CHAPTER 10

Applications to the dynamics of rational maps

In this chapter, we study the dynamics of a rational map $\varphi \in K(T)$ of degree $d \geq 2$ acting on $\mathbb{P}^1_{\mathrm{Berk}}$, where K is a complete and algebraically closed non-Archimedean field.

We begin by showing, in §10.1, that there is a probability measure μ_φ on $\mathbb{P}^1_{\mathrm{Berk}}$ which satisfies $\varphi_*(\mu_\varphi) = \mu_\varphi$ and $\varphi^*(\mu_\varphi) = d \cdot \mu_\varphi$. By analogy with the classical situation for $\mathbb{P}^1(\mathbb{C})$ (see [**54**], [**72**]), we call μ_φ the *canonical measure* associated to φ. Loosely speaking, the canonical measure is the negative of the Laplacian of the Call-Silverman local height function from [**32**] (extended to $\mathbb{P}^1_{\mathrm{Berk}}$ in a natural way).

We also show that the associated Arakelov-Green's function $g_{\mu_\varphi}(x, y)$ satisfies a certain energy minimization principle. This was used in [**5**] and [**9**] to investigate properties of algebraic points of small dynamical height.

In §10.2, we give an explicit formula for the Arakelov-Green's function $g_{\mu_\varphi}(x, y)$, and we establish some functorial properties satisfied by $g_{\mu_\varphi}(x, y)$, including the important functional equation in Theorem 10.18.

In §10.3, we prove an adelic equidistribution theorem (due independently to Baker and Rumely, to Chambert-Loir, and to Favre and Rivera-Letelier) for sequences of distinct points in $\mathbb{P}^1$ over a number field k whose dynamical heights tend to zero. Our proof is a simplified version of that in [**9**], and it applies not just to number fields but to arbitrary product formula fields.

In §10.4, following Favre and Rivera-Letelier [**46, 48**], we prove an equidistribution theorem for the preimages under φ of any nonexceptional point of $\mathbb{P}^1_{\mathrm{Berk}}$. This is the Berkovich space analogue of a classical result due independently to Lyubich [**72**] and Freire, Lopes, and Mañé [**54**]. Our proof, which differs from the one in [**46, 48**], makes use of properties of $g_{\mu_\varphi}(x, y)$. However, our proof is valid only when $\mathrm{char}(K) = 0$, whereas Favre and Rivera-Letelier's proof holds in positive characteristic as well.

In §10.5–§10.8, we develop Fatou-Julia theory on $\mathbb{P}^1_{\mathrm{Berk}}$ when $\mathrm{char}(K) = 0$. We recover most of the basic facts known classically over $\mathbb{C}$, as presented for example in Milnor's book [**73**].

The modern approach to complex dynamics was initiated by Fatou and Julia in the early twentieth century. The best understood and most classical part of the theory concerns the iteration of a rational function $\varphi \in \mathbb{C}(T)$ on the complex projective line $\mathbb{P}^1(\mathbb{C})$. Its primary goal is to understand the structure of the Fatou set and its complement, the Julia set.

In p-adic dynamics, there is an analogous, more recent theory for a rational function $\varphi \in K(T)$. The theory was initially developed when $K = \mathbb{C}_p$,[1] for $\varphi \in \mathbb{C}_p(T)$ acting on $\mathbb{P}^1(\mathbb{C}_p)$, by Silverman, Benedetto, Hsia, and others (see e.g. [**14**], [**15**], [**61**], [**74**], [**81**]). One finds both striking similarities and striking differences between rational dynamics on $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{C}_p)$. (For an overview of the "classical" results in this field, see [**93**].) The differences arise for the most part from the fact that the topology on $\mathbb{C}_p$ is totally disconnected and not locally compact. Even the most basic topological questions about the Fatou and Julia sets have a completely different flavor when working over $\mathbb{C}_p$ versus $\mathbb{C}$. In fact, it was not *a priori* clear what the Fatou and Julia sets in $\mathbb{P}^1(\mathbb{C}_p)$ should be, since (for example) the notions of equicontinuity and normality do not coincide. Definitions which seemed reasonable (such as defining the Fatou set in terms of equicontinuity of the family of iterates $\{\varphi^{(n)}\}$) turned out to behave differently than expected. For example, in $\mathbb{P}^1(\mathbb{C}_p)$ the Julia set of any map with good reduction is empty, in sharp contrast to the situation over $\mathbb{C}$, where one of the most fundamental facts is that the Julia set is always nonempty. It took considerable effort to give a satisfactory definition for Fatou components in $\mathbb{P}^1(\mathbb{C}_p)$ [**83**].

However, as Rivera-Letelier's thesis [**81**] and subsequent works [**82, 83, 84, 46, 47, 48**] have made abundantly clear, a rational map $\varphi \in K(T)$ should be thought of as acting not just on $\mathbb{P}^1(K)$, but on $\mathbb{P}^1_{\mathrm{Berk}}$. One advantage of working with $\mathbb{P}^1_{\mathrm{Berk}}$ is evident in the existence of the canonical probability measure μ_φ. In contrast to $\mathbb{P}^1(K)$, the compact connected Hausdorff space $\mathbb{P}^1_{\mathrm{Berk}}$ serves naturally as a support for measures.

In §10.5 we define the Berkovich Julia set J_φ as the support of the canonical measure and the Berkovich Fatou set F_φ as the complement of J_φ.[2] A direct consequence of this definition is that the Berkovich Julia set is always nonempty.

In Theorem 10.56, we show that the Berkovich Fatou set of φ coincides with the set of all points in $\mathbb{P}^1_{\mathrm{Berk}}$ having a neighborhood V whose forward iterates under φ omit at least one nonexceptional point of $\mathbb{P}^1_{\mathrm{Berk}}$ (or, equivalently, at least three points of $\mathbb{P}^1(K)$). This was Rivera-Letelier's definition of the Berkovich Fatou set for $K = \mathbb{C}_p$. We use this result to deduce information about the topological structure of the Berkovich Julia set, in the style of [**73**].

In §10.6, we compare our Berkovich Fatou and Julia sets with the corresponding sets defined in terms of equicontinuity, and we use this comparison to obtain structural information about the Berkovich Fatou and Julia sets.

[1]There are some subtle but important differences between dynamics over $\mathbb{C}_p$ and dynamics over an arbitrary complete and algebraically closed non-Archimedean field K of characteristic 0; see e.g. Example 10.70 below.

[2]As Rivera-Letelier pointed out to us (with examples like Example 10.70 below), over a general complete and algebraically closed non-Archimedean ground field K, pathologies arise if one tries to define the Berkovich Julia and Fatou sets in terms of equicontinuity or normality.

We also show (Theorem 10.67) that for any complete and algebraically closed non-Archimedean ground field K of characteristic 0, the intersection of the Berkovich Fatou set with $\mathbb{P}^1(K)$ coincides with the classical Fatou set of φ in $\mathbb{P}^1(K)$ considered by Silverman, Benedetto, Hsia, and others.

Using deep results of Rivera-Letelier, we prove (Theorem 10.72) that over $\mathbb{C}_p$, the Berkovich Fatou set consists of all points having a neighborhood on which the set of iterates $\{\varphi^{(n)}\}$ is equicontinuous (in the sense of uniform spaces; cf. §A.9). However, over a general complete and algebraically closed non-Archimedean ground field K of characteristic 0, this is no longer true, as shown by Example 10.70.

In §10.7, we give a proof of Rivera-Letelier's theorem that a rational map $\varphi(T) \in K(T)$ of degree at least two has at least one repelling fixed point in $\mathbb{P}^1_{\mathrm{Berk}}$. We also show that the repelling periodic points belong to and are dense in the Berkovich Julia set. The complex analogue of this result is well known; indeed, the complex Julia set of a rational map $\varphi(T) \in \mathbb{C}(T)$ is sometimes defined as the closure of the repelling periodic points.

In §10.8, we show that when $\varphi(T) \in K[T]$ is a polynomial, the Berkovich Julia set J_φ coincides with the boundary of the Berkovich filled Julia set K_φ. This is the analogue of another well-known result over $\mathbb{C}$.

In §10.9, we consider the dynamics of rational functions over $\mathbb{C}_p$, which are much better understood than over arbitrary K. We recall the main results of Rivera-Letelier from [**81, 82, 83, 84**] and translate them into the language of $\mathbb{P}^1_{\mathrm{Berk}}$.

Finally, in §10.10 we give examples illustrating the theory.

10.1. Construction of the canonical measure

Let $\varphi(T) \in K(T)$ be a rational function of degree $d \ge 2$. In this section we construct the canonical measure μ_φ, a non-Archimedean analogue of the measure constructed by Brolin [**31**] for a polynomial in $\mathbb{C}[T]$ and by Lyubich [**72**] and Freire, Lopes, and Mañé [**54**] for a rational function in $\mathbb{C}(T)$. Our construction follows [**9**]; other constructions have been given by Chambert-Loir [**35**] and by Favre and Rivera-Letelier [**46, 47, 48**].

Let $f_1(T), f_2(T) \in K[T]$ be coprime polynomials with

$$\varphi(T) \;=\; f_2(T)/f_1(T)\ ,$$

such that $\max(\deg(f_1), \deg(f_2)) = d$. Homogenizing $f_1(T)$ and $f_2(T)$, we obtain homogeneous polynomials $F_1(X,Y), F_2(X,Y) \in K[X,Y]$ of degree d such that $f_1(T) = F_1(1,T)$, $f_2(T) = F_2(1,T)$, and where $\mathrm{Res}(F_1,F_2) \neq 0$. Let $F(X,Y) = (F_1(X,Y), F_2(X,Y))$.

We will be interested in the dynamics of φ on $\mathbb{P}^1_{\mathrm{Berk}}$, so for each $n \ge 1$, let $\varphi^{(n)} = \varphi \circ \varphi \circ \cdots \circ \varphi$ (n times). Similarly, let $F^{(n)}$ be the n-fold composition of F with itself, and write $F^{(n)}(X,Y) = (F_1^{(n)}(X,Y), F_2^{(n)}(X,Y))$, where $F_1^{(n)}(X,Y), F_2^{(n)}(X,Y) \in K[X,Y]$ are homogeneous of degree d^n.

For the moment, regard $F_1(X,Y)$ and $F_2(X,Y)$ as functions on K^2, so that for $(x,y) \in K^2$ we have $F(x,y) = (F_1(x,y), F_2(x,y))$. Let

$$\|(x,y)\| \;=\; \max(|x|,|y|)$$

be the sup norm on K^2.

LEMMA 10.1. *There are numbers* $0 < B_1 < B_2$*, depending on* F*, such that for all* $(x,y) \in K^2$*,*

$$B_1 \cdot \|(x,y)\|^d \;\le\; \|F(x,y)\| \;\le\; B_2 \cdot \|(x,y)\|^d \;. \tag{10.1}$$

Furthermore, if F_1 *and* F_2 *have coefficients in the valuation ring of* K*, then we may take* $B_1 = |\operatorname{Res}(F_1,F_2)|$ *and* $B_2 = 1$.

PROOF. The upper bound is trivial, since if $F_1(X,Y) = \sum c_{1,ij}X^iY^j$, $F_2(X,Y) = \sum c_{2,ij}X^iY^j$, and if we let $B_2 = \max(|c_{1,ij}|,|c_{2,ij}|)$, then by the ultrametric inequality

$$|F_1(x,y)|,\ |F_2(x,y)| \;\le\; \max_{i,j}(|c_{1,ij}||x|^i|y|^j, |c_{2,ij}||x|^i|y|^j) \;\le\; B_2\|(x,y)\|^d \;.$$

Note that if F_1 and F_2 have coefficients in the valuation ring of K, then we may take $B_2 = 1$.

For the lower bound, note that since $\operatorname{Res}(F_1,F_2) \neq 0$, Lemma 2.11 shows that there are homogenous polynomials $G_1(X,Y), G_2(X,Y) \in K[X,Y]$ of degree $d-1$ for which

$$G_1(X,Y)F_1(X,Y) + G_2(X,Y)F_2(X,Y) \;=\; \operatorname{Res}(F_1,F_2)X^{2d-1} \;, \tag{10.2}$$

and there are homogeneous polynomials $H_1(X,Y), H_2(X,Y) \in K[X,Y]$ of degree $d-1$ such that

$$H_1(X,Y)F_1(X,Y) + H_2(X,Y)F_2(X,Y) \;=\; \operatorname{Res}(F_1,F_2)Y^{2d-1} \;. \tag{10.3}$$

By the upper bound argument applied to $G = (G_1,G_2)$ and $H = (H_1,H_2)$, there is an $A_2 > 0$ such that $\|G(x,y)\|, \|H(x,y)\| \le A_2\|(x,y)\|^{d-1}$ for all $(x,y) \in K^2$. By (10.2), (10.3), and the ultrametric inequality,

$$\begin{aligned} |\operatorname{Res}(F_1,F_2)||x|^{2d-1} &\;\le\; A_2\|(x,y)\|^{d-1}\|F(x,y)\| \;, \\ |\operatorname{Res}(F_1,F_2)||y|^{2d-1} &\;\le\; A_2\|(x,y)\|^{d-1}\|F(x,y)\| \;. \end{aligned}$$

Writing $A_1 = |\operatorname{Res}(F_1,F_2)|$, it follows that

$$A_1\|(x,y)\|^{2d-1} \;\le\; A_2\|(x,y)\|^{d-1}\|F(x,y)\| \;.$$

Thus, taking $B_1 = A_1/A_2$, we have $\|F(x,y)\| \ge B_1\|(x,y)\|^d$.

