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Part 2: Formal Aspects of QFT

Preface

The 1996–1997 Special Year at the Institute for Advanced Study was a program in Quantum Field Theory. It was a year of intense activity. There were about four lectures per week, each lecture lasting two hours. In addition, there were weekly problem sessions, as well as supplementary lectures and remedial study meetings at all hours. The students ranged from the obscenely young to those less so.

In these two volumes we share our struggles with a broader audience.

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We warmly thank Dottie Phares for her infinite patience and skill in dealing with a noxious mixture of TeX flavors. We are indebted to Arthur Greenspoon, who not only proofread the entire manuscript but also prepared the index. The reader will join us in saying: thank you, Arthur. As always, Robbert Dijkgraaf's drawings—enjoy the covers!—liven up the proceedings, and we thank him for his contribution.

Most of all, we thank our teachers. They worked hard to educate us about beautiful ideas in physics, and we hope that they will be rewarded when one day this all becomes a beautiful part of mathematics. We will be pleased if these volumes can aid in that development. Finally, our thanksgiving would be amiss if we did not single out Edward Witten (and so exclude him from the signatories below). He gave generously of his time, energy, and abilities while serving as lecturer, teaching assistant, tutor, grader, organizer, and editor.

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Introduction

Until recently the interplay between physics and mathematics followed a familiar pattern: physics provides problems and mathematics provides solutions to these problems. Of course at times this relationship has led to the development of new mathematics. In fact, sometimes this new development was initiated by physicists (the discovery of Fourier series is one of the success stories). But physicists did not traditionally attack problems of "pure" mathematics.

The situation has drastically changed during the last 15 years. Physicists have formulated a number of striking conjectures (such as the existence of mirror symmetry) and concepts (for example, the existence of "fusion" for representations of Kac-Moody algebras, i.e., the structure of a braided tensor category). In many cases mathematicians have been able to verify the conjectures of physicists, but the proofs have dealt with each individual case and ignore the bigger picture which governs the physicists' intuition. The basis of the physicists' intuition is their belief that underlying quantum field theory and string theory is a (as yet undiscovered) self-consistent mathematical framework. After all, physicists believe in the "reality" of these physical theories, and hence in the existence of mathematical foundations for them. A number of mathematical objects and notions, especially geometric ones, can be formulated in terms of quantum field theory and strings. Then the non-rigorous techniques of physics can be brought to bear on these objects, often providing new insight. The great advances come when this insight is not readily apparent from the purely mathematical perspective.

The mathematicians and the physicists at the Institute for Advanced Study decided that it was worthwhile to organize a special year-long program to teach a small group of mathematicians the basic physical concepts—renormalization group, infrared limit, dimensional transmutation, etc.—which are in constant use. The conception for the Special Year at the IAS was eloquently described in a letter by Robert MacPherson:

> The goal is to create and convey an understanding, in terms congenial to mathematicians, of some fundamental notions of physics, such as quantum field theory, supersymmetry and string theory. The emphasis will be on developing the intuition stemming from functional integrals.
>
> One way to define the goals of the program is by negation, excluding certain important subjects commonly pursued by mathematicians whose work is motivated by physics. In this spirit, it is not planned to treat except peripherally the magnificent new applications of field theory, such as Seiberg-Witten equations to Donaldson theory. Nor is the plan to consider fundamental new constructions within mathematics that were inspired by physics, such as quantum groups or vertex operator algebras. Nor is the aim to discuss how to

> provide mathematical rigor for physical theories. Rather, the goal is to develop the sort of intuition common among physicists for those who are used to thought processes stemming from geometry and algebra.

This book is a written record of that attempt. The extent to which the program, and by extension these volumes, meet the goals set down in MacPherson's letter is left for the reader to judge.

