Chapter 2

Basic notions of representation theory

2.1. What is representation theory?

In technical terms, representation theory studies representations of associative algebras. Its general content can be very briefly summarized as follows.

An **associative algebra** over a field k is a vector space A over k equipped with an associative bilinear multiplication $a, b \mapsto ab$, $a, b \in A$. We will always consider associative algebras with unit, i.e., with an element 1 such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$. A basic example of an associative algebra is the algebra $\mathrm{End}V$ of linear operators from a vector space V to itself. Other important examples include algebras defined by generators and relations, such as group algebras and universal enveloping algebras of Lie algebras.

A **representation** of an associative algebra A (also called a **left A-module**) is a vector space V equipped with a homomorphism $\rho : A \to \mathrm{End}V$, i.e., a linear map preserving the multiplication and unit.

A **subrepresentation** of a representation V is a subspace $U \subset V$ which is invariant under all operators $\rho(a)$, $a \in A$. Also, if V_1, V_2 are two representations of A, then the **direct sum** $V_1 \oplus V_2$ has an obvious structure of a representation of A.

A nonzero representation V of A is said to be **irreducible** if its only subrepresentations are 0 and V itself, and it is said to be **indecomposable** if it cannot be written as a direct sum of two nonzero subrepresentations. Obviously, irreducible implies indecomposable, but not vice versa.

Typical problems of representation theory are as follows:

(1) Classify irreducible representations of a given algebra A.

(2) Classify indecomposable representations of A.

(3) Do (1) and (2) restricting to finite dimensional representations.

As mentioned above, the algebra A is often given to us by generators and relations. For example, the universal enveloping algebra U of the Lie algebra $\mathfrak{sl}(2)$ is generated by h, e, f with defining relations

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h. \tag{2.1.1}$$

This means that the problem of finding, say, N-dimensional representations of A reduces to solving a bunch of nonlinear algebraic equations with respect to a bunch of unknown $N \times N$ matrices, for example system (2.1.1) with respect to unknown matrices h, e, f.

It is really striking that such, at first glance hopelessly complicated, systems of equations can in fact be solved completely by methods of representation theory! For example, we will prove the following theorem.

Theorem 2.1.1. *Let $k = \mathbb{C}$ be the field of complex numbers. Then:*

(i) The algebra U has exactly one irreducible representation V_d of each dimension, up to equivalence; this representation is realized in the space of homogeneous polynomials of two variables x, y of degree $d-1$ and is defined by the formulas

$$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad \rho(e) = x\frac{\partial}{\partial y}, \quad \rho(f) = y\frac{\partial}{\partial x}.$$

(ii) Any indecomposable finite dimensional representation of U is irreducible. That is, any finite dimensional representation of U is a direct sum of irreducible representations.

As another example consider the representation theory of quivers.

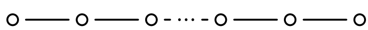
A **quiver** is an oriented graph Q (which we will assume to be finite). A **representation** of Q over a field k is an assignment of a k-vector space V_i to every vertex i of Q and of a linear operator $A_h : V_i \to V_j$ to every directed edge h going from i to j (loops and multiple edges are allowed). We will show that a representation of a quiver Q is the same thing as a representation of a certain algebra P_Q called the path algebra of Q. Thus one may ask: what are the indecomposable finite dimensional representations of Q?

More specifically, let us say that Q is **of finite type** if it has finitely many indecomposable representations.

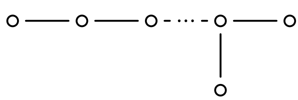
We will prove the following striking theorem, proved by P. Gabriel in early 1970s:

Theorem 2.1.2. *The finite type property of Q does not depend on the orientation of edges. The connected graphs that yield quivers of finite type are given by the following list:*

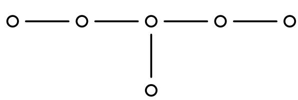
- A_n :



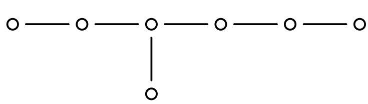
- D_n:



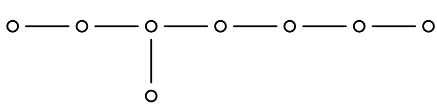
- E_6



- E_7



- E_8



The graphs listed in the theorem are called (simply laced) **Dynkin diagrams**. These graphs arise in a multitude of classification problems in mathematics, such as the classification of simple Lie algebras, singularities, platonic solids, reflection groups, etc. In fact, if we needed to make contact with an alien civilization and show them how sophisticated our civilization is, perhaps showing them Dynkin diagrams would be the best choice!

As a final example consider the representation theory of finite groups, which is one of the most fascinating chapters of representation theory. In this theory, one considers representations of the group algebra $A = \mathbb{C}[G]$ of a finite group G — the algebra with basis $a_g, g \in G$, and multiplication law $a_g a_h = a_{gh}$. We will show that any finite dimensional representation of A is a direct sum of irreducible representations, i.e., the notions of an irreducible and indecomposable representation are the same for A (Maschke's theorem). Another striking result discussed below is the Frobenius divisibility theorem: the dimension of any irreducible representation of A divides the order of G. Finally, we will show how to use the representation theory of finite groups to prove Burnside's theorem: any finite group of order $p^a q^b$, where p, q are primes, is solvable. Note that this theorem does not mention representations, which are used only in its proof; a purely group-theoretical proof of this theorem (not using representations) exists but is much more difficult!

2.2. Algebras

Let us now begin a systematic discussion of representation theory.

Let k be a field. Unless stated otherwise, we will always assume that k is algebraically closed, i.e., any nonconstant polynomial with coefficients in k has a root in k. The main example is the field of complex numbers $\mathbb{C}$, but we will also consider fields of characteristic p, such as the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ of p elements.

Definition 2.2.1. An **associative algebra** over k is a vector space A over k together with a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, such that $(ab)c = a(bc)$.

Definition 2.2.2. A **unit** in an associative algebra A is an element $1 \in A$ such that $1a = a1 = a$.

Proposition 2.2.3. *If a unit exists, it is unique.*

Proof. Let $1, 1'$ be two units. Then $1 = 11' = 1'$. □

From now on, by an algebra A we will mean an associative algebra with a unit. We will also assume that $A \neq 0$.

Example 2.2.4. Here are some examples of algebras over k:

1. $A = k$.

2. $A = k[x_1, \ldots, x_n]$ — the algebra of polynomials in variables $x_1, \ldots, x_n$.

3. $A = \mathrm{End} V$ — the algebra of endomorphisms of a vector space V over k (i.e., linear maps, or operators, from V to itself). The multiplication is given by composition of operators.

4. The **free algebra** $A = k\langle x_1, \ldots, x_n\rangle$. A basis of this algebra consists of words in letters $x_1, \ldots, x_n$, and multiplication in this basis is simply the concatenation of words.

5. The **group algebra** $A = k[G]$ of a group G. Its basis is $\{a_g, g \in G\}$, with multiplication law $a_g a_h = a_{gh}$.

Definition 2.2.5. An algebra A is **commutative** if $ab = ba$ for all $a, b \in A$.

For instance, in the above examples, A is commutative in cases 1 and 2 but not commutative in cases 3 (if $\dim V > 1$) and 4 (if $n > 1$). In case 5, A is commutative if and only if G is commutative.

Definition 2.2.6. A **homomorphism of algebras** $f : A \to B$ is a linear map such that $f(xy) = f(x)f(y)$ for all $x, y \in A$ and $f(1) = 1$.

2.3. Representations

Definition 2.3.1. A **representation** of an algebra A (also called a **left A-module**) is a vector space V together with a homomorphism of algebras $\rho : A \to \mathrm{End} V$.

Similarly, a **right A-module** is a space V equipped with an antihomomorphism $\rho : A \to \mathrm{End}V$; i.e., ρ satisfies $\rho(ab) = \rho(b)\rho(a)$ and $\rho(1) = 1$.

The usual abbreviated notation for $\rho(a)v$ is av for a left module and va for a right module. Then the property that ρ is an (anti)homomorphism can be written as a kind of associativity law: $(ab)v = a(bv)$ for left modules, and $(va)b = v(ab)$ for right modules.

Remark 2.3.2. Let M be a left module over a commutative ring A. Then one can regard M as a right A-module, with $ma := am$. Similarly, any right A-module can be regarded as a left A-module. For this reason, for commutative rings one does not distinguish between left and right A-modules and just calls them A-modules.

Here are some examples of representations.

Example 2.3.3. 1. $V = 0$.

2. $V = A$, and $\rho : A \to \mathrm{End}A$ is defined as follows: $\rho(a)$ is the operator of left multiplication by a, so that $\rho(a)b = ab$ (the usual product). This representation is called the **regular** representation of A. Similarly, one can equip A with a structure of a right A-module by setting $\rho(a)b := ba$.

3. $A = k$. Then a representation of A is simply a vector space over k.

