A Linear Ordering of Braids

In this chapter, we introduce the linear ordering of braids, sometimes called the Dehornoy ordering, that is the main subject of this book, and we list its main properties known so far. The construction starts with the notion of a $\sigma$-positive braid, and it relies on three basic properties, called $A$, $C$, and $S$, from which the $\sigma$-ordering can easily be constructed and investigated. In this chapter, we take Properties $A$, $C$, and $S$ for granted, and explore their consequences. The many different proofs of these statements will be found in subsequent chapters.

The chapter is organized as follows. In Section 1, we introduce the $\sigma$-ordering and its variant the $\sigma\Phi$-ordering starting from Properties $A$ and $C$. In Section 2, we give many examples of the sometimes surprising behaviour of the $\sigma$-ordering, and we introduce Property $S$. In Section 3, we develop global properties of the $\sigma$-ordering, involving Archimedian property, discreteness, density, and convex subgroups. Finally, in Section 4, we investigate the restriction of the $\sigma$-ordering to the monoid $B_n^+$ of positive braids, showing that this restriction is a well-ordering, and we give an inductive construction of the $\sigma$-ordering of $B_n^+$ from the $\sigma$-ordering of $B_{n-1}^+$.

Convention. In this chapter and everywhere in this book, when we speak of positive braids, we always mean those braids that lie in the monoid $B_\infty^+$, i.e., those braids that admit at least one expression by a word containing no letter $\sigma_i^{-1}$. Such braids are sometimes called Garside positive braids, but we shall not use that name here. So the word “positive” never refers to any of the specific linear orderings we shall investigate hereafter. For the latter case, we shall introduce specific names for the braids that are larger than 1, typically $\sigma$-positive and $\sigma\Phi$-positive in the case of the $\sigma$-ordering and of the $\sigma\Phi$-ordering.

1. The $\sigma$-ordering of $B_n$

In this section we give a first definition of the $\sigma$-ordering of braids, based on the notion of a $\sigma$-positive braid word—many alternative definitions will be given in subsequent chapters. We explain how to construct the $\sigma$-ordering from two specific properties of braids called $A$ and $C$. We also introduce a useful variant of the $\sigma$-ordering, called the $\sigma\Phi$-ordering, which is its image under the flip automorphism. Finally, we briefly discuss the algorithmic issues involving the $\sigma$-ordering.

1.1. Ordering a group. We start with preliminary remarks about what can be expected here. First, we recall that a strict ordering of a set $\Omega$ is a binary relation $\prec$ that is antireflexive ($x \prec x$ never holds) and transitive (the conjunction of $x \prec y$ and $y \prec z$ implies $x \prec z$). A strict ordering of $\Omega$ is called linear (or total) if, for all $x, x'$ in $\Omega$, one of $x = x'$, $x \prec x'$, $x' \prec x$ holds. Then, we recall the notion of an orderable group.
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Definition 1.1. (i) A left-invariant ordering, or left-ordering, of a group $G$ is a strict linear ordering $<$ of $G$ such that $g < h$ implies $fg < fh$ for all $f,g,h$ in $G$. A group $G$ is said to be left-orderable if there exists at least one left-invariant ordering of $G$.

(ii) A bi-invariant ordering, or bi-ordering, of a group $G$ is a left-ordering of $G$ that is also right-invariant, i.e., $g < h$ implies $gf < hf$ for all $f,g,h$ in $G$. A group $G$ is said to be bi-orderable if there exists at least one bi-invariant ordering of $G$.

Proposition 1.2. For $n \geq 3$, the group $B_n$ is not bi-orderable.

Proof. If $<$ is a bi-invariant ordering of a group $G$, then $g < h$ implies $\varphi(g) < \varphi(h)$ for each inner automorphism $\varphi$ of $G$. Now, in the case of $B_n$, the inner automorphism $\Phi_n$ associated with Garside’s fundamental braid $\Delta_n$ of (I.4.1) exchanges $\sigma_i$ and $\sigma_{n-i}$ for each $i$. Hence it is impossible to have $\sigma_1 < \sigma_{n-1}$ and $\Phi_n(\sigma_i) < \Phi_n(\sigma_{n-1})$ simultaneously.

Therefore, in the best case, we shall be interested in orders that are invariant under multiplication on one side. Then, both sides play symmetric roles, as an immediate verification gives

Lemma 1.3. Assume that $G$ is a group and $<$ is a left-invariant ordering of $G$. Define $g \prec h$ to mean $g^{-1} \prec h^{-1}$. Then $\prec$ is a right-invariant ordering of $G$.

We shall concentrate hereafter on left-invariant orderings. Specifying such an ordering is actually equivalent to specifying a subsemigroup of a certain type, called a positive cone.

Definition 1.4. A subset $P$ of a group $G$ is called a positive cone on $G$ if $P$ is closed under multiplication and $G \setminus \{1\}$ is the disjoint union of $P$ and $P^{-1}$.

Lemma 1.5. (i) Assume that $<$ is a left-invariant ordering of a group $G$. Then the set $P$ of all elements in $G$ that are larger than 1 is a positive cone on $G$, and $g < h$ is equivalent to $g^{-1}h \in P$.

(ii) Assume that $P$ is a positive cone on a group $G$. Then the relation $g^{-1}h \in P$ is a left-invariant ordering of $G$, and $P$ is then the set of all elements of $G$ that are larger than 1.

The verification is easy. Note that the formula $hg^{-1} \in P$ would define a right-invariant ordering.

1.2. The $\sigma$-ordering of braids. We now introduce on $B_n$ a certain binary relation that will turn out to be a left-invariant ordering. The construction involves particular braid words defined in terms of the letters they contain.

Definition 1.6. A braid word $w$ is said to be $\sigma$-positive (resp. $\sigma$-negative) if, among the letters $\sigma_i^{-1}$ that occur in $w$, the one with lowest index occurs positively only, i.e., $\sigma_i$ occurs but $\sigma_i^{-1}$ does not (resp. negatively only, i.e., $\sigma_i^{-1}$ occurs but $\sigma_i$ does not).

For instance, $\sigma_3 \sigma_2 \sigma_3^{-1}$ is a $\sigma$-positive braid word: the letter with lowest index is $\sigma_2$ (there is no $\sigma_2^{-1}$), and there is one $\sigma_2$ but no $\sigma_2^{-1}$. By contrast, the word $\sigma_2^{-1} \sigma_1 \sigma_2$, which is equivalent to $\sigma_2 \sigma_3^{-1}$, is neither $\sigma$-positive nor $\sigma$-negative: the letter with lowest index is $\sigma_2$ again, but, here, both $\sigma_2$ and $\sigma_2^{-1}$ appear.
DEFINITION 1.7. For \( \beta, \beta' \) in \( B_n \), we say that \( \beta <_n \beta' \) is true if \( \beta^{-1} \beta' \) admits an \( n \)-strand representative word that is \( \sigma \)-positive.

EXAMPLE 1.8. Let \( \beta = \sigma_2 \) and \( \beta' = \sigma_3 \sigma_2 \). Among the 4-strand braid words that represent the quotient \( (\sigma_2)^{-1}(\sigma_3 \sigma_2) \), there is the word \( \sigma_3^{-1} \sigma_3 \sigma_2 \), which is neither \( \sigma \)-positive nor \( \sigma \)-negative, but there is also the word \( \sigma_2 \sigma_3^{-1} \) and many others. As the latter word is a 4-strand braid word that is \( \sigma \)-positive, \( \beta <_4 \beta' \) is true.

Similarly, we have

\[
\sigma_1 >_\infty \sigma_2 >_\infty \sigma_3 >_\infty \ldots
\]

since, for each \( i \), the braid word \( \sigma_{i+1}^{-1} \sigma_i \) is \( \sigma \)-positive.

The central property is the following result of \([47]\) (see Remark 1.16) which implies the first part of the theorem mentioned in the Introduction:

**Proposition 1.9.** (i) For \( 2 \leq n \leq \infty \), the relation \( <_n \) is a left-invariant ordering of \( B_n \).

(ii) For each \( n \), the relation \( <_n \) is the restriction of \( <_\infty \) to \( B_n \).

Owing to (ii) above, we shall drop the subscripts and simply write \(<\) for \( <_n \).

The order \(<\) will be called the \( \sigma \)-ordering of braids, which is coherent with its definition in terms of the generators \( \sigma_i \).

By definition, the relation \( \beta >_n 1 \) is true if and only if \( \beta \) admits at least one \( \sigma \)-positive \( n \)-strand representative word. According to Lemma 1.5, proving Proposition 1.9(i) amounts to proving that the set of all such braids is a positive cone. The latter result is a consequence of the following two statements:

**Property A (Acyclicity).** A \( \sigma \)-positive braid word is not trivial.

**Property C (Comparison).** Every nontrivial braid of \( B_n \) admits an \( n \)-strand representative word that is \( \sigma \)-positive or \( \sigma \)-negative.

**Proof of Proposition 1.9 from Properties A and C.** (i) Let \( P_n \) be the set of all \( n \)-strand braids that admit a \( \sigma \)-positive \( n \)-strand representative word. We shall prove that \( P_n \) is a positive cone in \( B_n \). First, the concatenation of two \( \sigma \)-positive \( n \)-strand braids is a \( \sigma \)-positive \( n \)-strand braid word; hence \( P_n \) is closed under multiplication.

Then, we claim that \( B_n \setminus \{1\} \) is the disjoint union of \( P_n \) and \( P_n^{-1} \). Indeed, Property A implies \( 1 \notin P_n \), and therefore \( 1 \notin P_n^{-1} \) as \( 1^{-1} = 1 \) holds. So \( P_n \cup P_n^{-1} \) is included in \( B_n \setminus \{1\} \). Now assume \( \beta \in P_n \cap P_n^{-1} \). We deduce \( 1 = \beta \beta^{-1} \in P_n \cdot P_n \subseteq P_n \), which contradicts \( 1 \notin P_n \). So \( P_n \) and \( P_n^{-1} \) must be disjoint. Finally, Property C (for \( B_n \)) means that \( P_n \cup P_n^{-1} \) covers \( B_n \setminus \{1\} \).

(ii) Assume \( \beta, \beta' \in B_n \). Any \( \sigma \)-positive \( n \)-strand braid word representing \( \beta^{-1} \beta' \) *a fortiori* witnesses the relation \( \beta <_\infty \beta' \), so \( \beta <_n \beta' \) implies \( \beta <_\infty \beta' \). Conversely, assume \( \beta <_\infty \beta' \). As \( <_n \) is a linear ordering of \( B_n \), one of \( \beta <_n \beta' \) or \( \beta >_n \beta' \) holds. In the latter case, we would deduce \( \beta >_\infty \beta' \), which contradicts the hypothesis \( \beta <_\infty \beta' \). So \( \beta <_n \beta' \) is the only possibility.

Property A has four different proofs in this text: they can be found on pages 73, 175, 190, and 224. As for Property C, no fewer than eight proofs are given, on pages 60, 89, 116, 148, 164, 190, 201, and 205.
In addition to being invariant under left multiplication, the $\sigma$-ordering of braids is invariant under the shift endomorphism, defined as follows.

**Definition 1.10.** For $w$ a braid word, the shifting of $w$ is the braid word $sh(w)$ obtained by replacing each letter $\sigma_i$ with $\sigma_{i+1}$, and each letter $\sigma_i^{-1}$ with $\sigma_{i+1}^{-1}$.

The explicit form of the braid relations implies that the shift mapping induces an endomorphism of $B_\infty$, still denoted $sh$ and called the shift endomorphism. The same argument guaranteeing that the canonical morphism of $B_{n-1}$ into $B_n$ is an embedding shows that the shift endomorphism of $B_\infty$ is injective.

**Proposition 1.11.** For all braids $\beta, \beta'$, the relation $\beta < \beta'$ is equivalent to $sh(\beta) < sh(\beta')$.

**Proof.** The shifting of a $\sigma$-positive braid word is a $\sigma$-positive braid word, so $\beta < \beta'$ implies $sh(\beta) < sh(\beta')$. Conversely, as $<$ is a linear ordering, the only possibility when $sh(\beta) < sh(\beta')$ is true is that $\beta < \beta'$ is true as well, as $\beta \geq \beta'$ would imply $sh(\beta) \geq sh(\beta')$. \qed

It is straightforward to check that, conversely, the $\sigma$-ordering is the only partial ordering on $B_\infty$ that is invariant under multiplication on the left and under the shift endomorphism, and satisfies for all braids $\beta, \beta'$ the inequality

$$1 < sh(\beta) \sigma_i sh(\beta').$$

**1.3. Equivalent formulations.** Before proceeding, we introduce derived notions in order to restate Properties A and C in slightly different forms. First, we can refine the notion of a $\sigma$-positive braid word by taking into account the specific index $i$ that is involved.

**Definition 1.12.** A braid word is said to be $\sigma_i$-positive if it contains at least one letter $\sigma_i$, but no $\sigma_i^{-1}$ and no $\sigma_j^{\pm 1}$ with $j < i$. Similarly, it is said to be $\sigma_i$-negative if it contains at least one $\sigma_i^{-1}$, but no $\sigma_i$ and no $\sigma_j^{\pm 1}$ with $j < i$. It is said to be $\sigma_i$-free if it contains no $\sigma_j^{\pm 1}$ with $j \leq i$.

So a braid word is $\sigma$-positive if and only if it is $\sigma_i$-positive for some $i$. Note that, for $i \geq 2$, a word $w$ is $\sigma_i$-positive if and only if it is $sh^{-1}(w_1)$ for some $\sigma_i$-positive word $w_1$—we recall that $sh$ is the shift mapping of Definition 1.10. Similarly, a braid word $w$ is $\sigma_i$-free if and only if it is $sh^1(w_1)$ for some $w_1$.

Then Properties A and C can be expressed in terms of $\sigma_i$-positive, $\sigma_i$-negative, and $\sigma_i$-free words.

**Proposition 1.13.** Property A is equivalent to:

**Property A** (second form). A $\sigma_i$-positive braid word is not trivial.

**Proof.** Every $\sigma_i$-positive braid word is $\sigma$-positive, so the first form of Property A implies the second form.

Conversely, assume the second form of Property A. Let $w$ be a $\sigma$-positive word. Then $w$ is $\sigma_i$-positive for some $i$. As observed above, this means that we have $w = sh^{-1}(w_1)$ for some $\sigma_i$-positive word $w_1$. By the second form of Property A, the word $w_1$ is not trivial, i.e., it does not represent the unit braid. As the shift endomorphism of $B_\infty$ is injective, this implies that $w$ is not trivial either. So, the first form of Property A is satisfied. \qed
Proposition 1.14. Property C is equivalent to:

Property C (second form). Every braid of \( B_n \) admits an \( n \)-strand representative word that is \( \sigma_1 \)-positive, \( \sigma_1 \)-negative, or \( \sigma_1 \)-free.

Proof. A \( \sigma \)-positive braid word is either \( \sigma_1 \)-positive or \( \sigma_1 \)-free, so the first form of Property C implies the second form.

Conversely, assume the second form of Property C. We prove the first form using induction on \( n \geq 2 \). For \( n = 2 \), the two forms coincide. Assume \( n \geq 3 \). Let \( \beta \) be a nontrivial \( n \)-strand braid. By the second form of Property C, we find an \( n \)-strand braid word \( w \) representing \( \beta \) that is \( \sigma_1 \)-positive, \( \sigma_1 \)-negative, or \( \sigma_1 \)-free. In the first two cases, we are done. Otherwise, let \( w_1 = \text{sh}^{-1}(w) \), which makes sense as, by hypothesis, \( w \) contains no letter \( \sigma_n^{-1} \). As the shift endomorphism of \( B_\infty \) is injective, the word \( w_1 \) does not represent 1, so the induction hypothesis implies that \( w_1 \) is equivalent to some \((n-1)\)-strand braid word \( w'_1 \) that is \( \sigma \)-positive or \( \sigma \)-negative. By construction, the word \( \text{sh}(w'_1) \) represents \( \beta \) and it is \( \sigma \)-positive or \( \sigma \)-negative.

On the other hand, it will be often convenient in the sequel to have a name for the braids that admit a \( \sigma \)-positive word representative. So, we introduce the following natural terminology.

Definition 1.15. A braid \( \beta \) is said to be \( \sigma \)-positive inside \( B_n \) (resp. \( \sigma \)-negative, \( \sigma_1 \)-positive, \( \sigma_1 \)-negative, \( \sigma_1 \)-free) if, among all word representatives of \( \beta \), there is at least one \( n \)-strand braid word that is \( \sigma \)-positive (resp. \( \sigma \)-negative, \( \sigma \)-positive, \( \sigma \)-negative, \( \sigma \)-free).

We insist that, in Definition 1.15, we only demand that there exists at least one word representative with the considered property. So, for instance, the braid \( \sigma_2^{-1}\sigma_3\sigma_2 \) is \( \sigma_2 \)-positive since, among its many word representatives, there is one, namely \( \sigma_3\sigma_2\sigma_3^{-1} \), that is \( \sigma_2 \)-positive—there are many more: \( \sigma_3\sigma_2\sigma_3^{-1}\sigma_3\sigma_3^{-1} \) is another \( \sigma_2 \)-positive 4-strand braid word that represents the braid \( \sigma_2^{-1}\sigma_3\sigma_2 \).

With this terminology, \( \beta <_n \beta' \) is equivalent to \( \beta'^{-1}\beta \) being \( \sigma \)-positive inside \( B_n \). Similarly, Property A means that a \( \sigma \)-positive braid is not trivial, and Property C that every nontrivial braid of \( B_n \) is \( \sigma \)-positive or \( \sigma \)-negative inside \( B_n \).

