. Thus, there cannot be an arbitrage opportunity in the finite market model, and hence the model is viable.

We now turn to proving the “only if” part of the theorem. For this, suppose that the finite market model is viable. Since \( \Omega \) is a finite set, for any random variable \( Y \) defined on \( (\Omega, \mathcal{F}) \), by enumerating \( \Omega \) as \( \{\omega_1, \ldots, \omega_n\} \), we may view \( Y \) as \( (Y(\omega_1), \ldots, Y(\omega_n)) \in \mathbb{R}^n \). Since \( \mathcal{F} \) consists of all subsets of \( \Omega \), any point in \( \mathbb{R}^n \) can be thought of as representing a real-valued random variable on \( (\Omega, \mathcal{F}) \). Thus, there is a one-to-one correspondence between points in \( \mathbb{R}^n \) and (real-valued) random variables defined on \( \Omega \). Adopting this point of view for the terminal discounted gain random variables \( G_T^*(\phi) \), we define

\[
L = \{G_T^*(\phi) : \phi \text{ is a trading strategy such that } V_0(\phi) = 0\}.
\]

Note that \( L \) is a linear space, since \( G_T^*(\phi) \) is linear in \( \phi \) and any linear combination of trading strategies with initial values of zero is again a trading strategy with the same initial value. Also, \( L \) is non-empty because the origin is contained in \( L \). Let

\[
D = \{Y \in \mathbb{R}^n : Y_i \geq 0 \text{ for } i = 1, \ldots, n \text{ and } Y_j > 0 \text{ for some } j\}.
\]

Thus, \( D \) is the positive orthant in \( \mathbb{R}^n \) with the origin removed.

Since the market is assumed to be viable, \( L \cap D = \emptyset \). To see this, observe that if the latter were not true, there would be a trading strategy \( \phi \) with \( V_0(\phi) = 0 \), \( V_T^*(\phi) = G_T^*(\phi) \geq 0 \) and \( V_T^*(\phi)(\omega_i) > 0 \) for at least one \( i \), which would represent an arbitrage opportunity. Let

\[
F = \left\{ Y \in D : \sum_{i=1}^n Y_i = 1 \right\}.
\]

Then \( F \) is a convex, compact, non-empty subset of \( \mathbb{R}^n \) and \( L \cap F = \emptyset \) because \( L \cap D = \emptyset \).

By applying the Separating Hyperplane Theorem 3.6.1 (see Section 3.6 for a statement and proof of this theorem), we see that there is a vector \( Z \in \mathbb{R}^n \setminus \{0\} \) such that the hyperplane \( H = \{Y \in \mathbb{R}^n : Y \cdot Z = 0\} \) contains \( L \) and \( Z \cdot Y > 0 \) for all \( Y \in F \). By setting \( Y_i = 1 \) if \( i = j \) and \( Y_i = 0 \) if \( i \neq j \),

\[
E^{P^*}[V_T^*(\phi)] = E^{P^*}[V_0^*(\phi)].
\]
3.2. First Fundamental Theorem of Asset Pricing

we see that \( Z_j > 0 \) for each \( j \in \{1, \ldots, n\} \). Define

\[
P^*\{\omega_i\} = \frac{Z_i}{\sum_{j=1}^{n} Z_j}, \quad i = 1, \ldots, n.
\] (3.17)

Then \( P^* \) is a probability measure on \((\Omega, \mathcal{F})\) and it is equivalent to \( P \). Moreover, for any trading strategy \( \phi \) such that \( V_0(\phi) = 0 \), we have

\[
E^{P^*}[G^*_T(\phi)] = \sum_{i=1}^{n} G^*_T(\phi)(\omega_i) \frac{Z_i}{\sum_{j=1}^{n} Z_j} = \frac{G^*_T(\phi) \cdot Z}{\sum_{j=1}^{n} Z_j} = 0,
\] (3.18)

where the last line follows from the fact that \( Z \) is perpendicular to \( H \), which contains \( L \).

