

CHAPTER 5

Triangle Congruence Tests

In congruent triangles, respective angles and sides are congruent.

To determine whether two triangles are congruent, there is no need to check all the angles and sides for congruence. It is enough to use one of the *triangle congruence tests*.

The first test is the SAS (Side-Angle-Side) test. It refers to two sides and the angle between them. Triangles ABC and $A_1B_1C_1$ are congruent if $AB = A_1B_1$, $AC = A_1C_1$, and $\angle BAC = \angle B_1A_1C_1$.

The second test is the ASA (Angle-Side-Angle) test. It refers to a side and the two adjacent angles. Triangles ABC and $A_1B_1C_1$ are congruent if $AB = A_1B_1$, $\angle A = \angle A_1$, and $\angle B = \angle B_1$.

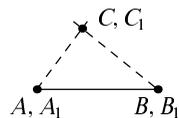
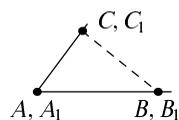
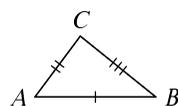
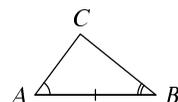
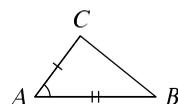
The third test is the SSS (Side-Side-Side) test. Triangles ABC and $A_1B_1C_1$ are congruent if $AB = A_1B_1$, $AC = A_1C_1$, and $BC = B_1C_1$.

The pictures show the elements of a triangle (sides, angles) that must be the same in a pair of triangles for that pair to be congruent.

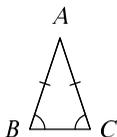
An intuitive explanation for why these triangle congruence tests work is very simple for the first and second test and slightly more difficult for the third test.

Explanation of the SAS test. We assume that $\angle A$ is congruent to $\angle A_1$. With this in mind, we align the two triangles so that segment AB goes along A_1B_1 and AC goes along A_1C_1 . Since $AB = A_1B_1$, points B and B_1 must coincide. Similarly, points C and C_1 coincide too. Hence, all the pairs of points in the triangles coincide ($A = A_1$, $B = B_1$, $C = C_1$).

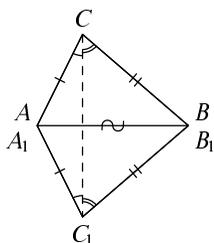
Explanation of the ASA test. We will superimpose AB on A_1B_1 (these sides are congruent) in a way that A aligns with A_1 , B aligns with B_1 , and points C and C_1 are on one side of AB . Since angles $\angle A$ and $\angle A_1$ are equal, the line AC will coincide with the line A_1C_1 , although we do not know yet whether or not points C and C_1 coincide. For the same reason, the line BC will coincide with the line B_1C_1 . This means that point C (at



which lines AC and BC intersect) will align with point C_1 , and the two triangles will coincide.



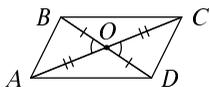
Explanation of the SSS test. The third test requires an additional comment: if there are two sides in a triangle which are congruent, the two angles opposite to these sides are congruent as well. (Such triangles are called *isosceles triangles*.) Indeed, suppose that triangle ABC has a pair of congruent sides, AB and AC . Turn this triangle around in such a way that point A stays at the same place but AB will go along AC and AC will go along AB . Since $AB = AC$, point B will align with point C and point C will align with point B . Therefore, angles B and C are congruent.



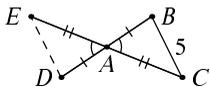
We are now ready to explain the third test. Begin by joining triangles ABC and $A_1B_1C_1$ along side AB and A_1B_1 (in a way that points C and C_1 are on opposite sides). Triangles ACC_1 and BCC_1 are isosceles, so they have equal (congruent)¹ angles that are marked in the same way in the picture. Observe that the sum of the angles on the top of the figure is equal to the sum of the angles at the bottom of the figure. Therefore, angles C and C_1 are congruent. It remains to use the SAS test.

These explanations are not, as mathematicians say, “solid proofs”, but they are nevertheless quite convincing.

76. In a quadrilateral, the intersection point of the diagonals divides each diagonal in half. Prove that opposite sides in this quadrilateral are congruent.



