Introduction

This book has several goals. The first goal is to develop the foundations of potential theory on $\mathbb{P}^1_{\text{Berk}}$, including the definition of a measure-valued Laplacian operator, capacity theory, and a theory of harmonic and subharmonic functions. A second goal is to give applications of potential theory on $\mathbb{P}^1_{\text{Berk}}$, especially to the dynamics of rational maps defined over an arbitrary complete and algebraically closed non-Archimedean field K. A third goal is to provide the reader with a concrete introduction to Berkovich's theory of analytic spaces by focusing on the special case of the Berkovich projective line.

We now outline the contents of the book.

The Berkovich affine and projective lines. Let K be an algebraically closed field which is complete with respect to a nontrivial non-Archimedean absolute value. The topology on K induced by the given absolute value is Hausdorff, but it is also totally disconnected and not locally compact. This makes it difficult to define a good notion of an analytic function on K. Tate dealt with this problem by developing the subject now known as *rigid analysis*, in which one works with a certain Grothendieck topology on K. This leads to a satisfactory theory of analytic functions, but since the underlying topological space is unchanged, difficulties remain for other applications. For example, using only the topology on K, there is no evident way to define a Laplacian operator analogous to the classical Laplacian on $\mathbb{C}$ or to work sensibly with probability measures on K.

However, these difficulties, and many more, can be resolved in a very satisfactory way using Berkovich's theory. The Berkovich affine line $\mathbb{A}^1_{\text{Berk}}$ over K is a locally compact, Hausdorff, and path-connected topological space which contains K (with the topology induced by the given absolute value) as a dense subspace. One obtains the Berkovich projective line $\mathbb{P}^1_{\text{Berk}}$ by adjoining to $\mathbb{A}^1_{\text{Berk}}$ in a suitable manner a point at infinity; the resulting space $\mathbb{P}^1_{\text{Berk}}$ is a compact, Hausdorff, path-connected topological space which contains $\mathbb{P}^1(K)$ (with its natural topology) as a dense subspace. In fact, $\mathbb{A}^1_{\text{Berk}}$ and $\mathbb{P}^1_{\text{Berk}}$ are more than just path-connected: they are *uniquely* path-connected, in the sense that any two distinct points can be joined by a unique arc. The unique path-connectedness is closely related to the fact that $\mathbb{A}^1_{\text{Berk}}$ and $\mathbb{P}^1_{\text{Berk}}$ are endowed with a natural tree structure. (More specifically, they are $\mathbb{R}$-*trees*, as defined in §1.4.) The tree structure on $\mathbb{A}^1_{\text{Berk}}$ (resp. $\mathbb{P}^1_{\text{Berk}}$) can be used to define a *Laplacian operator* in terms of the classical Laplacian on a finite graph. This in turn leads to a theory of harmonic and subharmonic functions which closely parallels the classical theory over $\mathbb{C}$.

The definition of $\mathbb{A}^1_{\mathrm{Berk}}$ is quite simple and makes sense with K replaced by an arbitrary field k endowed with a (possibly Archimedean or even trivial) absolute value. As a set, $\mathbb{A}^1_{\mathrm{Berk},k}$ consists of all multiplicative seminorms on the polynomial ring $k[T]$ which extend the usual absolute value on k. (A *multiplicative seminorm* on a ring A is a function $[\ ]_x : A \to \mathbb{R}_{\geq 0}$ satisfying $[0]_x = 0, [1]_x = 1, [fg]_x = [f]_x \cdot [g]_x$, and $[f+g]_x \leq [f]_x + [g]_x$ for all $f, g \in A$.) By an aesthetically desirable abuse of notation, we will identify seminorms $[\ ]_x$ with points $x \in \mathbb{A}^1_{\mathrm{Berk},k}$, and we will usually omit explicit reference to the field k, writing simply $\mathbb{A}^1_{\mathrm{Berk}}$. The topology on $\mathbb{A}^1_{\mathrm{Berk},k}$ is the weakest one for which $x \mapsto [f]_x$ is continuous for every $f \in k[T]$.

To motivate this definition, we observe that in the classical setting, every multiplicative seminorm on $\mathbb{C}[T]$ which extends the usual absolute value on $\mathbb{C}$ is of the form $f \mapsto |f(z)|$ for some $z \in \mathbb{C}$. (This can be deduced from the well-known Gelfand-Mazur theorem from functional analysis.) It is then easy to see that $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}}$ is homeomorphic to $\mathbb{C}$ itself and also to the Gelfand spectrum (i.e., the space of all maximal ideals) of $\mathbb{C}[T]$.

In the non-Archimedean world, K can once again be identified with the Gelfand space of maximal ideals in $K[T]$, but now there are many more multiplicative seminorms on $K[T]$ than just the ones given by evaluation at a point of K. The prototypical example arises by fixing a closed disc $D(a,r) = \{z \in K \ : \ |z-a| \leq r\}$ in K and defining $[\ ]_{D(a,r)}$ by

$$[f]_{D(a,r)} \ = \ \sup_{z \in D(a,r)} |f(z)| \ .$$

It is an elementary consequence of Gauss's lemma that $[\ ]_{D(a,r)}$ is *multiplicative*, and the other axioms for a seminorm are trivially satisfied. Thus each disc $D(a,r)$ gives rise to a point of $\mathbb{A}^1_{\mathrm{Berk}}$. Note that this includes discs for which $r \notin |K^\times|$, i.e., "irrational discs" for which the set $\{z \in K \ : \ |z-a| = r\}$ is empty. We may consider the point a as a "degenerate" disc of radius zero. (If $r > 0$, then $[\ ]_{D(a,r)}$ is not only a seminorm, but a norm.) It is not hard to see that distinct discs $D(a,r)$ with $r \geq 0$ give rise to distinct multiplicative seminorms on $K[T]$, and therefore the set of all such discs embeds naturally into $\mathbb{A}^1_{\mathrm{Berk}}$.