If F_1 and F_2 have coefficients in the valuation ring of K, then since the coefficients of G_1, G_2, H_1, H_2 are integral polynomials in the coefficients of F_1 and F_2 (see, for example, Proposition 2.13 in [**93**]), we find that G_1, G_2, H_1, H_2 have coefficients in the valuation ring of K as well. We may thus take $A_2 = 1$, in which case $B_1 = |\operatorname{Res}(F_1,F_2)|$. □

Taking logarithms in (10.1), for $C_1 = \max(|\log_v(B_1)|, |\log_v(B_2)|)$ we have

$$|\frac{1}{d}\log_v \|F(x,y)\| - \log_v \|(x,y)\|| \ \le \ \frac{C_1}{d} \tag{10.4}$$

for all $(x,y) \in K^2\backslash\{(0,0)\}$. Inserting $F^{(n-1)}(x,y)$ for (x,y) in (10.4) and dividing by d^{n-1}, we find that for each $n \ge 1$,

$$|\frac{1}{d^n}\log_v \|F^{(n)}(x,y)\| - \frac{1}{d^{n-1}}\log_v \|F^{(n-1)}(x,y)\|| \ \le \ \frac{C_1}{d^n} \ . \tag{10.5}$$

Put

$$\begin{aligned} h^{(n)}_{\varphi,v,(\infty)}(x) &= \frac{1}{d^n}\log_v \|F^{(n)}(1,x)\| \\ &= \frac{1}{d^n}\log_v \Big(\max(|F_1^{(n)}(1,x)|, |F_2^{(n)}(1,x)|)\Big) \ . \end{aligned}$$

The *Call-Silverman local height* for φ on $\mathbb{P}^1(K)$ (relative to the point ∞ and the dehomogenization $F_1(1,T), F_2(1,T)$) is defined for $x \in \mathbb{A}^1(K)$ by

$$h_{\varphi,v,(\infty)}(x) \ = \ \lim_{n\to\infty} h^{(n)}_{\varphi,v,(\infty)}(x) \ . \tag{10.6}$$

The fact that the limit exists follows from (10.5) using a standard telescoping sum argument, as does the bound

$$|h_{\varphi,v,(\infty)}(x) - \log_v(\max(1,|x|))| \ \le \ \sum_{n=1}^{\infty} \frac{C_1}{d^n} \ = \ \frac{C_1}{d-1} \ . \tag{10.7}$$

We will denote the constant on the right side by C. Taking logarithms in the identity

$$F^{(n-1)}(1,\varphi(x)) \ = \ F^{(n-1)}\left(1, \frac{F_2(1,x)}{F_1(1,x)}\right) \ = \ \frac{F^{(n)}(1,x)}{F_1(1,x)^{d^{n-1}}} \ ,$$

then dividing by d^{n-1} and letting $n \to \infty$, gives the functional equation

$$h_{\varphi,v,(\infty)}(\varphi(x)) \ = \ d \cdot h_{\varphi,v,(\infty)}(x) - \log_v(|F_1(1,x)|) \ , \tag{10.8}$$

valid on $\mathbb{P}^1(K)\backslash(\{\infty\} \cup \varphi^{-1}(\{\infty\}))$.

Properties (10.7) and (10.8) characterize the Call-Silverman local height: in general, given a divisor D on $\mathbb{P}^1(K)$, a function $h_{\varphi,v,D} : \mathbb{P}^1(K)\backslash \operatorname{supp}(D) \to \mathbb{R}$ is a Call-Silverman local height for D if it is a Weil local height associated to D and if there exists a rational function $f \in K(T)$ with $\operatorname{div}(f) = \varphi^*(D) - d \cdot D$, such that

$$h_{\varphi,v,D}(\varphi(z)) \ = \ d \cdot h_{\varphi,v,D}(z) - \log_v(|f(z)|_v)$$

for all $z \in \mathbb{P}^1(K)\backslash(\operatorname{supp}(D) \cup \operatorname{supp}(\varphi^*(D)))$. If f is replaced by cf for some $c \ne 0$, then $h_{\varphi,v,D}$ is changed by an additive constant.

We will now "Berkovich-ize" the local height. For each n and each $x \in \mathbb{P}^1_{\text{Berk}}$, put

$$\hat{h}^{(n)}_{\varphi,v,(\infty)}(x) \ = \ \frac{1}{d^n}\max\Big(\log_v[F_1^{(n)}(1,T)]_x, \log_v[F_2^{(n)}(1,T)]_x\Big) \ . \tag{10.9}$$

(By convention, we set $\hat{h}^{(n)}_{\varphi,v,\infty}(\infty) = \infty$.) Then $\hat{h}^{(n)}_{\varphi,v,(\infty)}$ coincides with $h^{(n)}_{\varphi,v,(\infty)}$ on $\mathbb{A}^1(K)$ and is continuous and strongly subharmonic on $\mathbb{A}^1_{\mathrm{Berk}}$. We claim that for all $x \in \mathbb{A}^1_{\mathrm{Berk}}$,

$$\left|\hat{h}^{(n)}_{\varphi,v,(\infty)}(x) - \hat{h}^{(n-1)}_{\varphi,v,(\infty)}(x)\right| \ \le\ \frac{C_1}{d^n} \ . \tag{10.10}$$

Indeed, this holds for all type I points in $\mathbb{A}^1(K)$; such points are dense in $\mathbb{A}^1_{\mathrm{Berk}}$ and the functions involved are continuous, so it holds for all $x \in \mathbb{A}^1_{\mathrm{Berk}}$.

It follows that the functions $\hat{h}^{(n)}_{\varphi,v,(\infty)}$ converge uniformly to a continuous subharmonic function $\hat{h}_{\varphi,v,(\infty)}$ on $\mathbb{A}^1_{\mathrm{Berk}}$ which extends the Call-Silverman local height $h_{\varphi,v,(\infty)}$. By the same arguments as before, for all $x \in \mathbb{A}^1_{\mathrm{Berk}}$ we have

$$|\hat{h}_{\varphi,v,(\infty)}(x) - \log_v(\max(1,[T]_x))| \ \le\ C \ , \tag{10.11}$$

and for all $x \in \mathbb{P}^1_{\mathrm{Berk}} \backslash (\infty \cup \varphi^{-1}(\infty))$,

$$\hat{h}_{\varphi,v,(\infty)}(\varphi(x)) \ =\ d \cdot \hat{h}_{\varphi,v,(\infty)}(x) - \log_v([F_1(1,T)]_x) \ . \tag{10.12}$$

Actually, (10.12) can be viewed as an identity for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, if at $x = \infty$ one takes the right side to be given by its limit as $x \to \infty$.

Let $V_1 = \mathbb{A}^1_{\mathrm{Berk}} = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\infty\}$. Since $\hat{h}_{\varphi,v,(\infty)}$ is subharmonic on V_1, there is a nonnegative measure μ_1 on V_1 such that $\Delta_{V_1}(\hat{h}_{\varphi,v,(\infty)}) = -\mu_1$. On the other hand, each $\hat{h}^{(n)}_{\varphi,v,(\infty)}$ belongs to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and satisfies

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(\hat{h}^{(n)}_{\varphi,v,(\infty)}) \ =\ \delta_\infty - \mu_1^{(n)} \ ,$$

where $\mu_1^{(n)} \ge 0$ has total mass 1. Since the $\hat{h}^{(n)}_{\varphi,v,(\infty)}$ converge uniformly to $\hat{h}_{\varphi,v,(\infty)}$ on V_1, it follows from Proposition 5.32 that $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(\hat{h}^{(n)}_{\varphi,v,(\infty)}) \to \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(\hat{h}_{\varphi,v,(\infty)})$. From this, one concludes from Proposition 8.51 that the $\mu_1^{(n)}$ converge weakly to μ_1 on simple subdomains of V_1, that $\hat{h}_{\varphi,v,(\infty)}$ belongs to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$, and that there is a nonnegative measure μ_φ on $\mathbb{P}^1_{\mathrm{Berk}}$ of total mass 1 such that

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(\hat{h}_{\varphi,v,(\infty)}) \ =\ \delta_\infty - \mu_\varphi \ . \tag{10.13}$$

(Note that functions in $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ are allowed to take the values $\pm\infty$ on points of $\mathbb{P}^1(K)$; the definition of the Laplacian only involves their restriction to $\mathbb{H}_{\mathrm{Berk}}$.)

In the affine patch $V_2 := \mathbb{P}^1_{\mathrm{Berk}} \backslash \{0\}$, relative to the coordinate function $U = 1/T$, the map φ is given by $F_1(U,1)/F_2(U,1)$. By a construction similar to the one above, using the functions $F_1(U,1)$ and $F_2(U,1)$, we obtain a function $\hat{h}_{\varphi,v,(0)}(x) \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ which is continuous and subharmonic on

V_2 and extends the Call-Silverman local height relative to the point 0. For all $x \in V_2$, it satisfies

$$|\hat{h}_{\varphi,v,(0)}(x) - \log_v(\max(1, [1/T]_x))| \ \le \ C \ , \tag{10.14}$$

$$\hat{h}_{\varphi,v,(0)}(\varphi(x)) \ = \ d \cdot \hat{h}_{\varphi,v,(0)}(x) - \log_v([F_2(1/T,1)]_x) \ , \tag{10.15}$$

where (10.15) holds in the same sense as (10.12). Using $F^{(n)}(U,1) = F^{(n)}(1,T)/T^{d^n}$, taking logarithms, dividing by d^n, and letting $n \to \infty$ gives

$$\hat{h}_{\varphi,v,(0)}(x) \ = \ \hat{h}_{\varphi,v,(\infty)}(x) - \log_v([T]_x) \ . \tag{10.16}$$

Applying the Laplacian and using (10.13) shows that

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(\hat{h}_{\varphi,v,(0)}) \ = \ \delta_0 - \mu_\varphi \ . \tag{10.17}$$

We will refer to the probability measure μ_φ on $\mathbb{P}^1_{\mathrm{Berk}}$ appearing in (10.13) and (10.17) as the *canonical measure* associated to φ.

THEOREM 10.2. *Let $\varphi(T) \in K(T)$ have degree $d \ge 2$. Then the Call-Silverman local height $\hat{h}_{\varphi,v,(\infty)}$ belongs to* $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ *and $\mathcal{SH}(\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\infty\})$, with*

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(\hat{h}_{\varphi,v,(\infty)}) \ = \ \delta_\infty - \mu_\varphi \ . \tag{10.18}$$

The canonical measure μ_φ is nonnegative and has total mass 1*. It satisfies the functional equations*

$$\varphi^*(\mu_\varphi) = d \cdot \mu_\varphi \ , \quad \varphi_*(\mu_\varphi) = \mu_\varphi \ . \tag{10.19}$$

PROOF. The assertions about $\hat{h}_{\varphi,v,(\infty)}$ have been established earlier in this section.

Put $\mu = \mu_\varphi$. To show that $\varphi^*(\mu) = d \cdot \mu$, we use (10.12) and (10.15). Let $x_1, \ldots, x_d$ be the zeros (not necessarily distinct) of $F_1(1,T)$, and put $U_1 = \varphi^{-1}(V_1) = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{x_1, \ldots, x_d\}$. By the pullback formula for subharmonic functions (Proposition 9.54), $H(x) := \hat{h}_{\varphi,v,(\infty)}(\varphi(x))$ is subharmonic on U_1, and

$$\Delta_{U_1}(H) \ = \ \varphi^*(\Delta_{V_1}(\hat{h}_{\varphi,v,(\infty)})) \ .$$

Taking Laplacians in (10.12) gives $\varphi^*(\mu|_{V_1}) \ = \ d \cdot \mu|_{U_1}$. Likewise, put $U_2 = \varphi^{-1}(V_2) = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{y_1, \ldots, y_d\}$, where $y_1, \ldots, y_d$ are the zeros of $F_2(1/T, 1) = F_2(1,T)/T^d$. Since $F_1(1,T)$ and $F_2(1,T)$ are coprime, the sets $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_d\}$ are disjoint. Then $G(x) := \hat{h}_{\varphi,v,(0)}(\varphi(x))$ is subharmonic on U_2 and satisfies

$$\Delta_{U_2}(G) \ = \ \varphi^*(\Delta_{V_2}(\hat{h}_{\varphi,v,(0)})) \ .$$

Taking Laplacians in (10.15) gives $\varphi^*(\mu|_{V_2}) \ = \ d \cdot \mu|_{U_2}$. Since $V_1 \cup V_2 = U_1 \cup U_2 = \mathbb{P}^1_{\mathrm{Berk}}$, it follows that $\varphi^*(\mu) = d \cdot \mu$.

The identity $\varphi_*(\mu) = \mu$ follows formally from $\varphi^*(\mu) = d \cdot \mu$. By Theorem 9.53(B), $\varphi_*(\varphi^*(\mu)) = d \cdot \mu$. Since $\varphi^*(\mu) = d \cdot \mu$, this gives $\varphi_*(\mu) = \mu$. □

We give the following definition (cf. Corollary 9.27):

DEFINITION 10.3. A rational function $\varphi(T) \in K(T)$ of degree $d \geq 1$ has *good reduction* if it can be written as $\varphi(T) = F_2(1,T)/F_1(1,T)$ where $F_1, F_2 \in K[X,Y]$ are homogeneous polynomials of degree d whose coefficients belong to the valuation ring $\mathcal{O}$ of K and where $\mathrm{Res}(F_1, F_2)$ is a unit in $\mathcal{O}$.