◁ Suppose $ABCD$ is the given quadrilateral and point O is the intersection of the diagonals. Let us mark congruent segments in the picture: one pair with single dashes and the other with double dashes. Additionally, we will mark the vertical (and therefore congruent) angles AOB and COD . Now we have all the required attributes to use the ASA test for triangles AOB and COD (two sides and the angle between them). Since these two triangles are congruent, $AB = CD$. By looking at two other triangles (which ones, by the way?) we see that $BC = AD$. ▷



77. Side AB in triangle ABC extends past point A . Take the point D on the extension of side AB past point A such that $AD = AB$. Take also the point E on the

¹Sometimes congruent angles (that have equal measure) are also called “equal”. For example, one could say “both angles α and β equal 60° , so they are equal” instead of the more pedantic “both angles α and β have equal measures of 60° , so these angles are congruent”. The same happens with segments: one could say “ $AB = CD = 1$ cm, so segments AB and CD are equal to each other” instead of “both AB and CD have length 1 cm, so AB and CD are congruent”. We hope that the meaning is clear from the context in such cases.

extension of side AC past point A such that $AE = AC$. Find the distance between points D and E if $BC = 5$.

◁ The solution of this problem is almost identical to the previous one. In triangles ABC and ADE (in which we need to draw segment ED), $AB = AD$ and $AC = AE$. Angles BAC and DAE are vertical, which means they are congruent. By the SAS congruence test the two triangles are congruent. Therefore, $ED = BC = 5$. ▷

78. In triangle ABC , segment AM is the median of the triangle (connecting point A to the midpoint M of side BC). AM is extended past point M for a length of AM . Find the distance between this new point and points B and C if $AB = 5$ and $AC = 4$.

◁ Let N be the end of the extended median, so that N lies on the extension of AM past point M and $AM = MN$. Then the diagonals in quadrilateral $ABNC$ are cut in half by the intersection point, and, referring to the solution of Problem 76 we see that the opposite sides of the quadrilateral are equal, so that $NB = AC = 4$ and $NC = AB = 5$. ▷

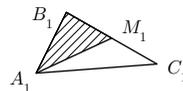
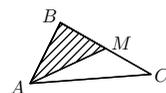
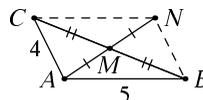
79. Prove that in any triangle ABC , the median AM is not greater than half the sum of sides AB and AC .

◁ We are trying to prove that twice the length of AM does not exceed the sum $AB + AC$. In the previous problem, segment AN was exactly twice the length of segment AM . Segment AN is a side of triangle ABN and the other two sides of the triangle are AB and $BN = AC$. To complete the solution, use the triangle inequality. ▷

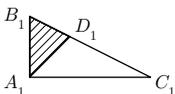
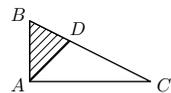
80. Using the SAS test of triangle congruence, prove that the (respective) medians in a pair of congruent triangles are congruent.

◁ Let ABC and $A_1B_1C_1$ be these two triangles (congruent angles and sides in these triangles are denoted by the same letters), and let AM and A_1M_1 be the medians (so that M is the midpoint of BC and M_1 is the midpoint of B_1C_1). Take a look at triangles ABM and $A_1B_1M_1$. In these two triangles, $\angle B = \angle B_1$, $AB = A_1B_1$, and $BM = B_1M_1$ (since they are halves of congruent segments BC and B_1C_1). Therefore, these triangles are congruent by the SAS test, and respective sides, one of which is the median, are congruent.

Note that we could have used triangles AMC and $A_1M_1C_1$ equally well. ▷

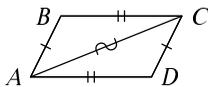


81. Using the ASA test for triangle congruence, prove that the bisectors of respective angles in congruent triangles are congruent.

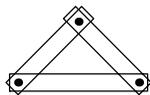


◁ (The problem refers to the segment of the bisector from the vertex to the opposite edge.) Let AD and A_1D_1 be two such segments. We can use a method similar to that used in the previous problem. In triangles ABD and $A_1B_1D_1$, sides AB and A_1B_1 are congruent, and so are angles B and B_1 . In addition, angles DAB and $D_1A_1B_1$ are congruent because they are halves of congruent angles CAB and $C_1A_1B_1$. To complete the solution, use the ASA test for triangle congruence. ▷

82. In the given quadrilateral, opposite pairs of sides are congruent. Prove that opposite pairs of angles are also congruent. (Such quadrilaterals are called *parallelograms*; we will come back to them later.)