Suppose $x, x' \in \mathbb{A}^1_{\mathrm{Berk}}$ are distinct points corresponding to the (possibly degenerate) discs $D(a,r), D(a',r')$, respectively. The unique path in $\mathbb{A}^1_{\mathrm{Berk}}$ between x and x' has a very intuitive description. If $D(a,r) \subset D(a',r')$, it consists of all points of $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to discs containing $D(a,r)$ and contained in $D(a',r')$. The set of all such "intermediate discs" is totally ordered by containment, and if $a = a'$, it is just $\{D(a,t) : r \leq t \leq r'\}$. If $D(a,r)$ and $D(a',r')$ are disjoint, the unique path between x and x' consists of all points of $\mathbb{A}^1_{\mathrm{Berk}}$ corresponding to discs which are either of the form $D(a,t)$ with $r \leq t \leq |a-a'|$ or of the form $D(a',t)$ with $r' \leq t \leq |a-a'|$.

The disc $D(a, |a - a'|) = D(a', |a - a'|)$ is the smallest one containing both $D(a, r)$ and $D(a', r')$, and if $x \vee x'$ denotes the point of $\mathbb{A}^1_{\text{Berk}}$ corresponding to $D(a, |a - a'|)$, then the path from x to x' is just the path from x to $x \vee x'$ followed by the path from $x \vee x'$ to x'.

In particular, if a, a' are distinct points of K, one can visualize the path in $\mathbb{A}^1_{\text{Berk}}$ from a to a' as follows: increase the "radius" of the degenerate disc $D(a, 0)$ until a disc $D(a, r)$ is reached which also contains a'. This disc can also be written as $D(a', s)$ with $s = |a - a'|$. Now decrease s until the radius reaches zero. This "connects" the totally disconnected space K by adding points corresponding to closed discs in K. In order to obtain a *compact* space, however, it is necessary in general to add even more points, for K may not be *spherically complete* (this happens, e.g., when $K = \mathbb{C}_p$): there may be decreasing sequences of closed discs with empty intersection. Intuitively, we need to add in points corresponding to such sequences in order to obtain a space which has a chance of being compact. More precisely, returning to the definition of $\mathbb{A}^1_{\text{Berk}}$ in terms of multiplicative seminorms, if $\{D(a_i, r_i)\}$ is any decreasing nested sequence of closed discs, then the map

$$f \mapsto \lim_{i \to \infty} [f]_{D(a_i, r_i)}$$

defines a multiplicative seminorm on $K[T]$ extending the usual absolute value on K. One can show that two sequences of discs with empty intersection define the same seminorm if and only if the sequences are *cofinal*. This yields a large number of additional points of $\mathbb{A}^1_{\text{Berk}}$. According to *Berkovich's classification theorem*, we have now described all the points of $\mathbb{A}^1_{\text{Berk}}$: each point $x \in \mathbb{A}^1_{\text{Berk}}$ corresponds to a decreasing nested sequence $\{D(a_i, r_i)\}$ of closed discs, and we can categorize the points of $\mathbb{A}^1_{\text{Berk}}$ into four types according to the nature of $D = \bigcap D(a_i, r_i)$:

- (I) D is a point of K.
- (II) D is a closed disc with radius belonging to $|K^\times|$.
- (III) D is a closed disc with radius *not* belonging to $|K^\times|$.
- (IV) $D = \emptyset$.

As a set, $\mathbb{P}^1_{\text{Berk}}$ can be obtained from $\mathbb{A}^1_{\text{Berk}}$ by adding a type I point denoted ∞. The topology on $\mathbb{P}^1_{\text{Berk}}$ is that of the one-point compactification.

Following Rivera-Letelier, we write $\mathbb{H}_{\text{Berk}}$ for the subset of $\mathbb{P}^1_{\text{Berk}}$ consisting of all points of type II, III, or IV (Berkovich "hyperbolic space"). Note that $\mathbb{H}_{\text{Berk}}$ consists of precisely the points in $\mathbb{P}^1_{\text{Berk}}$ for which $[\ ]_x$ is a norm. We also write $\mathbb{H}^{\mathbb{Q}}_{\text{Berk}}$ for the set of type II points and $\mathbb{H}^{\mathbb{R}}_{\text{Berk}}$ for the set of points of type II or III.

The description of points of $\mathbb{A}^1_{\text{Berk}}$ in terms of closed discs is very useful, because it allows one to visualize quite concretely the abstract space of multiplicative seminorms which we started with. It also allows us to understand in a more concrete way the natural partial order on $\mathbb{A}^1_{\text{Berk}}$ in which $x \leq y$ if and only if $[f]_x \leq [f]_y$ for all $f \in K[T]$. In terms of discs, if x, y are points of type I, II, or III, one can show that $x \leq y$ if and only if the disc

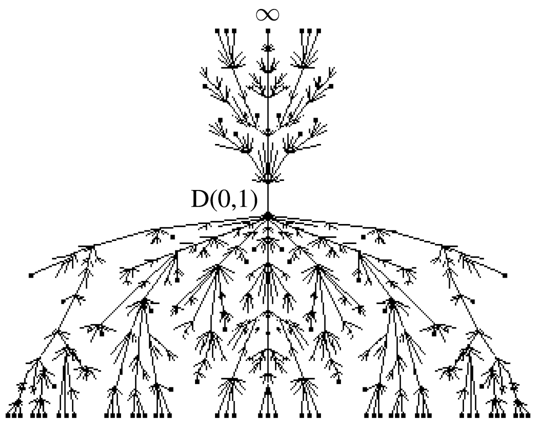

FIGURE 0.1. The Berkovich Projective Line

corresponding to x is contained in the disc corresponding to y. (We leave it as an exercise to the reader to extend this description of the partial order to points of type IV.) For any pair of points $x, y \in \mathbb{A}^1_{\text{Berk}}$, there is a unique least upper bound $x \vee y \in \mathbb{A}^1_{\text{Berk}}$ with respect to this partial order. We can extend the partial order to $\mathbb{P}^1_{\text{Berk}}$ by declaring that $x \le \infty$ for all $x \in \mathbb{A}^1_{\text{Berk}}$.