A rational function $\varphi(T) \in K(T)$ of degree $d \geq 1$ has *simple reduction*[3] if there is a Möbius transformation $M \in \mathrm{PGL}(2,K)$ such that the conjugate $M \circ \varphi \circ M^{-1}$ has good reduction.

EXAMPLE 10.4. Suppose $\varphi(T)$ has good reduction and degree $d \geq 2$. We claim that the canonical measure μ_φ coincides with the Dirac measure $\delta_{\zeta_{\mathrm{Gauss}}}$, where ζ_{Gauss} is the Gauss point of $\mathbb{P}^1_{\mathrm{Berk}}$.

To see this, recall formula (10.1) and note that under our hypotheses, we may take $B_1 = B_2 = 1$. Thus for all $(x,y) \in K^2$,

$$\|F(x,y)\| \;=\; \|(x,y)\|^d \ .$$

By iteration, for each n, $\|F^{(n)}(x,y)\| = \|(x,y)\|^{d^n}$. Examining the construction of $\hat{h}_{\varphi,v,(\infty)}$, one finds that

$$\hat{h}_{\varphi,v,(\infty)}(x) \;=\; \log_v \max(1, [T]_x) \;=\; \log_v(\delta(x, \zeta_{\mathrm{Gauss}})_\infty) \ ,$$

so that $\Delta(\hat{h}_{\varphi,v,(\infty)}) = \delta_\infty - \delta_{\zeta_{\mathrm{Gauss}}}$ by Example 5.19. Hence $\mu_\varphi = \delta_{\mathrm{Gauss}}$.

More generally, we have:

PROPOSITION 10.5 (Rivera-Letelier[4]). *A rational map $\varphi(T) \in K(T)$ of degree at least 2 has good reduction if and only if $\mu_\varphi = \delta_{\zeta_{\mathrm{Gauss}}}$.*

PROOF. We have already shown in Example 10.4 that if φ has good reduction, then $\mu_\varphi = \delta_{\zeta_{\mathrm{Gauss}}}$. Conversely, suppose that $\mu_\varphi = \delta_{\zeta_{\mathrm{Gauss}}}$. Then using the identities $\varphi_*(\mu_\varphi) = \mu_\varphi$ and $\varphi^*(\mu_\varphi) = \deg(\varphi) \cdot \mu_\varphi$ proved in Theorem 10.2, it follows that $\varphi(\zeta_{\mathrm{Gauss}}) = \zeta_{\mathrm{Gauss}}$ and $m_\varphi(\zeta_{\mathrm{Gauss}}) = \deg(\varphi)$. In particular, $\varphi^{-1}(\zeta_{\mathrm{Gauss}}) = \{\zeta_{\mathrm{Gauss}}\}$. By Corollary 9.27, this means that φ has good reduction. $\square$

REMARK 10.6. Later on (in Proposition 10.45), we will see that $\varphi(T)$ has simple reduction if and only if there is a point ζ (necessarily of type II) such that $\mu_\varphi = \delta_\zeta$. We will also see (in Corollary 10.47) that this is the only case in which μ_φ has point masses: μ_φ can never charge points of $\mathbb{P}^1(K)$, and it cannot charge a point of $\mathbb{H}_{\mathrm{Berk}}$ unless φ has simple reduction.

We will now show that the canonical measure is better behaved than an arbitrary measure, in that it has continuous potentials (Definition 5.40): for each $\zeta \in \mathbb{H}_{\mathrm{Berk}}$, the potential function $u_{\mu_\varphi}(z,\zeta) = \int -\log_v(\delta(z,y)_\zeta)\, d\mu_\varphi(y)$ is continuous (hence bounded).

---

[3]This is Rivera-Letelier's terminology.

[4]See [**46**, Proposition 0.1] and [**48**, Theorem E].

Proposition 10.7. *For any rational function $\varphi(T) \in K(T)$ of degree $d \ge 2$, the canonical measure μ_φ has continuous potentials.*

Proof. Apply Proposition 8.65, noting that in $V_1 = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\infty\}$, the Laplacian of $\hat{h}_{\varphi,v,(\infty)}$ is $\mu_\varphi|_{V_1}$ and that in $V_2 = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{0\}$ the Laplacian of $\hat{h}_{\varphi,v,(0)}$ is $\mu_\varphi|_{V_2}$. □

Since μ_φ has continuous potentials, the theory of Arakelov-Green's functions developed in §8.10 applies to it. For $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$, we define the *normalized Arakelov-Green's function attached to* μ_φ by

$$g_{\mu_\varphi}(x,y) \ = \ \int -\log_v(\delta(x,y)_\zeta)\, d\mu_\varphi(\zeta) + C \ , \tag{10.20}$$

where the constant C is chosen so that

$$\iint g_{\mu_\varphi}(x,y)\, d\mu_\varphi(x) d\mu_\varphi(y) \ = \ 0 \ .$$

(In §10.2, we will determine the constant C explicitly.) By Proposition 8.66, $g_{\mu_\varphi}(x,y)$ is continuous in each variable separately (as a function from $\mathbb{P}^1_{\mathrm{Berk}}$ to $\mathbb{R} \cup \{\infty\}$). As a function of two variables, it is symmetric, bounded below, and continuous off the diagonal and on the type I diagonal, but at points on the diagonal in $\mathbb{H}_{\mathrm{Berk}} \times \mathbb{H}_{\mathrm{Berk}}$ it is only lower semicontinuous.

Given a probability measure ν on $\mathbb{P}^1_{\mathrm{Berk}}$, define the μ_φ-energy integral

$$I_{\mu_\varphi}(\nu) \ = \ \iint g_{\mu_\varphi}(x,y)\, d\nu(x) d\nu(y) \ .$$

Theorem 8.67 yields the following energy minimization principle, which will be used in §10.3 below to prove an equidistribution theorem for algebraic points of small dynamical height:

Theorem 10.8 (μ_φ-Energy Minimization Principle).

(A) $I_{\mu_\varphi}(\nu) \ge 0$ *for each probability measure ν on $\mathbb{P}^1_{\mathrm{Berk}}$.*
(B) $I_{\mu_\varphi}(\nu) = 0$ *if and only if $\nu = \mu_\varphi$.*

Combining Proposition 10.5, Theorem 10.8, and Corollary 9.27, we obtain:

Corollary 10.9. *We have $g_{\mu_\varphi}(\xi,\xi) \ge 0$ for all $\xi \in \mathbb{H}_{\mathrm{Berk}}$, with equality iff $\mu_\varphi = \delta_\xi$. In particular, $g_{\mu_\varphi}(\zeta_{\mathrm{Gauss}}, \zeta_{\mathrm{Gauss}}) \ge 0$, with equality iff φ has good reduction.*

10.2. The Arakelov-Green's function $g_{\mu_\varphi}(x,y)$

Our goal in this section is to provide an explicit formula for the normalized Arakelov-Green's function $g_{\mu_\varphi}(x,y)$: we will show that $g_{\mu_\varphi}(x,y)$ coincides with the function $g_\varphi(x,y)$ given by (10.21) or (10.23) below. We also establish an important functional equation satisfied by $g_{\mu_\varphi}(x,y)$.

We continue the notation from §10.1. In particular, $\varphi(T) \in K(T)$ is a rational function of degree $d \geq 2$, and $F_1(X,Y), F_2(X,Y)$ are homogeneous polynomials of degree d such that $\varphi(T) = F_2(1,T)/F_1(1,T)$ and $\mathrm{Res}(F_1, F_2) \neq 0$. The polynomials F_1, F_2 are determined by φ up to multiplication by a common nonzero scalar $c \in K^\times$.

With the conventions that $-\log_v(\delta(x,x)_\infty) = \infty$ for all $x \in \mathbb{P}^1(K)$ and that $\hat{h}_{\varphi,v,(\infty)}(\infty) = \infty$, we make the following definition:

For $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$, let $R = |\,\mathrm{Res}(F_1,F_2)|^{-\frac{1}{d(d-1)}}$ and define

$$(10.21)\quad g_\varphi(x,y) \;=\; \begin{cases} -\log_v(\delta(x,y)_\infty) + \hat{h}_{\varphi,v,(\infty)}(x) & \\ \qquad +\hat{h}_{\varphi,v,(\infty)}(y) + \log_v R & \text{if } x,y \neq \infty \;, \\ \hat{h}_{\varphi,v,(\infty)}(x) + \hat{h}_{\varphi,v,(0)}(\infty) + \log_v R & \text{if } y = \infty \;, \\ \hat{h}_{\varphi,v,(\infty)}(y) + \hat{h}_{\varphi,v,(0)}(\infty) + \log_v R & \text{if } x = \infty \;. \end{cases}$$

For any fixed $y \neq \infty$, taking $a = 0$ in Corollary 4.2 and using Proposition 4.1(C) with $z = 0$ shows that $\delta(x,y)_\infty = \log_v([T]_x)$ for all x sufficiently near ∞. Hence by (10.16)

$$(10.22)\qquad \lim_{x\to\infty} \hat{h}_{\varphi,v,(\infty)}(x) - \log_v(\delta(x,y)_\infty) \;=\; \hat{h}_{\varphi,v,(0)}(\infty)\;.$$

From this, it follows that for each fixed $y \in \mathbb{P}^1_{\mathrm{Berk}}$, $g_\varphi(x,y)$ is continuous and belongs to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ as a function of x. It is also clear from the definition that $g_\varphi(x,y)$ is symmetric in x and y.

As we will see in Lemma 10.10, the reason for the choice of the normalizing constant $\log_v R$ in (10.21) is to make g_φ independent of the lift $F = (F_1, F_2)$ of φ. This has the magical consequence that the functional equation in Theorem 10.18 holds and that in (10.25) the normalizing constant C is equal to 0.

It is possible to give a more elegant description of $g_\varphi(x,y)$ when $x, y \in \mathbb{P}^1(K)$. To do this, given $\tilde{x} = (x_1, x_2),\ \tilde{y} = (y_1, y_2) \in K^2$, define $\tilde{x} \wedge \tilde{y}$ to be

$$\tilde{x} \wedge \tilde{y} \;=\; x_1 y_2 - x_2 y_1\;,$$

and let $\|\tilde{x}\| = \max(|x_1|, |x_2|)$.

Recall that $F = (F_1, F_2) : K^2 \to K^2$ is a lifting of φ to K^2 and that we defined $F^{(n)} : K^2 \to K^2$ to be the n^{th} iterate of F. For $\tilde{z} \in K^2\backslash\{0\}$, define the *homogeneous dynamical height* $H_F : K^2\backslash\{0\} \to \mathbb{R}$ by

$$H_F(\tilde{z}) \;:=\; \lim_{n\to\infty} \frac{1}{d^n} \log_v \|F^{(n)}(\tilde{z})\|\;.$$

(By convention, we put $H_F(0,0) := -\infty$.)

By a standard telescoping series argument (see [**9**]), it follows that the limit $\lim_{n\to\infty} \frac{1}{d^n} \log_v \|F^{(n)}(\tilde{z})\|$ exists for all $\tilde{z} \in K^2\backslash\{0\}$ and that the sequence $\frac{1}{d^n} \log_v \|F^{(n)}(\tilde{z})\|$ converges uniformly on $K^2\backslash\{0\}$ to $H_F(\tilde{z})$. The definition of H_F is independent of the norm used to define it; this follows easily from the equivalence of norms on K^2. Moreover, it is easy to check that $H_F(F(z)) = dH_F(z)$ for all $z \in K^2\backslash\{0\}$.

LEMMA 10.10. *Let $x, y \in \mathbb{P}^1(K)$, and let $\tilde{x}, \tilde{y}$ be arbitrary lifts of x, y to $K^2\backslash\{0\}$. Then*

$$g_\varphi(x, y) \;=\; -\log_v |\tilde{x} \wedge \tilde{y}| + H_F(\tilde{x}) + H_F(\tilde{y}) + \log_v R \;. \tag{10.23}$$

Moreover, the right-hand side of (10.23) *is independent of the choice of the lifts F_1, F_2 of F.*

PROOF. This is a straightforward computation (see [**9**, §3.4]). □

We will now show that $g_\varphi(x, y) = g_{\mu_\varphi}(x, y)$ for all $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$. The proof is based on the following observation.

LEMMA 10.11. *For each $y \in \mathbb{P}^1_{\mathrm{Berk}}$, we have $\Delta_x g_\varphi(x, y) = \delta_y - \mu_\varphi$.*

PROOF. This follows from the definition of $g_\varphi(x, y)$, together with the identities $\Delta_x(-\log_v(\delta(x, y)_\infty)) = \delta_y - \delta_\infty$ and $\Delta_x(\hat{h}_{\varphi,v,(\infty)}) = \delta_\infty - \mu_\varphi$ (see Theorem 10.2). □

It follows from Proposition 8.66 and Lemma 10.11 that for each $y \in \mathbb{P}^1_{\mathrm{Berk}}$,

$$\Delta_x(g_\varphi(x, y)) \;=\; \delta_y - \mu_\varphi \;=\; \Delta_x(g_{\mu_\varphi}(x, y)) \;.$$

By Proposition 5.28, there is a constant $C(y) \in \mathbb{R}$ such that

$$g_\varphi(x, y) \;=\; g_{\mu_\varphi}(x, y) + C(y) \tag{10.24}$$

for all $x \in \mathbb{H}_{\mathrm{Berk}}$. The continuity of $g_\varphi(x, y)$ and $g_{\mu_\varphi}(x, y)$ in x show that (10.24) in fact holds for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$.