4. $A = k\langle x_1, \ldots, x_n\rangle$. Then a representation of A is just a vector space V over k with a collection of arbitrary linear operators $\rho(x_1), \ldots, \rho(x_n) : V \to V$ (explain why!).

Definition 2.3.4. A **subrepresentation** of a representation V of an algebra A is a subspace $W \subset V$ which is invariant under all the operators $\rho(a) : V \to V$, $a \in A$.

For instance, 0 and V are always subrepresentations.

Definition 2.3.5. A representation $V \neq 0$ of A is **irreducible** (or **simple**) if the only subrepresentations of V are 0 and V.

Definition 2.3.6. Let V_1, V_2 be two representations of an algebra A. A **homomorphism** (or **intertwining operator**) $\phi : V_1 \to V_2$ is a linear operator which commutes with the action of A, i.e., $\phi(av) = a\phi(v)$ for any $v \in V_1$. A homomorphism ϕ is said to be an **isomorphism of representations** if it is an isomorphism of vector spaces. The set (space) of all homomorphisms of representations $V_1 \to V_2$ is denoted by $\mathrm{Hom}_A(V_1, V_2)$.

Note that if a linear operator $\phi : V_1 \to V_2$ is an isomorphism of representations, then so is the linear operator $\phi^{-1} : V_2 \to V_1$ (check it!).

Two representations between which there exists an isomorphism are said to be isomorphic. For practical purposes, two isomorphic representations may be regarded as "the same", although there could be subtleties related to the fact that an isomorphism between two representations, when it exists, is not unique.

Definition 2.3.7. Let V_1, V_2 be representations of an algebra A. Then the space $V_1 \oplus V_2$ has an obvious structure of a representation of A, given by $a(v_1 \oplus v_2) = av_1 \oplus av_2$. This representation is called the **direct sum** of V_1 and V_2.

Definition 2.3.8. A nonzero representation V of an algebra A is said to be **indecomposable** if it is not isomorphic to a direct sum of two nonzero representations.

It is obvious that an irreducible representation is indecomposable. On the other hand, we will see below that the converse statement is false in general.

One of the main problems of representation theory is to classify irreducible and indecomposable representations of a given algebra up to isomorphism. This problem is usually hard and often can be solved only partially (say, for finite dimensional representations). Below we will see a number of examples in which this problem is partially or fully solved for specific algebras.

We will now prove our first result — Schur's lemma. Although it is very easy to prove, it is fundamental in the whole subject of representation theory.

Proposition 2.3.9 (Schur's lemma)**.** *Let V_1, V_2 be representations of an algebra A over any field F (which need not be algebraically closed).*

Let $\phi : V_1 \to V_2$ be a nonzero homomorphism of representations. Then:

(i) If V_1 is irreducible, ϕ is injective.

(ii) If V_2 is irreducible, ϕ is surjective.

Thus, if both V_1 and V_2 are irreducible, ϕ is an isomorphism.

Proof. (i) The kernel K of ϕ is a subrepresentation of V_1. Since $\phi \neq 0$, this subrepresentation cannot be V_1. So by irreducibility of V_1 we have $K = 0$.

(ii) The image I of ϕ is a subrepresentation of V_2. Since $\phi \neq 0$, this subrepresentation cannot be 0. So by irreducibility of V_2 we have $I = V_2$. □

Corollary 2.3.10 (Schur's lemma for algebraically closed fields)**.** *Let V be a finite dimensional irreducible representation of an algebra A over an algebraically closed field k, and let $\phi : V \to V$ be an intertwining operator. Then $\phi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in k$ (a scalar operator).*

Remark 2.3.11. Note that this corollary is false over the field of real numbers: it suffices to take $A = \mathbb{C}$ (regarded as an $\mathbb{R}$-algebra) and $V = A$.

Proof. Let λ be an eigenvalue of ϕ (a root of the characteristic polynomial of ϕ). It exists since k is an algebraically closed field. Then the operator $\phi - \lambda\,\mathrm{Id}$ is an intertwining operator $V \to V$, which is not an isomorphism (since its determinant is zero). Thus by Proposition 2.3.9 this operator is zero, hence the result. □

Corollary 2.3.12. *Let A be a commutative algebra. Then every irreducible finite dimensional representation V of A is 1-dimensional.*

Remark 2.3.13. Note that a 1-dimensional representation of any algebra is automatically irreducible.

Proof. Let V be irreducible. For any element $a \in A$, the operator $\rho(a) : V \to V$ is an intertwining operator. Indeed,

$$\rho(a)\rho(b)v = \rho(ab)v = \rho(ba)v = \rho(b)\rho(a)v$$

(the second equality is true since the algebra is commutative). Thus, by Schur's lemma, $\rho(a)$ is a scalar operator for any $a \in A$. Hence every subspace of V is a subrepresentation. But V is irreducible, so 0 and V are the only subspaces of V. This means that $\dim V = 1$ (since $V \neq 0$). □

Example 2.3.14. 1. $A = k$. Since representations of A are simply vector spaces, $V = A$ is the only irreducible and the only indecomposable representation.

2. $A = k[x]$. Since this algebra is commutative, the irreducible representations of A are its 1-dimensional representations. As we discussed above, they are defined by a single operator $\rho(x)$. In the 1-dimensional case, this is just a number from k. So all the irreducible representations of A are $V_\lambda = k$, $\lambda \in k$, in which the action of A is defined by $\rho(x) = \lambda$. Clearly, these representations are pairwise nonisomorphic.

The classification of indecomposable representations of $k[x]$ is more interesting. To obtain it, recall that any linear operator on a finite dimensional vector space V can be brought to Jordan normal form. More specifically, recall that the Jordan block $J_{\lambda,n}$ is the operator on k^n which in the standard basis is given by the formulas $J_{\lambda,n}e_i = \lambda e_i + e_{i-1}$ for $i > 1$ and $J_{\lambda,n}e_1 = \lambda e_1$. Then for any linear operator $B : V \to V$ there exists a basis of V such that the matrix of B in this basis is a direct sum of Jordan blocks. This implies that all the indecomposable representations of A are $V_{\lambda,n} = k^n$, $\lambda \in k$, with $\rho(x) = J_{\lambda,n}$. The fact that these representations are indecomposable and pairwise nonisomorphic follows from the Jordan normal form theorem (which in particular says that the Jordan normal form of an operator is unique up to permutation of blocks).

This example shows that an indecomposable representation of an algebra need not be irreducible.

3. The group algebra $A = k[G]$, where G is a group. A representation of A is the same thing as a representation of G, i.e., a vector space V together with a group homomorphism $\rho : G \to \mathrm{Aut}(V)$, where $\mathrm{Aut}(V) = GL(V)$ denotes the group of invertible linear maps from the space V to itself (the **general linear group** of V).

Problem 2.3.15. Let V be a nonzero finite dimensional representation of an algebra A. Show that it has an irreducible subrepresentation. Then show by example that this does not always hold for infinite dimensional representations.

Problem 2.3.16. Let A be an algebra over a field k. The center $Z(A)$ of A is the set of all elements $z \in A$ which commute with all elements of A. For example, if A is commutative, then $Z(A) = A$.

(a) Show that if V is an irreducible finite dimensional representation of A, then any element $z \in Z(A)$ acts in V by multiplication by some scalar $\chi_V(z)$. Show that $\chi_V : Z(A) \to k$ is a homomorphism. It is called the **central character** of V.

(b) Show that if V is an indecomposable finite dimensional representation of A, then for any $z \in Z(A)$, the operator $\rho(z)$ by which z acts in V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible subrepresentation of V. Thus $\chi_V : Z(A) \to k$ is a homomorphism, which is again called the central character of V.

(c) Does $\rho(z)$ in (b) have to be a scalar operator?

Problem 2.3.17. Let A be an associative algebra, and let V be a representation of A. By $\mathrm{End}_A(V)$ one denotes the algebra of all homomorphisms of representations $V \to V$. Show that $\mathrm{End}_A(A) = A^{\mathrm{op}}$, the algebra A with opposite multiplication.

Problem 2.3.18. Prove the following "infinite dimensional Schur lemma" (due to Dixmier): Let A be an algebra over $\mathbb{C}$ and let V be an irreducible representation of A with at most countable basis. Then any homomorphism of representations $\phi : V \to V$ is a scalar operator.

Hint: By the usual Schur's lemma, the algebra $D := \mathrm{End}_A(V)$ is an algebra with division. Show that D is at most countably dimensional. Suppose ϕ is not a scalar, and consider the subfield $\mathbb{C}(\phi) \subset D$. Show that $\mathbb{C}(\phi)$ is a transcendental extension of $\mathbb{C}$. Derive from this that $\mathbb{C}(\phi)$ is uncountably dimensional and obtain a contradiction.

2.4. Ideals

A **left ideal** of an algebra A is a subspace $I \subseteq A$ such that $aI \subseteq I$ for all $a \in A$. Similarly, a **right ideal** of an algebra A is a subspace $I \subseteq A$ such that $Ia \subseteq I$ for all $a \in A$. A **two-sided ideal** is a subspace that is both a left and a right ideal.