Note that \( G^*_T(\phi) \) involves only \( (\phi^1, \ldots, \phi^d) \). From Lemma 3.2.5, proved below, given \( \hat{\phi}^1, \ldots, \hat{\phi}^d \), where for \( i = 1, \ldots, d \), \( \hat{\phi}^i = \{\hat{\phi}^i_t, t = 1, \ldots, T\} \) and \( \hat{\phi}^i_t \) is a real-valued, \( \mathcal{F}_{t-1} \)-measurable random variable for each \( t \), there is a unique time-ordered set of \( T \) real-valued random variables \( \tilde{\phi}^0 = \{\phi^0_t, t = 1, \ldots, T\} \) such that \( \phi \equiv \{(\phi^0_t, \hat{\phi}^1_t, \ldots, \hat{\phi}^d_t), t = 1, \ldots, T\} \) is a trading strategy with an initial value of zero. Upon substituting this in (3.18) and writing out the expression (cf. (3.12)) for \( G^*_T(\phi) \), we see that

\[
0 = E^{P^*}\{G^*_T(\phi)\} = E^{P^*}\left[\sum_{t=1}^{T} \phi_t \cdot \Delta S^*_t\right] = E^{P^*}\left[\sum_{t=1}^{T} \sum_{i=1}^{d} \hat{\phi}^i_t \Delta S^*_{t,i}\right].
\] (3.19)
For each fixed \( i \in \{1, \ldots, d\} \), if we set \( \hat{\phi}_i^t = 0 \) for all \( t \) and \( j \neq i \), we obtain

\[
0 = E^{P^*}\left[ \sum_{t=1}^{T} \phi_i^t \Delta S^*_i \right],
\]

(3.20)

for each \( \hat{\phi}_i^t = \{\hat{\phi}_i^t, t = 1, \ldots, T\} \) such that \( \hat{\phi}_i^t \) is a real-valued \( \mathcal{F}_{t-1} \)-measurable random variable for each \( t \). It then follows from Lemma 3.2.6, proved below, that for \( i = 1, \ldots, d \), \( S^*_i \) is a martingale under \( P^* \). Hence, \( P^* \) is an equivalent martingale measure. \( \square \)

The next two lemmas were used in the above proof of the first fundamental theorem of asset pricing. The first lemma shows that given (non-anticipating) holdings in the risky assets and an initial wealth, there is a unique sequence of holdings in the riskless asset that makes the associated trading strategy self-financing.

**Lemma 3.2.5.** For \( i = 1, \ldots, d \), let \( \hat{\phi}_i^t = \{\hat{\phi}_i^t, t = 1, \ldots, T\} \) where \( \hat{\phi}_i^t \) is a real-valued, \( \mathcal{F}_{t-1} \)-measurable random variable for \( t = 1, \ldots, T \). For each real-valued \( \mathcal{F}_0 \)-measurable random variable \( V_0 \), there exists a unique time-ordered set of \( T \) real-valued random variables \( \phi_0^t = \{\phi_0^t, t = 1, \ldots, T\} \) such that \( \phi \equiv \{\phi_0^t, \hat{\phi}_1^t, \ldots, \hat{\phi}_d^t, t = 1, \ldots, T\} \) is a trading strategy with an initial value of \( V_0 \).

**Proof.** Fix \( V_0 \in \mathcal{F}_0 \). For \( \phi \) to be self-financing at time zero, we must have (cf. (3.2)):

\[
\phi_1 \cdot S_0 = V_0
\]

(3.21)

and since \( \hat{\phi}_1^t, \ldots, \hat{\phi}_d^t \) are given, this will be satisfied if and only if

\[
\phi_0^t = (S_0^t)^{-1} \left( V_0 - \sum_{i=1}^{d} \phi_i^s S_0^i \right).
\]

(3.22)

Note that this \( \phi_0^t \in \mathcal{F}_0 \). Thus, \( \phi_0^t \) is uniquely determined. For an induction, suppose that for some \( 1 \leq s \leq T - 1 \), \( \phi_i^t, t = 1, \ldots, s \), have been determined uniquely such that \( \phi_i^t \in \mathcal{F}_{t-1} \) for each \( t = 1, \ldots, s \), (3.21) holds, and

\[
\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \quad t = 1, \ldots, s - 1.
\]

(3.23)

Then, the self-financing property (3.23) holds for \( t = s \) if and only if we have

\[
\phi_{s+1}^0 = (S_s^0)^{-1} \left( \phi_s \cdot S_s - \sum_{i=1}^{d} \hat{\phi}_i^s S_s^i \right).
\]

(3.24)

Note that this expression for \( \phi_{s+1}^0 \) is \( \mathcal{F}_s \)-measurable. This establishes the induction step, and it follows that there is a unique \( \phi_0^t \) that makes \( \phi \) a trading strategy with initial value \( V_0 \). \( \square \)
Lemma 3.2.6. Let $M = \{M_t, t = 0, 1, \ldots, T\}$ be a real-valued process such that $M_t \in F_t$ for each $t$. Then, $M$ is a martingale (relative to the filtration $\{F_t\}$) if and only if
\[
E \left[ \sum_{t=1}^{T} \eta_t \Delta M_t \right] = 0
\]
for all $\eta = \{\eta_t, t = 1, \ldots, T\}$ such that $\eta_t$ is a real-valued $F_{t-1}$-measurable random variable for $t = 1, \ldots, T$. Here, $\Delta M_t = M_t - M_{t-1}$ for $t = 1, \ldots, T$.