Writing

$$[x, x'] \ = \ \{z \in \mathbb{P}^1_{\text{Berk}} \ : \ x \le z \le x'\} \cup \{z \in \mathbb{P}^1_{\text{Berk}} \ : \ x' \le z \le x\} \ ,$$

it is easy to see that the unique path between $x, y \in \mathbb{P}^1_{\text{Berk}}$ is just

$$[x, x \vee y] \cup [x \vee y, y] \ .$$

There is a canonical metric ρ on $\mathbb{H}_{\text{Berk}}$ which is of great importance for potential theory. To define it, we first define a function $\text{diam} : \mathbb{A}^1_{\text{Berk}} \to \mathbb{R}_{\ge 0}$ by setting $\text{diam}(x) = \lim r_i$ if x corresponds to the nested sequence $\{D(a_i, r_i)\}$. This is easily checked to be well-defined, independent of the choice of nested sequence. If $x \in \mathbb{H}^{\mathbb{R}}_{\text{Berk}}$, then $\text{diam}(x)$ is just the diameter (= radius) of the corresponding closed disc. Because K is complete, if x is of type IV, then $\text{diam}(x) > 0$. Thus $\text{diam}(x) = 0$ for $x \in \mathbb{A}^1_{\text{Berk}}$ of type I, and $\text{diam}(x) > 0$ for $x \in \mathbb{H}_{\text{Berk}}$.

If $x, y \in \mathbb{H}_{\text{Berk}}$ with $x \le y$, we define

$$\rho(x, y) \ = \ \log_v \frac{\text{diam}(y)}{\text{diam}(x)} \ ,$$

where $\log_v$ denotes the logarithm to the base q_v, with $q_v > 1$ a fixed real number chosen so that $-\log_v |\cdot|$ is a prescribed normalized valuation on K.

More generally, for $x, y \in \mathbb{H}_{\text{Berk}}$ arbitrary, we define the *path distance metric* $\rho(x, y)$ by

$$\rho(x, y) \ = \ \rho(x, x \vee y) + \rho(y, x \vee y) \ .$$

It is not hard to verify that ρ defines a metric on $\mathbb{H}_{\text{Berk}}$. One can extend ρ to a singular function on $\mathbb{P}^1_{\text{Berk}}$ by declaring that if $x \in \mathbb{P}^1(K)$ and $y \in \mathbb{P}^1_{\text{Berk}}$, we have $\rho(x,y) = \infty$ if $x \neq y$ and 0 if $x = y$. However, we usually only consider ρ as being defined on $\mathbb{H}_{\rm Berk}$.

It is important to note that the topology on $\mathbb{H}_{\rm Berk}$ defined by the metric ρ is *not* the subspace topology induced from the Berkovich (or Gelfand) topology on $\mathbb{P}^1_{\rm Berk} \supset \mathbb{H}_{\rm Berk}$; it is strictly finer than the subspace topology.

The group $\mathrm{PGL}(2,K)$ of *Möbius transformations* acts continuously on $\mathbb{P}^1_{\rm Berk}$ in a natural way compatible with the usual action on $\mathbb{P}^1(K)$, and this action preserves $\mathbb{H}_{\rm Berk}, \mathbb{H}^{\mathbb{Q}}_{\rm Berk}$, and $\mathbb{H}^{\mathbb{R}}_{\rm Berk}$. Using the definition of $\mathbb{P}^1_{\rm Berk}$ in terms of multiplicative seminorms (and extending each $[\ ]_x$ to a seminorm on its local ring in the quotient field $K(T)$), we have $[f]_{M(x)} = [f \circ M]_x$ for each $M \in \mathrm{PGL}(2,K)$. The action of $\mathrm{PGL}(2,K)$ on $\mathbb{P}^1_{\rm Berk}$ can also be described concretely in terms of Berkovich's classification theorem, using the fact that each $M \in \mathrm{PGL}(2,K)$ takes closed discs to closed discs. An important observation is that $\mathrm{PGL}(2,K)$ acts *isometrically* on $\mathbb{H}_{\rm Berk}$, i.e.,

$$\rho(M(x), M(y)) \ = \ \rho(x,y)$$

for all $x, y \in \mathbb{H}_{\rm Berk}$ and all $M \in \mathrm{PGL}(2,K)$. This shows that the path distance metric ρ is "coordinate-free".

The diameter function diam can also be used to extend the usual distance function $|x-y|$ on K to $\mathbb{A}^1_{\rm Berk}$. We call this extension the *Hsia kernel* and denote it by $\delta(x,y)_\infty$. Formally, for $x, y \in \mathbb{A}^1_{\rm Berk}$ we have

$$\delta(x,y)_\infty \ = \ \mathrm{diam}(x \vee y) \ .$$

It is easy to see that if $x, y \in K$, then $\delta(x,y)_\infty = |x-y|$. More generally, one has the formula

$$\delta(x,y)_\infty \ = \ \limsup_{(x_0,y_0)\to(x,y)} |x_0 - y_0| \ ,$$

where $(x_0, y_0) \in K \times K$ and the convergence implicit in the lim sup is with respect to the product topology on $\mathbb{P}^1_{\rm Berk} \times \mathbb{P}^1_{\rm Berk}$. The Hsia kernel satisfies all of the axioms for an ultrametric with one exception: we have $\delta(x,x)_\infty > 0$ for $x \in \mathbb{H}_{\rm Berk}$.