The Special Year consisted of many short—and some long—courses covering various aspects of quantum field theory and perturbative string theory. Notes from these lectures form the bulk of this volume. We have fairly accurately kept the chronology of the program: lectures from the fall appear in Volume I and lectures from the spring in Volume II. (There is one exception: Gawędzki's lectures were given in the fall.) During the year there were many problem sets and even one exam. They also appear in these volumes together with solutions to most of the problems. There are also a few mathematical notes which supplement the lectures. Krzystof Gawędzki and Eric D'Hoker wrote their own notes, but otherwise the lecture notes in these volumes were written by mathematicians who participated in the program. In some cases they stick very closely to the content and style of the actual lecture, while in other cases the material is exposed with a mathematical spin.

Part 1 of these volumes is somewhat different in nature from the parts which follow. It is a mathematical text which is meant to be rigorous, fairly self-contained, and consistent in its notation and sign conventions. The topics covered are "classical" in the sense of "non-quantum," and so well within reach of standard notions in algebra and geometry. For the most part it is supplementary material not based on lectures from the Special Year. The exception is *Introduction to Supersymmetry*, which covers parts of lectures given by Joseph Bernstein, but the material has been substantially reworked by Pierre Deligne and John Morgan. This text lays the foundations of supersymmetry, or "$\mathbb{Z}/2$-graded mathematics." The algebraic manifestation is more well-known than the geometric one, where Grothendieck's "functor of points" is used to formalize the notorious "anticommuting variables" of classical supersymmetry. Pierre Deligne's treatment of *Spinors* is tailored to the needs of the rest of these volumes. For example, special facts relevant to Lorentz signature are covered in great detail. *Classical Field Theory*, by Pierre Deligne and Dan Freed, develops a mathematical framework for lagrangian field theory. Despite the word "classical" in the title, these ideas play an important role in quantum field theory as well. (For example, one often describes a quantum theory by an *effective lagrangian*, which one then manipulates according to the rules of classical field theory.) The basic examples are also described here, while more sophisticated examples appear throughout the rest of the volumes. Most modern developments in field theory and string theory are for supersymmetric theories, hence a thorough understanding of the classical supersymmetric lagrangians is a prerequisite. That motivation lay behind Witten's *Superhomework* (which appears here at the end of Part 2). Answers to the problems posed there are incorporated into the text *Supersolutions*, by Pierre Deligne and Dan Freed, which is a systematic development of classical supersymmetric field theories in various dimensions. The flavor of the subject changes with the dimension, in part due to the dimension-dependent behavior of spinors. An appendix, *Sign Manifesto*, collects and explains many of the sign conventions used in Part 1. We emphasize that there are many different sign choices made in the physics literature, and the lecture notes in subsequent parts of these volumes reflect the sign conventions of the particular lecturer.

Part 2 covers disparate topics in quantum field theory which we can loosely collect under the rubrics "axiomatics" and "perturbation theory." They are linked in that both are more amenable to current mathematics than the dynamical ideas of Part 4. Pierre Deligne's short note *Quantization* summarizes general mathematical notions which apply in both mechanics and field theory. *Introduction to QFT*, by David Kazhdan, begins with the Wightman axioms, a mathematical framework for rigorous quantum field theory. He treats scattering theory within that framework and also gives an exposition of Gaussian integrals and Feynman diagrams. A major component of Part 2 is Edward Witten's lectures *Perturbative Quantum Field Theory*. He begins with the relationship between the spectrum of the hamiltonian of a Wightman quantum field theory and the analytic behavior of its 2-point function. In the simplest example of ϕ^3 theory this quickly leads to a 1-loop Feynman diagram, and so to a general discussion about ultraviolet divergences and renormalization theory. Here the reader finds an explanation of basic concepts and tools—Feynman diagrams, quantum operators, operator product expansion, renormalizability—as well as discussions of many general issues related to other lectures in Part 2. Edward Witten's two lectures *Index of Dirac Operators* give applications of the path integral to Dirac operators, both on finite dimensional manifolds where he derives the index formula, and on the infinite dimensional loop space where he constructs the elliptic genus and derives some topological consequences of the field theoretic viewpoint. A somewhat different emphasis is given by Ludwig Faddeev in his *Elementary Introduction to Quantum Field Theory*. In particular, the first lecture explains very clearly how the path integral naturally arises from the usual formalism of quantum mechanics. After some comments about scattering, he goes on to describe the Faddeev-Popov method of quantization of gauge theories. Another important component of Part 2 is David Gross' *Renormalization Groups*. The renormalization group flow in quantum field theory explains how information from theories that describe the microscopic world is integrated into a few macroscopic parameters. There is a rigorous definition of the renormalization group flow in perturbation theory; a detailed treatment of the renormalization group differential equation, both for lagrangians and operators; the definition of anomalous dimension; a discussion of asymptotic freedom; and a first example of a well-defined nontrivial renormalizable quantum field theory, the Gross-Neveu model. Pavel Etingof's note *Dimensional Regularization* explains the mathematical structure behind a commonly used regularization technique. Edward Witten's *Homework* problems are a useful supplement to all of these lectures. We provide solutions to most of the problems.