Now fix x, and note that by symmetry, $g_\varphi(x, y)$ and $g_{\mu_\varphi}(x, y)$ are continuous and belong to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ as functions of y. Hence $C(y)$ is continuous and $C(y) \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$. Taking Laplacians with respect to y of both sides of (10.24), we find that $\Delta_y(C(y)) = 0$. By the same argument as before, $C(y) = C$ is constant on $\mathbb{P}^1_{\mathrm{Berk}}$, so

$$g_\varphi(x, y) \;=\; g_{\mu_\varphi}(x, y) + C \tag{10.25}$$

for all $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$.

We claim that $C = 0$. To see this, it suffices to show that for some y,

$$\int g_\varphi(x, y)\, d\mu_\varphi(x) \;=\; \int g_{\mu_\varphi}(x, y)\, d\mu_\varphi(x) \;=\; 0 \;.$$

Below, we will show that both integrals are 0 for each y.

For $g_{\mu_\varphi}(x, y)$, this follows easily from known facts:

PROPOSITION 10.12. *For each $y \in \mathbb{P}^1_{\mathrm{Berk}}$, we have*

$$\int_{\mathbb{P}^1_{\mathrm{Berk}}} g_{\mu_\varphi}(x, y)\, d\mu_\varphi(x) \;=\; 0 \;.$$

PROOF. By Proposition 8.68, the potential function

$$u_{\mu_\varphi}(y, \mu_\varphi) \;=\; \int g_{\mu_\varphi}(x, y)\, d\mu_\varphi(x)$$

has Laplacian equal to $\mu_\varphi - \mu_\varphi = 0$. By continuity, $\int g_{\mu_\varphi}(x,y)\,d\mu_\varphi(x)$ is a constant independent of y. Integrating against μ_φ with respect to y and using the normalization in the definition of $g_{\mu_\varphi}(x,y)$ shows that this constant is 0. □

It remains to show that for each fixed $y \in \mathbb{P}^1_{\mathrm{Berk}}$,

$$\int_{\mathbb{P}^1_{\mathrm{Berk}}} g_\varphi(x,y)\,d\mu_\varphi(x) \;=\; 0 \;.$$

We will prove this in Corollary 10.20 after first establishing a useful functional equation for $g_\varphi(x,y)$ (Theorem 10.18 below). For this we need several preliminary results.

LEMMA 10.13. *If $M \in \mathrm{GL}_2(K)$, let $F' = M^{-1} \circ F \circ M$, and let $\varphi' = M^{-1} \circ \varphi \circ M$. Then for all $z, w \in K^2\backslash\{0\}$,*

$$H_F(M(z)) \;=\; H_{F'}(z) \tag{10.26}$$

and

$$g_\varphi(M(z), M(w)) \;=\; g_{\varphi'}(z,w) \;. \tag{10.27}$$

PROOF. First of all note that, given M, there exist constants $C_1, C_2 > 0$ such that

$$C_1\|z\| \;\le\; \|M(z)\| \;\le\; C_2\|z\| \;. \tag{10.28}$$

Indeed, if we take C_2 to be the maximum of the absolute values of the entries of M, then clearly $\|M(z)\| \le C_2\|z\|$ for all z. By the same reasoning, if we let C_1^{-1} be the maximum of the absolute values of the entries of M^{-1}, then

$$\|M^{-1}(M(z))\| \;\le\; C_1^{-1}\|M(z)\| \;,$$

which gives the other inequality. (Alternatively, one can use the argument from Lemma 10.1.)

By the definition of H_F, we have

$$\begin{aligned} H_F(M(z)) \;&=\; \lim_{n\to\infty} \frac{1}{d^n} \log_v \|F^{(n)}(M(z))\| \\ &=\; \lim_{n\to\infty} \frac{1}{d^n} \log_v \|MF'^{(n)}(z)\| \;=\; H_{F'}(z) \;, \end{aligned}$$

where the last equality follows from (10.28). This proves (10.26).

Since $|M(z) \wedge M(w)| = |\det(M)| \cdot |z \wedge w|$, we have

$$-\log_v |M(z) \wedge M(w)| \;=\; -\log_v |\det(M)| - \log_v |z \wedge w| \;. \tag{10.29}$$

On the other hand, by (10.31) below and the fact that $|\mathrm{Res}(M)| = |\det(M)|$, it follows that

$$-\frac{1}{d(d-1)} \log_v |\mathrm{Res}(F')| \;=\; -\frac{1}{d(d-1)} \log_v |\mathrm{Res}(F)| - \log_v |\det(M)| \;. \tag{10.30}$$

Combining (10.26), (10.29), and (10.30) gives (10.27). □

Corollary 10.14. *Let $M \in \mathrm{PGL}_2(K)$ be a Möbius transformation. Then for all $z, w \in \mathbb{P}^1(K)$, we have*

$$g_\varphi(M(z), M(w)) \;=\; g_{M^{-1}\circ\varphi\circ M}(z,w)\ .$$

Corollary 10.15. *Let $M \in \mathrm{PGL}_2(K)$ be a Möbius transformation. Then $\mu_{M^{-1}\circ\varphi\circ M} = M^*\mu_\varphi$.*

Proof. This follows from Corollary 10.14, together with Lemma 10.11 and Proposition 9.56. □

Let k be a field. By [**71**, Theorem IX.3.13], if $F = (F_1, F_2)$ and $G = (G_1, G_2)$ where $F_1, F_2 \in k[X,Y]$ are homogeneous of degree d and $G_1, G_2 \in k[X,Y]$ are homogeneous of degree e, then

$$\mathrm{Res}(F\circ G) \;=\; \mathrm{Res}(F)^e\,\mathrm{Res}(G)^{d^2}. \tag{10.31}$$

In particular, it follows by induction on n that

$$\mathrm{Res}(F^{(n)}) \;=\; \mathrm{Res}(F)^{\frac{d^{n-1}(d^n-1)}{d-1}}\ . \tag{10.32}$$

Lemma 10.16. *For every positive integer n, we have $\mu_\varphi = \mu_{\varphi^{(n)}}$ and*

$$g_\varphi(z,w) \;=\; g_{\varphi^{(n)}}(z,w)$$

for all $z, w \in \mathbb{P}^1_{\mathrm{Berk}}$.

Proof. The second assertion follows easily from the definition of $g_\varphi(z,w)$ using the fact that

$$|\,\mathrm{Res}(F)|^{-\frac{1}{d(d-1)}} \;=\; |\,\mathrm{Res}(F^{(n)})|^{-\frac{1}{d^n(d^n-1)}}$$

for any homogeneous lifting F of φ, by (10.32). The first assertion follows from this by Lemma 10.11. □

As a consequence of Lemma 10.16 and Proposition 10.5, we obtain the following fact, originally proved by Benedetto [**14**] by a different method (see also [**81**, §7]):

Corollary 10.17. *For a rational map $\varphi \in K(T)$ of degree at least 2, the following are equivalent:*

(A) *φ has good reduction.*
(B) *$\varphi^{(n)}$ has good reduction for some integer $n \geq 2$.*
(C) *$\varphi^{(n)}$ has good reduction for every integer $n \geq 2$.*

For each $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$, define

$$g_\varphi(x, \varphi^*(y)) \;:=\; \sum_{i=1}^{d} g_\varphi(x, y_i)\ , \tag{10.33}$$

where $y_1, \ldots, y_d$ are the preimages of y under φ, counting multiplicities.

The function $g_\varphi(x,y)$ satisfies the following functional equation:

THEOREM 10.18. *Let $\varphi(T) \in K(T)$ be a rational function of degree $d \ge 2$. Then for all $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$,*

$$g_\varphi(\varphi(x), y) \ = \ g_\varphi(x, \varphi^*(y)) \ . \tag{10.34}$$

We first establish this in the special case where both $x, y \in \mathbb{P}^1(K)$:

LEMMA 10.19. *For all $x, y \in \mathbb{P}^1(K)$, we have*

$$g_\varphi(\varphi(x), y) \ = \ g_\varphi(x, \varphi^*(y)) \ .$$

PROOF. Using Corollary 10.14, we may assume without loss of generality that $y = \infty$. Let F be a homogeneous lifting of φ, and let $R = |\operatorname{Res}(F)|^{-\frac{1}{d(d-1)}}$. For $\tilde{z} \in K^2 \backslash \{0\}$, let $[\tilde{z}]$ denote the class of $\tilde{z}$ in $\mathbb{P}^1(K)$. If $\tilde{w} = (1, 0)$, then $[\tilde{w}] = \infty$. Let $w_1, \ldots, w_d$ be the preimages of ∞ under φ (counting multiplicities), and let $\tilde{w}_1, \ldots, \tilde{w}_d$ be any solutions to $F(\tilde{w}_i) = \tilde{w}$ with $[\tilde{w}_i] = w_i$.

It suffices to show that for all $\tilde{z} \in K^2 \backslash \{0\}$, we have

$$g_\varphi(\varphi([\tilde{z}]), [\tilde{w}]) \ = \ \sum_{i=1}^{d} g_\varphi([\tilde{z}], [\tilde{w}_i]) \ . \tag{10.35}$$

Since $H_F(F(\tilde{z})) = dH_F(\tilde{z})$ and $H_F(\tilde{w}_i) = \frac{1}{d} H_F(\tilde{w})$ for all i, (10.35) is equivalent to

$$-\log_v |F(\tilde{z}) \wedge \tilde{w}| \ = \ (d-1)\log_v R - \sum_{i=1}^{d} \log_v |\tilde{z} \wedge \tilde{w}_i| \ ,$$

which itself is equivalent to

$$|F(\tilde{z}) \wedge \tilde{w}| \ = \ |\operatorname{Res}(F)|^{1/d} \prod_{i=1}^{d} |\tilde{z} \wedge \tilde{w}_i| \ . \tag{10.36}$$

We verify (10.36) by an explicit calculation (compare with [**41**, Lemma 6.5]). Write $F_1(\tilde{z}) = \prod_{i=1}^d \tilde{z} \wedge \tilde{a}_i$, $F_2(\tilde{z}) = \prod_{i=1}^d \tilde{z} \wedge \tilde{b}_i$ with $\tilde{a}_i, \tilde{b}_i \in K^2$. Since $F(\tilde{w}_j) = \tilde{w} = (1, 0)$, it follows that $\prod_{i=1}^d \tilde{w}_j \wedge \tilde{a}_i = 1$, $\prod_{i=1}^d \tilde{w}_i \wedge \tilde{b}_i = 0$. Thus for each $j = 1, \ldots, d$, we can assume $\tilde{w}_j$ has been chosen so that

$$\tilde{w}_j \ = \ \frac{\tilde{b}_j}{(\prod_i \tilde{a}_i \wedge \tilde{b}_j)^{1/d}} \ ,$$

where for each j we fix some d^{th} root of $\prod_i \tilde{a}_i \wedge \tilde{b}_j$. Note that $|\tilde{w}_j|$ is independent of which d^{th} root we pick. It follows that

$$\prod_{j=1}^{d} |\tilde{z} \wedge \tilde{w}_j| \ = \ \frac{\prod_j |\tilde{z} \wedge \tilde{b}_j|}{\prod_{i,j} |\tilde{a}_i \wedge \tilde{b}_j|^{1/d}} \ = \ \frac{|F(\tilde{z}) \wedge \tilde{w}|}{|\operatorname{Res}(F)|^{1/d}} \ ,$$

which gives (10.36). □

Recall that by Proposition 9.13, for any continuous function $f : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R}$, the function $\varphi_*(f)$ defined by

$$\varphi_*(f)(y) \;=\; \sum_{\varphi(x)=y} m_\varphi(x) f(x) \tag{10.37}$$

is itself continuous. (Here $m_\varphi(x)$ denotes the analytic multiplicity of φ at x, as defined in §9.1.)

Using this, we can show that Theorem 10.18 holds for all $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$:

PROOF OF THEOREM 10.18. First, fix $y \in \mathbb{P}^1(K)$. Then as functions of x, $g_\varphi(\varphi(x), y)$ and $g_\varphi(x, \varphi^*(y))$ are continuous from $\mathbb{P}^1_{\mathrm{Berk}}$ to the extended reals. By Lemma 10.19, they agree on the dense subset $\mathbb{P}^1(K)$ of $\mathbb{P}^1_{\mathrm{Berk}}$, so (10.34) holds for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$, $y \in \mathbb{P}^1(K)$.

Next, fix $x \in \mathbb{H}^1_{\mathrm{Berk}}$. Then as a function of y, $g_\varphi(x, y)$ is bounded and continuous, and $g_\varphi(x, \varphi^*(y))$ coincides with the pushforward of the function $f(y) = g_\varphi(x, y)$ as in (10.37). By Proposition 9.13 it is continuous for all $y \in \mathbb{P}^1_{\mathrm{Berk}}$. Thus both $g_\varphi(\varphi(x), y)$ and $g_\varphi(x, \varphi^*(y))$ are continuous and real-valued for all $y \in \mathbb{P}^1_{\mathrm{Berk}}$. By what has been shown above, they agree for $y \in \mathbb{P}^1(K)$, so (10.34) holds for all $x \in \mathbb{H}_{\mathrm{Berk}}$ and $y \in \mathbb{P}^1_{\mathrm{Berk}}$.