The function $-\log_v \delta(x,y)_\infty$, which generalizes the usual potential theory kernel $-\log_v |x-y|$, leads to a theory of capacities on $\mathbb{P}^1_{\rm Berk}$ which generalizes that of [**88**] and which has many features in common with classical capacity theory over $\mathbb{C}$.

There is also a *generalized Hsia kernel* $\delta(x,y)_\zeta$ with respect to an arbitrary point $\zeta \in \mathbb{P}^1_{\rm Berk}$; we refer the reader to §4.4 for details.

We now come to an important description of $\mathbb{P}^1_{\rm Berk}$ as a *profinite* $\mathbb{R}$*-tree*. An $\mathbb{R}$*-tree* is a metric space (T,d) such that for each distinct pair of points $x, y \in T$, there is a unique arc in T from x to y, and this arc is a geodesic. (See Appendix B for a more detailed discussion of $\mathbb{R}$-trees.) A *branch point* is a point $x \in T$ for which $T \backslash \{x\}$ has either one or more than two connected components. A *finite* $\mathbb{R}$*-tree* is an $\mathbb{R}$-tree which is compact and has only finitely many branch points. Intuitively, a finite $\mathbb{R}$-tree is just a finite tree in the usual graph-theoretic sense, but where the edges are thought of as line segments having definite lengths. Finally, a *profinite* $\mathbb{R}$*-tree* is an inverse limit of finite $\mathbb{R}$-trees.

Let us consider how these definitions play out for $\mathbb{P}^1_{\text{Berk}}$. If $S \subset \mathbb{P}^1_{\text{Berk}}$, define the *convex hull* of S to be the smallest path-connected subset of $\mathbb{P}^1_{\text{Berk}}$ containing S. (This is the same as the set of all paths between points of S.) By abuse of terminology, a *finite subgraph* of $\mathbb{P}^1_{\text{Berk}}$ will mean the convex hull of a finite subset $S \subset \mathbb{H}^{\mathbb{R}}_{\text{Berk}}$. Every finite subgraph Γ, when endowed with the induced path distance metric ρ, is a finite $\mathbb{R}$-tree, and the collection of all finite subgraphs of $\mathbb{P}^1_{\text{Berk}}$ is a directed set under inclusion. Moreover, if $\Gamma \leq \Gamma'$, then by a basic property of $\mathbb{R}$-trees, there is a continuous *retraction map* $r_{\Gamma',\Gamma} : \Gamma' \to \Gamma$. In §1.4, we will show that $\mathbb{P}^1_{\text{Berk}}$ is homeomorphic to the inverse limit $\varprojlim \Gamma$ over all finite subgraphs $\Gamma \subset \mathbb{P}^1_{\text{Berk}}$. (Intuitively, this is just a topological formulation of Berkovich's classification theorem.) This description of $\mathbb{P}^1_{\text{Berk}}$ as a profinite $\mathbb{R}$-tree provides a convenient way to visualize the topology on $\mathbb{P}^1_{\text{Berk}}$: two points are "close" if they retract to the same point of a "large" finite subgraph. For each Γ, we let $r_{\mathbb{P}^1_{\text{Berk}},\Gamma}$ be the natural retraction map from $\mathbb{P}^1_{\text{Berk}}$ to Γ coming from the universal property of the inverse limit.

A fundamental system of open neighborhoods for the topology on $\mathbb{P}^1_{\text{Berk}}$ is given by the *open affinoid subsets*, which are the sets of the form $r^{-1}_{\mathbb{P}^1_{\text{Berk}},\Gamma}(V)$ for Γ a finite subgraph of $\mathbb{H}^{\mathbb{R}}_{\text{Berk}}$ and V an open subset of Γ. We will refer to a connected open affinoid subset of $\mathbb{P}^1_{\text{Berk}}$ as a *simple domain*. Simple domains can be completely characterized as the connected open subsets of $\mathbb{P}^1_{\text{Berk}}$ having a finite (nonzero) number of boundary points, all of which are contained in $\mathbb{H}^{\mathbb{R}}_{\text{Berk}}$. If U is an open subset of $\mathbb{P}^1_{\text{Berk}}$, a *simple subdomain* of U is defined to be a simple domain whose closure is contained in U.

Laplacians. The profinite $\mathbb{R}$-tree structure on $\mathbb{P}^1_{\text{Berk}}$ leads directly to the construction of a Laplacian operator. On a finite subgraph Γ of $\mathbb{P}^1_{\text{Berk}}$ (or, more generally, on any 'metrized graph'; see Chapter 3 for details), there is a natural Laplacian operator Δ_Γ generalizing the well-known combinatorial Laplacian on a weighted graph. If $f : \Gamma \to \mathbb{R}$ is continuous, and $\mathcal{C}^2$ except at a finite number of points, then there is a unique Borel measure $\Delta_\Gamma(f)$ of total mass zero on Γ such that

$$\int_\Gamma \psi\, \Delta_\Gamma(f) \;=\; \int_\Gamma f'(x)\psi'(x)\,dx \tag{0.1}$$

for all continuous, piecewise affine functions ψ on Γ. The measure $\Delta_\Gamma(f)$ has a discrete part and a continuous part. At each $P \in \Gamma$ which is either a branch point of Γ or a point where $f(x)$ fails to be $\mathcal{C}^2$, $\Delta_\Gamma(f)$ has a point mass equal to the negative of the sum of the directional derivatives of $f(x)$ on the edges emanating from P. On the intervening edges, it is given by $-f''(x)dx$. (See Chapter 3 for details.)