Left ideals are the same as subrepresentations of the regular representation A. Right ideals are the same as subrepresentations of the regular representation of the opposite algebra A^{op}.

Below are some examples of ideals:

- If A is any algebra, 0 and A are two-sided ideals. An algebra A is called **simple** if 0 and A are its only two-sided ideals.
- If $\phi : A \to B$ is a homomorphism of algebras, then $\ker \phi$ is a two-sided ideal of A.
- If S is any subset of an algebra A, then the two-sided ideal **generated** by S is denoted by $\langle S \rangle$ and is the span of elements of the form asb, where $a, b \in A$ and $s \in S$. Similarly, we can define $\langle S \rangle_\ell = \mathrm{span}\{as\}$ and $\langle S \rangle_r = \mathrm{span}\{sb\}$, the left, respectively right, ideal generated by S.

Problem 2.4.1. A **maximal** ideal in a ring A is an ideal $I \neq A$ such that any strictly larger ideal coincides with A. (This definition is made for left, right, or two-sided ideals.) Show that any unital ring has a maximal left, right, and two-sided ideal. (Hint: Use Zorn's lemma.)

2.5. Quotients

Let A be an algebra and let I be a two-sided ideal in A. Then A/I is the set of (additive) cosets of I. Let $\pi : A \to A/I$ be the quotient map. We can define multiplication in A/I by $\pi(a) \cdot \pi(b) := \pi(ab)$. This is well defined because if $\pi(a) = \pi(a')$, then

$$\pi(a'b) = \pi(ab + (a' - a)b) = \pi(ab) + \pi((a' - a)b) = \pi(ab)$$

because $(a' - a)b \in Ib \subseteq I = \ker \pi$, as I is a right ideal; similarly, if $\pi(b) = \pi(b')$, then

$$\pi(ab') = \pi(ab + a(b' - b)) = \pi(ab) + \pi(a(b' - b)) = \pi(ab)$$

because $a(b' - b) \in aI \subseteq I = \ker \pi$, as I is also a left ideal. Thus, A/I is an algebra.

Similarly, if V is a representation of A and $W \subset V$ is a subrepresentation, then V/W is also a representation. Indeed, let $\pi : V \to V/W$ be the quotient map, and set $\rho_{V/W}(a)\pi(x) := \pi(\rho_V(a)x)$.

Above we noted that left ideals of A are subrepresentations of the regular representation of A, and vice versa. Thus, if I is a left ideal in A, then A/I is a representation of A.

Problem 2.5.1. Let $A = k[x_1, \ldots, x_n]$ and let $I \neq A$ be any ideal in A containing all homogeneous polynomials of degree $\geq N$. Show that A/I is an indecomposable representation of A.

Problem 2.5.2. Let $V \neq 0$ be a representation of A. We say that a vector $v \in V$ is **cyclic** if it generates V, i.e., $Av = V$. A representation admitting a cyclic vector is said to be **cyclic**. Show the following:

(a) V is irreducible if and only if all nonzero vectors of V are cyclic.

(b) V is cyclic if and only if it is isomorphic to A/I, where I is a left ideal in A.

(c) Give an example of an indecomposable representation which is not cyclic.

Hint: Let $A = \mathbb{C}[x, y]/I_2$, where I_2 is the ideal spanned by homogeneous polynomials of degree ≥ 2 (so A has a basis $1, x, y$). Let $V = A^*$ be the space of linear functionals on A, with the action of A given by $(\rho(a)f)(b) = f(ba)$. Show that V provides such an example.

2.6. Algebras defined by generators and relations

If $f_1, \ldots, f_m$ are elements of the free algebra $k\langle x_1, \ldots, x_n\rangle$, we say that the algebra $A := k\langle x_1, \ldots, x_n\rangle/\langle\{f_1, \ldots, f_m\}\rangle$ is **generated** by $x_1, \ldots, x_n$ with **defining relations** $f_1 = 0, \ \ldots, \ f_m = 0$.

2.7. Examples of algebras

The following two examples are among the simplest interesting examples of noncommutative associative algebras:

(1) the **Weyl algebra**, $k\langle x, y\rangle/\langle yx - xy - 1\rangle$;

(2) the q-**Weyl algebra**, generated by x, x^{-1}, y, y^{-1} with defining relations $yx = qxy$ and $xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1$.

Proposition 2.7.1. *(i) A basis for the Weyl algebra A is $\{x^i y^j, i, j \geq 0\}$.*

(ii) A basis for the q-Weyl algebra A_q is $\{x^i y^j, i, j \in \mathbb{Z}\}$.

Proof. (i) First let us show that the elements $x^i y^j$ are a spanning set for A. To do this, note that any word in x, y can be ordered to have all the x's on the left of the y's, at the cost of interchanging some x and y. Since $yx - xy = 1$, this will lead to error terms, but these terms will be sums of monomials that have a smaller number of letters x, y than the original word. Therefore, continuing this process, we can order everything and represent any word as a linear combination of $x^i y^j$.

The proof that $x^i y^j$ are linearly independent is based on representation theory. Namely, let a be a variable, and let $E = t^a k[a][t, t^{-1}]$ (here t^a is just a formal symbol, so really $E = k[a][t, t^{-1}]$). Then E is a representation of A with action given by $xf = tf$ and $yf = \frac{df}{dt}$ (where $\frac{d(t^{a+n})}{dt} := (a + n)t^{a+n-1}$). Suppose now that we have a nontrivial linear relation $\sum c_{ij} x^i y^j = 0$. Then the operator

$$L = \sum c_{ij} t^i \left(\frac{d}{dt}\right)^j$$

acts by zero in E. Let us write L as

$$L = \sum_{j=0}^{r} Q_j(t) \left(\frac{d}{dt}\right)^j,$$

where $Q_r \neq 0$. Then we have

$$Lt^a = \sum_{j=0}^{r} Q_j(t) a(a - 1) \ldots (a - j + 1)t^{a-j}.$$

This must be zero, so we have $\sum_{j=0}^{r} Q_j(t)a(a-1)\dots(a-j+1)t^{-j} = 0$ in $k[a][t, t^{-1}]$. Taking the leading term in a, we get $Q_r(t) = 0$, a contradiction.

(ii) Any word in x, y, x^{-1}, y^{-1} can be ordered at the cost of multiplying it by a power of q. This easily implies both the spanning property and the linear independence. □

Remark 2.7.2. The proof of (i) shows that the Weyl algebra A can be viewed as the algebra of polynomial differential operators in one variable t.

The proof of (i) also brings up the notion of a faithful representation.

Definition 2.7.3. A representation $\rho : A \to \operatorname{End} V$ of an algebra A is **faithful** if ρ is injective.

For example, $k[t]$ is a faithful representation of the Weyl algebra if k has characteristic zero (check it!), but not in characteristic p, where $(d/dt)^p Q = 0$ for any polynomial Q. However, the representation $E = t^a k[a][t, t^{-1}]$, as we've seen, is faithful in any characteristic.

Problem 2.7.4. Let A be the Weyl algebra.

(a) If $\operatorname{char} k = 0$, what are the finite dimensional representations of A? What are the two-sided ideals in A?

Hint: For the first question, use the fact that for two square matrices B, C, $\operatorname{Tr}(BC) = \operatorname{Tr}(CB)$. For the second question, show that any nonzero two-sided ideal in A contains a nonzero polynomial in x, and use this to characterize this ideal.

Suppose for the rest of the problem that $\operatorname{char} k = p$.

(b) What is the center of A?

Hint: Show that x^p and y^p are central elements.

(c) Find all irreducible finite dimensional representations of A.

Hint: Let V be an irreducible finite dimensional representation of A, and let v be an eigenvector of y in V. Show that the collection of vectors $\{v, xv, x^2v, \dots, x^{p-1}v\}$ is a basis of V.

Problem 2.7.5. Let q be a nonzero complex number, and let A be the q-Weyl algebra over $\mathbb{C}$.

(a) What is the center of A for different q? If q is not a root of unity, what are the two-sided ideals in A?

(b) For which q does this algebra have finite dimensional representations?

Hint: Use determinants.

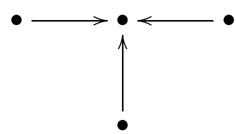
(c) Find all finite dimensional irreducible representations of A for such q.

Hint: This is similar to part (c) of the previous problem.

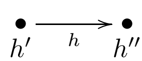
2.8. Quivers

Definition 2.8.1. A **quiver** Q is a directed graph, possibly with self-loops and/or multiple edges between two vertices.

Example 2.8.2.



We denote the set of vertices of the quiver Q as I and the set of edges as E. For an edge $h \in E$, let h', h'' denote the source and target of h, respectively:



Definition 2.8.3. A **representation of a quiver** Q is an assignment to each vertex $i \in I$ of a vector space V_i and to each edge $h \in E$ of a linear map $x_h : V_{h'} \longrightarrow V_{h''}$.