Remark 1.16. By Proposition 1.9(ii), a braid \( \beta \) of \( B_n \) satisfies \( \beta >_n 1 \) if and only if it satisfies \( \beta > \infty 1 \), hence \( \beta \) is \( \sigma \)-positive inside \( B_\infty \) if and only if it is \( \sigma \)-positive inside \( B_\infty \). In other words, if an \( n \)-strand braid admits a word representative that is \( \sigma \)-positive, then it admits a word representative that is \( \sigma \)-positive and is an \( n \)-strand braid word, an \textit{a priori} stronger property. Building on this result, we shall often drop the mention “inside \( B_n \)”, exactly as when we write \( < \) for \( <_n \). However, a careful distinction has to be made when proving Property C. It can be mentioned that the original argument of [47] only leads to a proof of Property C in \( B_\infty \): this is enough to order every braid group \( B_n \), but not to deduce Property C in \( B_n \); see Chapter IV.

1.4. The \( \sigma \)-ordering of braids. If \( \prec \) is an ordering of a group \( G \) and \( \varphi \) is an automorphism of \( G \), then the relation \( \varphi(g) \prec \varphi(h) \) defines a new ordering of \( G \) with the same invariance properties as \( \prec \). In the case of \( B_n \), the flip automorphism, \( \i.e. \), the inner automorphism \( \Phi_n \) associated with the braid \( \Delta_n \), plays an important role, and it is natural to introduce the image of the \( \sigma \)-ordering under \( \Phi_n \), \( \i.e. \), the flipped version of the \( \sigma \)-ordering. As will be seen in Section 4, the new ordering so
obtained has some nice properties not shared by the original version, particularly in terms of avoiding the infinite descending sequence of (1.1).

We recall from Lemma I.4.4 that $\Phi_n$ exchanges $\sigma_i$ and $\sigma_{n-i}$ for $1 \leq i < n$, thus corresponding to a symmetry in the associated braid diagrams.

**Definition 1.17.** For $2 \leq n < \infty$ and $\beta,\beta'$ in $B_n$, we declare that $\beta <^n \beta'$ is true if we have $\Phi_n(\beta) < \Phi_n(\beta')$.

**Proposition 1.18.** The relation $<^n$ is a left-invariant ordering of $B_n$. Moreover, for all $\beta,\beta'$ in $B_n$, the relations $\beta <^n \beta'$ and $\beta <^{n+1} \beta'$ are equivalent.

**Proof.** The first part is clear as $\Phi_n$ is an automorphism of $B_n$.

Assume $\beta,\beta' \in B_n$ and $\beta <^n \beta'$. By definition, we have $\Phi_n(\beta) < \Phi_n(\beta')$, hence $\text{sh}(\Phi_n(\beta)) < \text{sh}(\Phi_n(\beta'))$ by Proposition 1.11. By construction, we have

$$\Phi_{n+1}(\beta) = \text{sh}(\Phi_n(\beta)) \quad \text{and} \quad \Phi_{n+1}(\beta') = \text{sh}(\Phi_n(\beta')),$$

so $\beta <^{n+1} \beta'$ follows. As $<^n$ is a linear ordering, this is enough to conclude that $<^n$ coincides with the restriction of $<^{n+1}$ to $B_n$. $\square$

Owing to Proposition 1.18, we shall drop the subscripts and simply write $<^*$ for the ordering of $B_\infty$ whose restriction to $B_n$ is $<^n$. For instance, we have

$$1 <^* \sigma_1 <^* \sigma_2 <^* \ldots .$$

The flipped order $<^*$ is easily described in terms of word representatives.

**Definition 1.19.** (i) A braid word $w$ is said to be $\sigma^*$-positive (resp. $\sigma^*$-negative) if, among the letters $\sigma_i^{\pm 1}$ that occur in $w$, the one with highest index occurs positively only (resp. negatively only).

(ii) A braid $\beta$ is said to be $\sigma^*$-positive (resp. $\sigma^*$-negative) if it admits at least one braid word representative that is $\sigma^*$-positive (resp. $\sigma^*$-negative).

The only difference between a $\sigma$-positive and a $\sigma^*$-positive braid word is that, in the former case, we consider the letter $\sigma_i$ with lowest index, while, in the latter case, we consider the letter $\sigma_i$ with highest index.

**Proposition 1.20.** For all braids $\beta,\beta'$, the relation $\beta <^* \beta'$ holds if and only if $\beta^{-1} \beta'$ is $\sigma^*$-positive.

**Proof.** By construction, an $n$-strand braid word $w$ is $\sigma^*$-positive if and only if the $n$-strand braid word $\Phi_n(w)$ is $\sigma$-positive. $\square$

Thus the flipped order $<^*$ is the counterpart of the $\sigma$-order $<$ in which the highest index replaces the lowest index, and $\sigma^*$-positive words replace $\sigma$-positive words. It is therefore natural to call it the $\sigma^*$-ordering of braids.

As the flip $\Phi_n$ is an automorphism of the group $B_n$, the properties of $<$ and $<^*$ are similar. However, there are at least two reasons for considering both $<$ and $<^*$. First, there is no flip on $B_\infty$, and the two orderings differ radically on $B_\infty$: (1.1) shows that $(B_\infty^-, <)$ has infinite descending sequences, while we shall see in Section 4.1 below that $(B_\infty^+, <^*)$ is a well-ordering, and, therefore, it has no infinite descending chain. The second reason is that, in subsequent chapters, certain approaches demand that one specific version be used: the original version $<$ in Chapter IV, the flipped version $<^*$ in Chapters VII and VIII.
1.5. Algorithmic aspects. The $\sigma$-ordering of braids is a complicated object. However, it is completely effective in that there exist efficient comparison algorithms. In this section (and everywhere in the sequel) we denote by $\overline{w}$ the braid represented by a braid word $w$—but, as usual, we use $\sigma_i$ both for the letter and for the braid it represents.

**Proposition 1.21.** For each $n$, the $\sigma$-ordering of $B_n$ has at most a quadratic complexity: there exists an algorithm that, starting with two $n$-strand braid words $w$, $w'$ of length $\ell$, runs in time $O(\ell^2)$ and decides whether $\overline{w} < \overline{w'}$ holds.

At this early stage, we cannot yet describe the algorithms witnessing to the above upper complexity bound. It turns out that most of the proofs of Property C alluded to in Section 1.2 provide an effective comparison algorithm. Some of them are quite inefficient—typically the one of Chapter IV—but several lead to a quadratic complexity. This is particularly the case with those based on the $\Phi$-normal form of Chapter VII and on the $\phi$-normal form of Chapter VIII: in both cases, the normal form can be computed in quadratic time, and, then, the comparison itself can be made in linear time. This is also the case with the lamination method of Chapter XII: in this case, the coordinates of a braid can be computed in quadratic time, and the comparison (with the unit braid) can then be made in (sub)linear time. Similar results are conjectured in the case of the handle reduction method of Chapter V and the Tetris algorithm of Chapter XI—see Chapter XVI for further discussion.

Let us mention that, for a convenient definition for the RAM complexity of the input braids, the algorithm of Chapter XII even leads to a complexity upper bound which is quadratic independently of the braid index $n$, i.e., there exists an absolute constant $C$ so that the running time for complexity $\ell$ input braids in $B_\infty$ is bounded above by $C \cdot \ell^2$.

We also point out that every comparison algorithm for the $\sigma$-ordering of braids automatically gives a solution to the braid word problem, i.e., to the braid isotopy problem: indeed, we have $\overline{w} = \overline{w'}$ if and only if we have neither $\overline{w} < \overline{w'}$ nor $\overline{w} > \overline{w'}$. It also leads to a comparison for the flipped version $\prec^*$ of the $\sigma$-ordering, as, if $w$, $w'$ are $n$-strand braid words, $\overline{w} \prec^* \overline{w'}$ is equivalent to $\Phi_n(w) < \Phi_n(w')$, and the flip automorphism $\Phi_n$ can be computed in linear time.

Another related question is that of effectively finding $\sigma$-positive representative words, i.e., starting with a braid word $w$, finding an equivalent braid word $w'$ that is $\sigma$-positive, $\sigma$-negative, or empty. Property C asserts that this is always possible. Every algorithmic solution to that problem gives a comparison algorithm as, by Property A, $w'$ being $\sigma$-positive implies $\overline{w} = \overline{w'} > 1$, but, conversely, deciding $\overline{w} > 1$ does not require that we exhibit a $\sigma$-positive witness.

**Proposition 1.22.** The $\sigma$-positive representative problem has at most an exponential complexity: there exist a polynomial $P(n, \ell)$ and an algorithm that, starting with an $n$-strand braid word $w$ of length $\ell$, runs in time $2^{P(n, \ell)}$ and returns a braid word of length bounded by $2^{P(n, \ell)}$ that is equivalent to $w$ and is $\sigma$-positive, $\sigma$-negative, or empty.

The handle reduction approach of Chapter V gives the precise form of such a polynomial: $P(n, \ell) = n^4 \ell$. From the transmission-relaxation approach of Chapter XI, an asymptotically better estimate can be extracted: $P(n, \ell) \approx \text{const} \cdot n\ell$. However, the algorithm outlined in Chapter XI is just polynomial, but the output
of the algorithm is not a braid word in the standard sense but a zipped word, this meaning that, sometimes, instead of writing one and the same subword many times, the algorithm outputs the subword once and specifies the number of repetitions. This allows us to make the size of the output bounded above by a polynomial in \( n \) and \( \ell \) though the length of the word after unzipping is not known to be of polynomial size so far.

It is likely that the approach of Chapter VIII leads to much better results: very recently, J. Fromentin announced a new algorithm that solves the \( \sigma \)-positive representative problem with a quadratic time complexity and a linear space complexity, without zipping the output. We refer to Chapter XVI for further discussion.

2. Local properties of the \( \sigma \)-ordering

We shall now list—with or without proof—some properties of the \( \sigma \)-ordering of braids. In this section, we consider properties that can be called local in that they involve finitely many braids at a time.

2.1. Curious examples. We start with a series of examples, including some rather surprising ones, that illustrate the complexity of the \( \sigma \)-ordering. The reader should note that all examples below live in \( B_3 \). This shows that, despite its simple definition, even the \( \sigma \)-ordering of 3-strand braids is a quite complicated object.

The first example shows that the \( \sigma \)-ordering is not invariant under multiplication on the right, as was already known from Proposition 1.2.

Example 2.1. Let \( \beta = \sigma_1 \sigma_2^{-1} \), and \( \gamma = \sigma_1 \sigma_3 \), i.e., \( \gamma = \Delta_3 \). The word \( \sigma_1 \sigma_2^{-1} \) contains one occurrence of \( \sigma_1 \) and no occurrence of \( \sigma_1^{-1} \), so the braid \( \beta \) is \( \sigma \)-positive, and \( \beta > 1 \) is true. On the other hand, the braid \( \gamma^{-1} \beta \gamma \) is represented by the word \( \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_1 \sigma_3 \sigma_1 \), hence also by the equivalent word \( \sigma_2 \sigma_1^{-1} \), as, by Lemma I.4.4, we have \( \Delta_3^{-1} \sigma_i \Delta_3 = \sigma_{3-i} \) for \( i = 1, 2 \). The word \( \sigma_2 \sigma_1^{-1} \) contains one letter \( \sigma_1^{-1} \) and no letter \( \sigma_1 \). So, by definition, we have \( \gamma^{-1} \beta \gamma < 1 \), and, therefore, \( \beta \gamma < \gamma \). So \( 1 < \beta \) does not imply \( \gamma < \beta \gamma \).

A phenomenon connected with the noninvariance under right multiplication is that a conjugate of a braid that is larger than 1 may be smaller than 1. Example 2.1 actually gives us an illustration of this situation: in fact, in this case, the conjugate is the inverse.

Example 2.2. Let \( \beta = \sigma_1 \sigma_2^{-1} \) again. Then \( \beta \) is \( \sigma_1 \)-positive, hence larger than 1. By Lemma I.4.4, conjugating by \( \Delta_3 \) amounts to exchanging \( \sigma_1 \) and \( \sigma_2 \). So we have \( \Delta_3 \beta \Delta_3^{-1} = \sigma_2 \sigma_1^{-1} \), a \( \sigma_1 \)-negative braid, hence smaller than 1, i.e., we have \( \beta > 1 \) and \( \Delta_3 \beta \Delta_3^{-1} < 1 \); however, we shall see in Corollary 3.7 below that the conjugates of a braid \( \beta \) cannot be too far from \( \beta \).

An easy exercise is that every left-invariant ordering such that \( g < h \) implies \( g^{-1} > h^{-1} \) is also right-invariant. As the braid ordering is not right-invariant, there must exist counterexamples, i.e., braids \( \beta, \gamma \) satisfying \( \beta < \gamma \) and \( \beta^{-1} > \gamma^{-1} \). Here are examples of this situation.

Example 2.3. Let \( \beta = \Delta_3 \) and \( \gamma = \sigma_3^2 \sigma_1 \). Then we find \( \beta^{-1} \gamma = \sigma_1 \sigma_2^{-1} \), a \( \sigma_1 \)-positive word, and \( \beta \gamma^{-1} = \sigma_1 \sigma_2^{-1} \), again a \( \sigma_1 \)-positive word, So, in this case, we have \( 1 < \beta < \gamma \) and \( \beta^{-1} < \gamma^{-1} \).
EXAMPLE 2.4. Here is a stronger example. Let \( \beta = \sigma_2^{-1} \sigma_2^{1} \) and \( \gamma = \Delta_3 \). We now find \( \beta^{-1} \gamma = \sigma_2^{-1} \sigma_1 \) (see below), a \( \sigma_1 \)-positive word, and \( \beta \gamma^{-1} = \sigma_2^{-1} \sigma_2^{1} \), a \( \sigma_1 \)-positive word. So we obtain again \( 1 < \beta < \gamma \) and \( \beta^{-1} < \gamma^{-1} \). But there is more. We claim that \( \beta^{-p} \gamma = \sigma_1 \sigma_2^{-2p+1} \sigma_1 \) holds for \( p \geq 1 \). Indeed, for \( p = 1 \), we have
\[
\beta^{-1} \gamma = \sigma_2^{-1} \sigma_2^{1} \cdot \sigma_1^{-1} \sigma_2 \sigma_1 \cdot \sigma_1 = \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1 \cdot \sigma_1 = \sigma_1 \sigma_2^{-1} \sigma_1.
\]
For \( p \geq 2 \), applying the induction hypothesis, we find
\[
\beta^{-p} \gamma = \sigma_2^{-1} \sigma_2^{1} \cdot \beta^{-p+1} \gamma
= \sigma_2^{-1} \sigma_2^{1} \sigma_1 \sigma_2^{-2p} \gamma
= \sigma_2^{-1} \sigma_1 \sigma_2^{-2} \cdot \sigma_2^{-2p+3} \gamma
= \sigma_2^{-1} \sigma_2^{-2p+1} \sigma_1.
\]
As \( \sigma_1 \sigma_2^{-2p+1} \sigma_1 \) is a \( \sigma_1 \)-positive word for each \( p \), we have in this case \( 1 < \beta^{-p} < \gamma \) for each positive \( p \), and \( \beta^{-1} < \gamma^{-1} \).

Even more curious situations occur. Assume that \( \beta \) is a \( \sigma_1 \)-positive braid. Then the sequence \( 1, \beta, \beta^2, \ldots \) is strictly increasing, and its entries admit expressions in which more and more letters \( \sigma_1 \) occur. One might therefore expect that, eventually, the braid \( \beta \) dominates \( \sigma_1 \), which only contains one letter \( \sigma_1 \). The next example shows this is not the case.

EXAMPLE 2.5. Consider \( \beta = \sigma_2^{-1} \sigma_1 \). Then \( \beta^p < \sigma_1 \) holds for each \( p \). The inequality clearly holds for \( p \leq 0 \). For positive \( p \), we will show that \( \sigma_1^{-1} \beta^p \) is \( \sigma_1 \)-negative. To this end, we prove the equality
\[
(2.1) \quad \sigma_1^{-1} \beta^p = \sigma_1 (\sigma_2 \sigma_1^{-1})^{p-1} \sigma_1^{-1} \sigma_2^{-1}
\]
using induction on \( p \geq 1 \). For \( p = 1 \), \( (2.1) \) reduces to \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 = \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \), which directly follows from the braid relation. For \( p \geq 2 \), we find
\[
\sigma_1^{-1} \beta^p = (\sigma_1^{-1} \beta^{p-1}) \cdot \sigma_2^{-1} \sigma_1
= \sigma_1 (\sigma_2 \sigma_1^{-1})^{p-2} \sigma_1^{-1} \sigma_2^{-1} \cdot \sigma_1
= \sigma_1 (\sigma_2 \sigma_1^{-1})^{p-2} \sigma_1^{-2} \sigma_2^{-1} \sigma_1 = \sigma_2 (\sigma_2 \sigma_1^{-1})^{p-1} \sigma_1^{-1} \sigma_2^{-1},
\]
using the induction hypothesis and the equality \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 = \sigma_2 \sigma_1^{-2} \sigma_2^{-1} \).

It can be observed that, more generally, \( \beta^p < \sigma_2 \sigma_1 \) holds for all nonnegative \( p \) and \( q \). So the ascending sequence \( \beta^p \) does not even approach \( \sigma_1 \), as it remains below each entry in the descending sequence \( \sigma_2 \sigma_1 \).

Our last example will demonstrate that the \( \sigma \)-ordering of \( B_n \) is not Conradian.

DEFINITION 2.6. A left-invariant ordering \( \prec \) of a group \( G \) is Conradian if for all \( g, h \) in \( G \) that are greater than \( 1 \), there exists a positive integer \( p \) satisfying \( h < gh^p \).

Conrad used this property in [38] to show that such left-ordered groups share many of the properties of bi-orderable groups; see Section XV.5 for more details.

PROPOSITION 2.7. For \( n \geq 3 \), the \( \sigma \)-ordering of the braid group \( B_n \) is not Conradian.