We define BDV(Γ) to be the space of all continuous real-valued functions f on Γ for which the distribution defined by

$$\psi \ \mapsto \ \int_\Gamma f \, \Delta_\Gamma(\psi) \ , \tag{0.2}$$

for all ψ as above, is represented by a bounded signed Borel measure $\Delta_\Gamma(f)$. A simple integration by parts argument shows that this measure coincides with the one defined by (0.1) when f is sufficiently smooth. The name "BDV" is an abbreviation for "Bounded Differential Variation". We call the measure $\Delta_\Gamma(f)$ the *Laplacian* of f on Γ.

The Laplacian satisfies an important compatibility property with respect to the partial order on the set of finite subgraphs of $\mathbb{P}^1_{\rm Berk}$ given by containment: if $\Gamma \le \Gamma'$ and $f \in {\rm BDV}(\Gamma')$, then

$$\Delta_\Gamma(f|_\Gamma) \ = \ \left(r_{\Gamma',\Gamma}\right)_* \Delta_{\Gamma'}(f) \ . \tag{0.3}$$

We define ${\rm BDV}(\mathbb{P}^1_{\rm Berk})$ to be the collection of all functions $f : \mathbb{P}^1_{\rm Berk} \to \mathbb{R} \cup \{\pm\infty\}$ such that:

- $f|_\Gamma \in {\rm BDV}(\Gamma)$ for each finite subgraph Γ.
- The measures $|\Delta_\Gamma(f)|$ have uniformly bounded total mass.

Note that belonging to ${\rm BDV}(\mathbb{P}^1_{\rm Berk})$ imposes no condition on the values of f at points of $\mathbb{P}^1(K)$.

Using the compatibility property (0.3), one shows that if $f \in {\rm BDV}(\mathbb{P}^1_{\rm Berk})$, then the collection of measures $\{\Delta_\Gamma\}$ "cohere" to give a unique Borel measure $\Delta(f)$ of total mass zero on the inverse limit space $\mathbb{P}^1_{\rm Berk}$ satisfying

$$\left(r_{\mathbb{P}^1_{\rm Berk},\Gamma}\right)_* \Delta(f) \ = \ \Delta_\Gamma(f)$$

for all finite subgraphs Γ of $\mathbb{P}^1_{\rm Berk}$. We call $\Delta(f)$ the *Laplacian* of f on $\mathbb{P}^1_{\rm Berk}$.

Similarly, if U is a domain (i.e., a nonempty connected open subset) in $\mathbb{P}^1_{\rm Berk}$, one defines a class ${\rm BDV}(U)$ of functions $f : U \to \mathbb{R} \cup \{\pm\infty\}$ for which the Laplacian $\Delta_{\overline{U}}(f)$ is a bounded Borel measure of total mass zero supported on the closure of U. The measure $\Delta_{\overline{U}}(f)$ has the property that

$$\left(r_{\overline{U},\Gamma}\right)_* \Delta(f) \ = \ \Delta_\Gamma(f)$$

for all finite subgraphs Γ of $\mathbb{P}^1_{\rm Berk}$ contained in U.

As a concrete example, fix $y \in \mathbb{A}^1_{\rm Berk}$ and let $f : \mathbb{P}^1_{\rm Berk} \to \mathbb{R} \cup \{\pm\infty\}$ be defined by $f(\infty) = -\infty$ and

$$f(x) \ = \ -\log_v \delta(x,y)_\infty$$

for $x \in \mathbb{A}^1_{\rm Berk}$. Then $f \in {\rm BDV}(\mathbb{P}^1_{\rm Berk})$, and

$$\Delta f \ = \ \delta_y - \delta_\infty \tag{0.4}$$

is a discrete measure on $\mathbb{P}^1_{\rm Berk}$ supported on $\{y, \infty\}$. Intuitively, the explanation for the formula (0.4) is as follows. The function f is locally constant away from the path $\Lambda = [y, \infty]$ from y to ∞; more precisely, we have $f(x) = f(r_{\mathbb{P}^1_{\rm Berk},\Lambda}(x))$. Moreover, the restriction of f to Λ is linear (with respect to the distance function ρ) with slope -1. For every suitable test function ψ, we therefore have the "heuristic" calculation

$$\int_{\mathbb{P}^1_{\mathrm{Berk}}} \psi \Delta f = \int_\Lambda f'(x)\psi'(x)\,dx = -\int_y^\infty \psi'(x)\,dx = \psi(y) - \psi(\infty)\ .$$

(To make this calculation rigorous, one needs to exhaust $\Lambda = [y, \infty]$ by an increasing sequence of line segments $\Gamma \subset \mathbb{H}^{\mathbb{R}}_{\mathrm{Berk}}$ and then observe that the corresponding measures $\Delta_\Gamma f$ converge weakly to $\delta_y - \delta_\infty$.)

Equation (0.4) shows that $-\log_v \delta(x,y)_\infty$, like its classical counterpart $-\log|x-y|$ over $\mathbb{C}$, is a fundamental solution (in the sense of distributions) to the Laplace equation. This "explains" why $-\log_v \delta(x,y)_\infty$ is the correct kernel for doing potential theory.

More generally, let $\varphi \in K(T)$ be a nonzero rational function with zeros and poles given by the divisor $\mathrm{div}(\varphi)$ on $\mathbb{P}^1(K)$. The usual action of φ on $\mathbb{P}^1(K)$ extends naturally to an action of φ on $\mathbb{P}^1_{\mathrm{Berk}}$, and there is a continuous function $-\log_v[\varphi]_x : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ extending the usual map $x \mapsto -\log_v |\varphi(x)|$ on $\mathbb{P}^1(K)$. One derives from (0.4) the following version of the *Poincaré-Lelong formula*:

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(-\log_v[\varphi]_x) \ = \ \delta_{\mathrm{div}(\varphi)}\ .$$

Capacities. Fix $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$, and let E be a compact subset of $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{\zeta\}$. (For concreteness, the reader may wish to imagine that $\zeta = \infty$.) By analogy with the classical theory over $\mathbb{C}$ and also with the non-Archimedean theory developed in [**88**], one can define the *logarithmic capacity* of E with respect to ζ. This is done as follows.