It turns out that the theory of representations of quivers is a part of the theory of representations of algebras in the sense that for each quiver Q, there exists a certain algebra P_Q, called the path algebra of Q, such that a representation of the quiver Q is "the same" as a representation of the algebra P_Q. We shall first define the path algebra of a quiver and then justify our claim that representations of these two objects are "the same".

Definition 2.8.4. The **path algebra** P_Q of a quiver Q is the algebra whose basis is formed by oriented paths in Q, including the trivial paths p_i, $i \in I$, corresponding to the vertices of Q, and multiplication is the concatenation of paths: ab is the path obtained by first tracing b and then a. If two paths cannot be concatenated, the product is defined to be zero.

Remark 2.8.5. It is easy to see that for a finite quiver $\sum_{i \in I} p_i = 1$, so P_Q is an algebra with unit.

Problem 2.8.6. Show that the algebra P_Q is generated by p_i for $i \in I$ and a_h for $h \in E$ with the following defining relations:

(1) $p_i^2 = p_i$, $p_i p_j = 0$ for $i \neq j$,

(2) $a_h p_{h'} = a_h$, $a_h p_j = 0$ for $j \neq h'$,

(3) $p_{h''} a_h = a_h$, $p_i a_h = 0$ for $i \neq h''$.

We now justify our statement that a representation of a quiver is the same thing as a representation of the path algebra of a quiver.

Let $\mathbf{V}$ be a representation of the path algebra P_Q. From this representation, we can construct a representation of Q as follows: let $V_i = p_i \mathbf{V}$, and for any edge h, let $x_h = a_h|_{p_{h'}\mathbf{V}} : p_{h'}\mathbf{V} \longrightarrow p_{h''}\mathbf{V}$ be the operator corresponding to the one-edge path h.

Similarly, let (V_i, x_h) be a representation of a quiver Q. From this representation, we can construct a representation of the path algebra P_Q: let $\mathbf{V} = \bigoplus_i V_i$, let $p_i : \mathbf{V} \to V_i \to \mathbf{V}$ be the projection onto V_i, and for any path $p = h_1 \dots h_m$ let $a_p = x_{h_1} \dots x_{h_m} : V_{h'_m} \to V_{h''_1}$ be the composition of the operators corresponding to the edges occurring in p (and the action of this operator on the other V_i is zero).

It is clear that the above assignments $\mathbf{V} \mapsto (p_i \mathbf{V})$ and $(V_i) \mapsto \bigoplus_i V_i$ are inverses of each other. Thus, we have a bijection between isomorphism classes of representations of the algebra P_Q and of the quiver Q.

Remark 2.8.7. In practice, it is generally easier to consider a representation of a quiver as in Definition 2.8.3.

We lastly define several previous concepts in the context of quiver representations.

Definition 2.8.8. A **subrepresentation** of a representation (V_i, x_h) of a quiver Q is a representation (W_i, x'_h) where $W_i \subseteq V_i$ for all $i \in I$ and where $x_h(W_{h'}) \subseteq W_{h''}$ and $x'_h = x_h|_{W_{h'}} : W_{h'} \longrightarrow W_{h''}$ for all $h \in E$.

Definition 2.8.9. The **direct sum** of two representations (V_i, x_h) and (W_i, y_h) is the representation $(V_i \oplus W_i, x_h \oplus y_h)$.

As with representations of algebras, a nonzero representation (V_i) of a quiver Q is said to be irreducible if its only subrepresentations are (0) and (V_i) itself, and it is said to be indecomposable if it is not isomorphic to a direct sum of two nonzero representations.

Definition 2.8.10. Let (V_i, x_h) and (W_i, y_h) be representations of the quiver Q. A **homomorphism** $\varphi : (V_i) \longrightarrow (W_i)$ of quiver representations is a collection of maps $\varphi_i : V_i \longrightarrow W_i$ such that $y_h \circ \varphi_{h'} = \varphi_{h''} \circ x_h$ for all $h \in E$.

Problem 2.8.11. Let A be a $\mathbb{Z}_+$-graded algebra, i.e., $A = \bigoplus_{n \geq 0} A[n]$, and $A[n] \cdot A[m] \subset A[n+m]$. If $A[n]$ is finite dimensional, it is useful to consider the Hilbert series $h_A(t) = \sum \dim A[n] t^n$ (the generating function of dimensions of $A[n]$). Often this series converges to a rational function, and the answer is written in the form of such a function. For example, if $A = k[x]$ and $deg(x^n) = n$, then

$$h_A(t) = 1 + t + t^2 + \cdots + t^n + \cdots = \frac{1}{1-t}.$$

Find the Hilbert series of the following graded algebras:

(a) $A = k[x_1, \ldots, x_m]$ (where the grading is by degree of polynomials).

(b) $A = k\langle x_1, \ldots, x_m \rangle$ (the grading is by length of words).

(c) A is the exterior (= Grassmann) algebra $\wedge_k[x_1, \ldots, x_m]$ generated over some field k by $x_1, \ldots, x_m$ with the defining relations $x_i x_j + x_j x_i = 0$ and $x_i^2 = 0$ for all i, j (the grading is by degree).

(d) A is the path algebra P_Q of a quiver Q (the grading is defined by $\deg(p_i) = 0$, $\deg(a_h) = 1$).

Hint: The closed answer is written in terms of the adjacency matrix M_Q of Q.

2.9. Lie algebras

Let $\mathfrak{g}$ be a vector space over a field k, and let $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be a skew-symmetric bilinear map. (That is, $[a,a] = 0$, and hence $[a,b] = -[b,a]$.)

Definition 2.9.1. $(\mathfrak{g}, [\ ,\ ])$ is a **Lie algebra** if $[\ ,\ ]$ satisfies the Jacobi identity

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0. \tag{2.9.1}$$

Example 2.9.2. Some examples of Lie algebras are:

(1) Any space $\mathfrak{g}$ with $[\ ,\ ] = 0$ (abelian Lie algebra).

(2) Any associative algebra A with $[a,b] = ab - ba$, in particular, the endomorphism algebra $A = \mathrm{End}(V)$, where V is a vector space. When such an A is regarded as a Lie algebra, it is often denoted by $\mathfrak{gl}(V)$ (general linear Lie algebra).

(3) Any subspace U of an associative algebra A such that $[a,b] \in U$ for all $a, b \in U$.

(4) The space $\mathrm{Der}(A)$ of derivations of an algebra A, i.e. linear maps $D : A \to A$ which satisfy the Leibniz rule:

$$D(ab) = D(a)b + aD(b).$$

(5) Any subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ which is closed under the commutator map $[\ ,\ ]$, i.e., such that $[a,b] \in \mathfrak{a}$ if $a, b \in \mathfrak{a}$. Such a subspace is called a **Lie subalgebra** of $\mathfrak{g}$.

Remark 2.9.3. Ado's theorem says that any finite dimensional Lie algebra is a Lie subalgebra of $\mathfrak{gl}(V)$ for a suitable finite dimensional vector space V.

Remark 2.9.4. Derivations are important because they are the "infinitesimal version" of automorphisms (i.e., isomorphisms onto itself). For example, assume that $g(t)$ is a differentiable family of automorphisms of a finite dimensional algebra A over $\mathbb{R}$ or $\mathbb{C}$ parametrized by $t \in (-\epsilon, \epsilon)$ such that $g(0) = \mathrm{Id}$. Then $D := g'(0) : A \to A$ is a derivation (check it!). Conversely, if $D : A \to A$ is a derivation, then e^{tD} is a 1-parameter family of automorphisms (give a proof!).

This provides a motivation for the notion of a Lie algebra. Namely, we see that Lie algebras arise as spaces of infinitesimal automorphisms (= derivations) of associative algebras. In fact, they similarly arise as spaces of derivations of any kind of linear algebraic structures, such as Lie algebras, Hopf algebras, etc., and for this reason play a very important role in algebra.

Here are a few more concrete examples of Lie algebras:

(1) $\mathbb{R}^3$ with $[u,v] = u \times v$, the cross-product of u and v.

(2) $\mathfrak{sl}(n)$, the set of $n \times n$ matrices with trace 0.
For example, $\mathfrak{sl}(2)$ has the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with relations

$$[h,e] = 2e, \ [h,f] = -2f, \ [e,f] = h.$$

(3) The Heisenberg Lie algebra $\mathcal{H}$ of matrices $\left(\begin{smallmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{smallmatrix}\right)$.
It has the basis

$$x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with relations $[y,x] = c$ and $[y,c] = [x,c] = 0$.

(4) The algebra $\mathrm{aff}(1)$ of matrices $\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$.
Its basis consists of $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, with $[X,Y] = Y$.

(5) $\mathfrak{so}(n)$, the space of skew-symmetric $n \times n$ matrices, with $[a,b] = ab - ba$.

Exercise 2.9.5. Show that example (1) is a special case of example (5) (for $n = 3$).

Definition 2.9.6. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras. A **homomorphism of Lie algebras** $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a linear map such that $\varphi([a,b]) = [\varphi(a), \varphi(b)]$.