PROOF. Let \( \beta = \sigma_2^{-1} \sigma_1 \) and \( \gamma = \sigma_2^{-2} \sigma_1 \). Clearly, \( \beta \) and \( \gamma \) are \( \sigma_1 \)-positive, so \( \beta > 1 \) and \( \gamma > 1 \) hold. We claim that \( \gamma \beta^p < \beta \) holds for each \( p \geq 0 \). To see that, using induction on \( p \geq 0 \), we prove the equality
\[
(2.2) \quad \beta^{-1} \gamma \beta^p = \sigma_1^2 (\sigma_1^{-1} \sigma_2^{p-1} \sigma_1^{-2} \sigma_2^{-1}).
\]
For \( p = 0 \), using the braid relations, we find
\[
\beta^{-1} \gamma = \sigma_1^{-1} \sigma_2 \sigma_2^{-2} \sigma_1 = \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_1 = \sigma_2 \sigma_1^{-1} \sigma_2^{-1} = \sigma_2^2 (\sigma_1^{-1} \sigma_2)^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1}.
\]
For \( p = 1 \), we have
\[
\beta^{-1} \gamma \beta = \sigma_1^{-1} \sigma_2 \sigma_2^{-2} \sigma_1 \sigma_2^{-1} \sigma_1 = \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1 = \sigma_2 \sigma_1^{-1} \sigma_2^{-2} \sigma_1 = \sigma_2^2 \sigma_1^{-1} \sigma_2^{-1} \sigma_2^{-1}.
\]
For \( p \geq 1 \), again using the equality \( \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \) of Example 2.5, we find
\[
\beta^{-1} \gamma \beta^p = (\sigma_2^2 (\sigma_1^{-1} \sigma_2)^{p-2} \sigma_1^{-2} \sigma_2^{-1})(\sigma_2^{-1} \sigma_1) = \sigma_2^2 (\sigma_1^{-1} \sigma_2)^{p-2} \sigma_1^{-1} \sigma_2^{-1} \sigma_2^{-1} = \sigma_2^2 (\sigma_1^{-1} \sigma_2)^{p-1} \sigma_1^{-1} \sigma_2^{-1}.
\]
For \( p \geq 1 \), the right-hand side of (2.2) is \( \sigma_1 \)-negative, and, for \( p = 0 \), it is equivalent to the \( \sigma_1 \)-negative word \( \sigma_2 \sigma_1^{-1} \sigma_2 \), so, in each case, we obtain \( \beta < \gamma \beta^p \). \( \square \)

2.2. Property S. After the many counterexamples of Section 2.1, we turn to positive results.

We have seen in Example 2.1 that the \( \sigma \)-ordering of braids is not invariant under multiplication on the right, and, therefore, that a conjugate of a braid larger than 1 need not be larger than 1. This phenomenon cannot, however, occur with conjugates of positive braids, i.e., of braids that can be expressed using the generators \( \sigma_i \) only, and not their inverses. The core of the question is the last of the three fundamental properties of braids we shall develop here:

Property S (Subword). Every braid of the form \( \beta^{-1} \sigma \beta \) is \( \sigma \)-positive.

Property S was first proved by Richard Laver in [137]. In this text, proofs of Property S appear on pages 83, 152, 193, and 263.

Using the compatibility of \( \prec \) with multiplication on the left and a straightforward induction, we deduce the following result, which explains our terminology:

**Proposition 2.8.** Assume that \( \beta, \beta' \) are braids and some braid word representing \( \beta' \) is obtained by inserting positive letters \( \sigma_i \) in a braid word representing \( \beta \). Then we have \( \beta' > \beta \).

We recall that \( B^+_{\infty} \) denotes the submonoid of \( B_{\infty} \) generated by the braids \( \sigma_i \).

Another consequence of Property S is:

**Proposition 2.9.** If \( \beta \) belongs to \( B^+_{\infty} \) and is not 1, then \( \beta' > 1 \) is true for every conjugate \( \beta' \) of \( \beta \). More generally, \( \beta > 1 \) is true for every \( \sigma \)-positive braid \( \beta \), the latter being defined as a braid that can be expressed as a product of conjugates of positive braids.

**Proof.** Assume \( \beta' = \gamma^{-1} \beta \gamma \) with \( \beta \in B^+_{\infty} \). By definition, \( \beta \) is a product of finitely many braids \( \sigma_i \), so, in order to prove \( \beta' > 1 \), it suffices to establish that \( \gamma^{-1} \sigma_i \gamma > 1 \) holds for each \( i \), and this is Property S. \( \square \)

As was noted by Stepan Orevkov [166], the converse implication is not true: the braid \( \sigma_2^{-5} \sigma_1 \sigma_2 \sigma_1 \) is a non-quasi-positive braid but every conjugate of it is \( \sigma \)-positive.

By applying the flip automorphism \( \Phi_n \), we immediately deduce from Property S that every braid of the form \( \beta^{-1} \sigma \beta \) is also \( \sigma^n \)-positive, and that the counterpart of Proposition 2.8 involving the ordering \( \prec^* \) is true. A direct application is the following result, which is important for analysing the restriction of \( \prec^* \) to \( B^+_{\infty} \):

**Proposition 2.10.** For each \( n \), the set \( B^+_n \) is the initial segment of \( (B^+_{\infty}, \prec^*) \) determined by \( \sigma_n \), i.e., we have \( B^+_n = \{ \beta \in B^+_{\infty} \mid \beta \prec^* \sigma_n \} \).
3. Global properties of the $\sigma$-ordering

We turn to more global properties, involving infinitely many braids at a time. Here we successively consider the Archimedean property, the question of density and the associated topology, and convex subgroups.

3.1. The Archimedean property. We shall show that the $\sigma$-ordering and, more generally, any left-invariant ordering of $B_n$ fails to be Archimedean for $n \geq 3$. However, certain partial Archimedean properties involving the central elements $\Delta^2_n$ are satisfied.

Definition 3.1. A left-ordered group $(G, \prec)$ is said to be Archimedean if, for all $g, h$ larger than 1 in $G$, there exists a positive integer $p$ for which $g \prec h^p$ holds.

In other words, the powers of any nontrivial element are cofinal in the ordering. For example, an infinite cyclic group, with either of the two possible orderings, is Archimedean. On the other hand, $\mathbb{Z} \times \mathbb{Z}$ with lexicographic ordering is not Archimedean, whereas Archimedean orderings for the same group do exist, by embedding $\mathbb{Z} \times \mathbb{Z}$ in the additive real numbers, sending the generators to rationally independent numbers, and taking the induced ordering.
Proposition 3.2. The $\sigma$-ordering of $B_n$ is not Archimedean for $n \geq 3$.

Proof. For every positive integer $p$, we have $1 < \sigma_2^p < \sigma_1$. □

One can say more.

Proposition 3.3. For $n \geq 3$, every left-invariant ordering of $B_n$ fails to be Archimedean.

This follows from the fact that $B_n$ is not Abelian for $n \geq 3$ and from a result of P. Conrad [38] generalizing the classical theorem of Hölder [111]: any left-invariant Archimedean ordering of a group must also be right-invariant, and the group embeds, simultaneously in the algebraic and order senses, in the additive real numbers. In particular, such a group is Abelian.

By contrast to the previous negative result, there is a partial Archimedean property involving the central element $\Delta_n^2$, namely that every braid is dominated by some power of the braid $\Delta_n^2$.

The results we shall establish turn out to be true not only for the $\sigma$-ordering, but also for any left-invariant ordering of $B_n$. So, for the rest of this section, we consider this extended framework. When $\prec$ denotes a strict ordering, $\preceq$ denotes the corresponding nonstrict ordering, i.e., $x \preceq y$ stands for “$x < y$ or $x = y$”.

Lemma 3.4. Assume that $\prec$ is a left-invariant ordering of $B_n$. Then $\Delta_n^{2p} \prec \beta$ implies $\beta^{-1} \prec \Delta_n^{-2p}$, and the conjunction of $\Delta_n^{2p} \prec \beta$ and $\Delta_n^{2q} \prec \gamma$ implies $\Delta_n^{2p+2q} \prec \beta \gamma$. The same implications hold for $\preceq$.

Proof. Assume $\Delta_n^{2p} \prec \beta$. Multiplying by $\beta^{-1}$ on the left, we get $\beta^{-1} \Delta_n^{2p} \prec 1$, which is also $\Delta_n^{2p} \beta^{-1} \prec 1$. Multiplying by $\Delta_n^{-2p}$ on the left, we deduce $\beta^{-1} \prec \Delta_n^{-2p}$.

Now assume $\Delta_n^{2p} \prec \beta$ and $\Delta_n^{2q} \prec \gamma$. By multiplying the first inequality by $\Delta_n^{2q}$ on the left, we obtain $\Delta_n^{2p+2q} \prec \Delta_n^{2q} \beta = \beta \Delta_n^{2q}$. By multiplying the second inequality by $\beta$ on the left, we obtain $\beta \Delta_n^{2q} \prec \beta \gamma$. We deduce $\Delta_n^{2p+2q} \prec \beta \gamma$. □

Lemma 3.5. Assume that $\prec$ is a left-invariant ordering of $B_n$ satisfying $1 \prec \Delta_n$. Then, for each $i$ in $\{1, \ldots, n-1\}$, we have $\Delta_n^{-2} \prec \sigma_i \prec \Delta_n^2$.

Proof. By Lemma 1.4.4, we have $\delta_n^n = \Delta_n^2$, so the hypothesis $1 \prec \Delta_n$ implies $1 \prec \Delta_n^2 = \delta_n^n$, hence $1 \prec \delta_n$, and, therefore, $1 \prec \delta_n \prec \delta_n^2 \prec \cdots \prec \delta_n^n = \Delta_n^2$.

Assume that $\Delta_n^i \preceq \sigma_i$ holds for some $i$. Let $j$ be any element of $\{1, \ldots, n-1\}$. By formulas (1.4.3) and (1.4.4), we can find $p$ with $0 \leq p \leq n-1$ satisfying $\sigma_j = \delta_n^{n-p} \delta_n^{n} \sigma_i \delta_n^{-n}$. Then we obtain

$$1 \prec \delta_n^{n-p} = \delta_n^{-p} \Delta_n^2 \preceq \delta_n^{-p} \sigma_i \delta_n^{-n} \sigma_i \delta_n^n = \sigma_j.$$ 

So $1 \prec \sigma_j$ holds for each generator $\sigma_j$. Applying Lemma 3.4, we deduce that, if a braid $\beta$ can be represented by a positive braid word that contains at least one letter $\sigma_i$, then $\Delta_n^i \preceq \beta$ holds. This applies in particular to $\Delta_n$, and we deduce $\Delta_n^{-2} \preceq \Delta_n$, which contradicts the assumption $1 \prec \Delta_n$.

Similarly, assume that $\sigma_i \preceq \Delta_n^{-2}$ holds. Consider again any $\sigma_j$. If $p$ is as above, we also have $\sigma_j = \delta_n^{n-p} \sigma_i \delta_n^{-n}$, since $\delta_n^n$ lies in the center of $B_n$. Then we find

$$\sigma_j \preceq \delta_n^{n-p} \sigma_i \delta_n^{-n} \preceq \delta_n^{n-p} \sigma_i \Delta_n^{-2} = \delta_n^{-p} \preceq 1.$$ 

This time, $\sigma_j \preceq 1$ holds for each $j$. As $\Delta_n$ is a positive braid, this implies $\Delta_n \preceq 1$, which contradicts the assumption $1 \prec \Delta_n$. □

Gathering the results, we immediately deduce:
Proposition 3.6. Assume $\prec$ is a left-invariant ordering of $B_n$ and $1 \prec \Delta_n$ holds. Then, for each braid $\beta$ in $B_n$, there exists a unique integer $p$ for which $\Delta_n^{2p} \prec \beta \prec \Delta_n^{2p+2}$ is true. Moreover, if $\beta$ can be represented by a braid word of length $\ell$, we have $|p| \leq \ell$.

Proof. Lemma 3.5 implies that each generator $\sigma_i$ lies in the interval $(\Delta_n^{-2}, \Delta_n^2)$. Then Lemma 3.4 implies that every braid that can be represented by a word of length $\ell$ lies in the interval $[\Delta_n^{-2\ell}, \Delta_n^{2\ell})$. As this interval is the disjoint union of the intervals $[\Delta_n^{2p}, \Delta_n^{2p+2})$ for $-\ell \leq p < \ell$, the result of the proposition follows. □

In this way, we obtain a decomposition of $(B_n, \prec)$ into a sequence of disjoint intervals of size $\Delta_n^2$, as suggested in Figure 1.

As noted by A. Malyutin and N.Yu. Netsvetaev in [150], the previous result implies that the action of conjugacy cannot move a braid too far.

Corollary 3.7 (Figure 1). Assume that $\prec$ is a left-invariant ordering of $B_n$ satisfying $1 \prec \Delta_n$. Then, if $\beta$ and $\beta'$ are conjugate,

$$
\Delta_n^{2p} \prec \beta \prec \Delta_n^{2p+2} \implies \Delta_n^{2p-2} \prec \beta' \prec \Delta_n^{2p+4}.
$$

So, in particular, $\beta\Delta_n^{-2} \prec \beta' \prec \beta\Delta_n^4$ is always true.

Proof. Assume $\Delta_n^{2p} \prec \beta \prec \Delta_n^{2p+2}$ and $\beta' = \gamma\beta\gamma^{-1}$. By Proposition 3.6, we have $\Delta_n^{2q} \prec \gamma \prec \Delta_n^{2q+2}$ for some $q$. Lemma 3.4 first implies $\Delta_n^{-2q-2} \prec \gamma^{-1} \prec \Delta_n^{-2q}$, and then

$$
\Delta_n^{2q+2p-2q-2} \prec \gamma\beta\gamma^{-1} \prec \Delta_n^{2q+2p+2-2q},
$$

which gives $\Delta_n^{2p-2} \prec \beta' \prec \Delta_n^{2p+4}$.

□

![Figure 1. Powers of $\Delta_n^2$ and the action of conjugacy on $(B_n, \prec)$](image)

All the previous results apply to the $\sigma$-ordering, as it is a left-invariant ordering of $B_n$ and $1 \prec \Delta_n$ is satisfied. Note that, in this case, Corollary 3.7 is optimal in the sense that we cannot replace intervals of length $\Delta_n^2$ with intervals of length $\Delta_n$ in Lemma 3.4: for instance, we have $1 \prec \sigma_1\sigma_2 \prec \Delta_3$ and $\Delta_3^2 \prec \Delta_3$.

3.2. Discreteness and density. Left-invariant orderings of a group have a sort of homogeneity—the ordering near any two group elements has similar order properties, because of invariance under left translation. In particular, there is a basic dichotomy between discrete and dense orders.

Definition 3.8. A left-invariant ordering of a group is said to be **discrete** if its positive cone has a least element; it is said to be **dense** if the positive cone does not have a least element.
Equivalently, a left-invariant ordering of a group is discrete if every group element has an immediate successor and predecessor, and it is dense if between any two group elements one can find another element of the group. One verifies easily that, in a discretely left-ordered group, with least element \( \varepsilon \) larger than 1, the immediate successor of a group element \( g \) is \( g\varepsilon \) and its immediate predecessor is \( g^{-1}\varepsilon \).

The braid orderings display both types.

**Proposition 3.9.** The \( \sigma \)-ordering of \( B_n \) is discrete, with least \( \sigma \)-positive element \( \sigma_{n-1} \).

**Proof.** Clearly \( \sigma_{n-1} \) is \( \sigma \)-positive. Conversely, assume that \( \beta \) belongs to \( B_n \) and is \( \sigma \)-positive. If \( \beta \) is \( \sigma_i \)-positive for some \( i \) with \( i \leq n-2 \), then \( \sigma_{n-1}\beta \) is \( \sigma_i \)-positive as well, so \( \sigma_{n-1} < \beta \) holds. On the other hand, if \( \beta \) is \( \sigma_{n-1} \)-positive, it must be \( \sigma_{n-1}^p \) for some \( p \geq 1 \), and we find \( \sigma_{n-1}^{-1}\beta = \sigma_{n-1}^{-1} \), hence \( \sigma_{n-1} \leq \beta \).  

As the flip automorphism \( \Phi_n \) is an isomorphism of \( (B_n, \sigma) \) to \( (B_n, \sigma^+) \), the flipped version \( \sigma^+ \) of the \( \sigma \)-ordering is also discrete on \( B_n \), and \( \sigma_1 \) is the least \( \sigma^+ \)-positive element. In the inclusions \( B_n \subseteq B_{n+1} \), the \( \sigma^+ \)-ordering has the pleasant property that the same element \( \sigma_1 \) is least \( \sigma \)-positive in each braid group. For this reason, we see a difference in the two orderings in the limit. The reader may easily verify the following.

**Proposition 3.10.** The \( \sigma \)-ordering of \( B_\infty \) is dense, whereas the \( \sigma^+ \)-ordering of \( B_\infty \) is discrete, with \( \sigma_1 \) being the least element larger than 1.

**Corollary 3.11.** The ordered set \( (B_\infty, \sigma) \) is order-isomorphic to \( (Q, <) \).

**Proof.** A well-known result of Cantor says that any two countable linearly ordered sets that are dense—there always exists an element between any two elements—and unbounded—there is no minimal or maximal element—are isomorphic: assuming that the sets are \( \{a_n \mid n \in \mathbb{N}\} \) and \( \{b_n \mid n \in \mathbb{N}\} \), one alternatively defines \( f(a_0), f^{-1}(b_0), f(a_1), f^{-1}(b_1) \), etc. so as to keep \( f \) order-preserving.

Here the rationals are eligible, and the set \( B_\infty \) is countable. So, in order to apply Cantor’s criterion, it suffices to prove that \( (B_\infty, \sigma) \) is dense and unbounded. The former result is Proposition 3.10. The latter is clear: for every braid \( \beta \), we have \( \beta \sigma_1^{-1} < \beta < \beta \sigma_1 \).

Of course, the order-isomorphism of Corollary 3.11 could not be an isomorphism in the algebraic sense, as \( B_\infty \) is non-Abelian.

Every linearly ordered set has an order topology, with open intervals forming a basis for the topology. If the ordering is discrete, as is the case for the \( \sigma \)-ordering of \( B_n \) for \( n < \infty \), then the topology is also discrete. Since \( B_\infty \), with the \( \sigma \)-ordering, is order isomorphic with the rational numbers, its order topology is metrizable. In fact, it has a natural metric, as follows.