Remark. The sum $\sum_{t=1}^{T} \eta_t \Delta M_t$ is actually a discrete stochastic integral. If one extends $\eta$ to a continuous time process by making it constant on $(t-1, t]$ with a value equal to that of $\eta_t$ there, and one extends $M$ to be constant on $[t-1, t)$ with a value equal to that of $M_{t-1}$ there, then the sum is the same as the stochastic integral $\int_{0}^{T} \eta_t dM_t$.

Proof. Suppose $M$ is a martingale. Let $\eta = \{\eta_t, t = 1, \ldots, T\}$ where $\eta_t$ is a real-valued $F_{t-1}$-measurable random variable for each $t$. Then, since $\eta_t \in F_{t-1}$, we have
\[
E \left[ \sum_{t=1}^{T} \eta_t \Delta M_t \right] = \sum_{t=1}^{T} E \left[ \eta_t E \left[ \Delta M_t \mid F_{t-1} \right] \right].
\]
Now, because $M$ is a martingale we have $E \left[ \Delta M_t \mid F_{t-1} \right] = 0$, for $t = 1, \ldots, T$, and it follows that (3.25) holds.

Conversely, suppose that (3.25) holds for all $\eta = \{\eta_t, t = 1, \ldots, T\}$ where $\eta_t$ is a real-valued $F_{t-1}$-measurable random variable for each $t$. For fixed $s \in \{1, \ldots, T\}$ and $A \in F_{s-1}$, let
\[
\eta_t = \begin{cases} 
0 & \text{for } t \neq s, \\
1_A & \text{for } t = s.
\end{cases}
\]
Then $\eta_t \in F_{t-1}$ for each $t$. Upon substituting this into (3.25), we obtain
\[
E \left[ 1_A \Delta M_s \right] = 0.
\]
Since $A \in F_{s-1}$ was arbitrary, it follows that
\[
E \left[ M_s \mid F_{s-1} \right] = M_{s-1},
\]
and then since $s$ was arbitrary, it follows that $M$ is a martingale. □

3.3. Second Fundamental Theorem of Asset Pricing

A European contingent claim is represented by an $F_T$-measurable random variable $X$. This is interpreted to mean that the value (or payoff) of the contingent claim at the expiration time $T$ is given by $X$. For example, a
and so $P^*$-a.s.,
\begin{equation}
    dS_t^* = \sigma S_t^* d\tilde{W}_t, \quad t \in [0, T],
\end{equation}
where $\tilde{W}$ is a standard Brownian motion martingale under $P^*$. It is well known that the form (4.33) is that of an $L^2$-martingale with respect to $\{\mathcal{F}_t\}$, under $P^*$ (cf. [11], Theorem 6.2). Thus, $P^*$ is an equivalent martingale measure. In fact, $P^*$ is the unique equivalent martingale measure.

An admissible strategy is a self-financing trading strategy $\phi$ such that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under the equivalent martingale measure $P^*$.

Given the stochastic differential equation (4.34) satisfied by $S^*$ under $P^*$ and the form (4.14) of $V^*(\phi)$, a sufficient condition for a self-financing trading strategy $\phi$ to be admissible is that
\begin{equation}
    E^{P^*} \left[ \int_0^T |\alpha_s S_s^*|^2 ds \right] < \infty.
\end{equation}
In this case, the discounted value process will be an $L^2$-martingale under $P^*$ (cf. Appendix D). The condition (4.35) holds if $\alpha$ is a bounded process.

4.4. European Contingent Claims

A European contingent claim is represented by an $\mathcal{F}_T$-measurable random variable $X$. A replicating (or hedging) strategy for a European contingent claim $X \in \mathcal{F}_T$ is an admissible strategy $\phi$ such that $V_T(\phi) = X$.