Definition 2.9.7. A **representation** of a Lie algebra $\mathfrak{g}$ is a vector space V with a homomorphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \operatorname{End} V$.

Example 2.9.8. Some examples of representations of Lie algebras are:

(1) $V = 0$.
(2) Any vector space V with $\rho = 0$ (the trivial representation).
(3) The adjoint representation $V = \mathfrak{g}$ with $\rho(a)(b) := [a, b]$. That this is a representation follows from equation (2.9.1). Thus, the meaning of the Jacobi identity is that it is equivalent to the existence of the adjoint representation.

It turns out that a representation of a Lie algebra $\mathfrak{g}$ is the same thing as a representation of a certain associative algebra $\mathcal{U}(\mathfrak{g})$. Thus, as with quivers, we can view the theory of representations of Lie algebras as a part of the theory of representations of associative algebras.

Definition 2.9.9. Let $\mathfrak{g}$ be a Lie algebra with basis x_i and $[\ ,\ ]$ defined by $[x_i, x_j] = \sum_k c_{ij}^k x_k$. The **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ is the associative algebra generated by the x_i's with the defining relations $x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$.

Remark 2.9.10. This is not a very good definition since it depends on the choice of a basis. Later we will give an equivalent definition which will be basis-independent.

Exercise 2.9.11. Explain why a representation of a Lie algebra is the same thing as a representation of its universal enveloping algebra.

Example 2.9.12. The associative algebra $\mathcal{U}(\mathfrak{sl}(2))$ is the algebra generated by e, f, h, with relations

$$he - eh = 2e, \qquad hf - fh = -2f, \qquad ef - fe = h.$$

Example 2.9.13. The algebra $\mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is the Heisenberg Lie algebra, is the algebra generated by x, y, c with the relations

$$yx - xy = c, \qquad yc - cy = 0, \qquad xc - cx = 0.$$

Note that the Weyl algebra is the quotient of $\mathcal{U}(\mathcal{H})$ by the relation $c = 1$.

Remark 2.9.14. Lie algebras were introduced by Sophus Lie (see Section 2.10) as an infinitesimal version of **Lie groups** (in early texts they were called "infinitesimal groups" and were called Lie algebras by Hermann Weyl in honor of Lie). A Lie group is a group G which is also a manifold (i.e., a topological space which locally looks like $\mathbb{R}^n$) such that the multiplication operation is differentiable. In this case, one can define the algebra of smooth functions $C^\infty(G)$ which carries an action of G by right translations ($(g \circ f)(x) := f(xg)$), and the Lie algebra $\mathrm{Lie}(G)$ of G consists of derivations of this algebra which are invariant under this action (with the Lie bracket being the usual commutator of derivations). Clearly, such a derivation is determined by its action at the unit element $e \in G$, so $\mathrm{Lie}(G)$ can be identified as a vector space with the tangent space T_eG to G at e.

Sophus Lie showed that the attachment $G \mapsto \mathrm{Lie}(G)$ is a bijection between isomorphism classes of simply connected Lie groups (i.e., connected Lie groups on which every loop contracts to a point) and finite dimensional Lie algebras over $\mathbb{R}$. This allows one to study (differentiable) representations of Lie groups by studying representations of their Lie algebras, which is easier since Lie algebras are "linear" objects while Lie groups are "nonlinear". Namely, a finite dimensional representation of G can be differentiated at e to yield a representation of $\mathrm{Lie}(G)$, and conversely, a finite dimensional representation of $\mathrm{Lie}(G)$ can be exponentiated to give a representation of G. Moreover, this correspondence extends to certain classes of infinite dimensional representations.

The most important examples of Lie groups are **linear algebraic groups**, which are subgroups of $GL_n(\mathbb{R})$ defined by algebraic equations (such as, for example, the group of orthogonal matrices $O_n(\mathbb{R})$).

Also, given a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ (which, by Ado's theorem, can be any finite dimensional real Lie algebra), we can define G to be the subgroup of $GL_n(\mathbb{R})$ generated by the elements e^X, $X \in \mathfrak{g}$. One can show that this group has a natural structure of a connected Lie group, whose Lie algebra is $\mathfrak{g}$ (even though it is not always a closed subgroup). While this group is not always simply connected, its universal covering $\widetilde{G}$ is, and it is the Lie group corresponding to $\mathfrak{g}$ under Lie's correspondence.

For more on Lie groups and their relation to Lie algebras, the reader is referred to textbooks on this subject, e.g. [**Ki**].

2.10. Historical interlude: Sophus Lie's trials and transformations

To call Sophus Lie (1842–1899) an overachiever would be an understatement. Scoring first at the 1859 entrance examinations to the University of Christiania (now Oslo) in Norway, he was determined to finish first as well. When problems with his biology class derailed this project, Lie received only the second-highest graduation score. He became depressed, suffered from insomnia, and even contemplated suicide. At that time, he had no desire to become a mathematician. He began working as a mathematics tutor to support himself, read more and more on the subject, and eventually began publishing research papers. He was 26 when he finally decided to devote himself to mathematics.

The Norwegian government realized that the best way to educate their promising scientists was for them to leave Norway, and Lie received a fellowship to travel to Europe. Lie went straight to Berlin, a leading European center of mathematical research, but the mathematics practiced by local stars — Weierstrass and Kronecker — did not impress him. There Lie met young Felix Klein, who eagerly shared this sentiment. The two had a common interest in line geometry and became friends. Klein's and Lie's personalities complemented each other very well. As the mathematician Hans Freudenthal put it, "Lie and Klein had quite different characters as humans and mathematicians: the algebraist Klein was fascinated by the peculiarities of charming problems; the analyst Lie, parting from special cases, sought to understand a problem in its appropriate generalization" [**16**, p. 323].

Lie liked to bounce ideas off his friend's head, and Klein's returns were often quite powerful. In particular, Klein pointed out an analogy between Lie's research on the tetrahedral complex and the Galois theory of commutative permutation groups. Blissfully unaware of the difficulties on his path, Lie enthusiastically embraced this suggestion. Developing a continuous analog of the Galois theory of algebraic equations became Lie's *idée fixe* for the next several years.

Lie and Klein traveled to Paris together, and there Lie produced the famous contact transformation, which mapped straight lines into spheres. An application of this expertise to the Earth sphere, however, did not serve him well. After the outbreak of the Franco-Prussian war, Lie could not find a better way to return to Norway than by first hiking to Italy. With his peculiar hiking habits, such as taking off his clothes in the rain and putting them into his backpack, he was not able to flee very far. The French quickly apprehended him and found papers filled with mysterious symbols. Lie's efforts to explain the meaning of his mathematical notation did not dispel the authorities' suspicion that he was a German spy. A short stay in prison afforded him some quiet time to complete his studies, and upon return to Norway, Lie successfully defended his doctoral dissertation. Unable to find a job in Norway, Lie resolved to go to Sweden, but Norwegian patriots intervened, and the Norwegian National Assembly voted by a large majority to establish a personal extraordinary professorship for Lie at the University of Christiania. Although the salary offered was less than extraordinary, he stayed.

Lie's research on sphere mapping and his lively exchanges with Klein led both of them to think of more general connections between group theory and geometry. In 1872 Klein presented his famous Erlangen Program, in which he suggested unifying specific geometries under a general framework of projective geometry and using group theory to organize all geometric knowledge. Lie and Klein clearly articulated the notion of a transformation group, the continuous analog of a permutation group, with promising applications to geometry and differential equations, but they lacked a general theory of the subject. The Erlangen Program implied one aspect of this project — the group classification problem — but Lie had no intention of attacking this bastion at the time. As he later wrote to Klein, "[I]n your essay the problem of determining *all* groups is not posited, probably on the grounds that at the time such a problem seemed to you absurd or impossible, as it did to me" [**22**, pp. 41–42].

By the end of 1873, Lie's pessimism gave way to a much brighter outlook. After dipping into the theory of first order differential equations, developed by Jacobi and his followers, and making considerable advances with his *idée fixe*, Lie finally acquired the mathematical weaponry needed to answer the challenge of the Erlangen Program and to tackle the theory of continuous transformation groups.

Living on the outskirts of Europe, Lie felt quite marginalized in the European mathematics community. No students and very few foreign colleagues were interested in his research. He wrote his papers in German but published them almost exclusively in Norwegian journals, preferring publication speed over wide accessibility. A few years later he learned, however, that one French mathematician had won the *Grand Prix* from the Académie des Sciences for independently obtained results that yielded some special cases of Lie's work on differential equations. Lie realized that his Norwegian publications were not the greatest publicity vehicle, and that he needed to make his work better known in Europe. "If only I could collect together and edit all my results," he wistfully wrote to Klein [**22**, p. 77]. Klein's practical mind quickly found a solution. Klein, who then taught at Leipzig, arranged for the young mathematician Friedrich Engel, a recent doctoral student of his colleague, to go to Christiania and to render Lie a helping mathematical hand.