Part 3 treats the related subjects of conformal field theory and perturbative string theory. Gawedzki's *Lectures on Conformal Field Theory* begins with a discussion of free fields, including circle- and torus-valued free fields. Then he presents a general axiomatic approach to two-dimensional conformal models. Perturbative aspects of the two-dimensional sigma model—regularization and renormalization—are treated in his third lecture. Finally, he takes up the Wess-Zumino-Witten sigma model (the sigma model with target a compact group) which can be described non-perturbatively using harmonic analysis on the loop group of the compact group. D'Hoker's *String Theory* is a comprehensive introduction to the subject. He begins with a treatment of the Hilbert space associated to the free (non-interacting) bosonic string, showing among other things that $D = 26$ is the critical dimension in which the physical "Hilbert" space has no negative norm states. Next come interacting bosonic strings, where the topology of Riemann surfaces, the moduli space of conformal structures, and the space of Weyl rescalings all play a role. The bosonic discussion concludes with Faddeev-Popov ghosts and BRST quantization, which are used to properly treat the diffeomorphism group of the surface in the theory. Then the lectures turn to superstrings, introducing the various species: Type I, Type IIA, Type IIB, and the heterotic strings. There is a detailed study of the perturbative properties of these various theories to one-loop. Finally, the lectures end with a general introductory discussion of supersymmetry and supergravity. Two mathematical texts follow D'Hoker's lectures. Pierre Deligne's brief *Super Gravity* explains the two-dimensional example important in perturbative string theory. *Notes on 2D Conformal Field Theory and String Theory*, by Dennis Gaitsgory, grew out of an effort to record solutions to the Exercises in D'Hoker's lectures. It explains many aspects of conformal field theory in the language of D-modules. Part 3 ends with *Kaluza-Klein Compactifications, Supersymmetry, and Calabi-Yau Spaces* by Andrew Strominger. These lectures discuss compactifying a theory of gravity on a higher dimensional spacetime along the fibers of a fibration. As a warmup he presents the classical example of Kaluza-Klein, where the fibers are circles, and then moves to the more recent supersymmetric theories compactified along Calabi-Yau complex three-folds.