We shall use the following form of the martingale representation theorem for Brownian motion to obtain a replicating strategy for suitably integrable $X$. A more general form of this result is stated in Appendix D. A simpler version is stated here for convenience. For this theorem, it is important that a suitable filtration generated by a Brownian motion under the ambient probability measure is used. For this, note that $\tilde{W}$ is a standard one-dimensional Brownian motion under $P^*$. Now, since $W$ and $\tilde{W}$ differ only by a deterministic process, it follows that $\sigma\{W_s : 0 \leq s \leq t\}$ is the same as $\sigma\{\tilde{W}_s : 0 \leq s \leq t\}$ for each $t \in [0, T]$. Furthermore, since $P$ and $P^*$ have the same null sets, augmenting by the $P^*$-null sets has the same effect as augmenting by the $P$-null sets. Consequently, the filtration $\{\mathcal{F}_t\}$ that was originally defined using $W$ and augmentation by the $P$-null sets also satisfies $\mathcal{F}_t = \sigma\{\tilde{W}_s, s \in [0, t]\}$ for each $t \in [0, T]$, where the superscript $\sim$ denotes augmentation by the $P^*$-null sets. Consequently, $\{\mathcal{F}_t, t \in [0, T]\}$ is also the standard filtration generated by the Brownian motion $\tilde{W}$ on $(\Omega, \mathcal{F}, P^*)$. For a proof of the following martingale representation theorem, see Revuz and Yor [35], Theorem V.3.4.
Theorem 4.4.1. (Martingale Representation Theorem) Suppose that $M = \{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a right continuous martingale under $P^*$. Then there exists a $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable, adapted process $\eta = \{\eta_t, t \in [0, T]\}$ such that $P^*$-a.s., \[ \int_0^T \eta_s^2 \, ds < \infty \] and \[ M_t = M_0 + \int_0^t \eta_s \, d\tilde{W}_s \text{ for all } t \in [0, T]. \] (4.36)

Theorem 4.4.2. Suppose that $X$ is an $\mathcal{F}_T$-measurable random variable such that $E^{P^*}[|X|] < \infty$. Then there exists a replicating strategy $\phi$ for $X$. Moreover, for any replicating strategy $\phi$ and $X^* = X/B_T$, we have that for each $t \in [0, T], P^*$-a.s., \[ V_t^*(\phi) = E^{P^*}[X^* \mid \mathcal{F}_t]. \] (4.37)

Proof. We first prove that (4.37) holds if $\phi$ is a replicating strategy for $X$. Indeed, for such a $\phi$, we have for fixed $t \in [0, T]$ that $P^*$-a.s., \[ V_t^*(\phi) = E^{P^*}[V_T^*(\phi) \mid \mathcal{F}_t] = E^{P^*}[X^* \mid \mathcal{F}_t], \] where the first equality follows from the admissibility of $\phi$, which implies that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under $P^*$, and the second equality follows from the replicating property of $\phi$: $V_T(\phi) = X$.

In light of the above, to prove the existence of a replicating strategy $\phi$, it is natural to consider \[ M_t = E^{P^*}[X^* \mid \mathcal{F}_t], \quad t \in [0, T]. \] (4.39)

Note that $\{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a $P^*$-martingale and hence it has a right continuous modification. In fact, since the filtration is generated by Brownian motion, $M$ has a continuous modification (cf. Revuz and Yor [35], Theorem V.3.5). We again denote such a continuous modification by $M$. It follows from the martingale representation theorem, Theorem 4.4.1, that there is an adapted process $\eta = \{\eta_t, t \in [0, T]\}$ satisfying the conditions in Theorem 4.4.1 and such that $P^*$-a.s. for each $t \in [0, T]$, \[ M_t = M_0 + \int_0^t \eta_s \, d\tilde{W}_s. \] (4.40)

Define \[ \alpha_t = \frac{\eta_t}{\sigma S_t^*}, \quad t \in [0, T]. \] (4.41)

Then $\alpha = \{\alpha_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable, adapted and \[ \int_0^T \alpha_t^2 \, dt \leq \left( \sigma \inf_{t \in [0, T]} S_t^* \right)^2 \int_0^T \eta_t^2 \, dt. \] (4.42)
4.5. Pricing European Contingent Claims