Lie and Engel met twice daily for a polite conversation about transformation groups. As Engel recalled, Lie carried his theory almost entirely in his head and dictated to Engel an outline of each chapter, "a sort of skeleton, to be clothed by me with flesh and blood" [**22**, p. 77]. Lie read and revised Engel's notes, eventually producing the first draft of a book-length manuscript.

When Klein left Leipzig to take up a professorship at Göttingen, he arranged for the vacated chair of geometry to be offered to Lie. Lie somewhat reluctantly left his homeland and arrived at Leipzig with the intention of building "a healthy mathematical school" there [**22**, p. 226]. He continued his collaboration with Engel, which culminated in the publication of their joint three-volume work, *Theorie der Transformationsgruppen*.

Lie's ideas began to spread around Europe, finding a particularly fertile ground in Paris. Inspired by Lie, Henri Poincaré remarked that all mathematics was a tale about groups, and Émile Picard wrote to Lie, "Paris is becoming a center for groups; it is all fermenting in young minds, and one will have an excellent wine after the liquors have settled a bit". German mathematicians were less impressed. Weierstrass believed that Lie's theory lacked rigor and had to be rebuilt from the foundations, and Frobenius labeled it a "theory of methods" for solving differential equations in a roundabout way, instead of the natural methods of Euler and Lagrange [**22**, pp. 186, 188–189].

Students flocked to Lie's lectures on his own research, but this only exacerbated his heavy teaching load at Leipzig — 8–10 lectures per week — compared to the leisurely pace of his work in Christiania. An outdoor man, who was used to weeks-long hikes in Norway, Lie felt homesick, longing for the forests and mountains of his native country. All this began taking its toll on Lie. Most importantly, he felt underappreciated and became obsessed with the idea that others plundered his work and betrayed his trust. His relations with colleagues gradually deteriorated, particularly with those closest to him. He broke with Engel and eventually with Klein. Lie felt that his role in the development of the Erlangen Program was undervalued, and he publicly attacked Klein, claiming, "I am no pupil of Klein, nor is the opposite the case, although this might be closer to the truth. I value Klein's talent highly and will never forget the sympathetic interest with which he has always followed my scientific endeavors. But I do not feel that he has a satisfactory understanding of the difference between induction and proof, or between a concept and its application" [**56**, p. 371]. Whoever was right in this dispute, Lie's public accusations against widely respected and influential Klein reflected badly on Lie's reputation.

Eventually Lie suffered a nervous breakdown and was diagnosed with "neurasthenia", a popular mental disease dubbed the "American Nervousness", or "Americanitis". Its cause was ascribed to the stress of modern urban life and the exhaustion of an individual's "nervous energy". Lie spent some months in the supposedly less stressful environment of a psychiatric clinic and upon some reflection decided that he was better off in his mathematics department. His mathematical abilities returned, but his psyche never fully recovered. Rumors spread of his mental illness, possibly fueled by his opponents, who tried to invalidate his accusations.

In the meantime, trying to assert its cultural (and eventually political) independence from Sweden, Norway took steps to bring back its leading intellectuals. The Norwegian National Assembly voted to establish a personal chair in transformation group theory for Lie, matching his high Leipzig salary. Lie was anxious to return to his homeland, but his wife and three children did not share his nostalgia. He eventually returned to Norway in 1898 with only a few months to live.

Lie "thought and wrote in grandiose terms, in a style that has now gone out of fashion, and that would be censored by our scientific journals", wrote one commentator [**26**, p. iii]. Lie was always more concerned with originality than with rigor. "Let us reason with concepts!" he often exclaimed during his lectures and drew geometrical pictures instead of providing analytical proofs [**22**, p. 244]. "Without Phantasy one would never become a Mathematician", he wrote. "[W]hat gave me a Place among the Mathematicians of our Day, despite my Lack of Knowledge and Form, was the Audacity of my Thinking" [**56**, p. 409]. Hardly lacking relevant knowledge, Lie indeed had trouble putting his ideas into publishable form. Due to Engel's diligence, Lie's research on transformation groups was summed up in three grand volumes, but Lie never liked this ghost-written work and preferred citing his own earlier papers [**47**, p. 310]. He had even less luck with the choice of assistant to write up results on contact transformations and partial differential equations. Felix Hausdorff's interests led him elsewhere, and Lie's thoughts on these subjects were never completely spelled out [**16**, p. 324]. Thus we may never discover the "true Lie".

2.11. Tensor products

In this subsection we recall the notion of tensor product of vector spaces, which will be extensively used below.

Definition 2.11.1. The **tensor product** $V \otimes W$ of vector spaces V and W over a field k is the quotient of the space $V * W$ whose basis is given by formal symbols $v \otimes w$, $v \in V$, $w \in W$, by the subspace spanned by the elements

$$\begin{aligned}
&(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w, \\
&v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2, \\
&av \otimes w - a(v \otimes w), \\
&v \otimes aw - a(v \otimes w),
\end{aligned}$$

where $v \in V, w \in W, a \in k$.

Exercise 2.11.2. Show that $V \otimes W$ can be equivalently defined as the quotient of the free abelian group $V \bullet W$ generated by $v \otimes w$, $v \in V, w \in W$ by the subgroup generated by

$$\begin{aligned}
&(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w, \\
&v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2, \\
&av \otimes w - v \otimes aw,
\end{aligned}$$

where $v \in V, w \in W, a \in k$.

The elements $v \otimes w \in V \otimes W$, for $v \in V, w \in W$ are called pure tensors. Note that in general, there are elements of $V \otimes W$ which are not pure tensors.

This allows one to define the tensor product of any number of vector spaces, $V_1 \otimes \cdots \otimes V_n$. Note that this tensor product is associative, in the sense that $(V_1 \otimes V_2) \otimes V_3$ can be naturally identified with $V_1 \otimes (V_2 \otimes V_3)$.

In particular, people often consider tensor products of the form $V^{\bigotimes n} = V \otimes \cdots \otimes V$ (n times) for a given vector space V, and, more generally, $E := V^{\bigotimes n} \otimes (V^*)^{\bigotimes m}$. This space is called **the space of tensors of type** (m, n) on V. For instance, tensors of type $(0, 1)$ are vectors, tensors of type $(1, 0)$ — linear functionals (covectors), tensors of type $(1, 1)$ — linear operators, of type $(2, 0)$ — bilinear forms, tensors of type $(2, 1)$ — algebra structures, etc.

If V is finite dimensional with basis e_i, $i = 1, \ldots, N$, and e^i is the dual basis of V^*, then a basis of E is the set of vectors

$$e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e^{j_1} \otimes \cdots \otimes e^{j_m},$$

and a typical element of E is

$$\sum_{i_1,\dots,i_n,j_1,\dots,j_m=1}^{N} T^{i_1\dots i_n}_{j_1\dots j_m} e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes e^{j_1}\otimes\cdots\otimes e^{j_m},$$

where T is a multidimensional table of numbers.

Physicists define a tensor as a collection of such multidimensional tables T_B attached to every basis B in V, which change according to a certain rule when the basis B is changed (derive this rule!). Here it is important to distinguish upper and lower indices, since lower indices of T correspond to V and upper ones to V^*. The physicists don't write the sum sign, but remember that one should sum over indices that repeat twice — once as an upper index and once as lower. This convention is called the *Einstein summation*, and it also stipulates that if an index appears once, then there is no summation over it, while no index is supposed to appear more than once as an upper index or more than once as a lower index.

One can also define the tensor product of linear maps. Namely, if $A : V \to V'$ and $B : W \to W'$ are linear maps, then one can define the linear map $A \otimes B : V \otimes W \to V' \otimes W'$ given by the formula $(A \otimes B)(v \otimes w) = Av \otimes Bw$ (check that this is well defined!). The most important properties of tensor products are summarized in the following problem.

Problem 2.11.3. (a) Let U be any k-vector space. Construct a natural bijection between bilinear maps $V \times W \to U$ and linear maps $V \otimes W \to U$ ("natural" means that the bijection is defined without choosing bases).

(b) Show that if $\{v_i\}$ is a basis of V and $\{w_j\}$ is a basis of W, then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$.

(c) Construct a natural isomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ in the case when V is finite dimensional.

(d) Let V be a vector space over a field k. Let S^nV be the quotient of $V^{\bigotimes n}$ (n-fold tensor product of V) by the subspace spanned by the tensors $T - s(T)$ where $T \in V^{\bigotimes n}$ and s is a transposition. Also let $\wedge^n V$ be the quotient of $V^{\bigotimes n}$ by the subspace spanned by the tensors T such that $s(T) = T$ for some transposition s. These spaces are called the nth **symmetric power**, respectively **exterior power** of V. If $\{v_i\}$ is a basis of V, can you construct a basis of $S^nV, \wedge^n V$? If $\dim V = m$, what are their dimensions?

(e) If k has characteristic zero, find a natural identification of S^nV with the space of $T \in V^{\bigotimes n}$ such that $T = sT$ for all transpositions s, and find a natural identification of $\wedge^n V$ with the space of $T \in V^{\bigotimes n}$ such that $T = -sT$ for all transpositions s.