**Proposition 3.12.** For \( \beta \neq \beta^\prime \) in \( B_\infty \), define \( d(\beta, \beta^\prime) \) to be \( 2^{-p} \) where \( p \) is the greatest integer satisfying \( \beta^{-1}\beta^\prime \in sh^p(B_\infty) \), completed with \( d(\beta, \beta) = 0 \). Then \( d \) is a distance on \( B_\infty \), and the topology of \( B_\infty \) associated with the linear order \( < \) is the topology associated with \( d \).

**Proof.** It is routine to verify that \( d \) is a distance. The open disk of radius \( 2^{-p} \) centered at \( \beta \) is the left coset \( \beta sh^p(B_\infty) \), i.e., the set of all braids of the form \( \beta sh^p(\gamma) \).
Assume now that $\beta_1, \beta, \beta_2$ lie in $B_n$ and $\beta_1 < \beta < \beta_2$ holds. We will show that the open $d$-disk around $\beta$ of radius $2^{-n+1}$ is included in the interval $(\beta_1, \beta_2)$. Indeed, if $d(\beta, \gamma) < 2^{-n+1}$, then $\beta^{-1} \gamma$ belongs to $\text{sh}^n(B_{\infty})$. The hypothesis $\beta_1 < \beta$ implies that $\beta_1^{-1} \beta$ is $\sigma_i$-positive for some $i < n - 1$. Writing $\beta_1^{-1} \gamma = (\beta_1^{-1} \beta)(\beta^{-1} \gamma)$, we see that $\beta_1^{-1} \gamma$ is also $\sigma_i$-positive and, therefore, $\beta_1 < \gamma$ is true. A similar argument gives $\gamma < \beta_2$.

Conversely, let us start with an arbitrary open $d$-disk $\beta \text{sh}^p(B_{\infty})$. Let $\beta'$ be a braid in this disk; we have to find an open $< \beta$-interval containing $\beta'$ which lies entirely in the disk. By hypothesis, we have $\beta' = \beta \text{sh}(\gamma)$ for some $\gamma$ of $B_{\infty}$. Let $\gamma_1$ and $\gamma_2$ be any braids satisfying $\gamma_1 < \gamma < \gamma_2$. Then the interval $(\beta \text{sh}(\gamma_1), \beta \text{sh}(\gamma_2))$ contains $\beta \text{sh}(\gamma)$ and is included in the disk, because $\text{sh}(B_{\infty})$ is convex—see Proposition 3.17 below. This completes the proof that the topologies associated with $<$ and with $d$ coincide.

3.3. Dense subgroups. It is clear that densely ordered groups can have subgroups which are discretely ordered (by the same ordering)—witness $\mathbb{Z}$ in $\mathbb{Q}$. But the reverse can happen, too. For example, the lexicographic ordering on $\mathbb{Q} \times \mathbb{Z}$ is discrete—wit the least positive element $(0, 1)$—whereas the subgroup $\mathbb{Q} \times \{0\}$ is densely ordered. This latter phenomenon happens quite naturally also for the braid groups.

Note that, if one allows the generators $\sigma_i$ to commute, the braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ implies that $\sigma_i$ and $\sigma_{i+1}$ become equal. From this one sees that the Abelianization of $B_n$ is infinite cyclic, and the Abelianization map $B_n \rightarrow \mathbb{Z}$ can be identified with the sum of the exponents of a word in the $\sigma_i$ generators. The commutator subgroup $[B_n, B_n]$ consists exactly of braids expressed in the generators $\sigma_i$ with exponent sum zero.

**Proposition 3.13 ([37]).** For $n \geq 3$, the commutator subgroup $[B_n, B_n]$ is densely ordered under the $\sigma$-ordering.

**Proof.** For simplicity, we will prove this just for $n = 3$, referring the reader to [37] for the general case, whose proof is similar.

For contradiction, suppose $[B_3, B_3]$ has a least $\sigma$-positive element $\beta$. We consider the braid $\beta \sigma_2 \beta^{-1}$. There are three possibilities:

Case 1: $\beta \sigma_2 \beta^{-1}$ is $\sigma_1$-positive. Then $\beta$ must be $\sigma_1$-positive. So is $\beta \sigma_2 \beta^{-1} \sigma_2^{-1}$ and we have $1 < \beta \sigma_2 \beta^{-1} \sigma_2^{-1}$. On the other hand, as $\beta$ is $\sigma_1$-positive, $\sigma_2 \beta^{-1} \sigma_2^{-1}$ is $\sigma_1$-negative, and we have $\sigma_2 \beta^{-1} \sigma_2^{-1} < 1$ and $\beta \sigma_2 \beta^{-1} \sigma_2^{-1} < \beta$. So the commutator $\beta \sigma_2 \beta^{-1} \sigma_2^{-1}$ is a smaller $\sigma$-positive element of $[B_3, B_3]$ than $\beta$, contradicting the hypothesis on $\beta$.

Case 2: $\beta \sigma_2 \beta^{-1}$ is $\sigma_1$-negative. A similar argument gives $1 < \beta \sigma_2^{-1} \beta^{-1} \sigma_2 < \beta$, again a contradiction.

Case 3: $\beta \sigma_2 \beta^{-1}$ is $\sigma_2^p$ for some $p$. Counting the exponents, we see that the only possibility is $p = 1$, i.e., $\beta$ commutes with $\sigma_2$. It is shown in [84] that the centralizer of the subgroup of $B_3$ generated by $\sigma_2$ is the subgroup (isomorphic to $\mathbb{Z} \times \mathbb{Z}$) generated by $\sigma_2$ and $\Delta_2^2$, so we must have $\beta = (\sigma_1 \sigma_2 \sigma_1)^q \sigma_2^r$ for some integers $q, r$. But, since $\beta$ is $\sigma_1$-positive and a commutator, we have $q > 0$ and $6q + r = 0$. Now, consider $\beta' = \sigma_1 \sigma_2^{-1}$. We have $\beta' > 1$ and $\beta' \in [B_3, B_3]$, and an easy calculation gives $\beta' < \beta$, again contradicting the hypothesis on $\beta$.

Other subgroups of $B_n$ with $n \geq 3$ which are shown to be densely ordered by the $\sigma$-ordering in [37] include the following:
• \([PB_n, PB_n]\), the commutator subgroup of the pure braid group; but \(PB_n\) itself is discretely ordered, with least positive element \(\sigma_{n-1}^2\);
• the subgroup of Brunnian braids—defined as braids such that, for every strand, its removal results in a trivial braid;
• the subgroup of homotopically trivial braids, as considered in [99];
• kernels of the Burau representation for those \(n\) for which this representation is unfaithful—it is known to be unfaithful for \(n \geq 5\) and faithful for \(n \leq 3\).

The method of proof is to identify explicitly which braids can possibly be the least \(\sigma\)-positive elements of a given normal subgroup of \(B_n\).

3.4. Convex subgroups. Convex subgroups play an important role in the theory of orderable groups.

Definition 3.14. If \((G, \prec)\) is a left-ordered group, a subgroup \(H\) of \(G\) is said to be convex if, for all \(h, h'\) in \(H\) and \(g\) in \(G\) satisfying \(h \prec g \prec h'\), one has \(g \in H\).

An equivalent criterion for convexity of \(H\) is the conjunction of \(1 \prec g \prec h\), \(g \in G\), and \(h \in H\) implies \(g \in H\). It is easy to verify that the collection of convex subgroups of a given group is linearly ordered by inclusion. Moreover, if \(N\) is a normal convex subgroup of the left-ordered group \(G\), then the quotient group \(G/N\) is left-orderable by ordering cosets according to their representatives.

If the ordering of \(G\) is discrete, and \(H\) is a convex subgroup distinct from \(\{1\}\), then the ordering on \(H\) is also discrete, and \(H\) contains the minimal positive element of \(G\), which is also minimal positive in \(H\).

We shall see that there are rather few convex subgroups in the braid groups under the \(\sigma\)-ordering.

Proposition 3.15. The group \(B_n\) has no proper normal convex subgroup.

Proof. Suppose \(H\) is a normal and convex subgroup of \(B_n\) distinct of \(\{1\}\). As remarked above, the minimal positive element \(\sigma_{n-1}\) of \(B_n\) belongs to \(H\) by convexity. Since \(H\) is normal, \(\sigma_1\) also belongs to \(H\), as the Garside braid \(\Delta_n\) conjugates it to \(\sigma_{n-1}\). All the other \(\sigma_i\) generators are positive and less than \(\sigma_1\), so they must also be in \(H\), and therefore we have \(H = B_n\), alternatively, we can observe that all generators \(\sigma_i\) are conjugated to \(\sigma_{n-1}\) in \(B_n\), as seen in Lemma I.4.4.

Proposition 3.16. For \(i\) in \(\{1, \ldots, n-1\}\), let \(H_i\) be the subgroup of \(B_n\) generated by \(\sigma_1, \ldots, \sigma_{n-1}\). Then each subgroup \(H_i\) is convex in \(B_n\) and these are the only nontrivial convex subgroups.

Proof. First, we verify that \(H_i\) is convex. Suppose \(1 < \gamma < \beta\) with \(\beta \in H_i\) and \(\gamma \in B_n\). Note that the \(\sigma\)-positive elements of \(H_i\) are exactly the \(\sigma_j\)-positive braids in \(B_n\) with \(j \geq i\). So \(\beta\) is \(\sigma_j\)-positive for some \(j \geq i\). By hypothesis, \(\gamma\) is \(\sigma_k\)-positive for some \(k \in \{1, \ldots, n-1\}\). If we had \(k < j\), then \(\beta^{-1}\gamma\) would be \(\sigma_j\)-positive, implying \(\beta < \gamma\) and contradicting the hypothesis. Therefore, we have \(k \geq j \geq i\) and \(\gamma\) lies in \(H_i\).

It remains to show that there are no other nontrivial convex subgroups. Assume that \(C\) is a convex subgroup of \(B_n\) distinct of \(\{1\}\). Let \(i\) be the least positive integer such that \(C\) contains a \(\sigma_i\)-positive braid, say \(\beta\). We will show that \(C = H_i\). First note that \(C\) contains each \(\sigma_j\) with \(j > i\), because \(\sigma_i^{-1}\beta\) is \(\sigma_j\)-positive and we have \(1 < \sigma_j < \beta \in C\).
4. The $\sigma$-ordering of positive braids

Now we may write $\beta = \beta_0 \sigma_1 \beta_1 \ldots \sigma_i \beta_i$ for some $m \geq 1$ and some $\beta_i$ belonging to $H_{i+1}$, hence to $C$. Since $C$ is a subgroup and $\beta_0$ belongs to $C$, the braid $\beta'$ defined by $\beta' = \sigma_i \beta_1 \ldots \sigma_i \beta_m$ also belongs to $C$. In case $m > 1$, we conclude $\sigma_i^{-1} \beta'$ is also $\sigma_i$-positive and therefore we have $1 < \sigma_i < \beta'$. On the other hand, if $m = 1$ holds, we have $\beta' = \sigma_i \beta_1$. In either case, we conclude that $\sigma_i$ belongs to $C$. We have shown that $C$ is included in $H_i$. If the inclusion were proper, then $C$ would contain a braid which is $\sigma_i$-positive for some $j < i$, contradicting our choice of $i$. □

Almost exactly the same argument shows the following.

**Proposition 3.17.** The nontrivial convex subgroups of $B_\infty$ are exactly those of the form $sh^i(B_\infty)$. None of these is normal.

Finally, using the flip automorphism $\Phi_n$, we see that, when the $\sigma^k$-ordering $<^k$ replaces the $\sigma$-ordering, then the convex subgroups of $B_n$ are the groups $B_i$ with $i \leq n$. The same holds for $B_\infty$.

4. The $\sigma$-ordering of positive braids

In this section, we review some results about the restriction of the orderings $<$ and $<^k$ to the braid monoids $B_n^+$, most of which will be further developed in Chapters VII and VIII. As the many examples of Section 2.1 showed, the $\sigma$-ordering is a quite complicated ordering. By contrast, its restriction to the monoid $B_n^+$ is a simple ordering, namely a well-ordering. In particular, every nonempty set of positive braids has a least element, and, if it is bounded, it has a least upper bound.

We give two proofs of the well-order property for the $\sigma$-ordering of $B_n^+$. Due to Laver [137] and based on Property S, the first one uses Higman's subword lemma, and it is not constructive. Then, we give another argument, which is constructive and much more precise. It is based on Serge Burckel's approach in [27]. Here we follow the new description of [62], which relies on an operation called the $\Phi_n$-splitting of a braid. It shows that the ordering of $B_n^+$ is a sort of lexicographical extension of the ordering of $B_{n-1}^+$.

Most of the properties described in this section for the monoids $B_n^+$ extend to the case of the so-called dual braid monoids $B_n^*$. Introduced by Birman, Ko, and Lee in [15], the dual monoid $B_n^*$ is a submonoid of $B_n$ that properly includes $B_n^+$. Interestingly, the proofs turn out to be easier in the case of $B_n^*$ than in the case of $B_n^+$. We refer to Chapter VIII for details.

4.1. The well-order property. Restricting a linear ordering to a proper subset always gives a linear ordering, but the properties of the initial ordering and of its restriction may be very different—we already saw examples in Section 3.3. This is what happens with the $\sigma$-ordering of $B_n$ and its restriction to $B_n^+$. For instance, we saw in Proposition 3.9 that $(B_n^+, <)$ is discrete, and therefore every braid $\beta$ has an immediate predecessor, namely $\beta \sigma_n^{-1}$. The situation is radically different with $B_n^+$. In particular, $(B_n^+, <)$ has limit points: for instance, in $(B_n^+, <)$, the braid $\sigma_1$ is the least upper bound of the increasing sequence $(\sigma_1^p)_{p \geq 0}$; see Figure 2.

We recall that a linear ordering is called a well-ordering if every nonempty subset has a least element, or, equivalently, provided some very weak form of the Axiom of Choice is assumed, if it admits no infinite descending sequence. A direct consequence of Property S is the following important result.
An infinite set of words over a finite alphabet necessarily contains two elements \( w \) such that \( wB \) many braids, each of which is a conjugate of some \( \sigma \). A positive braid word \( w \) is a sequence of braids, as we have an infinite descending sequence \( \sigma_1 > \sigma_2 > \ldots \). Such phenomena already occur inside \( B_3 \): for instance, the submonoid of \( B_3 \) generated by all conjugates \( \sigma_2^{-p} \sigma_1 \sigma_2^p \) of \( \sigma_1 \) and, more generally, the submonoid of all quasi-positive \( n \)-strand braids, defined to be the submonoid

\[
1 \quad \sigma_2 \sigma_2^2 \ldots \sigma_2^{-1} \sigma_2^{-1} \sigma_1 \ldots \sigma_2^{-1} \sigma_2 \ldots \sigma_1 \sigma_2^2 \sigma_2^2 \ldots (B_3, <)
\]

Figure 2. Restricting to positive braids completely changes the ordering: for instance, in \((B_3^+, <)\), the braid \( \sigma_1 \) is the limit of \( \sigma_3^k \) whereas, in \((B_3, <)\), it is an isolated point with immediate predecessor \( \sigma_3^{-1} \sigma_1 \); the grey part in \( B_3 \) includes infinitely many braids, such as \( \sigma_2^{-1} \sigma_1 \) and its neighbours—and much more—but none of them lies in \( B_3^+ \).

Proposition 4.1. For every \( n \), the restriction of \( < \) to \( B_n^+ \) is a well-ordering.

Proof. A theorem of Higman [108], known as Higman’s subword lemma, says: An infinite set of words over a finite alphabet necessarily contains two elements \( w, w' \) such that \( w' \) can be obtained from \( w \) by inserting intermediate letters (in not necessarily adjacent positions). Let \( \beta_1, \beta_2, \ldots \) be an infinite sequence of braids in \( B_n^+ \). Our aim is to prove that this sequence is not strictly decreasing. For each \( p \), choose a positive braid word \( w_p \) representing \( \beta_p \). There are only finitely many \( n \)-strand braid words of a given length, so, for each \( p \), there exists \( p' > p \) such that \( w_{p'} \) is at least as long as \( w_p \). So, inductively, we can extract a subsequence \( w_{p_1}, w_{p_2}, \ldots \) in which the lengths are nondecreasing. If the set \( \{w_{p_1}, w_{p_2}, \ldots \} \) is finite, there exist \( k, k' \) such that \( w_{p_k} = w_{p_{k'}} \). Otherwise, by Higman’s theorem, there exist \( k, k' \) such that \( w_{p_k} \) is a subword of \( w_{p_{k'}} \), and, by construction, we must have \( p_k < p_{k'} \). By Property S, this implies \( \beta_{p_k} \leq \beta_{p_{k'}} \) in \( B_n^+ \).

The previous proof actually shows more.

Proposition 4.2. Assume that \( M \) is a submonoid of \( B_{\infty} \) generated by finitely many braids, each of which is a conjugate of some \( \sigma_i \)—hence of \( \sigma_1 \). Then the restriction of \( < \) to \( M \) is a well-ordering.

Proof. In the proof of Proposition 4.1, Property S is used to ensure that, if a word \( w \) in the generators \( \sigma_i \) of \( B_n \) is a subword of another word \( w' \), then we have \( \overline{w} \leq \overline{w'} \), where \( \overline{w} \) denotes the braid represented by \( w \). Now the same property holds for the generators of \( M \), as each of them is a conjugate of some \( \sigma_i \). Indeed, inserting a pattern of the form \( v \sigma_i v^{-1} \) after \( w_1 \) in a braid word \( w_1 w_2 \) amounts to inserting \( \sigma_i \) in the equivalent braid word \( w_1 v \sigma_i v^{-1} w_2 \), and, therefore, the braid represented by \( w_1 \cdot v \sigma_i v^{-1} \cdot w_2 \) is larger than the braid represented by \( w_1 w_2 \).

Typically, the dual braid monoids investigated in Chapter VIII are eligible for Proposition 4.2.