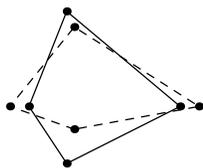
◁ Let $ABCD$ be a quadrilateral in which opposite pairs of sides are congruent. AC is a diagonal which divides quadrilateral $ABCD$ into two triangles— ABC and ADC . These two triangles share side AC , and it is a given that their other two pairs of sides are congruent. Therefore, they are congruent (due to the SSS test) and their respective angles must be congruent. Therefore, angles B and D (which are opposite to each other) are congruent. Similarly, angles A and C are congruent as well. ▷



83. Three wooden planks are connected by loose bolts, as shown in the picture. However, even though the three bolts are loose, the resulting structure is rigid. Which triangle congruence test explains this phenomenon?

◁ In this problem, the planks are the sides of a triangle, and the bolts are the vertices. The bolts are loose, which means the angles are not fixed. However, the lengths of the planks are constant, so, due to the SSS test, the angles cannot change as well. ▷

84. Is the following statement true or false? Two quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are congruent if their respective sides are congruent ($AB = A_1B_1$, $BC = B_1C_1$, and so on).



◁ Thinking back to the previous problem, we can reformulate this problem as follows: would a quadrilateral joint be rigid? The answer is clearly no: if we grab the opposite corners of the quadrilateral joint and pull or push, they will move. The picture illustrates two possible positions of the joint, i.e., distinct quadrilaterals with congruent respective sides. ▷

All three triangle congruence tests have the same general form: a triangle is defined uniquely by three certain elements (in the first test, two sides and the angle between them; in the second test, a side and two adjacent angles; and in the third test, three sides). You should not assume, however, that these three elements can be arbitrary. For example three angles do not determine a triangle uniquely: there are triangles that have the same angles but are different in size (see the picture). Such triangles are called *similar triangles*; we will talk more about them in Chapter 29.

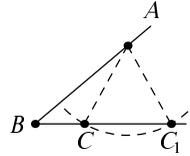


A somewhat trickier example where three elements do not define a unique triangle is given in the following problem.

85. Give an example of triangles ABC and $A_1B_1C_1$ in which $AB = A_1B_1$, $AC = A_1C_1$, $\angle ABC = \angle A_1B_1C_1$, and yet ABC and $A_1B_1C_1$ are not congruent.

This problem demonstrate that in the first test, “two sides and the angle between them”, the fact that the angle has to be between the two sides is crucial. If the congruent angle is opposite to a pair of congruent sides, congruence is not guaranteed.

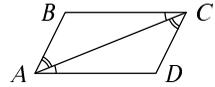
◁ Take an acute angle with vertex B . On one side of the angle, mark point A somewhere on the line. On the other side, find points C and C_1 that are at equal distance from point A . This demonstrates that triangles ABC and ABC_1 are not congruent even though they have three congruent elements. ▷



Additional Problems

86. A diagonal divides a convex quadrilateral into two congruent triangles. Prove that the other diagonal will divide the quadrilateral either into two congruent triangles or into two isosceles triangles.

87. Let AC be a diagonal in quadrilateral $ABCD$. It turns out that the angles “cross-opposite” to each other (see the picture) are equal: $\angle BAC = \angle ACD$ and $\angle CAD = \angle ACB$. Prove that opposite pairs of sides in this quadrilateral are congruent.



Afterword

The objectives of school geometry

Elementary geometry is, all in all, rather strange. Traditionally, philosophers used Euclidean geometry as a standard of rigor; however, it isn't possible to give a proof of the tests for triangle congruence that would be both rigorous and understandable. Geometry has been taught all across the world for several thousands of years; however, it's difficult to say why someone would need to know that the three bisectors of a triangle intersect at one point.

The famous French mathematician Jean Dieudonné wrote with great fervor and sharp wit in the introduction to his book *Linear Algebra and Geometry* (1969): “In ‘elementary geometry’, [linear algebra opened] . . . a ‘royal road’ by which, starting from extremely simply-stated axioms everything can be obtained in a very straightforward manner by few lines of trivial calculations, whereas previously it was necessary to start off by setting up a whole complex and artificial framework of constructions of auxiliary triangles in order to use the sacrosanct cases of ‘congruence’ or ‘similarity’ of triangles, which are the cornerstones of traditional teaching.”

This is from another part of the introduction: “But when it comes to actually building up such constructions, is it better to have acquired some of the basic principles about elasticity or rather to know that the perpendicular heights of a triangle are concurrent? It is also quite true to say that trigonometrical formulae are completely indispensable to astronomers, surveyors, and authors of books on trigonometry.” And then he wrote: “One of the results of progress in Mathematics is that results which were achieved only after a tremendous amount of hard work and tortuous, even obscure, means often will be proved 50 or 100 years later, in a few lines and with almost no effort involved. A universally well-known example is that of the discovery of infinitesimal calculus which, at one stroke,

brought the solution of problems which had taxed the wisdom of an Eudoxus or Archimedes, to within the range of almost automatic calculations.”