(f) Let $A : V \to W$ be a linear operator. Then we have an operator $A^{\bigotimes n} : V^{\bigotimes n} \to W^{\bigotimes n}$ and its symmetric and exterior powers $S^nA : S^nV \to S^nW$, $\wedge^n A : \wedge^n V \to \wedge^n W$ which are defined in an obvious way. Suppose that $V = W$ and that $\dim V = N$, and that the eigenvalues of A are $\lambda_1, \ldots, \lambda_N$. Find $\mathrm{Tr}(S^nA)$ and $\mathrm{Tr}(\wedge^n A)$.

(g) Show that $\wedge^N A = \det(A)\mathrm{Id}$, and use this equality to give a one-line proof of the fact that $\det(AB) = \det(A)\det(B)$.

Remark 2.11.4. Note that a similar definition to the above can be used to define the tensor product $V \otimes_A W$, where A is any ring, V is a right A-module, and W is a left A-module. Namely, $V \otimes_A W$ is the abelian group which is the quotient of the group $V \bullet W$ freely generated by formal symbols $v \otimes w$, $v \in V$, $w \in W$, modulo the relations

$$\begin{aligned}
&(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w,\\
&v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2,\\
&va \otimes w - v \otimes aw, \quad a \in A.
\end{aligned}$$

Exercise 2.11.5. Let K be a field, and let L be an extension of K. If A is an algebra over K, show that $A \otimes_K L$ is naturally an algebra over L. Show that if V is an A-module, then $V \otimes_K L$ has a natural structure of a module over the algebra $A \otimes_K L$.

Problem 2.11.6. Throughout this problem, we let k be an arbitrary field (not necessarily of characteristic zero and not necessarily algebraically closed).

If A and B are two k-algebras, then an (A, B)**-bimodule** will mean a k-vector space V with both a left A-module structure and a right B-module structure which satisfy $(av)\,b = a\,(vb)$ for any $v \in V$, $a \in A$, and $b \in B$. Note that both the notions of "left A-module" and "right A-module" are particular cases of the notion of bimodules; namely, a left A-module is the same as an (A, k)-bimodule, and a right A-module is the same as a (k, A)-bimodule.

Let B be a k-algebra, W a left B-module, and V a right B-module. We denote by $V \otimes_B W$ the k-vector space $(V \otimes_k W) / \langle vb \otimes w - v \otimes bw \mid v \in V,\ w \in W,\ b \in B \rangle$. We denote the projection of a pure tensor $v \otimes w$ (with $v \in V$ and $w \in W$) onto the space $V \otimes_B W$ by $v \otimes_B w$. (Note that this tensor product $V \otimes_B W$ is the one defined in Remark 2.11.4.)

If, additionally, A is another k-algebra and if the right B-module structure on V is part of an (A, B)-bimodule structure, then $V \otimes_B W$ becomes a left A-module by $a (v \otimes_B w) = av \otimes_B w$ for any $a \in A$, $v \in V$, and $w \in W$.

Similarly, if C is another k-algebra, and if the left B-module structure on W is part of a (B, C)-bimodule structure, then $V \otimes_B W$ becomes a right C-module by $(v \otimes_B w) c = v \otimes_B wc$ for any $c \in C$, $v \in V$, and $w \in W$.

If V is an (A, B)-bimodule and W is a (B, C)-bimodule, then these two structures on $V \otimes_B W$ can be combined into one (A, C)-bimodule structure on $V \otimes_B W$.

(a) Let A, B, C, D be four algebras. Let V be an (A, B)-bimodule, W a (B, C)-bimodule, and X a (C, D)-bimodule. Prove that $(V \otimes_B W) \otimes_C X \cong V \otimes_B (W \otimes_C X)$ as (A, D)-bimodules. The isomorphism (from left to right) is given by the formula

$$(v \otimes_B w) \otimes_C x \mapsto v \otimes_B (w \otimes_C x)$$

for all $v \in V$, $w \in W$, and $x \in X$.

(b) If A, B, C are three algebras and if V is an (A, B)-bimodule and W an (A, C)-bimodule, then the vector space $\mathrm{Hom}_A(V, W)$ (the space of all left A-linear homomorphisms from V to W) canonically becomes a (B, C)-bimodule by setting $(bf)(v) = f(vb)$ for all $b \in B$, $f \in \mathrm{Hom}_A(V, W)$, and $v \in V$ and setting $(fc)(v) = f(v)c$ for all $c \in C$, $f \in \mathrm{Hom}_A(V, W)$ and $v \in V$.

Let A, B, C, D be four algebras. Let V be a (B, A)-bimodule, W a (C, B)-bimodule, and X a (C, D)-bimodule. Prove that

$$\mathrm{Hom}_B(V, \mathrm{Hom}_C(W, X)) \cong \mathrm{Hom}_C(W \otimes_B V, X)$$

as (A, D)-bimodules. The isomorphism (from left to right) is given by

$$f \mapsto (w \otimes_B v \mapsto f(v)\,w)$$

for all $v \in V$, $w \in W$ and $f \in \mathrm{Hom}_B(V, \mathrm{Hom}_C(W, X))$.

Exercise 2.11.7. Show that if M and N are modules over a commutative ring A, then $M \otimes_A N$ has a natural structure of an A-module.

2.12. The tensor algebra

The notion of tensor product allows us to give more conceptual (i.e., coordinate-free) definitions of the free algebra, polynomial algebra, exterior algebra, and universal enveloping algebra of a Lie algebra.

Namely, given a vector space V, define its **tensor algebra** TV over a field k to be $TV = \bigoplus_{n \geq 0} V^{\otimes n}$, with multiplication defined by $a \cdot b := a \otimes b$, $a \in V^{\otimes n}$, $b \in V^{\otimes m}$. Observe that a choice of a basis $x_1, \ldots, x_N$ in V defines an isomorphism of TV with the free algebra $k\langle x_1, \ldots, x_n \rangle$.

Also, one can make the following definition.

Definition 2.12.1. (i) The **symmetric algebra** SV of V is the quotient of TV by the ideal generated by $v \otimes w - w \otimes v$, $v, w \in V$.

(ii) The **exterior algebra** $\wedge V$ of V is the quotient of TV by the ideal generated by $v \otimes v$, $v \in V$.

(iii) If V is a Lie algebra, the **universal enveloping algebra** $\mathcal{U}(V)$ of V is the quotient of TV by the ideal generated by $v \otimes w - w \otimes v - [v, w]$, $v, w \in V$.

It is easy to see that a choice of a basis $x_1, \ldots, x_N$ in V identifies SV with the polynomial algebra $k[x_1, \ldots, x_N]$, $\wedge V$ with the exterior algebra $\wedge_k(x_1, \ldots, x_N)$, and the universal enveloping algebra $\mathcal{U}(V)$ with one defined previously.

Moreover, it is easy to see that we have decompositions

$$SV = \bigoplus_{n\geq 0} S^n V, \ \wedge V = \bigoplus_{n\geq 0} \wedge^n V.$$

2.13. Hilbert's third problem

Problem 2.13.1. It is known that if A and B are two polygons of the same area, then A can be cut by finitely many straight cuts into pieces from which one can make B (check it — it is fun!). David Hilbert asked in 1900 whether it is true for polyhedra in three dimensions. In particular, is it true for a cube and a regular tetrahedron of the same volume?

The answer is "no", as was found by Dehn in 1901. The proof is very beautiful. Namely, to any polyhedron A, let us attach its "Dehn invariant" $D(A)$ in $V = \mathbb{R} \otimes (\mathbb{R}/\mathbb{Q})$ (the tensor product of $\mathbb{Q}$-vector spaces). Namely,

$$D(A) = \sum_a l(a) \otimes \frac{\beta(a)}{\pi},$$

where a runs over edges of A and $l(a), \beta(a)$ are the length of a and the angle at a.

(a) Show that if you cut A into B and C by a straight cut, then $D(A) = D(B) + D(C)$.

(b) Show that $\alpha = \arccos(1/3)/\pi$ is not a rational number.

Hint: Assume that $\alpha = 2m/n$, for integers m, n. Deduce that roots of the equation $x + x^{-1} = 2/3$ are roots of unity of degree n. Then show that $x^k + x^{-k}$ has denominator 3^k and get a contradiction.

(c) Using (a) and (b), show that the answer to Hilbert's question is negative. (Compute the Dehn invariant of the regular tetrahedron and the cube.)

2.14. Tensor products and duals of representations of Lie algebras

Definition 2.14.1. The **tensor product** of two representations V, W of a Lie algebra $\mathfrak{g}$ is the space $V \otimes W$ with

$$\rho_{V\otimes W}(x) = \rho_V(x) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho_W(x).$$

Definition 2.14.2. The **dual representation** V^* to a representation V of a Lie algebra $\mathfrak{g}$ is the dual space V^* to V with $\rho_{V^*}(x) = -\rho_V(x)^*$.