Finally, let $y \in \mathbb{H}_{\mathrm{Berk}}$. Then $g_\varphi(\varphi(x), y)$ and $g_\varphi(x, \varphi^*(y))$ are continuous and real-valued functions of $x \in \mathbb{P}^1_{\mathrm{Berk}}$, and they agree on the dense subset $\mathbb{H}_{\mathrm{Berk}}$, so they agree for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$.

The cases above cover all possibilities for $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$. □

Using Theorem 10.18, we deduce the following:

COROLLARY 10.20. *For all $y \in \mathbb{P}^1_{\mathrm{Berk}}$, we have*

$$\int_{\mathbb{P}^1_{\mathrm{Berk}}} g_\varphi(x, y)\, d\mu_\varphi(x) \;=\; 0\ . \tag{10.38}$$

PROOF. By (10.25) and Proposition 10.12, there is a constant C such that $\int g_\varphi(x, y)\, d\mu_\varphi(x) = C$ for each $y \in \mathbb{P}^1_{\mathrm{Berk}}$, and we need to show that $C = 0$.

Fix $y \in \mathbb{P}^1_{\mathrm{Berk}}$. Since $\varphi_*\mu_\varphi = \mu_\varphi$, it follows from Theorems 10.2 and 10.18 and formula (9.44) that

$$\begin{aligned} C \;&=\; \int g_\varphi(x, y)\, d\mu_\varphi(x) \;=\; \int g_\varphi(x, y)\, d(\varphi_*\mu_\varphi)(x) \\ &=\; \int g_\varphi(\varphi(x), y)\, d\mu_\varphi(x) \;=\; \int g_\varphi(x, \varphi^*(y))\, d\mu_\varphi(x) \;=\; d \cdot C\ , \end{aligned}$$

and thus $C = 0$ as desired. □

From (10.25), using Proposition 10.12 and Corollary 10.20, we finally conclude:

THEOREM 10.21. $g_\varphi(x, y) = g_{\mu_\varphi}(x, y)$ *for all* $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$.

We close this section by remarking that Theorem 10.18 shows that $g_\varphi(x,y)$ is a one-parameter family of Call-Silverman local heights, normalized by the condition (10.38). Indeed, for each fixed $y \in \mathbb{P}^1(K)$,

$$\Delta_x\big(g_\varphi(x,y) - \log_v(\|x,y\|)\big) \ = \ \delta_{\zeta_{\text{Gauss}}} - \mu_\varphi \ = \ \Delta_x(g_\varphi(x,\zeta_{\text{Gauss}})) \ ,$$

so since $g_\varphi(x,y) - \log_v(\|x,y\|)$ and $g_\varphi(x,\zeta_{\text{Gauss}})$ both have continuous extensions to $\mathbb{P}^1_{\text{Berk}}$, there is a constant $C(y)$ such that

$$g_\varphi(x,y) - \log_v(\|x,y\|) \ = \ g_\varphi(x,\zeta_{\text{Gauss}}) + C(y) \ .$$

Here $g_\varphi(x,\zeta_{\text{Gauss}})$ is continuous and bounded since μ_φ has continuous potentials (Proposition 10.7). Hence $|g_\varphi(x,y) - \log_v(\|x,y\|)|$ is bounded, and $g_\varphi(x,y)$ is a Weil height with respect to the divisor (y). Furthermore, if $f_y(T) \in K(T)$ is a function with $\text{div}(f_y) = d(y) - \varphi^*((y))$, then

$$\begin{aligned} \Delta_x(g_\varphi(\varphi(x),y)) \ &= \ \Delta_x(g_\varphi(x,\varphi^*(y))) \ = \ \sum_{\varphi(y_i)=y} m_\varphi(y_i)\delta_{y_i}(x) - d\cdot\mu_\varphi \\ &= \ \Delta_x(d\cdot g_\varphi(x,y) - \log_v([f_y]_x)) \ . \end{aligned}$$

After replacing f_y by a constant multiple, if necessary, we obtain

$$g_\varphi(\varphi(x),y) \ = \ d\cdot g_\varphi(x,y) - \log_v([f_y]_x) \ .$$

This holds for all $x \in \mathbb{P}^1_{\text{Berk}} \backslash (\text{supp}(D) \cup \text{supp}(\varphi^*(D)))$.

Thus $g_\varphi(x,y)$ satisfies both conditions in the definition of a Call-Silverman local height for (y).

10.3. Adelic equidistribution of dynamically small points[5]

Let k be a number field. In this section, we prove an adelic equidistribution theorem (Theorem 10.24) for sequences of distinct points $P_n \in \mathbb{P}^1(\overline{k})$ for which $\widehat{h}_\varphi(P_n) \to 0$, where $\widehat{h}_\varphi$ is the canonical height associated to a rational function $\varphi(T) \in k(T)$. This result is due (independently) to Baker and Rumely [**9**], to Chambert-Loir [**35**], and to Favre and Rivera-Letelier [**47**]. Here we follow the proof from [**9**], with simplifications. Since it is not much more difficult to prove, we actually formulate our equidistribution theorem in a more general context, with Galois invariant subsets instead of Galois orbits of points and with k an arbitrary product formula field, of any characteristic. Our exposition has been influenced by the treatment of this material in [**34**] and [**47**]. We quote without proof the necessary results from potential theory over $\mathbb{C}$ in the Archimedean case, but otherwise we provide a complete and self-contained argument.

Our proof of the adelic equidistribution theorem depends on (part (A) of) the following purely local result:

[5]Parts of §10.3 are taken from “A lower bound for average values of dynamical Green’s functions”, Math. Res. Lett. 13(2):245–257, 2006. Copyright International Press, 2006. Used with permission.

THEOREM 10.22. *Let K be a field with a nontrivial (Archimedean or non-Archimedean) absolute value. Let $\varphi \in K(T)$ be a rational function of degree $d \geq 2$, and let g_φ be the normalized Arakelov-Green's function associated to φ. For $z_1, \ldots, z_N$ in $\mathbb{P}^1(K)$ $(N \geq 2)$, define the* discrepancy of $z_1, \ldots, z_N$ relative to φ *to be*

$$D_\varphi(z_1, \ldots, z_N) \; = \; \frac{1}{N(N-1)} \sum_{\substack{1 \leq i,j \leq N \\ i \neq j}} g_\varphi(z_i, z_j) \; .$$

(By convention, we set $D_\varphi(z_1, \ldots, z_N) = +\infty$ if $z_i = z_j$ for some $i \neq j$.) Then:

(A) *The sequence $D_N := \inf_{z_1,\ldots,z_N \in \mathbb{P}^1(K)} D_\varphi(z_1, \ldots, z_N)$ is nondecreasing, and $\lim_{N\to\infty} D_N \geq 0$.*

(B) *If K is complete and algebraically closed, then*

$$\lim_{N\to\infty} D_N = 0 \; .$$

We will prove Theorem 10.22 at the end of this section. Following [**3**], we will also prove the following quantitative version of the nonnegativity assertion in Theorem 10.22(A).

THEOREM 10.23. *Let K be a field with a nontrivial (Archimedean or non-Archimedean) absolute value, and let $\varphi \in K(T)$ be a rational function of degree $d \geq 2$. Then there is an effective constant $C > 0$, depending only on φ and K, such that*

$$D_\varphi(z_1, \ldots, z_N) \; \geq \; -C \frac{\log N}{N-1} \tag{10.39}$$

for all $z_1, \ldots, z_N \in \mathbb{P}^1(K)$ with $N \geq 2$.

Theorem 10.23 is a dynamical analogue of classical results due to Mahler and Elkies and can be used to make the equidistribution in Theorem 10.24 "quantitative".

In order to state the adelic equidistribution theorem for points of small dynamical height, we first need some definitions and notation. Let k be a product formula field (Definition 7.51), and let $\overline{k}$ (resp. k^{sep}) denote a fixed algebraic closure (resp. separable closure) of k. For $v \in \mathcal{M}_k$, let k_v be the completion of k at v, let $\overline{k}_v$ be an algebraic closure of k_v, and let $\mathbb{C}_v$ denote the completion of $\overline{k}_v$. For each $v \in \mathcal{M}_k$, we fix an embedding of $\overline{k}$ in $\mathbb{C}_v$ extending the canonical embedding of k in k_v and view this embedding as an identification. As discussed in §7.9, if v is Archimedean, then $\mathbb{C}_v \cong \mathbb{C}$.

Let $\varphi \in k(T)$ be a rational map of degree $d \geq 2$ defined over k, so that φ acts on $\mathbb{P}^1(\overline{k})$ in the usual way. As before, we define a *homogeneous lifting* of φ to be a choice of homogeneous polynomials $F_1, F_2 \in k[X,Y]$ of degree $d \geq 2$ having no common linear factor in $\overline{k}[X,Y]$ such that

$$\varphi([z_0 : z_1]) \; = \; [F_1(z_0, z_1) : F_2(z_0, z_1)]$$

for all $[z_0 : z_1] \in \mathbb{P}^1(\overline{k})$. The polynomials F_1, F_2 are uniquely determined by φ up to multiplication by a common scalar $c \in k^\times$, and the condition that F_1, F_2 have no common linear factor in $\overline{k}[X, Y]$ is equivalent to saying that the homogeneous resultant $\mathrm{Res}(F) = \mathrm{Res}(F_1, F_2) \in k$ is nonzero.

The mapping

$$F \;=\; (F_1, F_2) : \overline{k}^2 \;\to\; \overline{k}^2$$

is a lifting of φ to $\overline{k}^2$, and we denote by $F^{(n)} : \overline{k}^2 \to \overline{k}^2$ the iterated map $F \circ F \circ \ldots \circ F$ (n times).

The global canonical height function $\hat{H}_F$ is defined for $P \in k^2 \setminus \{0\}$ by

$$\widehat{H}_F(P) \;=\; \lim_{n\to\infty} \frac{1}{d^n} \sum_{v \in \mathcal{M}_k} N_v \log \|F^{(n)}(P)\|_v \ ,$$

where $\|(x, y)\|_v = \max\{|x|_v, |y|_v\}$. Equivalently, in the terminology of §10.2, we have

$$\widehat{H}_F(P) \;=\; \sum_{v \in \mathcal{M}_k} N_v H_{F,v}(P) \ ,$$

where $H_{F,v} : \mathbb{C}_v^2 \setminus \{0\} \to \mathbb{R}$ is the homogeneous dynamical height function associated to F, relative to the ground field $K = \mathbb{C}_v$. By convention, we set $\hat{H}_F(0) = 0$.

One sees easily from the product formula that $\hat{H}_F(P) \geq 0$ for all P, that $\hat{H}_F(P)$ depends only on the class of P in $\mathbb{P}^1(k)$, and that $\hat{H}_{cF} = \hat{H}_F$ for $c \in k^\times$. Thus $\widehat{H}_F$ descends to a well-defined global canonical height function $\widehat{h}_\varphi : \mathbb{P}^1(k) \to \mathbb{R}_{\geq 0}$ depending only on φ.

More generally, if $S \subset \mathbb{P}^1(k^{\mathrm{sep}})$ is any finite set which is invariant under $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, we define $\widehat{h}_\varphi(S)$ by

$$\widehat{h}_\varphi(S) \;=\; \sum_{v \in \mathcal{M}_k} N_v \left(\frac{1}{|S|} \sum_{P \in S} H_{F,v}(\tilde{P}) \right) \ , \tag{10.40}$$

where $\tilde{P}$ is an arbitrary lifting of P to $\overline{k}^2 \setminus \{0\}$. Using Galois-invariance, one sees easily that $\sum_{P \in S} H_{F,v}(\tilde{P})$ does not depend on the choice of an embedding of $\overline{k}$ into $\mathbb{C}_v$. By the product formula, $\widehat{h}_\varphi(S)$ does not depend on the choice of $\tilde{P}$ in (10.40).

If $P \in \mathbb{P}^1(k^{\mathrm{sep}})$, let $S_k(P) = \{P_1, \ldots, P_n\}$ denote the set of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-conjugates of P over k, where $n = [k(P) : k]$. We define $\widehat{h}_\varphi(P) = \widehat{h}_\varphi(S_k(P))$. This extends $\widehat{h}_\varphi : \mathbb{P}^1(k) \to \mathbb{R}_{\geq 0}$ in a canonical way to a function $\widehat{h}_\varphi : \mathbb{P}^1(k^{\mathrm{sep}}) \to \mathbb{R}_{\geq 0}$.

For all $P \in \mathbb{P}^1(k^{\mathrm{sep}})$, $\widehat{h}_\varphi$ satisfies the functional equation

$$\widehat{h}_\varphi(\varphi(P)) \;=\; d\,\widehat{h}_\varphi(P) \ , \tag{10.41}$$

which is easily verified using the fact that $H_{F,v} \circ F = d\, H_{F,v}$ for each $v \in \mathcal{M}_k$.

If $h : \mathbb{P}^1(\overline{k}) \to \mathbb{R}_{\geq 0}$ denotes the standard logarithmic Weil height on $\mathbb{P}^1(\overline{k})$, defined for $P = [x : y] \in \mathbb{P}^1(k)$ by

$$h(P) \ = \ \sum_{v \in \mathcal{M}_k} N_v \log \max\{|x|_v, |y|_v\}$$

and extended to $\mathbb{P}^1(k^{\rm sep})$ as above, then there is a constant $C > 0$ such that

$$(10.42) \qquad |\widehat{h}_\varphi(P) - h(P)| \ \leq \ C \quad \text{ for all } P \in \mathbb{P}^1(k^{\rm sep}) \ .$$

This follows from the easily verified fact that

$$\widehat{h}_\varphi(P) \ = \ \lim_{n \to \infty} \frac{1}{d^n} h(\varphi^{(n)}(P))$$

for all $P \in \mathbb{P}^1(k^{\rm sep})$. If k is a global field (i.e., a number field or a function field in one variable over a finite field), then from the corresponding fact for the standard logarithmic Weil height h, it follows from (10.42) that $\widehat{h}_\varphi$ satisfies the *Northcott finiteness property*: For any real number $M > 0$ and any integer $D \geq 1$, the set

$$(10.43) \qquad \{P \in \mathbb{P}^1(k^{\rm sep}) \ : \ [k(P) : k] \leq D, \ \widehat{h}_\varphi(P) \leq M\}$$

is finite (see [**93**, Theorem 3.12]).