There were others like Dieudonné who held a similar opinion, not just in France: just about the same motives led to the replacement of “arithmetic solutions for word problems” and simple fractions in earlier grades with “solving for x ” and calculators. Additionally, it led to including “elements of calculus” into school courses.

Unfortunately, these reforms (now, it seems, this has already become apparent to everyone) turned out to be dramatically unsuccessful¹ despite the participation of prominent mathematicians. For example, in practical use the geometry textbook written by Kolmogorov and his colleagues² (as well as a subsequent textbook by Pogorelov³) was less successful than the old-fashioned textbooks of Kiselev⁴ (who had no great achievements in mathematics) in practice. I believe that the main mistake of the reformers was the following: the goal of a mathematics course is not to communicate a certain set of facts chosen in advance in the most logically concise way possible,⁵ but in solving problems and enjoying the process. Jogging through the park for exercise doesn’t lose its meaning even when you jog alongside a bus route. And however much limited the student’s capacity to walk independently might be (by preparation, or natural ability, etc.), replacing it with rides in an express bus can only do harm.⁶

¹Here I speak mainly about the Russian (Soviet) experience. I have heard that this is not an exception and similar things happened in other countries, but I have no firsthand information.

²A. N. Kolmogorov, A. F. Semenovich, R. S. Cherkasov, *Geometry*, textbook for grades 6–8, Prosveshchenie, Moscow, 1980 (in Russian).

³A. V. Pogorelov. *Geometry*, textbook for grades 6–10, Prosveshchenie, Moscow, 1985 (in Russian).

⁴A. P. Kiselev, *Elementary Geometry*, textbook for middle school, Tipografiya Ryabushinskogo, Moscow, 1914 (in Russian). There were many subsequent editions and an English translation: Kiselev’s *Geometry*, adapted from Russian by Alexander Givental, published by Samizdat, 2006.

⁵Which is what education officials [in Russia] think up to this day, announcing a contest to produce a textbook according to a pre-determined curriculum.

⁶Dieudonné seemed to understand this. In the same introduction he writes: “Some people will admit that the statements of theorems taught to children in the secondary schools are destined to be forgotten in favour of more important concepts; but it will be claimed that

Problem book and textbook. Looking at what was said above, one can come to the conclusion that the most important thing in the school course of geometry is the practice of solving problems. Memorizing theorems and proofs is meaningless unless you also try solving problems. On the other hand, a large part of the theoretical materials can be decomposed into a series of problems⁷.

In contrast to the textbook situation, in the last decade many good problem books on geometry have been appearing in Russia (including the problem books by Gordin, Prasolov, and Sharygin⁸). A multitude of interesting and difficult problems is collected in these books. Of course, this advantage could become a disadvantage if we need a source of problems for a basic geometry course where it is important to have simple problems (but diverse and not just repetitive drill exercises) that are accessible to beginners.

The purpose of this collection is an attempt to collect such problems (simple, but diverse) and arrange them in such an order that they can serve as a foundation for gradual learning of planar geometry. Of course, the collection cannot be an adequate replacement for a textbook (such an undertaking would require, at minimum, many years of experience of using the text in teaching this material and making appropriate corrections) but is an attempt to include the basic theoretical materials in a sequence of

in the course of racking their brains over these artificial topics, they will at least have acquired the method of inquiry and habits of mind which will later be of great help to them.” But later he replies to this in a strange way: “This was probably true before Descartes’ time but was no longer so for Newton’s contemporaries. One of the results of progress in Mathematics is that results which were achieved only after a tremendous amount of hard work and tortuous, even obscure, means often will be proved 50 or 100 years later, in a few lines . . .” (we already presented the end of this quote). It is not clear though how these “few lines” can replace acquiring critical thinking skills (even if they are acquired using artificial examples).

⁷This is what happens, for example, in the classes of R. K. Gordin in Moscow School 57—students are given sheets of paper with problems which they solve during classes and at home. They only open the textbook in order to prepare for the exam at the very end. It must be taken into account that this is a school with more in-depth mathematical study and students of the corresponding ability.