It is easy to check that these are indeed representations.

Problem 2.14.3. Let V, W, U be finite dimensional representations of a Lie algebra $\mathfrak{g}$. Show that the space $\mathrm{Hom}_{\mathfrak{g}}(V \otimes W, U)$ is isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(V, U \otimes W^*)$. (Here $\mathrm{Hom}_{\mathfrak{g}} := \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}$.)

2.15. Representations of $\mathfrak{sl}(2)$

This subsection is devoted to the representation theory of $\mathfrak{sl}(2)$, which is of central importance in many areas of mathematics. It is useful to study this topic by solving the following sequence of exercises, which every mathematician should do, in one form or another.

Problem 2.15.1. According to the above, a representation of $\mathfrak{sl}(2)$ is just a vector space V with a triple of operators E, F, H such that $HE - EH = 2E$, $HF - FH = -2F$, $EF - FE = H$ (the corresponding map ρ is given by $\rho(e) = E, \rho(f) = F$, $\rho(h) = H$).

Let V be a finite dimensional representation of $\mathfrak{sl}(2)$ (the ground field in this problem is $\mathbb{C}$).

(a) Take eigenvalues of H and pick one with the biggest real part. Call it λ. Let $\bar{V}(\lambda)$ be the generalized eigenspace corresponding to λ. Show that $E|_{\bar{V}(\lambda)} = 0$.

(b) Let W be any representation of $\mathfrak{sl}(2)$ and let $w \in W$ be a nonzero vector such that $Ew = 0$. For any $k > 0$ find a polynomial $P_k(x)$ of degree k such that $E^k F^k w = P_k(H)w$. (First compute $EF^k w$; then use induction in k.)

(c) Let $v \in \bar{V}(\lambda)$ be a generalized eigenvector of H with eigenvalue λ. Show that there exists $N > 0$ such that $F^N v = 0$.

(d) Show that H is diagonalizable on $\bar{V}(\lambda)$. (Take N to be such that $F^N = 0$ on $\bar{V}(\lambda)$, and compute $E^N F^N v$, $v \in \bar{V}(\lambda)$, by (b). Use the fact that $P_k(x)$ does not have multiple roots.)

(e) Let N_v be the smallest N satisfying (c). Show that $\lambda = N_v - 1$.

(f) Show that for each $N > 0$, there exists a unique up to isomorphism irreducible representation of $\mathfrak{sl}(2)$ of dimension N. Compute the matrices E, F, H in this representation using a convenient basis. (For V finite dimensional irreducible take λ as in (a) and $v \in V(\lambda)$ an eigenvector of H. Show that $v, Fv, \dots, F^\lambda v$ is a basis of V, and compute the matrices of the operators E, F, H in this basis.)

Denote the $(\lambda+1)$-dimensional irreducible representation from (f) by V_λ. Below you will show that any finite dimensional representation is a direct sum of V_λ.

(g) Show that the operator $C = EF + FE + H^2/2$ (the so-called **Casimir operator**) commutes with E, F, H and equals $\frac{\lambda(\lambda+2)}{2}\operatorname{Id}$ on V_λ.

Now it is easy to prove the direct sum decomposition. Namely, assume the contrary, and let V be a reducible representation of the smallest dimension, which is not a direct sum of smaller representations.

(h) Show that C has only one eigenvalue on V, namely $\frac{\lambda(\lambda+2)}{2}$ for some nonnegative integer λ (use the fact that the generalized eigenspace decomposition of C must be a decomposition of representations).

(i) Show that V has a subrepresentation $W = V_\lambda$ such that $V/W = nV_\lambda$ for some n (use (h) and the fact that V is the smallest reducible representation which cannot be decomposed).

(j) Deduce from (i) that the eigenspace $V(\lambda)$ of H is $(n+1)$-dimensional. If $v_1, \dots, v_{n+1}$ is its basis, show that $F^j v_i$, $1 \le i \le n+1$, $0 \le j \le \lambda$, are linearly independent and therefore form a basis of V (establish that if $Fx = 0$ and $Hx = \mu x$ for $x \neq 0$, then $Cx = \frac{\mu(\mu-2)}{2}x$ and hence $\mu = -\lambda$).

(k) Define $W_i = \operatorname{span}(v_i, Fv_i, \dots, F^\lambda v_i)$. Show that W_i are subrepresentations of V and derive a contradiction to the fact that V cannot be decomposed.

(l) (Jacobson-Morozov lemma) Let V be a finite dimensional complex vector space and $A : V \to V$ a nilpotent operator. Show that there exists a unique, up to an isomorphism, representation of $\mathfrak{sl}(2)$

on V such that $E = A$. (Use the classification of the representations and the Jordan normal form theorem.)

(m) (Clebsch-Gordan decomposition) Find the decomposition of the representation $V_\lambda \otimes V_\mu$ of $\mathfrak{sl}(2)$ into irreducibles components.

Hint: For a finite dimensional representation V of $\mathfrak{sl}(2)$ it is useful to introduce the character $\chi_V(x) = Tr(e^{xH})$, $x \in \mathbb{C}$. Show that $\chi_{V \oplus W}(x) = \chi_V(x) + \chi_W(x)$ and $\chi_{V \otimes W}(x) = \chi_V(x)\chi_W(x)$. Then compute the character of V_λ and of $V_\lambda \otimes V_\mu$ and derive the decomposition. This decomposition is of fundamental importance in quantum mechanics.

(n) Let $V = \mathbb{C}^M \otimes \mathbb{C}^N$ and $A = J_{0,M} \otimes \mathrm{Id}_N + \mathrm{Id}_M \otimes J_{0,N}$, where $J_{0,n}$ is the Jordan block of size n with eigenvalue zero (i.e., $J_{0,n}e_i = e_{i-1}$, $i = 2, \ldots, n$, and $J_{0,n}e_1 = 0$). Find the Jordan normal form of A using (l) and (m).

2.16. Problems on Lie algebras

Problem 2.16.1 (Lie's theorem)**.** The **commutant** $K(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the linear span of elements $[x, y]$, $x, y \in \mathfrak{g}$. This is an ideal in $\mathfrak{g}$ (i.e., it is a subrepresentation of the adjoint representation). A finite dimensional Lie algebra $\mathfrak{g}$ over a field k is said to be **solvable** if there exists n such that $K^n(\mathfrak{g}) = 0$. Prove the Lie theorem: if $k = \mathbb{C}$ and V is a finite dimensional irreducible representation of a solvable Lie algebra $\mathfrak{g}$, then V is 1-dimensional.

Hint: Prove the result by induction in dimension. By the induction assumption, $K(\mathfrak{g})$ has a common eigenvector v in V; that is, there is a linear function $\chi : K(\mathfrak{g}) \to \mathbb{C}$ such that $av = \chi(a)v$ for any $a \in K(\mathfrak{g})$. Show that $\mathfrak{g}$ preserves common eigenspaces of $K(\mathfrak{g})$. (For this you will need to show that $\chi([x, a]) = 0$ for $x \in \mathfrak{g}$ and $a \in K(\mathfrak{g})$. To prove this, consider the smallest subspace U containing v and invariant under x. This subspace is invariant under $K(\mathfrak{g})$ and any $a \in K(\mathfrak{g})$ acts with trace $\dim(U)\chi(a)$ in this subspace. In particular $0 = \mathrm{Tr}([x, a]) = \dim(U)\chi([x, a])$.)

Problem 2.16.2. Classify irreducible finite dimensional representations of the two-dimensional Lie algebra with basis X, Y and commutation relation $[X, Y] = Y$. Consider the cases of zero and positive characteristic. Is the Lie theorem true in positive characteristic?

Problem 2.16.3. (Hard!) For any element x of a Lie algebra $\mathfrak{g}$ let $\mathrm{ad}(x)$ denote the operator $\mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]$. Consider the Lie algebra $\mathfrak{g}_n$ generated by two elements x, y with the defining relations $\mathrm{ad}(x)^2(y) = \mathrm{ad}(y)^{n+1}(x) = 0$.

(a) Show that the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ are finite dimensional and find their dimensions.

(b) (Harder!) Show that the Lie algebra $\mathfrak{g}_4$ has infinite dimension. Construct explicitly a basis of this algebra.

Problem 2.16.4. Classify irreducible representations of the Lie algebra $\mathfrak{sl}(2)$ over an algebraically closed field k of characteristic $p > 2$.

Problem 2.16.5. Let k be an algebraically closed field of characteristic zero, and let $q \in k^\times, q \neq \pm 1$. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ is the algebra generated by e, f, K, K^{-1} with relations

$$KeK^{-1} = q^2 e, \; KfK^{-1} = q^{-2} f, \; [e, f] = \frac{K - K^{-1}}{q - q^{-1}}$$

(if you formally set $K = q^h$, you'll see that this algebra, in an appropriate sense, "degenerates" to $\mathcal{U}(\mathfrak{sl}(2))$ as $q \to 1$). Classify irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(2))$. Consider separately the cases of q being a root of unity and q not being a root of unity.