It is easy to see using (10.41) and the Northcott finiteness property that if k is a global field and $P \in \mathbb{P}^1(k)$, then $\widehat{h}_\varphi(P) = 0$ if and only if P is *preperiodic* for φ (that is, the orbit of P under iteration of φ is finite).

For $v \in \mathcal{M}_k$, we let $\mathbb{P}^1_{\rm Berk,\it v}$ denote the Berkovich projective line over $\mathbb{C}_v$ (which we take to mean $\mathbb{P}^1(\mathbb{C})$ if v is Archimedean). We let $\mu_{\varphi,v}$ be the canonical measure on $\mathbb{P}^1_{\rm Berk,\it v}$ associated to the iteration of φ over $\mathbb{C}_v$, as defined in §10.1. Finally, we denote by $g_{\varphi,v}$ the corresponding normalized Arakelov-Green's function on $\mathbb{P}^1_{\rm Berk,\it v} \times \mathbb{P}^1_{\rm Berk,\it v}$. Our main result is:

Theorem 10.24 (Adelic Equidistribution of Small Points). *Let k be a product formula field, and let $\varphi \in k(T)$ be a rational function of degree $d \geq 2$. Suppose S_n is a sequence of* $\mathrm{Gal}(k^{\rm sep}/k)$*-invariant finite subsets of* $\mathbb{P}^1(k^{\rm sep})$ *with $|S_n| \to \infty$ and $\widehat{h}_\varphi(S_n) \to 0$. Fix $v \in \mathcal{M}_k$, and for each n let δ_n be the discrete probability measure on $\mathbb{P}^1_{\rm Berk,\it v}$ supported equally on the elements of S_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to the canonical measure $\mu_{\varphi,v}$ on $\mathbb{P}^1_{\rm Berk,\it v}$ associated to φ.*

For concreteness, we note the special case of Theorem 10.24 where k is a number field and S_n is the set of Galois conjugates of a point $P_n \in \mathbb{P}^1(\overline{k})$:

Corollary 10.25. *Let k be a number field, and let $\varphi \in k(T)$ be a rational function of degree $d \geq 2$. Suppose $\{P_n\}$ is a sequence of distinct points of $\mathbb{P}^1(\overline{k})$ with $\widehat{h}_\varphi(P_n) \to 0$. Fix a place v of k, and for each n let δ_n be the discrete probability measure on $\mathbb{P}^1_{\rm Berk,\it v}$ supported equally on the* $\mathrm{Gal}(\overline{k}/k)$*-conjugates of P_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to the canonical measure $\mu_{\varphi,v}$ on $\mathbb{P}^1_{\rm Berk,\it v}$ associated to φ.*

Note that if S_n is the set of $\mathrm{Gal}(\overline{k}/k)$-conjugates of P_n in the statement of Corollary 10.25, then $|S_n| \to \infty$ by the Northcott finiteness property (10.43). Thus Corollary 10.25 is a special case of Theorem 10.24.

REMARK 10.26. If $\varphi(T)$ is a polynomial, then Theorem 10.24 is a special case of Theorem 7.52. Indeed, for $v \in M_k$ let E_v be the Berkovich filled Julia set of $\varphi(T)$ over $\mathbb{C}_v$ (Definition 10.90), and let $\mathbb{E}$ be the corresponding compact Berkovich adelic set. Then by Theorem 10.91 below, $\widehat{h}_\varphi = h_{\mathbb{E}}$ and $\mu_{\varphi,v}$ coincides with the equilibrium distribution μ_v for E_v.

Before giving the proof of Theorem 10.24, we need some preliminary results. Let $S = \{P_1, \ldots, P_n\}$ be a finite subset of $\mathbb{P}^1(k^{\mathrm{sep}})$, with $n \geq 2$. For each $v \in \mathcal{M}_k$, define the *v-adic discrepancy* of S relative to φ by

$$D_v(S) \;=\; D_{\varphi,v}(S) \;=\; \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} g_{\varphi,v}(P_i, P_j) \ .$$

LEMMA 10.27. *For each* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-invariant finite subset* S *of* $\mathbb{P}^1(k^{\mathrm{sep}})$*, we have*

$$\widehat{h}_\varphi(S) \;=\; \frac{1}{2} \sum_{v \in \mathcal{M}_k} N_v D_v(S) \ .$$

PROOF. Assume first that $\infty \notin S$. Write $S = \{P_1, \ldots, P_n\}$, and choose a homogeneous lifting $\tilde{P}_i = (1, z_i)$ of each P_i with $z_i \in k^{\mathrm{sep}}$. By definition of g_φ, we have

$$\begin{aligned} D_v(S) \;=\; & -\frac{1}{n(n-1)} \sum_{i\neq j} \log |z_i - z_j|_v \\ & + \frac{2}{n} \sum_{i=1}^{n} H_{F,v}(\tilde{P}_i) - \frac{1}{d(d-1)} \log |\mathrm{Res}(F)|_v \ . \end{aligned}$$

By Galois theory, $\prod_{i\neq j}(z_i - z_j) \in k^\times$, and also $\mathrm{Res}(F) \in k^\times$ by construction. The product formula therefore implies that

$$\frac{1}{2} \sum_{v \in \mathcal{M}_k} N_v D_v(S) \;=\; \sum_{v \in \mathcal{M}_k} N_v \left(\frac{1}{n} \sum_{i=1}^{n} H_{F,v}(\tilde{P}_i) \right) \;=\; \widehat{h}_\varphi(S) \ .$$

This proves the result when $\infty \notin S$. If $\infty \in S$, the result follows from a similar calculation (taking $\tilde{\infty} = (0, 1)$). □

PROPOSITION 10.28. *Let* $\{S_n\}$ *be a sequence of* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-invariant finite subsets of* $\mathbb{P}^1(k^{\mathrm{sep}})$ *such that* $|S_n| \to \infty$ *and* $\lim_{n\to\infty} \widehat{h}_\varphi(S_n) = 0$. *Then for each* $w \in \mathcal{M}_k$*, we have*

$$\liminf_{n\to\infty} D_w(S_n) \;=\; 0 \ .$$

PROOF. For each $w \in \mathcal{M}_k$ and $n \geq 1$, define $b_{w,n} = N_w D_w(S_n)$. We make the following observations:

(1) If w is non-Archimedean and φ has good reduction at w (which is the case for all but finitely many $w \in \mathcal{M}_k$), then by Example 10.4 and (10.20), we have (setting $N_n = |S_n|$)

$$b_{w,n} = N_w \left(\frac{1}{N_n(N_n-1)} \sum_{\substack{x,y \in S_n \\ x \neq y}} -\log \|x,y\|_w \right) \geq 0 \ .$$

(2) For all $w \in \mathcal{M}_k$, we have $\liminf_{n\to\infty} b_{w,n} \geq 0$ by Theorem 10.22(A).

(3) For each n, we have $b_{w,n} = 0$ for all but finitely many $w \in \mathcal{M}_k$, and by Lemma 10.27 we have $\sum_{w\in\mathcal{M}_k} b_{w,n} = 2\widehat{h}_\varphi(S_n)$. In particular, $\lim_{n\to\infty} \sum_{w\in\mathcal{M}_k} b_{w,n} = 2\lim_{n\to\infty} \widehat{h}_\varphi(S_n) = 0$.

Thus the hypotheses of Lemma 7.55 are satisfied, and we conclude that $\liminf_{n\to\infty} D_w(S_n) = 0$ for every $w \in \mathcal{M}_k$. □

We can now give the proof of the Adelic Equidistribution of Small Points theorem:

Proof of Theorem 10.24. We may assume without loss of generality that $|S_n| \geq 2$. Extracting a subsequence of S_n if necessary, we may assume by Prohorov's theorem (Theorem A.11) that δ_n converges weakly to a probability measure ν on $\mathbb{P}^1_{\mathrm{Berk},v}$. We want to show that $\nu = \mu_{\varphi,v}$. To accomplish this, consider the energy of ν relative to $g_{\varphi,v}$, defined by

$$I_v(\nu) \ = \ \iint_{\mathbb{P}^1_{\mathrm{Berk},v}\times\mathbb{P}^1_{\mathrm{Berk},v}} g_{\varphi,v}(x,y)\, d\nu(x)\, d\nu(y) \ .$$

Recall from Theorem 10.8 and its Archimedean counterpart (proved in [**9**, §5]; see also [**34**, §3.2/2]) that $I_v(\nu) \geq 0$, with equality if and only if $\nu = \mu_{\varphi,v}$. On the other hand, we claim that $I_v(\nu) \leq 0$, which will immediately yield the desired result.

To prove the claim, let Δ be the diagonal in $\mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v}$, and note that by definition of the v-adic discrepancy, we have

$$D_v(S_n) \ = \ \iint_{\mathbb{P}^1_{\mathrm{Berk},v}\times\mathbb{P}^1_{\mathrm{Berk},v}\backslash\Delta} g_{\varphi,v}(x,y)\, d\delta_n(x)\, d\delta_n(y) \ .$$

Since $g_{\varphi,v}$ is lower semicontinuous, Lemma 7.54 and Proposition 10.28 show that

$$I_v(\nu) \ \leq \ \liminf_{n\to\infty} D_v(S_n) \ = \ 0$$

as claimed. □

We now give a proof of the local result, Theorem 10.22, which was used in our proof of Theorem 10.24. As in the statement of that result, we let K be a field endowed with a nontrivial (Archimedean or non-Archimedean) absolute value, and we let $\varphi \in K(T)$ be a rational function of degree $d \geq 2$. We begin with the following lemma [**9**, Lemma 3.48]:

LEMMA 10.29. *For $N \geq 2$, define*

$$D_N \ := \ \inf_{z_1,\dots,z_N \in \mathbb{P}^1(K)} D_\varphi(z_1,\dots,z_N) \ .$$

Then the sequence D_N is nondecreasing.

PROOF. If $z_0,\dots,z_N \in \mathbb{P}^1(K)$, we have

$$\sum_{\substack{i,j=0 \\ i\neq j}}^{N} g_\varphi(z_i,z_j) \ = \ \frac{1}{N-1}\sum_{k=0}^{N}\left(\sum_{\substack{i\neq j \\ i,j\neq k}} g_\varphi(z_i,z_j)\right)$$

$$\geq \ \frac{1}{N-1}\sum_{k=0}^{N} N(N-1)D_N \ = \ N(N+1)D_N \ ,$$

from which we obtain $D_{N+1} \geq D_N$. □

The following result is taken from [**9**, Proof of Theorem 3.28]:

LEMMA 10.30. *Fix $N \geq 2$, and assume that K is complete and algebraically closed. Then for all $z_1,\dots,z_N \in \mathbb{P}^1_{\mathrm{Berk},K}$ we have*

$$\frac{1}{N(N-1)}\sum_{i\neq j} g_\varphi(z_i,z_j) \ \geq \ D_N \ . \tag{10.44}$$

PROOF. If K is Archimedean, then this is trivial, since $\mathbb{P}^1_{\mathrm{Berk},K} = \mathbb{P}^1(K)$ in that case. We may therefore assume that K is non-Archimedean. By (8.62) and Proposition 4.7, we have

$$g_\varphi(z,w) \ = \ \liminf_{\substack{(x,y)\to(z,w) \\ x,y\in\mathbb{P}^1(K)}} g_\varphi(x,y)$$

for all $z,w \in \mathbb{P}^1_{\mathrm{Berk}}$. Thus for any $\varepsilon > 0$, there are points $x_1,\dots,x_N \in \mathbb{P}^1(K)$ such that

$$|g_\varphi(z_i,z_j) - g_\varphi(x_i,x_j)| \ < \ \varepsilon$$

for all $i \neq j$. By definition, we have

$$\frac{1}{N(N-1)}\sum_{i\neq j} g_\varphi(x_i,x_j) \ \geq \ D_N \ ,$$

so letting $\varepsilon \to 0$ gives (10.44). □

We can now prove Theorem 10.22:

PROOF OF THEOREM 10.22. We will assume in the proof which follows that K is non-Archimedean. The Archimedean case follows from the same argument if we apply the complex counterpart of Theorem 10.8 (proved in [**9**, §5]; see also [**34**, §3.2/2]).

The sequence D_N is nondecreasing by Lemma 10.29, and in particular the limit $D_\infty := \lim_{N\to\infty} D_N$ exists. Replacing K by the completion of its algebraic closure if necessary, it therefore suffices to prove assertion (B). Let

$\mu = \mu_\varphi$ be the canonical measure for φ supported on $\mathbb{P}^1_{\mathrm{Berk},K}$. We already know by (10.38) that $I_\mu(\mu) := \iint g_\varphi(x,y)d\mu(x)d\mu(y) = 0$, so it suffices to prove that $\lim_{N\to\infty} D_N = \iint g_\varphi(x,y)d\mu(x)d\mu(y)$. We do this following the argument of [**9**, Theorem 3.28].