⁸R. K. Gordin, *Problems in Plane Geometry*, grades 7–9, MCCME, 2006; V. V. Prasolov, *Problems in Plane Geometry*, Nauka, Moscow, 1986, MCCME, 2006; I. F. Sharygin, *Problems in Geometry. Plane Geometry*, Nauka, Moscow, 1982.

naturally developing problems. A certain portion of more complex problems is also included to make it interesting for more advanced students (and teachers). Problems come with brief explanations which assume that students are somewhat familiar with the terminology (from explanations of teachers, from textbooks, etc.).

While selecting the problems, I tried to include first and foremost those problems that have interesting and easily remembered formulations and solutions, as well as problems that connect geometry with the world around us (including some informal questions). Of course, in the practice of teaching, one should use many similar problems, and sometimes even boring computational problems—simply because they are easier to grade (sometimes just looking at the answer). In the collection, there are hardly any of these and it's clearly necessary to look at other textbooks and collections of problems.

Mathematical rigor

In my opinion, geometric proofs must (at least initially) be considered as conclusive arguments about properties of real objects: triangles cut out of paper, segments measured with a ruler, etc. Therefore, for example, the “proof” of the congruence of triangles based on placing one on top of another or on rotating the triangle is perfectly acceptable, although calling this a proof might be far-fetched (in the textbook by Pogorelov, there is an axiom that says that one can construct a triangle congruent to a given one, on a given side of a given ray). On the other hand, in those cases when the arguments have a certain clear logical structure, it is better to emphasize it.

We can more or less make such a distinction: if we argue that some segments are equal, this must be proven—but justifying that some lines will indeed intersect or some point will indeed appear on the given side of a line is not necessary (unless it is possible to do so in a way that is particularly beautiful or demonstrative). In this way, all the arguments related to the “axioms of order” (terminology from *The Foundations of Geometry* by Hilbert) almost entirely disappear. In my opinion, such a distinction between that which should be proved and that which shouldn't is understood sufficiently well by students (and at the end of the course, after acquiring some mathematical culture,

the best of them will be able to comprehend gaps in arguments).

Concerning measuring segments and angles, Euclid⁹ and some old textbooks (Hadamard¹⁰ and, to a certain extent, Kiselev) pay a lot of attention to this, whereas modern textbooks assume from the very beginning that the length of a segment and the measure of an angle are numbers (whatever this may mean) and have all the necessary properties. The second approach seems to be more practically sound.

As a de facto axiom it is good to use statements that are often needed in solving problems: the tests of triangle congruence, the criteria for parallel lines (equality of angles at a secant), maintaining the ratio of segments during parallel projection (having proved it for the case of commensurable segments, we can use the general statement), the formula for the area of a rectangle (see the corresponding section of the book). Discussing the problems of “rigorous” proofs of these facts is clearly outside of the scope of the basic knowledge of the subject of geometry. (This is discussed in a Russian-language booklet¹¹ I wrote.)

How this book came about

These problems were mainly collected in the mid-1990s when I visited I. M. Gelfand to collaborate on assignments for the School by Correspondence that he organized at Rutgers University, as well as on an algebra book for high school students.¹² Under his observation I made a list of problems on planar geometry but did not finish it in time. That text was a draft, subsequently fleshed out (he is in no way responsible for mistakes in the final version).

I would like to believe that his ideas and comments about these problems (and about teaching mathematics in general) can be seen in this collection.

I would like to use this opportunity to thank my teachers (particularly those who taught me geometry) and my colleagues whose suggestions and conversations (about this collection of problems, about teaching geometry, and more) had a great influence on me, including G. A. Chuvakhina,

⁹*Elements*, Books I–IV, Dover, 1956.

¹⁰J. Hadamard, *Lessons in Geometry: I. Plane Geometry*, American Mathematical Society, Providence, RI, 2008.

¹¹A. Shen, About “mathematical rigor” and teaching mathematics at school, MCCME, Moscow, 2011.

¹²I. M. Gelfand and A. Shen, *Algebra*, Birkhauser, Boston, 2003.

T. M. Velikanova, B. P. Geidman, R. K. Gordin, S. V. Markelov, V. V. Prasolov, V. M. Sapozhnikov, I. F. Sharygin. This book would not exist if it were not for P. R. Goldshtein, who read the preliminary draft of the text, pointed out various mistakes (now fixed), and convinced me to finish the task.

I would also like to thank Olga Gagarkina and Dmitry Shcherbakov for creating METAPOST programs that generated most of the pictures in this book (the rest were made by me), and all of the readers of the preliminary draft, who pointed out the shortcomings of the book. Tatiana Korobkova discovered a multitude of mistakes and typographical errors that I missed, and Victor Shuvalov corrected them (or bid me to do so) and also perfected the layout and pictures in the Russian editions.