Given a probability measure ν on $\mathbb{P}^1_{\mathrm{Berk}}$ with support contained in E, we define the energy integral

$$I_\zeta(\nu) \ = \ \iint_{E\times E} -\log_v \delta(x,y)_\zeta\, d\nu(x) d\nu(y)\ .$$

Letting ν vary over the collection $\mathbb{P}(E)$ of all probability measures supported on E, one defines the *Robin constant*

$$V_\zeta(E) \ = \ \inf_{\nu \in \mathbb{P}(E)} I_\zeta(\nu)\ .$$

The *logarithmic capacity* of E relative to ζ is then defined to be

$$\gamma_\zeta(E) \ = \ q_v^{-V_\zeta(E)}\ .$$

For an arbitrary set H, the logarithmic capacity $\gamma_\zeta(H)$ is defined by

$$\gamma_\zeta(H) \ = \ \sup_{\text{compact } E \subset H} \gamma_\zeta(E)\ .$$

A countably supported probability measure must have point masses, and $\delta(x,x)_\zeta = 0$ for $x \in \mathbb{P}^1(K)\backslash\{\zeta\}$; thus $V_\zeta(E) = +\infty$ when $E \subset \mathbb{P}^1(K)$ is countable, so every countable subset of $\mathbb{P}^1(K)$ has capacity zero. On the other hand, for a "nonclassical" point $x \in \mathbb{H}_{\mathrm{Berk}}$ we have $V_\zeta(\{x\}) < +\infty$, since $\delta(x,x)_\zeta > 0$, and therefore $\gamma_\zeta(\{x\}) > 0$. In particular, a singleton set can have positive capacity, a phenomenon which has no classical analogue. More generally, if $E \cap \mathbb{H}_{\text{Berk}} \neq \emptyset$, then $\gamma_\zeta(E) > 0$.

As a more elaborate example, if $K = \mathbb{C}_p$ and $E = \mathbb{Z}_p \subset \mathbb{A}^1(\mathbb{C}_p) \subset \mathbb{A}^1_{\text{Berk},\mathbb{C}_p}$, then $\gamma_\infty(E) = p^{-1/(p-1)}$. Since $\delta(x,y)_\infty = |x-y|$ for $x, y \in K$, this follows from the same computation as in [**88**, Example 4.1.24].

For fixed E, the property of E having capacity 0 relative to ζ is independent of the point $\zeta \notin E$.

If E is compact and $\gamma_\zeta(E) > 0$, we show that there is a *unique* probability measure $\mu_{E,\zeta}$ on E, called the *equilibrium measure* of E with respect to ζ, which minimizes energy (i.e., for which $I_\zeta(\mu_{E,\zeta}) = V_\zeta(E)$). As in the classical case, $\mu_{E,\zeta}$ is always supported on the boundary of E.

Closely linked to the theory of capacities is the theory of *potential functions.* For each probability measure ν supported on $\mathbb{P}^1_{\text{Berk}} \backslash \{\zeta\}$, one defines the potential function $u_\nu(z,\zeta)$ by

$$u_\nu(z,\zeta) \ = \ \int -\log_v \delta(z,w)_\zeta \, d\nu(w) \ .$$

As in classical potential theory, potential functions need not be continuous, but they do share several of the distinguishing features of continuous functions. For example, $u_\nu(z,\zeta)$ is lower semicontinuous, and it is continuous at each $z \notin \text{supp}(\nu)$. Potential functions on $\mathbb{P}^1_{\text{Berk}}$ satisfy the following analogues of Maria's theorem and Frostman's theorem from complex potential theory:

Theorem (Maria). *If $u_\nu(z,\zeta) \le M$ on* supp(ν)*, then $u_\nu(z,\zeta) \le M$ for all $z \in \mathbb{P}^1_{\text{Berk}} \backslash \{\zeta\}$.*

Theorem (Frostman). *If a compact set E has positive capacity, then the equilibrium potential $u_E(z,\zeta)$ satisfies $u_E(z,\zeta) \le V_\zeta(E)$ for all $z \in \mathbb{P}^1_{\text{Berk}} \backslash \{\zeta\}$, and $u_E(z,\zeta) = V_\zeta(E)$ for all $z \in E$ outside a set of capacity zero.*

As in capacity theory over $\mathbb{C}$, one can also define the *transfinite diameter* and the *Chebyshev constant* of E, and they both turn out to be equal to the logarithmic capacity of E. (In fact, we define three different variants of the Chebyshev constant and prove that they are all equal.)

As an arithmetic application of the theory of capacities on $\mathbb{P}^1_{\text{Berk}}$, we formulate generalizations to $\mathbb{P}^1_{\text{Berk}}$ of the Fekete and Fekete-Szegö theorems from [**88**]. The proofs are easy, since they go by reducing the general case to the special case of RL-domains, which was already treated in [**88**]. Nonetheless, the results are aesthetically pleasing because in their statement, the simple notion of compactness replaces the awkward concept of "algebraic capacitability". The possibility for such a reformulation is directly related to the fact that $\mathbb{P}^1_{\text{Berk}}$ is compact, while $\mathbb{P}^1(K)$ is not.

Harmonic functions. If U is a domain in $\mathbb{P}^1_{\mathrm{Berk}}$, a real-valued function $f : U \to \mathbb{R}$ is called *strongly harmonic* on U if it is continuous, belongs to $\mathrm{BDV}(U)$, and if $\Delta_{\overline{U}}(f)$ is supported on ∂U. The function f is *harmonic* on U if every point $x \in U$ has a connected open neighborhood on which f is strongly harmonic.