By (10.44), for $N \geq 2$ we have

$$\frac{1}{N(N-1)} \sum_{i \neq j} g_\varphi(z_i, z_j) \;\geq\; D_N$$

for all $z_1, \ldots, z_N \in \mathbb{P}^1_{\mathrm{Berk},K}$. Integrating this against $d\mu(z_1)\cdots d\mu(z_N)$ gives

$$\begin{aligned} I_\mu(\mu) &= \frac{1}{N(N-1)} \sum_{i \neq j} I_\mu(\mu) \\ &= \frac{1}{N(N-1)} \sum_{i \neq j} \iint g_\varphi(z_i, z_j) d\mu(z_i) d\mu(z_j) \geq D_N \end{aligned}$$

for all $N \geq 2$. It follows that $I_\mu(\mu) \geq D_\infty$.

For the other direction, for each $N \geq 2$ choose distinct points $w_1, \ldots, w_N$ in $\mathbb{P}^1(K)$ such that

$$\frac{1}{N(N-1)} \sum_{i \neq j} g_\varphi(w_i, w_j) \;\leq\; D_N + \frac{1}{N} \;,$$

and let $\nu_N := \frac{1}{N} \sum \delta_{w_i}$ be the discrete probability measure supported equally on each of the points w_i. Passing to a subsequence if necessary, we may assume by Prohorov's theorem (Theorem A.11) that the ν_N converge weakly to some probability measure ν on $\mathbb{P}^1_{\mathrm{Berk},K}$. Since

$$\begin{aligned} \frac{N-1}{N}\left(D_N + \frac{1}{N}\right) \geq\; & \frac{1}{N^2} \sum_{i \neq j} g_\varphi(w_i, w_j) \\ =\; & \iint_{\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathrm{Diag})} g_\varphi(x,y) d\nu_N(x) d\nu_N(y) \;, \end{aligned}$$

it follows from Theorem 10.8 and Lemma 7.54 that

$$D_\infty \;=\; \lim_{N\to\infty} D_N \;\geq\; I_\mu(\nu) \geq I_\mu(\mu) \;. \qquad \square$$

Finally, we give a proof of Theorem 10.23, a strengthening of Theorem 10.22(A) which can be used to make the equidistribution in Theorem 10.24 quantitative.[6]

By Lemma 10.29, in order to prove Theorem 10.23, it will suffice to give lower bounds for D_N as N runs through some conveniently chosen subsequence of natural numbers. In the argument which follows, we will assume that N belongs to the subset Σ of $\mathbb{N}$ consisting of those positive integers which can be written as $N = td^k$ with $k \geq 0$ and $2 \leq t \leq 2d-1$.

[6] A different quantitative version of Theorem 10.24 is given in [**47**, Corollary 1.6].

We now recall a definition from [**9**]. The *homogeneous filled Julia set* K_F of F is defined to be the set of all $z \in K^2$ for which the iterates $F^{(n)}(z)$ remain bounded. Clearly $F^{-1}(K_F) = K_F$, and the same is true for each $F^{(-n)}$. Since all norms on K^2 are equivalent, the set K_F is independent of which norm is used to define it.

The filled Julia set K_F can be thought of as the 'unit ball' with respect to the homogeneous dynamical height H_F:

LEMMA 10.31. *We have*

$$K_F \;=\; \{z \in K^2 \;:\; H_F(z) \le 0\} \;.$$

PROOF. If $z \in K_F$, then there exists $M > 0$ such that $\|F^{(n)}(z)\| \le M$ for all n, and therefore $H_F(z) \le \lim_{n\to\infty} \frac{1}{d^n} \log M = 0$.

Conversely, suppose $z \notin K_F$. Then by Lemma 10.1, there exist $\alpha, \beta > 1$ and n_0 such that

$$\|F^{(n+n_0)}(z)\|_v \;>\; \beta \cdot \alpha^{d^n - 1}$$

for all $n \ge 0$. Therefore

$$H_F(z) \;\ge\; \lim_{n\to\infty} \frac{1}{d^{n+n_0}} \left((d^n - 1)\log\alpha + \log\beta\right) \;=\; \frac{1}{d^{n_0}} \log\alpha \;>\; 0 \;.$$

□

Let ϵ_K be zero if the absolute value on K is non-Archimedean and 1 if it is Archimedean. We now prove the following somewhat technical result, which will easily imply Theorem 10.23:

THEOREM 10.32. *Let $N = td^k \in \Sigma$, and let $z_1, \ldots, z_N$ be nonzero elements of the filled Julia set K_F whose images in $\mathbb{P}^1(K)$ are all distinct. Choose a real number $R(F) > 0$ so that $K_F \subseteq \{z \in K^2 : \|z\| \le R(F)\}$. Then*

$$\begin{aligned}\textstyle\sum_{i\ne j} -\log|z_i \wedge z_j| \;\ge\;& r(F)N^2 - \epsilon_K N \log N \\ & -2\left(\log R(F)\right)\alpha N - r(F)(1+\alpha)N \;,\end{aligned}$$

where $r(F) = \frac{1}{d(d-1)} \log|\mathrm{Res}(F)|$ and $\alpha = t - 1 + (d-1)k > 0$ satisfies $2 \le \alpha \le (d-1)(\log_d N + 2)$.

An outline of the proof of Theorem 10.32 is as follows. First, we express $\prod_{i\ne j} |(x_i, y_i) \wedge (x_j, y_j)|$ as the determinant of a Vandermonde matrix S. We then replace this matrix with a new matrix H whose entries involve $F_1^{(k)}(x_i, y_i)$ and $F_2^{(k)}(x_i, y_i)$ for various $k \ge 0$, rather than the standard monomials $x_i^a y_i^b$. Replacing S by H amounts to choosing a different basis for the space of homogeneous polynomials in x and y of degree $N-1$, and we are able to explicitly calculate the determinant of the change of basis matrix under the assumption that $N \in \Sigma$. We then use Hadamard's inequality to estimate the determinant of H, using the fact that $\|F^{(k)}(x_i, y_i)\| \le R(F)$ for all $k \ge 0$.

Let $\Gamma^0(m)$ denote the vector space of homogeneous polynomials of degree m in $K[x,y]$, which has dimension $N = m+1$ over K. If $N \in \Sigma$, that is, if $m = td^k - 1$ with $2 \le t \le 2d-1$ and $k \ge 1$, we consider the collection $H(m)$ of polynomials in $\Gamma^0(m)$ given by

$$\begin{aligned} H(m) \;=\; & \{x^{a_0}y^{b_0}F_1(x,y)^{a_1}F_2(x,y)^{b_1}\cdots F_1^{(k)}(x,y)^{a_k}F_2^{(k)}(x,y)^{b_k} \;: \\ & a_i + b_i = d-1 \text{ for } 0 \le i \le k-1 \text{ and } a_k + b_k = t-1\} \;. \end{aligned}$$

The cardinality of $H(m)$ is easily seen to be $N = \dim \Gamma^0(m)$. The following proposition shows that $H(m)$ forms a basis for $\Gamma^0(m)$ and explicitly calculates the determinant of the change of basis matrix between $H(m)$ and the standard monomial basis $S(m)$ given by

$$S(m) \;=\; \{x^a y^b \mid a+b = m\} \;.$$

Proposition 10.33. *Let A be the matrix expressing the polynomials $H(m)$ (in some order) in terms of some ordering of the standard basis $S(m)$. Then* $\det(A) = \pm\operatorname{Res}(F)^r$, *where*

$$r \;=\; \frac{N^2}{2d(d-1)} - \frac{N}{2d(d-1)}\,(t + k(d-1))\,.$$

In particular, since $\operatorname{Res}(F) \ne 0$, *$H(m)$ is a basis for $\Gamma^0(m)$.*

Proof. Let $H_1,\ldots,H_N$ be an ordering of the elements of $H(m)$, and let $S_1,\ldots,S_N$ be an ordering of the elements of $S(m)$, so that A is the $N\times N$ matrix whose $(i,j)^{\text{th}}$ entry is the coefficient of the monomial S_i in the expansion of H_j as a polynomial in x and y. We have $\det(A) = 0$ if and only if some nontrivial linear combination of the elements of $H(m)$ is zero.

Suppose $\det(A) = 0$. Then there exist homogeneous polynomials $h_1 \in H(d^k-1)$ and $h_2 \in H((t-1)d^k-1)$, not both zero, such that

$$h_1\left(F_1^{(k)}(x,y)\right)^{t-1} + h_2F_2^{(k)}(x,y) \;=\; 0\;.$$

We may assume that neither of h_1, h_2 is the zero polynomial. Thus $F_2^{(k)}$ divides $h_1\left(F_1^{(k)}\right)^{t-1}$. Since $\deg(h_1) < \deg(F_2^{(k)})$, it follows that $F_1^{(k)}$ and $F_2^{(k)}$ have a common irreducible factor. Thus $\operatorname{Res}(F^{(k)}) = 0$. But $\operatorname{Res}(F^{(k)})$ is a power of $\operatorname{Res}(F)$ by (10.32), so $\operatorname{Res}(F) = 0$ as well.

Conversely, suppose $\operatorname{Res}(F) = 0$. By Lemma 2.11, there is a nontrivial relation of the form

$$h_1F_1 + h_2F_2 \;=\; 0 \tag{10.45}$$

with $h_1, h_2 \in K[x,y]$ homogeneous of degree $d-1$. If $k=1$, this already implies that $\det(A) = 0$. If $k \ge 2$, then multiplying both sides of (10.45) by $G(x,y)$, where

$$G(x,y) \;=\; F_1^{d-2}(F_1^{(2)})^{d-1}\cdots(F_1^{(k)})^{t-1}\;,$$

gives a linear relation which shows that $\det(A) = 0$.

Now expand both $\det(A)$ and $\mathrm{Res}(F)$ as polynomials in the coefficients of F_1 and F_2. Since $\mathrm{Res}(F)$ is irreducible (see [**98**, §5.9]), we find that

$$\det(A) \;=\; C \cdot \mathrm{Res}(F)^r$$

for some $C \in K^\times$ and some natural number r. Now $\mathrm{Res}(F)$ is homogeneous of degree $2d$ in the coefficients of F_1 and F_2, and a straightforward calculation shows that $F_1^{(j)}$ and $F_2^{(j)}$ are each homogeneous of degree $(d^j - 1)/(d - 1)$ in the coefficients of F_1 and F_2. It follows that $\det(A)$ is homogeneous of degree

$$r' \;=\; N\left(\sum_{j=1}^{k-1}(d^j - 1) + (t-1)\frac{d^k - 1}{d - 1}\right)$$

in the coefficients of F_1 and F_2. Comparing degrees and performing some straightforward algebraic manipulations, we find that

$$r \;=\; \frac{r'}{2d} \;=\; \frac{N^2}{2d(d-1)} - \frac{N}{2d}\left(\frac{t}{d-1} + k\right) \, .$$

Finally, to compute C, we set $F_1 = x^d$ and $F_2 = y^d$, in which case $H(m)$ is just a permutation of the standard monomial basis $S(m)$. It follows that $C = \pm 1$. $\square$

If $v = (v_1, \ldots, v_N)^{\mathrm{T}} \in K^N$, define $\|v\|$ to be the L^2-norm $\|v\| = (\sum |v_i|^2)^{1/2}$ if K is Archimedean and to be the sup-norm $\|v\| = \sup |v_i|$ if K is non-Archimedean. If H is a matrix with columns $h_1, \ldots, h_N \in K^N$, then *Hadamard's inequality*[7] states that

$$|\det(H)| \;\le\; \prod_{i=1}^{N} \|h_i\| \, . \tag{10.46}$$

Proof of Theorem 10.32. Let $\{S_1, \ldots, S_N\}$ and $\{H_1, \ldots, H_N\}$ be as in the proof of Proposition 10.33. Using the homogeneous version of the standard formula for the determinant of a Vandermonde matrix, we have

$$\prod_{i \neq j} |x_i y_j - x_j y_i| \;=\; |\det(S)|^2 \, , \tag{10.47}$$

where S is the matrix whose $(i,j)^{\mathrm{th}}$ entry is $S_j(x_i, y_i)$. If H is the matrix whose $(i,j)^{\mathrm{th}}$ entry is $H_j(x_i, y_i)$, then $H = SA$, so that by Proposition 10.33, we have

$$\det(S)^2 \;=\; \det(H)^2(\det(A))^{-2} \;=\; \det(H)^2 \,\mathrm{Res}(F)^{-2r} \, . \tag{10.48}$$

[7] See the discussion in [**3**] and the references therein.

On the other hand, we can estimate $|\det(H)|^2$ using Hadamard's inequality (10.46). Letting h_i be the i^{th} column of H, we obtain

$$\begin{aligned} |\det(H)|^2 &\le \textstyle\prod_{i=1}^N \|h_i\| \\ &\le N^{\epsilon_K N} \textstyle\prod_i R(F)^{2(k(d-1)+(t-1))} \\ &= N^{\epsilon_K N} R(F)^{(2t-2+k(2d-2))N} \ . \end{aligned} \tag{10.49}$$

Putting together (10.47), (10.48), and (10.49) gives

$$\begin{aligned} \textstyle\prod_{i\neq j} |x_i y_j - x_j y_i| &= |\det(S)|^2 \\ &= |\det(H)|^2 \cdot |\operatorname{Res}(F)|^{-2r} \\ &\le N^{\epsilon_K N} R(F)^{2\alpha N} |\operatorname{Res}(F)|^{-2r} \ , \end{aligned} \tag{10.50}$$

where $-2r = -\frac{N^2}{d(d-1)} + \frac{N}{d(d-1)}(t + k(d-1))$.