Remark 4.3. The hypothesis that the monoid \( M \) is finitely generated is crucial in Proposition 4.2. For instance, we already observed that the submonoid \( B_{\infty}^+ \) of \( B_{\infty} \) is not well-ordered by the \( \sigma \)-ordering, as we have an infinite descending sequence \( \sigma_1 > \sigma_2 > \ldots \). Such phenomena already occur inside \( B_3 \): for instance, the submonoid of \( B_3 \) generated by all conjugates \( \sigma_2^{-p} \sigma_1 \sigma_2^p \) of \( \sigma_1 \)—and, more generally, the submonoid of all quasi-positive \( n \)-strand braids, defined to be the submonoid...
of \( B_n \) generated by all conjugates of \( \sigma_1, \ldots, \sigma_{n-1} \) contains the infinite descending sequence \( \sigma_1 > \sigma_2^{-1} \sigma_1 \sigma_2 > \sigma_2^{-2} \sigma_1 \sigma_2^2 > \ldots \).

Being a well-ordering has strong consequences. In particular, in contrast to what the examples of Section 2.1 showed, the well-order property implies the most general form of the phenomenon observed in Figure 2:

**Corollary 4.4.** Every nonempty subset of \( B_n^+ \) is either cofinal or it has a least upper bound inside \( (B_n^+, <) \).

Indeed, for \( X \) included in \( B_n^+ \), unless \( X \) is unbounded in \( B_n^+ \), the set of all upper bounds of \( X \) is nonempty, hence it admits a least element.

### 4.2. The recursive construction of the ordering on \( B_n^+ \)

We gave above a quick proof for Proposition 4.1, but the latter is not constructive, and it gives no direct description of the well-ordering \((B_n^+, <)\). We shall now give such a description, based on a recursive construction that connects \((B_{n-1}^+, <)\) and \((B_n^+, <)\). This approach leads in particular to considering the ordering of \( B_n^+ \) as an iterated extension of the ordering of \( B_2^+ \), i.e., of the standard ordering of natural numbers.

To explain the results, it is crucial to use the flipped version of the \( \sigma \)-ordering, i.e., the ordering \( <^\sigma \) defined from \( \sigma^2 \)-positive braids. The reason is that, although \((B_n^+, <)\) and \((B_n^+, <^\sigma)\) are isomorphic, the pairs \((B_n^+, B_{n-1}^+, <)\) and \((B_n^+, B_{n-1}^+, <^\sigma)\) are not, and the connection between \( B_n^+ \) and \( B_{n-1}^+ \) is more easily described in the case of \( <^\sigma \).

The starting point of the approach is the following result from [62]. We recall that \( \Phi_n \) denotes the flip automorphism (both of \( B_n \) and of \( B_n^+ \)) that exchanges \( \sigma_i \) and \( \sigma_{n-i} \) for \( 1 \leq i \leq n-1 \).

**Proposition 4.5.** Assume \( n \geq 3 \). Then, for each braid \( \beta \) in \( B_n^+ \), there exists a unique sequence \( (\beta_p, \ldots, \beta_1) \) in \( B_{n-1}^+ \) such that \( \beta \) admits the decomposition

\[
\beta = \Phi_n^{p-1}(\beta_p) \cdot \cdots \cdot \Phi_n(\beta_2) \cdot \beta_1, \tag{4.1}
\]

and for each \( r \) the only generator \( \sigma_i \) that right divides \( \Phi_n^{p-r}(\beta_p) \cdot \cdots \cdot \beta_r \) is \( \sigma_1 \). The sequence \( (\beta_p, \ldots, \beta_1) \) is called the \( \Phi_n \)-splitting of \( \beta \).

The result easily follows from the fact that every positive braid \( \beta \) of \( B_n^+ \) admits a unique maximal right divisor that lies in \( B_{n-1}^+ \). The unusual enumeration of the sequence from the right emphasizes that the construction starts from the right and involves right divisors.

Now, the main result says that, through the \( \Phi_n \)-splitting, the ordering of \( B_n^+ \) is just a lexicographical extension of the ordering of \( B_{n-1}^+ \), more exactly a **ShortLex-extension** in the sense of [77], i.e., the variant of the lexicographical extension in which the length is first taken into account.

**Proposition 4.6.** Assume \( n \geq 3 \). Let \( \beta, \beta' \) belong to \( B_n^+ \), and let \((\beta_p, \ldots, \beta_1)\) and \((\beta'_p, \ldots, \beta'_1)\) be their \( \Phi_n \)-splittings. Then \( \beta <^\sigma \beta' \) holds if and only if \((\beta_p, \ldots, \beta_1)\) is smaller than \((\beta'_p, \ldots, \beta'_1)\) for the ShortLex-extension of \((B_{n-1}^+, <^\sigma)\), i.e., we have either \( p < p' \), or \( p = p' \) and there exists \( q \leq p \) satisfying \( \beta_r = \beta'_r \) for \( r > q \) and \( \beta_q <^\sigma \beta'_q \).

The result appears as Corollary VII.4.6, and it is also a consequence of Corollary VIII.3.3, with a disjoint argument.
The $\Phi_n$-splitting of a positive braid can be computed easily, and a direct outcome of Proposition 4.6 is the existence, already mentioned in Section 1.5, of a quadratic upper bound for the complexity of the $\sigma$- and $\sigma^k$-orderings.

**Corollary 4.7.** For each $n$, the orderings $<^*$ and $<$ of $B_n$ can be recognized in quadratic time.

**Proof.** We use induction on $n \geq 2$. Let $w$ be an $n$-strand braid word of length $\ell$. By Proposition I.4.6, we can obtain in time $O(\ell)$ two positive $n$-strand braid words $w_1, w_2$ such that $w$ is equivalent to $w_1^{-1}w_2$. Then $\overrightarrow{w} >^* 1$ is equivalent to $\overrightarrow{w_1} >^* \overrightarrow{w_2}$. The $\Phi_n$-splittings of the braids $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ can be computed in time $O(\ell^2)$; see Chapter VII. The induction hypothesis implies that the comparison of the sequences so obtained can be done in time $O(\ell^2)$ as well. The argument is similar for the $\sigma$-ordering as the shift automorphism $\Phi_n$ is computable in linear time. □

### 4.3. The length of $(B_n^+, <^*)$

Contrary to an arbitrary linear ordering, a well-ordering is completely determined up to isomorphism by a unique parameter, namely its length, usually specified by an ordinal number. In the case of the braid ordering on $B_n^+$, the length easily follows from the recursive characterization of Proposition 4.6.

We recall that ordinals are a transfinite continuation of the sequence of natural numbers: after the natural numbers comes $\omega$, the first infinite ordinal, then $\omega + 1$, $\omega + 2$, etc. For our purposes, it is enough to know that ordinals come equipped with a well-ordering and with arithmetic operations (addition, multiplication, exponentiation) that extend those of $\mathbb{N}$. For more background information about ordinals, we refer to any textbook in set theory, for instance [138].

**Proposition 4.8.** For each $n$, the ordered set $(B_n^+, <^*)$ has ordinal type $\omega^{n-2}$.

In other words, the length of $(B_n^+, <^*)$ is the ordinal $\omega^{n-2}$. The proof is an easy induction on $n$.

![Figure 3.](image)

By Proposition 2.10, the ordered set $(B_n^+, <^*)$ is the increasing union of the sets $(B_n^+, <^*)$, each set $B_n^+$ being an initial segment of the next one; see Figure 3. It is easy to deduce

**Proposition 4.9.** The ordered set $(B_{\infty}^+, <^*)$ is a well-ordering with ordinal type $\omega^{\omega^2}$.

As the flip automorphism $\Phi_n$ preserves $B_n^+$ globally, the results about $(B_n^+, <^*)$ translate into similar results about $(B_n^+, <)$. In particular, Proposition 4.8 implies
Corollary 4.10. For each \( n \), the well-ordering \((B^+_n, <)\) has ordinal type \( \omega^{\omega^{n-2}} \).

However, we have no counterpart of Proposition 4.9 for \(<\): the set \( B^+_\infty \) is not an initial segment of \((B^+_{\infty}, <)\), and the latter is not a well-ordered set since it contains the infinite descending sequence of \((1.1)\).

4.4. The rank of a positive braid. One of the nice features when an ordering \(<\) of a set \( \Omega \) is a well-ordering is that, for \( x \in \Omega \), the position of \( x \) in \((\Omega, <)\) is unambiguously specified by an ordinal number, called the rank of \( x \), namely the order type of the initial segment \( \{ y \in \Omega \mid y < x \} \). The rank function establishes an isomorphism between \((\Omega, <)\) and an initial segment of the sequence of ordinals: by construction, \( x < x' \) is true if and only if the rank of \( x \) is smaller than the rank of \( x' \).

So, in our current case, every positive braid \( \beta \) in \( B^+_n \) is associated with a well-defined ordinal number, the rank of \( \beta \), that specifies its position in \((B^+_n, <)\). Moreover, Proposition 2.10 (or simply Figure 3) shows that the rank of \( \beta \) in \((B^+_n, <)\) coincides with its rank in \((B_{\infty}, <)\), and we can forget about the braid index.

Some values of the rank function are easily computed. For instance, the rank of the braid \( \sigma_i \) is the ordinal \( \omega^{\omega^{i-2}} \) for \( i \geq 2 \): indeed, it is the ordinal type of the initial interval determined by \( \sigma_i \). By Proposition 2.10, the latter is \( B_1 \), which, by Proposition 4.8, has ordinal type \( \omega^{\omega^{i-2}} \). More values can be read in Figure 4.

```
1     \sigma_1     \sigma_2     \sigma_3     \sigma_4     \sigma_5     \sigma_6
0     1          2          \omega      \omega+1  \omega+2    \omega+2+1

\omega^2  \omega^2+1  \omega^3+\omega  \omega^3+\omega+1  \omega^3+\omega+2  \omega^3+\omega^2+1
\omega^3  \omega^3+1  \omega^3+\omega  \omega^3+\omega+1  \omega^3+\omega+2  \omega^3+\omega^2+1
\omega^4  \omega^4+1  \omega^4+\omega  \omega^4+\omega+1  \omega^4+\omega+2  \omega^4+\omega^2+1
\omega^5  \omega^5+1  \omega^5+\omega  \omega^5+\omega+1  \omega^5+\omega+2  \omega^5+\omega^2+1
\omega^6  \omega^6+1  \omega^6+\omega  \omega^6+\omega+1  \omega^6+\omega+2  \omega^6+\omega^2+1
\omega^7  \omega^7+1  \omega^7+\omega  \omega^7+\omega+1  \omega^7+\omega+2  \omega^7+\omega^2+1
\omega^8  \omega^8+1  \omega^8+\omega  \omega^8+\omega+1  \omega^8+\omega+2  \omega^8+\omega^2+1
\omega^9  \omega^9+1  \omega^9+\omega  \omega^9+\omega+1  \omega^9+\omega+2  \omega^9+\omega^2+1
\omega^{10} \omega^{10+1} \omega^{10+\omega} \omega^{10+\omega+1} \omega^{10+\omega+2} \omega^{10+\omega^2+1}
```

**Figure 4.** Ranks in the well-ordering \((B^+_{\infty}, <)\): the position of each braid is unambiguously specified by an ordinal number that measures the length of the initial interval it determines.

**Remark 4.11.** By construction, the rank mapping provides an order-isomorphism between positive braids and ordinals. Except for 2-strand braids, this mapping is not an algebraic homomorphism with respect to the ordinal sum: in general, the rank of \( \beta_1 \beta_2 \) is not the sum of the ranks of \( \beta_1 \) and \( \beta_2 \). This happens to be true
for $\beta_2 = \sigma_1$, which has rank 1, but, for instance, we can read in Figure 4 that the rank of $\sigma_2$ is $\omega$, while that of $\sigma_1 \sigma_2$ is $\omega^2$, which is not $1 + \omega$.

Arguably, an optimal description of $(B_3^\infty, <^\omega)$ would consist of a closed formula explicitly computing, for each positive braid $\beta$, the rank of $\beta$, i.e., determining the absolute position of $\beta$ in $(B_3^\infty, <^\omega)$. An algorithmic method has been described in [28], but, so far, it leads to no closed formula in the general case. However, in the case of 3-strand braids, such a formula exists. It relies on identifying distinguished word representatives called $\Phi$-normal, from which the rank can be directly read.

**Definition 4.12.** A nonempty positive 3-strand braid word $\sigma_1^{\varepsilon_1} \sigma_2^{\varepsilon_2} \sigma_3^{\varepsilon_3}$ is said to be $\Phi$-normal if the inequalities $e_p \geq 1$ and $e_r \geq e^\min_p$ for $r < p$ are satisfied, where we set $e_1^\min = 0$, $e_2^\min = 1$, and $e_r^\min = 2$ for $r \geq 3$, and use $[p]$ to denote 1 for odd $p$, and 2 for even $p$.

So the criterion is that a positive 3-strand braid word $w$ is $\Phi$-normal if the successive blocks of letters $\sigma_1$ and $\sigma_2$ in $w$, enumerated from the right, and insisting that the rightmost block is a (possibly empty) block of $\sigma_1$, have a minimal legal size prescribed by the absolute numbers $e^\min_r$. It is easy to check that every nontrivial braid $\beta$ of $B_3^+$ is represented by a unique $\Phi$-normal word, naturally called its $\Phi$-normal form. Then we have the following explicit formula for the rank.

**Proposition 4.13.** For each braid $\beta$ in $B_3^+$, the rank of $\beta$ in $(B_3^+, <^\omega)$ is

$$\omega^{p-1} \cdot e_p + \sum_{p > r \geq 1} \omega^{r-1} \cdot (e_r - e^\min_r),$$

where $\sigma_1^{\varepsilon_1} \sigma_2^{\varepsilon_2} \sigma_3^{\varepsilon_3}$ is the $\Phi$-normal form of $\beta$.

This makes the description of the ordered set $(B_3^+, <^\omega)$ complete.

**Example 4.14.** The $\Phi$-normal form of $\Delta_3$ is $\sigma_1 \sigma_2 \sigma_1$, as the latter word satisfies the defining inequalities, contrary to $\sigma_2 \sigma_1 \sigma_2$, i.e., $\sigma_2 \sigma_1 \sigma_2 \sigma_1\sigma_1$, in which the third exponent from the right, namely 1, is smaller than the minimal legal value $e^\min_3 = 2$.

So, in this case, the sequence $(e(\cdot), . . . , e_1)$ is $(1, 1, 1)$, and, applying (4.2), we deduce that the rank of $\Delta_3$ in $(B_3^+, <^\omega)$ is $\omega^2 \cdot 1 + \omega \cdot (1 - 1) + 1 \cdot (1 - 0)$, i.e., $\omega^2 + 1$. The reader can check that, more generally, the flip normal form of $\Delta_3$ corresponds to the length $d + 2$ exponent sequence $(1, 2, . . . , 2, 1, d)$, implying that the rank of $\Delta_3$ is the ordinal $\omega^{d+1} + d$. More values can be read in Figure 4.

### 4.5. Connection between positive and arbitrary braids.

By Proposition 4.1.6, every braid is a quotient of two positive braids. It follows that, in theory, the ordering of arbitrary braids is determined by its restriction to positive braids.

**Proposition 4.15.** Let $\beta_1, . . . , \beta_p$ be a finite family of braids in $B_n$. Then, for $d$ large enough, $\Delta_3^d \beta_1, . . . , \Delta_3^d \beta_p$ lie in $B_3^+$, and the mutual positions of $\beta_1, . . . , \beta_p$ in $(B_3^+, <)$ are the same as the mutual positions of the positive braids $\Delta_3^d \beta_1, . . . , \Delta_3^d \beta_p$ in $(B_3^+, <)$.

The result is clear, as the braid ordering $<$ is left-invariant. A similar result holds for $<^\omega$.

However, it turns out that this result is of little help in establishing global properties of the braid ordering, and so far there is not much to say about the connection. We just mention two easy remarks involving the left numerators and denominators introduced in Proposition 4.1.9 and their right counterparts.
Proposition 4.16. For each braid $\beta$, the right denominator $D_R(\beta)$ (resp. the left denominator $D_L(\beta)$) is the $<$-minimal positive braid $\beta_1$ such that $\beta\beta_1$ (resp. $\beta_1\beta$) is positive.

Proof. By construction, we have $\beta \cdot D_R(\beta) = N_R(\beta)$ and $D_L(\beta) \cdot \beta = N_L(\beta)$, and both $N_R(\beta)$ and $N_L(\beta)$ are positive braids.

Conversely, assume that $\beta_1$ and $\beta\beta_1$ lie in $B^+\infty$. Then we have $\beta = (\beta\beta_1)\beta_1^{-1}$. By the right counterpart of Proposition I.4.9, we have $\beta_1 = D_R(\beta)\gamma$ for some $\gamma$ in $B^+\infty$. Necessarily $\gamma$ is trivial or $\sigma$-positive, and, therefore, we have both $\beta_1 \geq D_R(\beta)$ and $\beta_1 \geq^* D_L(\beta)$.

Symmetrically, assume that $\beta_1$ and $\beta_1\beta$ lie in $B^+\infty$. Then we have $\beta = \beta_1^{-1}(\beta_1\beta)$. By Proposition I.4.9, there exists $\gamma$ in $B^+\infty$ satisfying $\beta_1 = \gamma D_L(\beta)$. As $\gamma$ belongs to $B^+\infty$, Property S implies both $\beta_1 \geq^* D_L(\beta)$ and $\beta_1 \geq D_R(\beta)$.

□

Proposition 4.17. For each braid $\beta$, the relations $\beta > 1$ and $N_L(\beta) > D_L(\beta)$ are equivalent. Similarly, $\beta >^* 1$ and $N_L(\beta) >^* D_L(\beta)$ are equivalent.

The verification is straightforward as $<$ and $<$ are left-invariant. Note that no such relation exists with the right numerators and denominators: for instance, for $\beta = \sigma_2^{-1}\sigma_1$, we have $\beta > 1$, but $N_R(\beta) = \sigma_1\sigma_2 < D_R(\beta) = \sigma_2\sigma_1$.