Harmonic functions on domains $U \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ satisfy many properties analogous to their classical counterparts over $\mathbb{C}$. For example, a harmonic function which attains its maximum or minimum value on U must be constant. There is also an analogue of the Poisson formula: if f is a harmonic function on an open affinoid U, then f extends uniquely to the boundary ∂U, and the values of f on U can be computed explicitly in terms of $f|_{\partial U}$. A version of Harnack's principle holds as well: the limit of a monotonically increasing sequence of nonnegative harmonic functions on U is either harmonic or identically $+\infty$. Even better than the classical case (where a hypothesis of uniform convergence is required), a *pointwise* limit of harmonic functions is automatically harmonic. As is the case over $\mathbb{C}$, harmonicity is preserved under pullbacks by meromorphic functions.

Fix $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$, and let E be a compact subset of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$. We define the *Green's function of E relative to ζ* to be

$$G(z, \zeta; E) \ = \ V_\zeta(E) - u_E(z, \zeta)$$

for all $z \in \mathbb{P}^1_{\mathrm{Berk}}$. We show that the Green's function is everywhere nonnegative and that it is strictly positive on the connected component U_ζ of $\mathbb{P}^1_{\mathrm{Berk}} \backslash E$ containing ζ. Also, $G(z, \zeta; E)$ is finite on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$, with a logarithmic singularity at ζ, and it is harmonic on $U_\zeta \backslash \{\zeta\}$. Additionally, $G(z, \zeta; E)$ is identically zero on the complement of U_ζ outside a set of capacity zero. The Laplacian of $G(z, \zeta; E)$ on $\mathbb{P}^1_{\mathrm{Berk}}$ is equal to $\delta_\zeta - \mu_{E,\zeta}$. As in the classical case, the Green's function is symmetric as a function of z and ζ: we have

$$G(z_1, z_2; E) \ = \ G(z_2, z_1; E)$$

for all $z_1, z_2 \notin E$. In a satisfying improvement over the theory for $\mathbb{P}^1(\mathbb{C}_p)$ in [**88**], the role of $G(z, \zeta; E)$ as a reproducing kernel for the Berkovich space Laplacian becomes evident.

As an arithmetic application of the theory of Green's functions and capacities on $\mathbb{P}^1_{\mathrm{Berk}}$, we prove a Berkovich space generalization of Bilu's equidistribution theorem for a rather general class of adelic heights.

Subharmonic functions. We give two characterizations of what it means for a function on a domain $U \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ to be subharmonic. The first, which we take as the definition, is as follows. We say that a function $f : U \to \mathbb{R} \cup \{-\infty\}$ is *strongly subharmonic* if it is upper semicontinuous, satisfies a further technical semicontinuity hypothesis at points of $\mathbb{P}^1(K)$, and if the positive part of $\Delta_{\overline{U}}(f)$ is supported on ∂U. We say that f is *subharmonic* on U if every point of U has a connected open neighborhood on which f is strongly subharmonic. We also say that f is *superharmonic* on U if $-f$ is subharmonic on U. As an example, if ν is a probability measure on $\mathbb{P}^1_{\text{Berk}}$ and $\zeta \notin \text{supp}(\nu)$, the potential function $u_\nu(x,\zeta)$ is strongly superharmonic on $\mathbb{P}^1_{\text{Berk}}\backslash\{\zeta\}$ and is strongly subharmonic on $\mathbb{P}^1_{\text{Berk}}\backslash \text{supp}(\nu)$. A function f is harmonic on U if and only if it is both subharmonic and superharmonic on U.

As a second characterization of subharmonic functions, we say that $f : U \to \mathbb{R} \cup \{-\infty\}$ (not identically $-\infty$) is *domination subharmonic* on the domain U if it is upper semicontinuous and if for each simple subdomain V of U and each harmonic function h on V for which $f \leq h$ on ∂V, we have $f \leq h$ on V. A fundamental fact, proved in §8.2, is that f is subharmonic on U if and only if it is domination subharmonic on U.

Like harmonic functions, subharmonic functions satisfy the Maximum Principle: if U is a domain in $\mathbb{P}^1_{\text{Berk}}$ and f is a subharmonic function which attains its maximum value on U, then f is constant. In addition, subharmonic functions on domains in $\mathbb{P}^1_{\text{Berk}}$ are stable under many of the same operations (e.g., convex combinations, maximum, monotone convergence, uniform convergence) as their classical counterparts. There is also an analogue of the Riesz Decomposition Theorem, according to which a subharmonic function on a simple subdomain $V \subset U$ can be written as the difference of a harmonic function and a potential function. We also show that subharmonic functions can be well-approximated by continuous functions of a special form, which we call *smooth* functions.

In §8.10, we define the notion of an Arakelov-Green's function on $\mathbb{P}^1_{\text{Berk}}$ and establish an *energy minimization principle* used in the proof of the main result in [**9**]. We give two proofs of the Energy Minimization Principle, one using the theory of subharmonic functions and another using the Dirichlet pairing.