Although non-perturbative aspects enter in several places to this point, it is only in Part 4 that they are the focus. The two lecture series presented here include results that were discovered in the mid '90s. *Dynamics of Quantum Field Theory*, by Edward Witten, is a systematic semester-length course which presents important old examples as well as new ones. The first lectures cover some formal aspects—spontaneous symmetry breaking and Goldstone's theorem, gauge symmetry breaking, BRST quantization of gauge theories—which for the most part do not appear elsewhere. The non-perturbative discussion begins with two dimensional bosonic sigma models, particularly in the large N limit. Next come various aspects of two- and three-dimensional theories: the Bose-Fermi correspondence in dimension two, gauge theories with Wilson and 't Hooft lines, confinement, abelian duality (including four dimensions), and solitons. There follows a mathematical application of two-dimensional gauge theories to certain moduli spaces attached to surfaces. Here the Cartan model of equivariant cohomology is related to path integrals in gauge theory. At this point supersymmetric field theories are introduced. Several lectures are devoted to two-dimensional supersymmetric gauge theories, particularly with $N = 2$ supersymmetry. Here we encounter topological twisting, the chiral ring of operators, quantum cohomology, and Landau Ginzburg models. The last part of the course deals with gauge theories in four dimensions. After a general discussion, the focus is again $N = 2$ supersymmetric theories, culminating with the Seiberg-Witten relationship between $SU(2)$-gauge theory (including Donaldson theory of four-manifolds) and $N = 2$ supersymmetric $U(1)$-gauge theory with matter (including the Seiberg-Witten invariants of four-manifolds). At the end of the lectures are Exercises and some solutions by Dan Freed; the Superhomework was also given as an accompaniment to the course. The other text in Part 4 is a series of three lectures by Nathan Seiberg called *Dynamics of $N = 1$ Supersymmetric Field Theories*. He reviews some general features—effective actions and nonrenormalization theorems, anomalies, confinement—for Wess-Zumino models and gauge theories. The main subject is supersymmetric quantum chromodynamics, with varying amount of matter. There is a wide variety of nonperturbative behavior exhibited by these theories, which provide a nice body of examples for many of the concepts quantum field theory. One notable feature is a nonabelian version of electric-magnetic duality. In the lectures the gauge group is the unitary group. In his solution to the Exercises, Siye Wu explains the theories with symplectic and orthogonal gauge groups.

The *Glossary* collects terms from physics which may be unfamiliar to the mathematical reader. Our capsule definitions should help bridge the gap between mathematics and physics vocabulary.

Quantum field theory and string theory are vast subjects, and it did not make sense to collect a comprehensive bibliography for these volumes. Rather, at the end of each lecture we often give a few references which are most relevant, or most accessible, and which are an entry into the literature. To our colleagues who feel slighted by omission we offer an apology: your work has become so standard that an explicit reference is superfluous.

As we did not set out to write a textbook, which would have been inappropriate for a myriad of reasons, a diagram of dependencies of various sections would not embed in a space of less than 10 dimensions. To help the reader make his way we give cross references at the end of each lecture and in many places in the text. These cross-references include the volume number. A list of cross-reference codes is printed facing the back cover of each volume.

For the most part this is physics written by mathematicians for mathematicians (read: the blind leading the blind). For that reason, throughout these volumes we have assumed without reference standard mathematical facts which are covered in textbooks and other literature. On the other hand, some very basic concepts in physics are explained. Certainly we did not attempt to explain everything, and the reader is well-advised to learn/review standard topics like special relativity, classical electromagnetism, etc. One basic prerequisite is a familiarity with at least the general framework of quantum mechanics.

There are many topics not covered in the Special Year which would have been natural to include. For example, the perturbative quantization of gauge theories is not treated systematically. Our mathematical discussion of supersymmetry in Part 1 contains nothing about supergravity (though there is a short piece about it in Part 3). Nor is there anything here about statistical mechanics. Although the lectures on quantum field theory reached quite modern developments, the exciting work over the past few years in nonperturbative string theory and beyond are outside the scope of these volumes. We offer no apologies for these omissions, though; the books are weighty enough as they are and should stimulate ample exercise of one sort or another.

A casual browse through these pages confirms what the reader must have gleaned from our blow-by-blow description: there is a great diversity of material here. The diversity of subject matter reflects the nature of the field; the rich mix of mathematical topics is quite typical. On the other hand, the diversity of quality is the predictable outcome of writing a book while learning the material. Most of all there is a diversity in the level of mathematical rigor and understanding. The task of absorbing physical intuition into mainstream mathematics is very much a work in progress. We offer this collection as a contribution to that effort.