The previous observations are rather trivial and do not shed much light on the structure of $(B_n, <)$. The point is that the fractionary decompositions defines two injections $\iota_\ell$ and $\iota_\ell$ of $B_n$ into a subset of $B^+_n \times B^+_n$, but neither of them preserves the ordered structure. On the other hand, we can easily define a well-ordering on $B^+_n \times B^+_n$ by using a lexicographical extension of the ordering of $B^+_n$, and, appealing to $\iota_\ell$ or $\iota_\ell$, deduce a well-ordering of $B_n$, but the latter will not be invariant under left (or right) multiplication.
In this chapter we mention further results and discuss open questions connected with the various aspects of braid orderings considered in this book.

We should start, however, with a very general remark. There are many approaches to braid groups that have not been considered in this book. In fact, braid groups play a role in many areas of mathematics that have not even been mentioned here—e.g., algebraic geometry or mathematical physics. We can therefore still hope that new, illuminating perspectives on braid orderings will emerge in the future.

The chapter is organized as follows. In Section 1, we list some general questions about the $\sigma$-ordering and related topics. Then in Section 2, we discuss more specific questions that arise in the context of the successive chapters of this book, taken in the order in which they appear. Finally, we address in Section 3 some of the many extensions of braid groups from the point of view of order properties.

1. General questions

We begin with three types of questions involving the $\sigma$-ordering in general, namely its uses, its structure, and the problem of finding $\sigma$-positive representatives.

1.1. Uses of the braid ordering. In Chapter III we listed several applications of the orderability of braid groups and of the more specific properties of the $\sigma$-ordering of braids. However, up to now, the applications are not so plentiful and not so strong. This situation contrasts with the seemingly deep and, at the least, sophisticated properties of the $\sigma$-ordering explained in this text, which may appear as a promising sign for potentially powerful applications. So, although vague, the first open question is the following.

**Question 1.1.** How to use the braid ordering?

In particular, one of the deepest properties of the $\sigma$-ordering of $B_n$ known so far is the fact that its restriction to the braid monoid $B_n^+$, and even to the dual braid monoid $B_n^{**}$, is a well-ordering. As emphasized in Section III.3.1, the well-order property is a very strong condition which enables one to distinguish one element in each nonempty subset—so, typically, in each conjugacy class or each Markov class. But, so far, this observation was of no use because we had no effective way to identify such minimal elements in practice, for instance in the case of the conjugacy problem. Thus, a special case of Question 1.1 is

**Question 1.2.** How to take advantage of the fact that the $\sigma$-ordering restricted to $B_n^+$ and $B_n^{**}$ is a well-ordering?

To raise less fuzzy questions, we may think more specifically of the conjugacy problem. Let us say that two positive braids $\beta, \beta'$ are positively conjugate if there
exists a positive braid \( \gamma \) satisfying \( \beta \gamma = \gamma \beta' \). As \( \Delta_2^n \) is central and multiplying any \( n \)-strand braid with a sufficient power of \( \Delta_2^n \) yields a positive braid, solving the conjugacy problem of the group \( B_n \) is algorithmically equivalent to solving the positive conjugacy problem of the monoid \( B^+_n \). Now, for each positive braid \( \beta \), the positive conjugacy class of \( \beta \) is a nonempty subset of \( B^+_n \), hence, by the well-order property, it admits a \(<\)-least element.

**Question 1.3.** Can one effectively compute the \(<\)-least element in a positive conjugacy class?

A similar question can be raised with “Markov equivalence class” replacing “conjugacy class”; a solution would typically associate a computable, well-defined ordinal number with each knot.

The recent developments described in Chapters VII and VIII around the alternating and cycling normal forms of braids have not been exploited thus far, and they might be useful here.

**1.2. Structure of the braid ordering.** To a large extent, the structure of the \( \sigma \)-ordering of braids remains mysterious. Even in the case of \( B_3 \), the examples of Section II.2.1 show that the order \(<\) is a complicated object. By contrast, the results of Chapters VII and VIII give a much simpler description for the restriction of \(<\) to the submonoids \( B^+_n \) and \( B^*_n \) of \( B_n \). The reason why the description is more satisfactory for \( B^+_n \) than for \( B_n \) is that we have a simple recursive definition describing how the ordering of \( B^+_n \) can be obtained from that of \( B^+_{n-1} \). It is natural to raise the question of finding similar constructions for \( B_n \), i.e., more precisely, to raise

**Question 1.4.** Does there exist a simple recursive definition of the \( \sigma \)-ordering on \( B_n \) from the \( \sigma \)-ordering on \( B_{n-1} \)?

**Question 1.5.** Does there exists a (computable) unique normal form on \( B_n \) so that, for any two braids \( \beta, \beta' \), whether \( \beta < \beta' \) holds can be read directly from the normal forms of \( \beta \) and \( \beta' \)?

The handle reduction algorithm of Chapter V does not answer Question 1.5, because it does not lead to a unique normal form, and because the result can be used to compare a braid with 1, but not directly to compare two braids. It is natural to wonder whether Bressaud’s normal form of Section XI.1 might be useful here. Note that the algorithm based on the Mosher normal form presented in Section XII.3 yields a positive answer to the above question, except that the normal forms are not braid words but sequences of edge flips of singular triangulations.

**1.3. Sigma-positive representatives.** We mentioned in Proposition II.1.21 that the algorithmic complexity of the \( \sigma \)-ordering is at most quadratic. Several proofs have been given—in particular the stronger version of Chapter XII involving random access machine (RAM) complexity. It seems unlikely that there exist subquadratic algorithms, and the current result might be close to be optimal.

The situation is different with the stronger question of finding \( \sigma \)-positive representatives. Property \( \text{C} \) says that every nontrivial braid admits at least one representative braid word that is \( \sigma \)-positive or \( \sigma \)-negative. The exponential upper bound of Proposition II.1.22 is certainly far from optimal. Actually, all proofs of Property \( \text{C} \) sketched in this text lead to algorithmic methods for finding \( \sigma \)-positive representatives. Some solutions are inefficient: the only upper bound proved for the
method of Chapter IV is a tower of exponentials of exponential height. Similarly, the proof of Chapter VII relies on a transfinite induction, and deriving a complexity statement is not easy. On the other hand, some methods, like handle reduction of Chapter V or transmission-relaxation of Chapter XII, seem quite efficient for producing short \( \sigma \)-positive expressions, but no result has been proved so far.

**Conjecture 1.6.** For every \( n > 3 \), there exist numbers \( C_n, C'_n \) such that every nontrivial \( n \)-strand braid represented by a word of length \( \ell \) has a \( \sigma \)-positive or \( \sigma \)-negative representative of length at most \( C_n \cdot \ell \). Moreover, such a representative word can be found by an algorithm whose running time is bounded by \( C'_n \cdot \ell^2 \).

Very recently, J. Fromentin announced a proof of Conjecture 1.6, with the explicit bounds \( C_n = C'_n \leq 12n^2 \). His method consists in turning the main proof of Section VIII.3 into an algorithm that, running on an arbitrary \( n \)-strand braid word \( w \), translates it into a fractionary expression in the \( a_{ij}^{\pm 1} \) letters, then puts the numerator and the denominator in cycling normal form, and returns an explicit \( \sigma^2 \)-positive or \( \sigma^2 \)-negative expression for the braid represented by \( w \).

**Remark 1.7.** The braid \( \sigma_1 \sigma_2 \sigma_3^{-1} \sigma_2 \sigma_1^{-1} \) has no \( \sigma \)-positive representative of length 6 or less [83, Theorem 5.1], so \( C_4 \geq 7/5 \) must hold. Moreover, the word

\[
\sigma_1 \sigma_2^{-2} \sigma_3^2 \sigma_4^{-2} \ldots \sigma_{n-1}^{2e} \sigma_{n-2}^{-2e} \ldots \sigma_2 \sigma_1^{-1},
\]

with \( e = \pm 1 \) according to the parity of \( n \), has no \( \sigma_1 \)-positive or \( \sigma_1 \)-negative representative with fewer than \((n - 2)(n + 1)\) crossings. As the above \( n \)-strand braid word has length \( 4(n - 2) \), it seems that \( C_4 \) needs to satisfy \( C_4 \geq (n + 1)/4 \).

Finally, let us mention possible connections with the problem of finding geodesics in braid groups. For \( \beta \) a braid, let us denote by \( \ell_\sigma(\beta) \) the minimal length of a braid word representing \( \beta \). The *geodesic problem* is the question of effectively finding, for each braid word \( w \), an equivalent braid word \( w' \) satisfying \( \ell(w') = \ell_\sigma(\beta) \), i.e., finding a shortest representative of \( \beta \).

It is shown in [169] that the \( B_\infty \)-version of the geodesic problem is co-NP-complete. However, this result says nothing about the problem in a fixed group \( B_n \), nor about the problem of finding quasi-geodesics, i.e., about algorithms that, starting with a braid word \( w \), would produce an equivalent word of length \( O(\ell_\sigma(w)) \). As for the latter problem, the symmetric version of the greedy normal form provides, for each \( n \), a quadratic algorithm returning for each \( n \)-strand braid word \( w \) an equivalent braid word of length at most \( n^2 \ell_\sigma(w) \). A *priori*, the problem of finding short representatives is unconnected with the problem of finding \( \sigma \)-positive representatives. In particular, the examples of (1.1) show that, when \( n \) is unbounded, the ratio between the length of the shortest \( \sigma \)-positive representative and the length of the shortest representative may be at least \( n/4 \). However, the algorithms solving the latter problem often turn out to also solve the former, at least in part.

### 2. More specific questions

We turn to more specific questions involving the \( \sigma \)-ordering of braids and the various approaches that have been developed in the text. For simplicity, we organize the questions according to the chapters they refer to.

#### 2.1. Self-distributivity

Many questions about self-distributivity in general, and about the self-distributive structure of braids in particular, remain open. We shall mention one such question here, and refer to [52] and [51] for many more.
We saw in Section IV.2 that, under the hypothesis that \((S, \ast)\) is a left-cancellative LD-system, there exists a partial action of \(B_n\) on \(S^n\). The action is partial in that \(x \ast \beta\) need not exist for each \(x\) in \(S^n\) and each braid \(\beta\). In Proposition IV.2.5, we proved that, for every braid \(\beta\) in \(B_n\), there exist \(x\) in \(S^n\) such that \(x \ast \beta\) is defined. Reversing the point of view, let us introduce, for \(x\) in \(S^n\),
\[
D_S(x) = \{ \beta \in B_n \mid x \ast \beta \text{ is defined} \}.
\]
As the action of positive braids is always defined, we have \(B_n^+ \subseteq D_S(x) \subseteq B_n\). If \((S, \ast)\) is a rack, the braid action is defined everywhere, and so, for each \(x\), we have \(D_S(x) = B_n\). On the other hand, if \(S\) is the LD-system \((B_\infty, \ast)\) of Definition IV.1.7, it is easy to see that \(D_{B_\infty}(1, \ldots, 1)\) never contains \(\sigma_i^{-1}\), and, therefore, it is a proper subset of \(B_n\). In some cases studied in [132], \(D_S(1, \ldots, n)\) coincides with \(B_n^+\), and, then, the restriction of the braid order \(<\) to \(D_S(1, \ldots, n)\) is a well-ordering.

Conjecture 2.1 (Laver). For all braids \(\beta_1, \ldots, \beta_n\), the subset \(D_{B_\infty}(\beta_1, \ldots, \beta_n)\) of \(B_n\) is well ordered by the \(\sigma\)-ordering.

Note that the question is a pure problem of braids, in that it involves no objects other than braids.

2.2. Handle reduction. We have seen that handle reduction, as described in Chapter V, is a very efficient solution to the braid word problem in practice—actually, the most efficient known so far; see, for instance, [35] for a comparison with the Tetris algorithm of Chapter XI. However, there remains a large gap between the complexity bound established in Proposition V.1.5 and the experimental values of Tables V.1 and V.2. This suggests that the argument of Section V.2 is far from optimal. One may hope that this is the manifestation of some deep and as yet unknown aspect of the geometry of braids.

Conjecture 2.2. For each \(n\), the handle reduction algorithm for \(B_n\) has a quadratic time complexity, and a linear space complexity: starting from a braid word of length \(\ell\), the running time lies in \(O(\ell^2)\) and all words produced during the algorithm have length in \(O(\ell)\)—so does, in particular, the final reduced word.

Clearly, Conjecture 2.2 implies Conjecture 1.6. The second statement in Conjecture 2.2 would be a consequence of a positive solution to the following more general conjecture about the subword reversing method—which extends without change to many group presentations; see [53] and [64].

Conjecture 2.3. If \(w\) is an \(n\)-strand braid word length \(\ell\), and if \(w'\) is a freely reduced braid word obtained from \(w\) by a sequence of special transformations in the sense of Definition V.2.7, each immediately followed by a free reduction, then the length of \(w'\) is at most \(C_n \cdot \ell\), where \(C_n\) is some constant that depends on \(n\).

It has been demonstrated experimentally in [161] that, by combining two handle reductions, namely, starting from a braid word \(w\), first reducing \(w\) to \(w'\), and then reducing \(\Phi_n(w')\) to \(\Phi_n(w'')\), one obtains a final word \(w''\) that is a short representative of \(w\). A. Myasnikov conjectured a positive answer to

Question 2.4. Does the above double handle reduction yield quasi-geodesics in \(B_n\)? In particular, does there exist a constant \(C_n\) such that, for \(w, w''\) as above, one obtains \(\ell(w'') \leq C_n \cdot \ell_{\sigma}(w)\)?
To conclude with a perhaps easier question, let us come back to the coarse handle reduction briefly alluded to at the end of Chapter V. This variant of handle reduction consists in replacing a handle of the form $\sigma_1^e \cdot sh_i(v) \cdot \sigma_1^{-e}$ with $\sigma_1^{e+1} \cdot sh_i^{-1}(v) \cdot \sigma_1^{-e} \cdots \sigma_1^{-1}$, as illustrated in Figure 1.

**Question 2.5. Does coarse handle reduction converge?**

The arguments of Sections V.2.4 and V.2.5 are still valid, but those of Section V.2.2 are not, as the words obtained using coarse reduction from a word that is drawn in some set $\text{Div}(|\beta|)$ may escape from $\text{Div}(|\beta|)$. Experiments suggest that coarse reduction always converges, but the proof is still to be found.

### 2.3. Connection with the Garside structure

The results of Section VI.3 remain partial, and it is an obvious question to ask for a complete description of the $<\cdot$-increasing enumeration $S_{n,d}$ of the divisors of $\Delta_n^d$ similar to the one given in Section VI.2 for the case $n = 3$. The general case is probably difficult, but the case of 4-strand braids should be doable. Owing to the recursive rule of Definition VI.1.6, one can expect the generic entry of $S_{4,d-1}$ to have six copies in $S_{4,d}$, but some entries from $S_{4,d-2}$ have three copies in $S_{4,d}$ only.

More promising might be the questions of braid combinatorics to which the approach of Chapter VI leads. Counting problems involving braids have been little investigated, and a number of questions remain open. We saw in Section VI.1 that a crucial role in counting problems connected with the greedy normal form is played by a certain $n! \times n!$ matrix $M_n$, whose rows and columns are indexed by permutations of $\{1, \ldots, n\}$, and the $(\pi, \pi')$-entry of $M_n$ is 1 if and only if all descents of $\pi'$ are descents of $\pi$, and is 0 otherwise. In particular, the number of positive $n$-strand braids that divide $\Delta_n^d$ is directly connected with the eigenvalues of $M_n$—and of an equivalent smaller matrix $\hat{M}_n$ whose size is the number of partitions of $n$.

Table 1 below shows the associated characteristic polynomials for small values of $n$, immediately leading to

**Conjecture 2.6.** For each $n$, the characteristic polynomial of $M_{n-1}$ divides that of $M_n$. More precisely, the spectrum of $M_n$ is the spectrum of $M_{n-1}$, plus $p(n) - p(n - 1)$ nonzero eigenvalues.

Very recently, a proof of the first part of the conjecture has been announced by F. Hivert, J.C. Novelli, and J.Y. Thibon in [109]. They use the framework of non-commutative symmetric and quasi-symmetric functions connected with combinatorial Hopf algebras, and they construct an explicit derivation that connects $M_{n-1}$ and $M_n$. 
Table 1. Characteristic polynomial of $M_n$ up to a power of $x$—together with the corresponding spectral radius $\rho_n$ and its relative growth.

<table>
<thead>
<tr>
<th>$P_{M_n}(x)$</th>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tr>
<td>$x - 1$</td>
<td>$\rho_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5.449</td>
<td>18.717</td>
<td>77.405</td>
<td>373.990</td>
<td>2,066.575</td>
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<tr>
<td>$P_{M_2}(x)$</td>
<td>$\rho_n/(n\rho_{n-1})$</td>
<td>0.5</td>
<td>0.667</td>
<td>0.681</td>
<td>0.687</td>
<td>0.689</td>
<td>0.690</td>
<td>0.691</td>
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<td>$P_{M_3}(x)$</td>
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<td>$P_{M_4}(x)$</td>
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<td>$P_{M_5}(x)$</td>
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<td>$P_{M_6}(x)$</td>
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<td>$P_{M_7}(x)$</td>
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Furthermore, for small values of $n$, all numbers $b_{n,d}^\beta$—except $b_{n,d}^\Delta_n$, which is 1—grow like $\rho_n^d$, where $\rho_n$ is the spectral radius of $M_n$. Whether this is always true is unknown, but it makes it natural to investigate $\rho_n$. The trivial upper bound $\#\text{Div}(\Delta_n^d) \leq (n!)^d$ suggests that we compare $\rho_n$ with $n\rho_{n-1}$. The values listed in Table 1 may suggest that this ratio tends to $\log 2$.

Finally, it should be clear that all the above questions involving the symmetric groups can be extended to other finite Coxeter groups and to the corresponding braid groups, i.e., the spherical Artin–Tits groups of Section 3.1.

2.4. Alternating decompositions. The recursive characterization of the $\sigma$-ordering of $B_n^+$ by means of the $\Phi_n$-splitting provides a very simple description of this ordering. However, in the current exposition, this description, as well as all results of Section VII.4, is deduced from Burckel’s delicate combinatorial methods, which involve in particular transfinite inductions.