Note that $S_t^* > 0$ for each $t \in [0, T]$, and $S_t^*$ is a continuous function of $t \in [0, T]$. Therefore, $\inf_{t \in [0, T]} S_t^* > 0$. So it follows from the integrability property of $\eta$ that $P^*$-a.s.,

$$\int_0^T \alpha_t^2 \, dt < \infty. \quad (4.43)$$

Define

$$\beta_t = M_t - \alpha_t S_t^*, \quad t \in [0, T]. \quad (4.44)$$

Then $\beta = \{\beta_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable and adapted, and using the Cauchy-Schwarz inequality we have

$$\int_0^T |\beta_t| \, dt \leq \int_0^T |M_t| \, dt + \int_0^T |\alpha_t| S_t^* \, dt$$

$$\leq T \max_{t \in [0, T]} |M_t| + \left( \max_{t \in [0, T]} S_t^* \right) \left( \int_0^T \alpha_t^2 \, dt \right)^{\frac{1}{2}} T^{\frac{1}{2}}.$$

The last line above is finite $P^*$-a.s. by (4.43) and the fact that $M$ and $S^*$ are continuous processes. Let $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$. Then,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = M_t, \quad t \in [0, T]. \quad (4.45)$$

This, together with (4.36), the definition of $\alpha$, and (4.34), gives $P^*$-a.s.,

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \eta_s \, d\tilde{W}_s$$

$$= V_0(\phi) + \int_0^t \alpha_s \sigma S_s^* \, d\tilde{W}_s$$

$$= V_0(\phi) + \int_0^t \alpha_s \, dS_s^*,$$

for each $t \in [0, T]$. Therefore, since $P$ and $P^*$ have the same null sets and $V^*(\phi) = M$ is continuous and adapted, $\phi$ is a self-financing trading strategy by Lemma 4.2.1. Combining this with (4.45) and the martingale property of $M$, we see that $\phi$ is admissible. Finally, taking $t = T$ in (4.45) implies that $\phi$ replicates $X$ since $M_T = X^*$. \qed

4.5. Pricing European Contingent Claims

Consider a European contingent claim $X \in \mathcal{F}_T$ such that $E^{P^*}[|X|] < \infty$. To determine the arbitrage free price process for this claim, we need a notion of admissible strategy for trading in stock, bond and the contingent claim.

We assume that any given price process $C = \{C_t, t \in [0, T]\}$ for the European contingent claim is a right continuous, adapted process, where $C_t$ represents the price of the European contingent claim at time $t$. (Adaptedness
Multi-dimensional Black-Scholes Model

In this chapter, we focus on a continuous financial market model that generalizes the Black-Scholes model of Chapter 4 in several directions. The model considered here has finitely many assets. One of these is a “riskless” money market asset and the others can be viewed as “risky” stocks. The price processes for the assets are given by an Itô process (cf. Appendix D) where the money market asset price process is non-decreasing with a zero volatility coefficient. Apart from this constraint, the drift and volatility coefficients of the Itô process can be stochastic and time dependent.

Following usage by some other authors [32], we call the (primary) financial market model considered here a multi-dimensional Black-Scholes model. This may be viewed as a continuous analogue of the finite market model studied in Chapter 3. One can consider more general financial market models using continuous semimartingales, or even general semimartingales with jumps, for the price processes. However, consideration of more general models would take us beyond the introductory level of this book. Moreover, the multi-dimensional Black-Scholes model considered here does allow us to illustrate how the fundamental theorems of asset pricing developed for the finite market model in Chapter 3 can be extended to a class of continuous models.

We begin by defining the multi-dimensional Black-Scholes model and the notion of a self-financing trading strategy. There are several different ways of defining such a model depending on the degree of measurability and integrability that one assumes for the coefficients in the Itô process and for the trading strategies. Our integrability assumptions on the trading
strategies are similar to those used in Karatzas and Shreve [28], Section 1.2. This form will be useful when proving some results related to completeness of the model.

After defining the financial market model, we describe the first fundamental theorem of asset pricing. Extrapolating from the finite market model case, we would expect this to say essentially that absence of arbitrage opportunities in the market model is equivalent to the existence of a risk neutral probability. The precise formulation of a suitably weak notion of arbitrage and of a risk neutral probability (as an equivalent local martingale measure) are subtle in the continuous setting. Although versions of the first fundamental theorem of asset pricing go back to the seminal works of Harrison, Kreps and Pliska [20, 21, 22, 29], a full understanding of this equivalence in a very general setting for price processes was developed only in the 1990s. Here we state a form of the fundamental theorem of asset pricing that derives from a 1994 result of Delbaen and Schachermayer [14]. Following that, we characterize the form of equivalent local martingale measures for the multi-dimensional Black-Scholes model. This exploits the Itô process form of the asset price processes and the (local) martingale representation theorem for Brownian motion.