Multiplicities. If $\varphi \in K(T)$ is a nonconstant rational function, then as discussed above, the action of φ on $\mathbb{P}^1(K)$ extends naturally to an action of φ on $\mathbb{P}^1_{\text{Berk}}$. We use the theory of Laplacians to give an analytic construction of *multiplicities* for points in $\mathbb{P}^1_{\text{Berk}}$ which generalize the usual multiplicity of φ at a point $a \in \mathbb{P}^1(K)$ (i.e., the multiplicity of a as a preimage of $b = \varphi(a) \in \mathbb{P}^1(K)$). Using the theory of multiplicities, we show that the extended map $\varphi : \mathbb{P}^1_{\text{Berk}} \to \mathbb{P}^1_{\text{Berk}}$ is a surjective open mapping. We also obtain a purely topological interpretation of multiplicities, which shows that our multiplicities coincide with those defined by Rivera-Letelier. For each $a \in \mathbb{P}^1_{\text{Berk}}$, the multiplicity of φ at a is a positive integer, and if $\text{char}(K) = 0$, it is equal to 1 if and only if φ is locally injective at a. For each $b \in \mathbb{P}^1_{\text{Berk}}$, the sum of the multiplicities of φ over all preimages of b is equal to the degree of φ.

Using these multiplicities, we define the pushforward and pullback of a bounded Borel measure on $\mathbb{P}^1_{\text{Berk}}$ under φ. The pushforward and pullback measures satisfy the expected functoriality properties; for example, if f is subharmonic on U, then $f \circ \varphi$ is subharmonic on $\varphi^{-1}(U)$ and the Laplacian of $f \circ \varphi$ is the pullback under φ of the Laplacian of f.

Applications to the dynamics of rational maps. Though Berkovich introduced his theory of analytic spaces with rather different goals in mind, Berkovich spaces are well adapted to the study of non-Archimedean dynamics. The fact that the topological space $\mathbb{P}^1_{\mathrm{Berk}}$ is both compact and connected means in practice that many of the difficulties encountered in "classical" non-Archimedean dynamics disappear when one defines the Fatou and Julia sets as subsets of $\mathbb{P}^1_{\mathrm{Berk}}$. For example, the notion of a connected component is straightforward in the Berkovich setting, so one avoids the subtle issues involved in defining Fatou components in $\mathbb{P}^1(\mathbb{C}_p)$ (e.g., the D-components versus analytic components in Benedetto's paper [**14**], or the definition by Rivera-Letelier in [**83**]).

Suppose $\varphi \in K(T)$ is a rational function of degree $d \geq 2$. In §10.1, we construct a *canonical probability measure* μ_φ on $\mathbb{P}^1_{\mathrm{Berk}}$ attached to φ, whose properties are analogous to the well-known measure on $\mathbb{P}^1(\mathbb{C})$ first defined by Lyubich and by Freire, Lopes, and Mañé. The measure μ_φ is φ-invariant (i.e., satisfies $\varphi_*(\mu_\varphi) = \mu_\varphi$) and also satisfies the functional equation $\varphi^*(\mu_\varphi) = d \cdot \mu_\varphi$.

In §10.2, we prove an explicit formula and functional equation for the Arakelov-Green's function $g_{\mu_\varphi}(x, y)$ associated to μ_φ. These results, along with the Energy Minimization Principle mentioned earlier, play a key role in applications of the theory to arithmetic dynamics over global fields (see [**5**] and [**9**]). In §10.3, we use these results to prove an adelic equidistribution theorem (Theorem 10.24) for the Galois conjugates of algebraic points of small dynamical height over a number field k.

We then discuss analogues for $\mathbb{P}^1_{\mathrm{Berk}}$ of classical results in the Fatou-Julia theory of iteration of rational maps on $\mathbb{P}^1(\mathbb{C})$. In particular, we define the Berkovich Fatou and Julia sets of φ and prove that the Berkovich Julia set J_φ (like its complex counterpart, but unlike its counterpart in $\mathbb{P}^1(K)$) is always nonempty. We give a new proof of the Favre–Rivera-Letelier equidistribution theorem for iterated pullbacks of Dirac measures attached to nonexceptional points, and using this theorem, we show that the Berkovich Julia set shares many properties with its classical complex counterpart. For example, it is either connected or has uncountably many connected components, repelling periodic points are dense in it, and the "Transitivity Theorem" holds.

In $\mathbb{P}^1(K)$, the notion of equicontinuity leads to a good definition of the Fatou set. In $\mathbb{P}^1_{\mathrm{Berk}}$, as was pointed out to us by Rivera-Letelier, this remains true when $K = \mathbb{C}_p$ but fails for general K. We explain the subtleties regarding equicontinuity in the Berkovich case and give Rivera-Letelier's proof that over $\mathbb{C}_p$ the Berkovich equicontinuity locus coincides with the Berkovich Fatou set. We also give an overview (mostly without proof) of some of Rivera-Letelier's fundamental results concerning rational dynamics over $\mathbb{C}_p$. While some of Rivera-Letelier's results hold for arbitrary K, others make special use of the fact that the residue field of $\mathbb{C}_p$ is a union of finite fields.

Appendices. In Appendix A, we review some facts from real analysis and point-set topology which are used throughout the text. Some of these (e.g., the Riesz Representation Theorem) are well known, while others (e.g., the Portmanteau theorem) are hard to find precise references for. We have provided self-contained proofs for the latter. We also include a detailed discussion of nets in topological spaces: since the space $\mathbb{P}^1_{\mathrm{Berk},K}$ is not in general metrizable, sequences do not suffice when discussing notions such as continuity.

In Appendix B, we discuss $\mathbb{R}$-trees and their relation to Gromov's theory of hyperbolic spaces. This appendix serves two main purposes. On the one hand, it provides references for some basic definitions and facts about $\mathbb{R}$-trees which are used in the text. On the other hand, it provides some intuition for the general theory of $\mathbb{R}$-trees by exploring the fundamental role played by the Gromov product, which is closely related to our generalized Hsia kernel.

Appendix C gives a brief overview of some basic definitions and results from Berkovich's theory of non-Archimedean analytic spaces. This material is included in order to give the reader some perspective on the relationship between the special cases dealt with in this book (the Berkovich unit disc, affine line, and projective line) and the general setting of Berkovich's theory.