Question 2.7. Does there exist for the recursive characterization of the $\sigma$-ordering of $B_n^+$, and for the other results of Section VII.4, a direct proof in the vein of the one described in Chapter VIII?

The two approaches developed in Chapters VII and VIII are quite similar, and answering Question 2.7 in the positive should not be impossible. However, the Artin relations differ from the Birman–Ko–Lee relations in that some of them involve words of length 3, and this small technical difference might make the solution more difficult in the case of $B_n^+$.

Another natural question—that is probably connected with the previous one—involves the computation of the ordinal rank. With Corollary VII.2.22, we have a simple closed formula that expresses the rank of any positive 3-strand braid in the well-ordering of $B_n^+$ in terms of its $\Phi$-normal form, which itself is very easily computed. In this way, we arguably obtain an optimal description of the ordering, as we identify the position of any element in an absolute way.

Question 2.8. Does there exist a similar method for determining the ordinal rank of an arbitrary braid in $(B_n^+, \prec)$?

A general solution is proposed in [28]. It relies on Burckel’s notion of reducible words, and consists in counting how many irreducible braid words precede a given
one in the tree ordering. The method is algorithmically efficient only in the case of 3 strands, and further investigation is certainly needed in the general case.

2.5. Dual braid monoids. The results of Chapter VIII are quite recent and many open questions remain, in particular the counterpart of Question 2.8. Another natural question would be to determine the ranks of the elements of $B_n^+$ inside $(B_n^+, <^*)$, hence to compare the ranks of a positive braid in $(B_n^+, <^*)$ and in $(B_n^{++}, <^*)$.

Another problem is to study the action of conjugacy on $B_n^{++}$, in particular in view of Question 1.3. The definition of the cycling normal form suggests the introduction of a cycling operation similar to that used in the Garside-based solution to the conjugacy problem, and one may hope for progress in this direction. A similar approach is of course possible with the alternating decompositions of Chapter VII, but the fact that the family of generators $a_{i,j}$ is closed under conjugacy might make the context of Chapter VIII more suitable.

Other types of questions connected with the $\Phi$- and the $\phi$-normal forms involve random walks on the monoids $B_n^+$ or $B_n^{++}$ and possible stabilization phenomena, as studied for instance in [149]. To state a simple question, we may ask

**Question 2.9.** Assume that $X$ is a random walk on $B_n^+$ (resp. $B_n^{++}$). What is the expectation for the $\Phi_n$-breadth (resp. the $\phi_n$-breadth) of $X$?

In other words, what is the average $\Phi_n$-breadth of a random positive $n$-strand braid of length $\ell$? Experiments suggest a connection with $\sqrt{\ell}$ that is not explained so far.

Another natural question is whether the $\Phi$- and $\phi$-normal forms might be connected with an automatic structure on $B_n$. It is known that the languages of normal words are regular languages, but it is unclear whether any form of the fellow traveler property might be satisfied.

Finally, we saw in Proposition II.4.2 that the restriction of the $\sigma$-ordering to every submonoid of $B_n$ generated by finitely many conjugates of the generators $\sigma_i$ is a well-ordering. So, the results about $B_n^+$ and $B_n^{++}$ might extend to more general monoids.

**Question 2.10.** Let $B_n^{++}$ be the submonoid of $B_n$ generated by all braids of the form $\beta \sigma_i \beta^{-1}$ with $\beta$ a simple $n$-strand braid. What is the order type of the restriction of the $\sigma$-ordering to $B_n^{++}$?

More generally, the algebraic study of the monoid $B_n^{++}$ is a natural question that has not yet been addressed. It is known that this monoid is not a Garside monoid in the usual sense, but it seems nevertheless to satisfy much of the interesting properties of a Garside monoid, and, in particular, it might be associated with a new automatic structure on $B_n$.

2.6. Automorphisms of a free group. The study of automorphism groups and outer automorphism groups of free groups is currently an area of intense activity; see for instance [191] for an excellent survey. The analogy between $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ on the one hand and mapping class groups on the other is one of the driving forces behind this research. Now, it is very well known, and explained in Chapter IX, that the braid group $B_n$ is a subgroup of $\text{Aut}(F_n)$, so it is natural to ask the following questions.
2.11. Which subgroups of $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ are left-orderable? Which ones are bi-orderable?

There is certainly no shortage of torsion-free subgroups, i.e., of candidates for being (left-)orderable. Indeed, let us consider the natural homomorphisms of $\text{Aut}(F_n)$ to $\text{GL}(n, \mathbb{Z})$ and of $\text{Out}(F_n)$ to $\text{GL}(n, \mathbb{Z})$. Using a result of Baumslag and Taylor, one can show that the preimage of any torsion-free finite-index subgroup of $\text{GL}(n, \mathbb{Z})$ under either of these homomorphisms is a torsion-free finite-index subgroup of $\text{Aut}(F_n)$ and of $\text{Out}(F_n)$.

2.7. Curve diagrams. In Chapter X we gave a proof of Property C by using a relaxation algorithm for curve diagrams, in the sense explained in Chapter XI. More precisely, the algorithm works by repeatedly sliding a puncture along a so-called useful arc, and relaxing the diagram after each slide. We saw that the length of the $\sigma$-consistent output braid could grow exponentially with the length of the input braid word. The reason for this is that the length of each of the useful arcs can grow exponentially with the length of the input, whereas the length of the relaxing braid (the puncture slide) is proportional to the length of the useful arc. So the algorithm in question is inefficient, but it is so for an obvious reason, and it is easy to invent improvements of the algorithm.

In fact, it seems that the idea of relaxing curve diagrams, as explained in Chapter XI, rather tends to lead to algorithms which are very efficient, but whose efficiency is difficult to prove. Hence we have the following very vague problem.

2.12. Is there a precise, provable statement which expresses the idea that any relaxation type algorithm which does not have an obvious obstruction to being of polynomial complexity has a quadratic time complexity and a linear space complexity?

2.8. Relaxation algorithms. Question 2.12 applies in particular to the two types of relaxation algorithms discussed in detail in Chapter XI, namely Bressaud’s relaxation algorithm and the transmission-relaxation schemes from [74].

A more concrete conjecture specifically involves the Tetris algorithm of Section XI.1. Note that the truth of the following conjecture would imply the truth of Conjecture 1.6:

2.13. For each $n$, the Tetris algorithm for $B_n$ has a quadratic time complexity, and a linear space complexity. Starting from a braid word of length $\ell$ in the generators $\sigma_{i,j,p}$, the running time lies in $O(\ell^2)$, and all words produced during the algorithm have length in $O(\ell)$. Moreover, the linear constants in these bounds depend linearly on the braid index $n$.

As mentioned in Section XI.1, the language of braid words in normal form is recognized by a finite-state automaton, but it fails to be (synchronously) automatic.

2.14. Is there an automatic structure on the braid group that is conceptually close to the approach of Section XI.1?

Bressaud’s original motivation was related to the study of random walks on the braid group $B_n$ and of its Poisson boundary. This boundary has been identified by Kaimanovich and Masur [117] as the space of uniquely ergodic measured foliations on the disk $D_n$. Bressaud’s relaxation procedure from Section XI.1 may be applied to such a foliation just as well as to a curve diagram, yielding an infinite braid
word—this is like a continued fraction expansion of the measured foliation [157]. Thus, in the context of trying to find a more combinatorial description of the Poisson boundary, one can ask

**Question 2.15.** Is Bressaud’s normal form stable for random walks on $B_n$?

In the context of Section XI.2, the braid group $B_n$ is equipped with the metric which gives to the braid $\Delta_{i,j}^d$, the length $\log_2(|d| + 1)$. This approach provides a combinatorial model of the thick part of the Teichmüller space $T$, equipped with the Teichmüller metric $d_{\text{Teich}}$. Further squashing this metric by giving length one to any nonzero power of a Garside-like braid $\Delta_{i,j}$ yields a combinatorial model of the Teichmüller space, equipped with the Weil–Peterson metric $d_{\text{WP}}$—for both of the above statements; see Rafi [178]. Now any word whose letters are of the form $\Delta_{i,j}^d$ represents a path in all three spaces: the Cayley graph of the braid group, the combinatorial model of $(T, d_{\text{Teich}})$, and the combinatorial model of $(T, d_{\text{WP}})$.

**Conjecture 2.16.** The set of braid words produced by the transmission-relaxation algorithm forms a family of parametrized uniform quasi-geodesics in all three spaces.

If this were true, then this normal form could serve as a very concrete and algorithmically efficient tool in the exploration of these spaces.

**2.9. Triangulations.** In Chapter XII we studied the Mosher normal form of a braid, which is a sequence of combinatorial types of triangulations on the surface $D_n$, where each element of the sequence is obtained from the preceding one by an edge flip. The connection between braid groups and triangulation sequences comes from the fact that the complex of triangulations of $D_n$, where triangulations are adjacent in the complex if they differ by an edge flip, is a refinement of the Cayley graph of the braid group, and, more precisely, is quasi-isometric to it. We saw in Chapter XII that the Mosher normal form is a useful tool for understanding the $\sigma$-ordering.

Since Mosher’s discovery of the (automatic) normal form, other complexes which are quasi-isometric to the Cayley graph of $B_n$ have greatly contributed to our understanding of braid groups, for instance the train track complex [104] and the marking complex [151]. Other complexes, like the pants complex [24] and the curve complex [151], have also been studied in great depth.

**Question 2.17.** Can any of the above-mentioned complexes contribute to our understanding of braid orderings?

**2.10. Hyperbolic geometry.** We have seen in Chapter XIII how to define an infinite family of distinct left-invariant orderings on $B_n$ whose restriction to $B_n^+$ is a well-ordering, but we did not address the determination of the length of that well-ordering. The following question may well be quite easy to answer.

**Question 2.18.** What are the possible ordinal types for the restriction of $\prec$ to $B_n^+$ when $\prec$ is an ordering of Nielsen–Thurston type?

**2.11. The space of all orderings of $B_n$.** We have seen in Chapter XIV that there exist many orders on $B_n$. In connection with Questions 2.18 above and 2.24 below, it is natural to raise:
**Question 2.19.** Assume that $\prec$ is a left-invariant ordering of $B_n$ that satisfies the subword property. What are the possible ordinal types for the restriction of $\prec$ to $B_n^+$?

As $B_n$ is not bi-orderable, one could imagine that only long orders may exist on it. This is not the case, as is shown in the following ordering, which was already considered in Remark XIII.1.9. We recall that $\epsilon$ denotes the exponent sum, i.e., the homomorphism of $B_\infty$ to $\mathbb{Z}$ that maps every $\sigma_i$ to 1.

**Proposition 2.20.** For $\beta, \beta'$ in $B_n$, declare that $\beta <_{\epsilon} \beta'$ is true if we have either $\epsilon(\beta) < \epsilon(\beta')$, or $\epsilon(\beta) = \epsilon(\beta')$ and $\beta < \beta'$. Then $<_{\epsilon}$ is a left-invariant ordering of $B_n$ whose restriction to $B_n^+$ is a well-ordering of ordinal type $\omega$.

**Proof.** For each (positive) braid $\beta$, each braid $\gamma$ satisfying $\gamma <_{\epsilon} \beta$ must satisfy $\epsilon(\gamma) \leq \epsilon(\beta)$. For fixed $n$, there exist only finitely many positive braids $\gamma$ satisfying this condition. □

Note that the ordering $<_{\epsilon}$ is the lexicographical ordering deduced from the exact sequence

$$1 \rightarrow [B_n, B_n] \rightarrow B_n \rightarrow \mathbb{Z} \rightarrow 1$$

using the $\sigma$-ordering of the commutator subgroup and the usual ordering of $\mathbb{Z}$.

Other natural questions involve convex subgroups. As already mentioned, the family of convex subgroups of a left-ordered group is linearly ordered under inclusion and closed under unions and intersections. According to Section II.3.4, the $\sigma$-ordering of $B_n$ has exactly $n$ convex subgroups, including $\{1\}$ and $B_n$ itself, and only the latter two subgroups are normal.

On the other hand, with respect to the ordering of Proposition 2.20, the commutator subgroup $[B_n, B_n]$ is both convex and normal. Other convex subgroups are $H_k \cap [B_n, B_n]$, where $H_k$ denotes the subgroup of $B_n$ generated by $\sigma_k, \sigma_{k+1}, \ldots, \sigma_{n-1}$, but they are not normal.

For a third example, consider the special case $n = 3$. The commutator subgroup $[B_3, B_3]$ is free on two generators, hence it is bi-orderable. In fact, using the Magnus ordering of this free group, one obtains infinitely many convex subgroups, namely the inverse images of the ideals $1 + O(X^k)$. Using this ordering of $[B_3, B_3]$ and the lexicographic ordering as described in the previous paragraph, one can construct a left-ordering of $B_3$ which has infinitely many distinct convex subgroups.

**Question 2.21.** What convex subgroups must a left-invariant ordering of $B_n$ admit? Is there a left-invariant ordering of $B_n$ which has no convex subgroups at all, other than $\{1\}$ and $B_n$? What about bi-invariant orderings of $PB_n$?

**2.12. Pure braid groups.** The Magnus ordering of the pure braid group $PB_n$ shares with the $\sigma$-ordering of $B_n$ the property that its restriction to the monoid $B_n^+$ of positive braids—and, similarly, to the dual braid monoid $B_n^{\ast}$—is well-ordered. This leads to several problems.

First, every positive pure braid receives a unique ordinal rank that describes its position in the well-ordered set $(PB_n^+, <_{\sigma})$. As in the case of $B_n^+$ and the $\sigma$-ordering, we can raise

**Question 2.22.** Does there exist a practical method for determining the rank of a pure braid in $(PB_n^+, <_{\sigma})$?
We observed that the Magnus ordering extends to the pure braid group $PB_n$, and we can consider its restriction to the positive monoid $PB_n^+$. It is easy to see that $(PB_n^+,<_{\text{st}})$ is not a well-ordering as it admits the infinite descending sequence $\sigma_1^2 >_{\text{st}} \sigma_2^2 >_{\text{st}} \ldots$. The situation resembles that of the $\sigma$-ordering. In the latter case, we obtained a well-ordering of $B_n^+$ by considering a flipped version so as to reverse the problematic inequalities. The point is that $\text{sh}(B_n^+)$ is the initial segment of $(B_n^+,<)$ determined by $\sigma_1$, implying that, after the flip, $B_n^+$ is the initial segment of $(B_n^-,<^*)$ determined by $\sigma_{n-1}$. The counterpart of that property fails for the Magnus ordering of $PB_n$: every pure braid in $\text{sh}(PB_n^+)$ is smaller than $\sigma_1^2$, but the converse is false, as we have for instance $1 <_{\text{st}} \sigma_2 \sigma_1^2 \sigma_2 <_{\text{st}} \sigma_1^2$. The example of $1 <_{\text{st}} \sigma_3 \sigma_2 \sigma_1^2 \sigma_2 \sigma_3 <_{\text{st}} \sigma_3 \sigma_2^2 \sigma_3$ shows that $\text{sh}(PB_n^+)$ is not even convex in $PB_n^+$. This does not discard the possibility of defining a flipped version of the Magnus ordering on $PB_n^+$, and then on $PB_n^*$, but the structure of the latter is unclear.

**Question 2.23.** Let $\sigma_{3,n}$ denote the image of the Magnus ordering of $PB_n$ under the flip automorphism $\Phi_n$. Is the induced ordering of $PB_n^*$ a well-ordering? If it is, what is its order type? Would some variant of the Magnus ordering be more suitable for such constructions?

There seems to be a large difference of complexity between the $\sigma$-ordering of $B_n$ and the Magnus ordering of $PB_n$. In particular, this difference is visible in the gap between the order types of the restrictions to $B_n^+$, namely the relatively large ordinal $\omega^{n-2}$ for the former, to be compared with the modest ordinal $\omega^{n-1}$ for the latter. It is natural to wonder whether this difference is essential.

**Question 2.24.** Can there exist a bi-invariant ordering of $PB_n$ whose restriction to $PB_n^+$ is a well-ordering of order type larger than $\omega^{n-1}$?

The point here is that we consider bi-invariant orderings of $PB_n$: the restriction of the $\sigma$-ordering to $PB_n$ is a left-invariant ordering of $PB_n$, whose restriction to $PB_n^+$ is a well-ordering whose order type is easily checked to be $\omega^{n-2}$.

More generally, one could wonder whether a bi-invariant ordering on a group $G$ can be as complicated—in a sense to be made precise—as a left-invariant ordering of $G$.

### 3. Generalizations and extensions

The braid groups can be generalized in many respects, so extending the results mentioned in this text to other groups is an obvious task. Of course, several types of extensions may be considered: extending orderability, extending the specific $\sigma$-ordering of braids, extending the various approaches that lead to that ordering, extending the associated algorithms, etc. Here, we shall briefly review a few results and conjectures involving such extensions, but we shall not try to be exhaustive.

#### 3.1. Artin–Tits groups

Starting from the presentation of $B_n$, rather than from any geometric description, we can construct braid groups in a larger framework of a completely different nature: they are special cases of Artin–Tits groups, and more specifically spherical Artin–Tits groups, as introduced in [65, 21].

An Artin–Tits group is, by definition, a group admitting a presentation with finitely many generators $s_1,\ldots,s_n$ and relations of the form $s_i s_j s_i s_j \ldots = s_j s_i s_j s_i \ldots$, where the words on both sides of the equality sign have the same length (finite and at least 2) depending on $i$ and $j$, and there is at most one relation for each pair $\{i,j\}$. 
For instance, finitely generated free groups (no relations) and free Abelian groups (commutation relations between all pairs of generators) are Artin–Tits groups. An Artin–Tits group is said to be **spherical** if the associated Coxeter group, namely the group obtained by adding the relations $s_i^2 = 1$ for $i = 1, \ldots, n$, is finite [112]. The braid group $B_n$ is then the spherical Artin–Tits group associated with the symmetric group $S_n$, thus corresponding with the so-called Coxeter type $A_{n-1}$.