Assuming that the market model is viable, which by the first fundamental theorem amounts to assuming that there exists an equivalent local martingale measure, we then describe the second fundamental theorem of asset pricing. This asserts that all suitably integrable European contingent claims are replicable (i.e., the market is complete) if and only if there is a unique equivalent local martingale measure. Assuming a certain non-degeneracy (i.e., there are no more stocks than there are degrees of freedom in the underlying Brownian motion driving the asset price processes), we show that the market is complete if and only if there are exactly as many stocks as there are degrees of freedom in the driving Brownian motion and almost surely the volatility matrix for the stocks is invertible at almost every time. There are some subtleties associated with these results on completeness that relate to integrability assumptions on self-financing strategies (see Harrison and Pliska [21], Section 3.3, for one indication of this). Our use of a similar setup to that of Karatzas and Shreve [28] is convenient for handling these subtleties. For more general market models (not treated here), completeness need not imply uniqueness of an equivalent local martingale measure (cf. Artzner and Heath [1] for an example), and it becomes a challenge to formulate a suitable variant of the second fundamental theorem (cf. Jarrow, Jin and Madan [26] for an approach to this and for a description of some of the related history).
5.1. Preliminaries

Assuming the market model is viable and complete, we illustrate how the unique arbitrage free price for a suitably integrable European contingent claim can be obtained. We conclude the chapter with some references to the literature on incomplete markets.

5.1. Preliminaries

We use a similar basic setup to that of the Black-Scholes model treated in Chapter 4, except that now we have a multi-dimensional Brownian motion as a driving noise source.

We consider a finite time interval \([0, T]\), for some \(0 < T < \infty\), as the interval during which trading may take place. The Borel \(\sigma\)-algebra of subsets of \([0, T]\) will be denoted by \(B_T\).

We assume as given a complete probability space \((\Omega, \mathcal{F}, P)\) now which is defined a standard \(n\)-dimensional Brownian motion \(W = \{W_t, t \in [0, T]\}\) for some positive integer \(n\). In particular, \(W = (W^1, \ldots, W^n)\) is an \(n\)-dimensional process defined on the time interval \([0, T]\); the one-dimensional coordinate processes \(W^1, \ldots, W^n\) are mutually independent; and for each \(i = 1, \ldots, n\), \(W^i = \{W^i_t, t \in [0, T]\}\) is a standard one-dimensional Brownian motion. Let \(\{\mathcal{F}_t, t \in [0, T]\}\) denote the standard filtration generated by the \(n\)-dimensional Brownian motion \(W\) under \(P\) (cf. Appendix D, Section D.1). It is well known that this filtration is right continuous; i.e., for each \(t \in [0, T]\), \(\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s \in (t, T]} \mathcal{F}_s\) (cf. Chung [9], Section 2.3, Theorem 4).

The \(\sigma\)-algebra \(\mathcal{F}_0\) is trivial in the sense that any \(\mathcal{F}_0\)-measurable random variable is \(P\)-a.s. constant. All random variables considered in this chapter will be assumed to be defined on \((\Omega, \mathcal{F}, \mathcal{F}_T, P)\), and so without loss of generality we assume that \(\mathcal{F} = \mathcal{F}_T\). We shall frequently write \(\{\mathcal{F}_t\}\) instead of the more cumbersome \(\{\mathcal{F}_t, t \in [0, T]\}\). The filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) satisfies the usual conditions (cf. Appendix C). It is well known that under this assumption, every martingale has a modification whose paths are all right continuous with finite left limits (cf. Chung [9], Theorem 3, page 29, and Corollary 1, page 26). In fact, since the filtration is generated by the Brownian motion \(W\), every (local) martingale has a continuous modification (cf. Theorems II.2.9, V.3.5 in Revuz and Yor [35]). Expectations with respect to \(P\) will be denoted by \(E[\cdot]\) unless there is more than one probability measure under consideration, in which case we shall use \(E^P[\cdot]\) in place of \(E[\cdot]\).

In the following, we shall encounter integrals of the form \(\int_0^t R_s \, ds\), for \(t \in [0, T]\), where \(R = \{R_t, t \in [0, T]\}\) is an \(m\)-dimensional, adapted process (for some \(m \geq 1\)) satisfying