**Question 3.1.** Which Artin–Tits groups are left-orderable or bi-orderable?

Currently, the only Artin–Tits groups known to be left-orderable are those that embed in mapping class groups. Among the spherical ones, these are all but those of type $E_6, E_7,$ and $E_8$ [193, 173]. Let us mention that, if the Artin–Tits group of type $E_8$ is left-orderable, then, due to embedding properties, all spherical Artin–Tits groups are; see [159]. Among the nonspherical ones, there is one well-known family of groups that are bi-orderable [72], namely the right-angled Artin–Tits groups (also called partially commutative groups), which have only commutation relations. Indeed, these groups embed in pure surface braid groups; see Section 3.2.

The more specific question of extending the $\sigma$-ordering of braid groups to other Artin–Tits groups seems rather artificial and is not very promising in general. It is well known that sending $s_1$ to $\sigma_1^2$ and $s_i$ to $\sigma_i$ for $i \geq 2$ defines an embedding of the type $B_n$ Artin–Tits group into the corresponding type $A_n$ group, i.e., into the braid group $B_{n+1}$. In this way, one obtains an exact counterpart of the $\sigma$-ordering for each type $B_n$ Artin–Tits group and, more generally, for every Artin–Tits group that is a product of type $A$ and type $B$ Artin–Tits groups.

In [186], Hervé Silber proves

**Proposition 3.2.** The counterpart of Property A is true in every Artin–Tits group. The counterpart of Property C is true only for those groups that are products of type $A$ and type $B$ groups.

Thus, except in the special cases of types $A$ and $B$, extending the definition of the $\sigma$-ordering leads to a partial ordering only.

Many algebraic properties of spherical Artin–Tits groups extend to a larger class of groups called Garside groups [53, 55, 63, 175, 174, 185]. In particular, the latter are known to be torsion-free.

**Question 3.3.** Is every Garside group left-orderable?

3.2. Mapping class groups and surface braid groups. We defined in Section 1.3 the mapping class group $\text{MCG}(S, \mathcal{P})$ of any compact surface $S$ relative to a finite set of punctures $\mathcal{P}$. Closely related is the $n$-strand braid group $B_n(S)$ of a surface $S$. It can be defined as the fundamental group of the configuration space of $n$ unlabelled points in $S$. More geometrically, we can fix arbitrarily $n$ distinguished points $P_1, \ldots, P_n$ in the interior of $S$. Then $B_n(S)$ is the group of isotopy classes of braids in $[0, 1] \times S$, where each strand starts at one of the points $\{0\} \times P_i$ and ends at one of the points $\{1\} \times P_j$. For instance, we have $B_1(S) = \pi_1(S)$ for every surface $S$. It is a simple fact [13] that for all compact surfaces $S$, the braid group $B_n(S)$ is in a natural way a subgroup of $\text{MCG}(S, \{P_1, \ldots, P_n\})$, except if $S$ is one of the following: the sphere $S^2$, the sphere with one or with two points removed, the torus, or the Klein bottle.

**Proposition 3.4.** Let $S$ be any compact surface with nonempty boundary. Then $\text{MCG}(S)$ is left-orderable.
In this statement the surface may or may not have punctures, and may or may not be orientable. A proof of this fact appeared in [182]. It uses a simple generalisation of the curve diagram construction from Section X.1.2. Since subgroups of left-orderable groups are also left-orderable, we deduce the following.

**Corollary 3.5.** Let $S$ be any compact surface with or without punctures, orientable or nonorientable, but necessarily with $\partial S \neq \emptyset$. Then $B_n(S)$ is left-orderable.

However, no interesting analogue of the notion of $\sigma$-positivity is known in this case. Nevertheless, it would be interesting to generalize the classification of Nielsen–Thurston type orderings encountered in Chapter XIII to this setting.

The situation is much more subtle if $S$ is a compact surface without boundary. The mapping class groups of such surfaces have torsion, and consequently are not left-orderable.

**Question 3.6.** If $S$ is a compact orientable surface without boundary, is the surface braid group $B_n(S)$ left-orderable?

Other interesting questions occur when we consider the pure braid groups of a surface. By definition, the pure $n$-strand braid group in a surface $S$, denoted $PB_n(S)$, is the fundamental group of the configuration space of $n$ labelled points in the surface $S$—or, equivalently the group of pure braids in $S \times [0,1]$ where each strand has one endpoint in $S \times \{0\}$ and the other in $S \times \{1\}$.

Some of these braid groups are quite obviously not bi-orderable. For instance, for $n \geq 3$, the pure $n$-strand braid group of the sphere $PB_n(S^2)$ has torsion. Indeed, if $\Delta^2$ denotes the usual full-twist braid inside an embedded disk in $S^2$ containing all the punctures, then $\Delta^2$ is nontrivial whereas its square $\Delta^4$ is trivial—this is the famous belt trick.

Similarly, if the surface $S$ is nonorientable, then, for $n \geq 2$, a generator $\sigma_i^2$ of $PB_n(S)$ is conjugate to its own inverse—the conjugating element being a pure braid which pushes the two strands involved in $\sigma_i$ once around an embedded Möbius band. So $PB_n(S)$ has generalized torsion.

J. González-Meneses proved [100] that these obvious obstructions to bi-orderability are the only ones:

**Proposition 3.7.** If $S$ is an orientable closed surface of genus $g \geq 1$, then, for $n \geq 1$, the pure braid group $PB_n(S)$ is bi-orderable.

The proof works by developing the ideas of Section XV.3, and combining them with some delicate combinatorics in surface braid groups [101].

As an immediate consequence of the theorem, we have that all right-angled Artin groups are bi-orderable, because, according to [39], they embed in pure surface braid groups. As a further corollary we have that all subgroups of right-angled Artin groups are bi-orderable, and this class of groups is surprisingly rich: it contains, for instance, all graph braid groups, all surface groups except the three simplest nonorientable ones, and certain 3-manifold groups.

Beyond the question of orderability, one may also wish to extend other techniques developed in this book to the more general context of mapping class groups. In this respect, it is especially tempting to try and generalize the ideas of Section XI.2 to other situations. This leads to the following ambitious claim.
Conjecture 3.8. All results and techniques mentioned in Section XI.2 can be generalized to mapping class groups of higher genus surfaces.

One could even speculate whether similar techniques might be applied to the outer automorphism group of free groups, possibly with applications to geometry of the outer space.

3.3. Torelli groups. The Torelli group of a surface $S$ is defined to be the subgroup of $\text{MCG}(S)$ consisting of those elements which act trivially on the homology $H_1(S,\mathbb{Z})$, i.e., on the Abelianization of $\pi_1(S)$. For a good recent survey on what is known and not known about Torelli groups, see [81].

Proposition 3.9. For each compact surface $S$, the Torelli group of $S$ is residually nilpotent, and hence bi-orderable.

The proof follows from the deep structure theory of Torelli groups, whose fundamental results are due to Dennis Johnson [114]. A crucial role in this theory is played by the so-called Johnson filtration, a certain infinite sequence of subgroups different from the lower central series of the Torelli group, such that the quotient of two successive terms is always torsion-free Abelian. The structure of the Johnson filtration is in fact a manifestation of a more general phenomenon; see [9].

Question 3.10. Is the mapping class group $\text{MCG}(S)$ virtually orderable or even virtually bi-orderable? In particular, what are the orderability properties of the kernel of the action on $H_1(S,\mathbb{Z}/p\mathbb{Z})$, where $p$ is a prime?

This subgroup of elements acting trivially on homology with $\mathbb{Z}/p\mathbb{Z}$-coefficients is torsion-free [113, Chapter 1], but by a result of Hain [103, 153] its Abelianization is finite, at least when the genus of $S$ is 3 or more. These results are related to the well-known question whether the mapping class group of a closed surface virtually surjects to $\mathbb{Z}$, i.e., whether it has a finite index subgroup which has an infinite Abelian quotient.

3.4. Surface groups and 3-manifold groups. It is shown in [181] that the fundamental group—or, equivalently, the one-string braid group—of every compact surface, except for the projective plane $\mathbb{RP}^2$, is left-orderable. Moreover, with the further exception of the Klein bottle, all surface fundamental groups are actually bi-orderable.

The situation is more subtle when considering the case of fundamental groups of compact 3-manifolds, which we will refer to simply as 3-manifold groups. A study of these groups is initiated in the paper [18], where necessary and sufficient conditions are derived for the left-orderability and bi-orderability of fundamental groups of the important class of Seifert-fibred 3-manifolds (manifolds which are foliated by topological circles). It is also shown there that for each of the eight 3-dimensional geometries, there exist manifolds modelled on that geometry which have left-orderable groups and also there exist examples whose groups are not left-orderable.

Recall that a 3-manifold is called irreducible if every smooth 2-sphere bounds a 3-ball in the manifold. An important general result of [18] is that all compact irreducible orientable 3-manifolds with a positive first Betti number have left-orderable groups. In particular, all knot and link groups are left-orderable.

Question 3.11. Which knot groups are bi-orderable?
The group of the figure eight knot \(4_1\) is bi-orderable. The first unknown case is the knot \(5_2\) in knot tables.

**Question 3.12.** Given an automorphism \(\varphi\) of a surface group \(G\) (or more generally of any bi-orderable group), under what conditions does there exist a bi-invariant ordering of \(G\) which is \(\varphi\)-invariant, meaning \(x < y\) implies \(\varphi(x) < \varphi(y)\)?

This is relevant to the study of 3-manifolds which are bundles over \(S^1\), with surface fibres. If \(\varphi\) is the monodromy associated with such a fibration, then a \(\varphi\)-invariant bi-invariant ordering of the fibre’s group naturally leads to a bi-invariant ordering of the fundamental group of the total space, and vice versa. In [172] this observation, as well as the techniques described in Chapter XV, are used to prove that certain fibred knots with pseudo-Anosov monodromy have bi-orderable groups. By contrast, the group of any torus knot cannot be bi-ordered, because it contains elements which do not commute, while a power of one of those elements commutes with the other, which cannot occur in a bi-orderable group.

**Conjecture 3.13.** If \(G\) is the fundamental group of a closed orientable (irreducible) 3-manifold, then \(G\) is virtually bi-orderable, i.e., there exists a subgroup of finite index which is bi-orderable.

It is shown in [18] that Conjecture 3.13 holds for Seifert-fibred 3-manifolds, and more generally for all manifolds with a geometric structure, except possibly hyperbolic manifolds. We do not even know if hyperbolic manifold groups are virtually left-orderable.

To put the difficulty of these questions into perspective, we point out that from general properties of orderable groups and covering space theory one can show that any 3-manifold satisfying Conjecture 3.13 also satisfies a certain well-known conjecture in 3-manifold theory; this conjecture states that any closed, orientable, irreducible 3-manifold \(M\) with an infinite fundamental group has a finite-sheeted cover \(\tilde{M}\) with positive first Betti number. This conjecture remains open despite Perelman’s recent proof of the geometrization conjecture.

In another direction, the pure braid group can be regarded as the fundamental group of the complement of the family of hyperplanes \(z_i = z_j\) in the space \(\mathbb{C}^n\) with coordinates \(z_1, \ldots, z_n\). The analysis of orderability for \(PB_n\) applies to many other (but not all) complex hyperplane arrangements.

**Proposition 3.14.** The fundamental group of the complement of every hyperplane arrangement of fibre type is bi-orderable.

For further details and a recent discussion of the fundamental groups of hyperplane arrangements, see [167].

### 3.5. A topological completion.

Let us now come back to the specific case of braids and their \(\sigma\)-ordering. Another line of research consists in looking for extensions of that particular ordering to larger spaces. Here we start with a topological completion.

As mentioned in Section II.3.2, the topology on \(B_\infty\) associated with the \(\sigma\)-ordering is metrizable, the radius \(2^{-n}\) ball centered at \(1\) being the shifted subgroup \(\text{sh}_n(B_\infty)\). With respect to that topology, a sequence \(\beta_1, \beta_2, \ldots\) converges to the trivial braid if for each integer \(n\) there exists an integer \(p\) such that, for \(q > p\), all braids \(\beta_q\) belong to \(\text{sh}_n(B_\infty)\). The order topology renders \(B_\infty\) homeomorphic to \(\mathbb{Q}\)—in particular, not completely metrizable.
Very recently, P. Fabel announced in [79] the following construction of a completion of \( B_\infty \). Let \( D_\infty \) be the closed unit disk in \( \mathbb{C} \) centered at 0 with punctures on the real line at \( 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \), and let \( H_\infty \) be the group of homeomorphisms of \( D_\infty \) fixing the boundary pointwise. Let \( M(H_\infty) \) be the group of isotopy classes of \( H_\infty \). Viewing it as the mapping class group of a disk containing the punctures at \( 0, \frac{1}{2}, \frac{1}{3}, 1 - \frac{1}{4}, \ldots \), one embeds \( B_n \) in \( M(H_\infty) \). One can equip \( M(H_\infty) \) with a metric \( d \) by declaring

\[
d(\beta_1, \beta_2) = \inf_{h_1, h_2 \in D_\infty} d_C(h_1(z), h_2(z)) + \inf_{h_1, h_2 \in D_\infty} d_C(h_1^{-1}(z), h_2^{-1}(z)),
\]

where \( h_i \) is a homeomorphism of \( D_\infty \) representing the isotopy class \( \beta_i \). The result announced by Fabel is

**Proposition 3.15.** The group \( (M(H_\infty), d) \) is complete as a metric space, it contains \( B_\infty \) as a dense subgroup, and it is left-ordered by an ordering that extends the \( \sigma \)-ordering of \( B_\infty \).

**Question 3.16.** Are all completions extending the \( \sigma \)-ordering of \( B_\infty \) essentially equivalent?

### 3.6. Parenthesized braids

We shall conclude with another seemingly promising extension of the braids and their \( \sigma \)-ordering—one that will at least enable us to end with a nice figure.

Thompson’s group \( F \) is a finitely presented group which, in many respects, is a cousin of the braid groups. Like the latter, it can be introduced in many different ways, and it has very rich properties involving geometric group theory and dynamical systems; see [32] for an introduction. The open question of its possible amenability has provided a strong motivation for studying the group \( F \) in recent years. Let us mention that \( F \) is nothing but the counterpart of the group \( G_{LD} \) of Section IV.3 when the associativity law replaces the self-distributivity law.

For our current purposes, it is enough to know that \( F \) admits the presentation

\[
\langle a_1, a_2, \ldots | a_ja_{j-1} = a_ja_i \text{ for } j \geq i + 2 \rangle
\]

It has recently been observed that the groups \( B_\infty \) and \( F \) can be married in a natural way. This was done independently by M. Brin in [22, 23] by constructing what was seen as a braided version of the Thompson group, and in [58, 59] by constructing what was seen as a Thompson version of \( B_\infty \). The groups so obtained are essentially similar; variants also appeared in [120] and [92].

From our current point of view, the most natural description is probably the one involving *parenthesized braids*.

**Definition 3.17.** The group of *parenthesized braids* \( B_* \) is defined by two infinite series of generators \( \sigma_1, \sigma_2, \ldots, a_1, a_2, \ldots \), subject to the following relations for \( i \geq 1 \) and \( j \geq i + 2 \):

\[
\begin{aligned}
&\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \sigma_i a_j = a_j\sigma_i, \quad a_i a_{j-1} = a_j a_i, \quad a_i a_{j-1} = a_j a_i, \\
&\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_{i+1} \sigma_i a_{i+1} = a_i \sigma_i, \quad \sigma_{i+1} \sigma_i a_i = a_{i+1} \sigma_i.
\end{aligned}
\]

The elements of \( B_* \) can be visualized using braid diagrams in which the distances between strands are not uniform. An ordinary braid diagram connects an initial sequence of equidistant positions to a similar final sequence. A parenthesized
braid diagram connects a parenthesized sequence of positions to another possibly different parenthesized sequence of positions, the intuition being that grouped positions are (infinitely) closer than ungrouped ones. A typical example is shown in Figure 2. The generator $\sigma_i$ corresponds to the usual crossing operator, with the difference that it involves all strands that start in the vicinity of $i$ and $i+1$. The generator $a_i$ corresponds to shrinking all strands that start in the vicinity of $i$ and translating the next ones so as to avoid a gap.

The relations of (3.2) correspond to the isotopies displayed in Figure 3. As can be expected, the elements $\sigma_i$ generate a copy of $B_\infty$, while the elements $a_i$ generate a copy of Thompson's group $F$. More precisely, $B_\bullet$ is a group of fractions for a monoid that is a bi-crossed product of $B_\infty^+$ and of the monoid $F^+$ defined by the presentation of (3.1).

**Proposition 3.18.** The group $B_\bullet$ is left-orderable, by an ordering that extends the $\sigma$-ordering of $B_\infty$.

An application of that result—more exactly, of the specific form of the elements larger than 1 in terms of $\sigma$-positive expressions—is that the Artin representation
of braids extends to $B_\bullet$, and that the latter embeds in the mapping class group of a sphere with a Cantor set of punctures (Figure 4).

Let us also mention that the natural subgroup of $B_\bullet$ corresponding to pure braids was recently shown to be bi-orderable, by an ordering that extends the ordering of $PB_\infty$ constructed in Chapter XV [30].

These results do not prove that the parenthesized braid group is an extraordinary object. After all, that two groups can be glued together in a somewhat tricky way is nothing exceptional, so $B_\bullet$ might very well be just an amusing example. However, we think that $B_\bullet$ is really an important object. Once again, the variety of approaches that lead to $B_\bullet$ or to close variants, and, mainly, the unexpected way in which the technical properties fit together, suggest that something interesting is hidden there. In particular, the self-distributive structure of $B_\infty$ described in Chapter IV extends to $B_\bullet$, a surprising result that certainly reflects deep properties. We hope for—and even predict—future applications.

**Figure 4.** Embedding parenthesized braids in the mapping class group of a sphere with a Cantor set of punctures: $\sigma_i$ acts by the usual half-twist, while $a_i$ acts as a dilatation-translation along the equator.