Taking the negative logarithm of both sides of (10.50) gives the desired lower bound

$$\begin{aligned} \textstyle\sum_{i\neq j} -\log|x_i y_j - x_j y_i| \ \ge\ & -\epsilon_K N \log N - 2N\alpha \log R(F) \\ & + \tfrac{N^2}{d(d-1)} \log|\operatorname{Res}(F)| \\ & - \tfrac{N}{d(d-1)}(t + k(d-1)) \log|\operatorname{Res}(F)| \ . \end{aligned}$$

□

We can now give the proof of Theorem 10.23:

Proof of Theorem 10.23. Without loss of generality, we may assume that $N \ge 2d$. Let N' be the smallest integer less than or equal to N which belongs to Σ. One deduces easily from the definition of Σ that

$$\frac{N-1}{2} \ \le\ N' - 1 \ \le\ N - 1 \ . \tag{10.51}$$

Using the fact that $N' \in \Sigma$, we claim that there is a constant $C' > 0$ (independent of N and N') such that

$$D_{N'} \ \ge\ -C' \frac{\log N'}{N'-1} \ . \tag{10.52}$$

From this, Lemma 10.29, and (10.51), it follows that

$$D_\varphi(z_1, \ldots, z_N) \ \ge\ D_{N'} \ \ge\ -C \frac{\log N}{N-1}$$

as desired, where $C = 2C'$.

It remains to prove the claim. Replacing K by $\overline{K}$ if necessary, we may assume without loss of generality that the value group $|K^\times|$ is dense in the group $\mathbb{R}_{>0}$ of nonnegative reals. Since $H_F(cz) = H_F(z) + \log|c|$ for all $z \in K^2 \setminus \{0\}$ and all $c \in K^\times$, given $\varepsilon > 0$, we can choose coordinates (x_i, y_i) for z_i so that $-\varepsilon \le H_F(x_i, y_i) \le 0$. Then $(x_i, y_i) \in K_F$ by Lemma 10.31, and we can apply Theorem 10.32 to $(x_1, y_1), \ldots, (x_N, y_N)$. Simplifying the resulting expression and letting $\varepsilon \to 0$ gives the desired inequality (10.52). □

Remark 10.34. As an example of an application of Theorem 10.23, we note the following global result (see [**3**, §3] for a proof) concerning fields of definition for preperiodic points:

Theorem 10.35. *Let k be a number field, and let $\varphi \in k(T)$ be a rational map of degree at least 2. Then there is a constant $C = C(\varphi, k)$ such that if $P_1, \ldots, P_N \in \mathbb{P}^1(\overline{k})$ are distinct preperiodic points of φ, then*

$$[k(P_1, \ldots, P_N) : k] \ \geq \ C \frac{N}{\log N} \ .$$

10.4. Equidistribution of preimages

In this and the following three sections, we develop the fundamental facts of Fatou-Julia theory when K has *characteristic* 0. This theory is very similar to the classical theory over $\mathbb{C}$, whereas the theory over fields of positive characteristic is different in some respects.

For the remainder of this section, we assume that K is a complete and algebraically closed non-Archimedean field of characteristic 0. Let $\varphi \in K(T)$ be a rational function of degree $d \geq 2$. Our goal is to prove an equidistribution theorem for the preimages under φ of any nonexceptional point of $\mathbb{P}^1_{\mathrm{Berk}}$. This result is originally due to Favre and Rivera-Letelier [**46, 48**] and is the Berkovich space analogue of a classical result due independently to Lyubich [**72**] and Freire, Lopes, and Mañé [**54**].

A point $x \in \mathbb{P}^1_{\mathrm{Berk}}$ is called *exceptional* for φ if the set of all forward and backward iterates of x is finite. In other words, if the *grand orbit of* x is

$$GO(x) \ = \ \{y \in \mathbb{P}^1_{\mathrm{Berk}} \ : \ \varphi^{(m)}(x) \ = \ \varphi^{(n)}(y) \text{ for some } m, n \geq 0\} \ ,$$

then x is exceptional if and only if $GO(x)$ is finite. It is easy to see that if x is exceptional, then $GO(x)$ forms a finite cyclic orbit under φ and each $y \in GO(x)$ has multiplicity $m_\varphi(y) = d$. By analogy with the situation over $\mathbb{C}$, we define the *exceptional locus* E_φ of φ to be the set of all exceptional points of $\mathbb{P}^1_{\mathrm{Berk}}$. We write $E_\varphi(K)$ for $E_\varphi \cap \mathbb{P}^1(K)$.

It is well known that if K has characteristic 0, then $|E_\varphi(K)| \leq 2$ (see Lemma 10.40 below). We will see below (see Proposition 10.45) that φ has at most one exceptional point in $\mathbb{H}_{\mathrm{Berk}}$.

We will now prove the following result of Favre and Rivera-Letelier when $\mathrm{char}(K) = 0$. In [**48**], it is established in positive characteristic as well.

Theorem 10.36 (Favre, Rivera-Letelier). *Let $\varphi \in K(T)$ be a rational function of degree $d \geq 2$, and let y be any point in $\mathbb{P}^1_{\mathrm{Berk}} \backslash E_\varphi(K)$. For each integer $n \geq 1$, consider the discrete probability measure*

$$\mu^y_{\varphi^{(n)}} \ = \ \frac{1}{d^n} \sum_{\varphi^{(n)}(y_i) = y} m_{\varphi^{(n)}}(y_i) \delta_{y_i} \ .$$

Then the sequence of measures $\{\mu^y_{\varphi^{(n)}}\}_{1 \leq n < \infty}$ converges weakly to the canonical measure μ_φ for φ.

In other words, the preimages under φ of any $y \in \mathbb{P}^1_{\mathrm{Berk}} \backslash E_\varphi(K)$ are equidistributed with respect to the canonical measure μ_φ.

REMARK 10.37. Using the version of Theorem 10.36 proved by Favre and Rivera-Letelier in [**48**], one can establish characteristic p analogues of the results about Berkovich Fatou and Julia sets proved in later sections of this chapter. However, it should be noted that those results are not always identical with the ones in characteristic 0. For example, in characteristic p the exceptional set $E_\varphi(K)$ need not have at most two elements but can be countably infinite.

We begin with the following lemma:

LEMMA 10.38. *Suppose* $g_n \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ *and* $g_n \to 0$ *pointwise on* $\mathbb{H}_{\mathrm{Berk}}$. *Then* $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(g_n) \to 0$ *weakly.*

PROOF. It suffices to prove that $\int f\Delta(g_n) \to 0$ for every simple function f on $\mathbb{P}^1_{\mathrm{Berk}}$. Write $\Delta(f) = \sum_{i=1}^t a_i \delta_{\zeta_i}$ with $\zeta_i \in \mathbb{H}^{\mathbb{R}}_{\mathrm{Berk}}$. By Corollary 5.39,

$$\begin{aligned} \lim_{n\to\infty} \int f\Delta(g_n) &= \lim_{n\to\infty} \int g_n\Delta(f) = \lim_{n\to\infty} \sum_{i=1}^t a_i g_n(\zeta_i) \\ &= \sum_{i=1}^t a_i (\lim_{n\to\infty} g_n(\zeta_i)) = 0 , \end{aligned}$$

as desired. □

LEMMA 10.39. *The function* $g_\varphi(x,y) = g_{\mu_\varphi}(x,y)$ *differs from the function* $-\log_v(\delta(x,y)_{\zeta_{\mathrm{Gauss}}})$ *by a bounded function on* $\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}}$.

PROOF. Since μ_φ has continuous potentials (cf. Definition 5.40 and Proposition 10.7), this follows directly from formula (8.62). □

Recall that a point $z \in \mathbb{P}^1(K)$ is a critical point of φ iff $m_\varphi(z) > 1$. The following result is an immediate consequence of the classical Riemann-Hurwitz formula (which depends on our assumption that $\mathrm{char}(K) = 0$):

LEMMA 10.40. *Assume* $\mathrm{char}(K) = 0$. *Then the map* φ *has only finitely many critical points in* $\mathbb{P}^1(K)$. *More precisely,*

$$\sum_{z \in \mathbb{P}^1(K)} (m_\varphi(z) - 1) = 2d - 2 .$$

Let $E_\varphi(K)$ be the exceptional locus of φ in $\mathbb{P}^1(K)$. By Lemma 10.40, we have $|E_\varphi(K)| \le 2$. It follows easily that if $E_\varphi(K)$ is nonempty, then it is a union of periodic cycles having length 1 or 2.

A periodic point $w \in \mathbb{P}^1(K)$ of period k for φ is called *attracting* if its multiplier $|(\varphi^{(k)})'(w)|$ (computed in any coordinate where $w \ne \infty$) is less than 1. It is *superattracting* if its multiplier is 0. A periodic cycle C of period k in $\mathbb{P}^1(K)$ is attracting if every point of C is an attracting periodic point. It is a well-known consequence of the chain rule that the multiplier is the same for every $w \in C$, so C is attracting if and only if some point of C is attracting. A periodic cycle composed of exceptional points for φ is necessarily attracting, since the multiplier is 0 at such points.

The *attracting basin* of an attracting periodic cycle $C \subseteq \mathbb{P}^1(K)$ for φ is defined to be the set of all $x \in \mathbb{P}^1_{\text{Berk}}$ for which, given any open neighborhood U of C, there exists an N such that $\varphi^{(n)}(x) \in U$ for $n \geq N$. The *attracting basin of* $E_\varphi(K)$ is defined to be empty if $E_\varphi(K)$ is empty and to be the union of the attracting basins of the cycles comprising $E_\varphi(K)$ otherwise.

LEMMA 10.41. *The attracting basin of any attracting periodic cycle $C \subseteq \mathbb{P}^1(K)$ is open in $\mathbb{P}^1_{\text{Berk}}$. In particular, the attracting basin of $E_\varphi(K)$ is open in $\mathbb{P}^1_{\text{Berk}}$.*

PROOF. Let $C \subseteq \mathbb{P}^1(K)$ be an attracting periodic cycle of period k. Replacing φ by $\varphi^{(k)}$, we may assume without loss of generality that $C = \{w\}$ with w a fixed point of φ. After a change of coordinates, we can assume that $w \neq \infty$. By classical non-Archimedean dynamics, there is a descending chain of open discs $D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$ in $\mathbb{P}^1(K)$ with $\varphi(D_i) \subseteq D_{i+1}$ for all i, such that $\bigcap D_i = \{w\}$. Indeed, if $\varphi(T)$ has multiplicity m at w, then by expanding $\varphi(T)$ as a power series and using the theory of Newton polygons (see Corollary A.17 in §A.10), one finds an $r_0 > 0$ and a constant $B > 0$ (with $B = |\varphi'(w)| < 1$ if $m = 1$) such that $\varphi(D(w,r)^-) = D(w, Br^m)^-$ for each $0 < r \leq r_0$. Letting $\mathcal{D}_i$ be the Berkovich open disc corresponding to D_i, we will still have $\mathcal{D}_0 \supseteq \mathcal{D}_1 \supseteq \cdots$, $\varphi(\mathcal{D}_i) \subseteq \mathcal{D}_{i+1}$ for all i and $\bigcap \mathcal{D}_i = \{w\}$. Let $\mathcal{A}$ be the attracting basin of w. If $x \in \mathcal{A}$, then $\varphi^{(N)}(x) \in \mathcal{D}_0$ for some N. By continuity, there is an open neighborhood V of x such that $\varphi^{(N)}(V) \subseteq \mathcal{D}_0$. Then for $n \geq N$, we have $\varphi^{(n)}(V) \subseteq \varphi^{n-N}(\mathcal{D}_0) \subseteq \mathcal{D}_{n-N}$. Since the Berkovich open discs $\mathcal{D}_i$ form a fundamental system of open neighborhoods for $\{w\}$, this shows that $V \subset \mathcal{A}$, and thus $\mathcal{A}$ is open. □

PROOF OF THEOREM 10.36. Recall that μ_φ satisfies the functional equation

$$\Delta_x g_\varphi(x,y) \ = \ \delta_y - \mu_\varphi$$

for all $y \in \mathbb{P}^1_{\text{Berk}}$. Pulling back by $\varphi^{(n)}$ and applying Proposition 9.56 and Theorems 10.2 and 10.18, we obtain

$$\mu^y_{\varphi^{(n)}} - \mu_\varphi \ = \ \frac{1}{d^n}\Delta_x g_\varphi(x, (\varphi^{(n)})^*(y)) \ = \ \frac{1}{d^n}\Delta_x g_\varphi(\varphi^{(n)}(x), y) \ .$$

By Lemma 10.38, to prove Theorem 10.36 it suffices to show that

$$\frac{1}{d^n} g_\varphi(\varphi^{(n)}(x), y) \ \to \ 0$$

as $n \to \infty$, for each fixed $x \in \mathbb{H}_{\text{Berk}}$ and each $y \in \mathbb{P}^1_{\text{Berk}} \backslash E_\varphi(K)$.

Since g_φ is bounded below, we automatically have

$$\liminf \frac{1}{d^n} g_\varphi(\varphi^{(n)}(x), y) \ \geq \ 0 \ ,$$

so it suffices to prove that

$$(10.53)\qquad \limsup \frac{1}{d^n} g_\varphi(\varphi^{(n)}(x), y) \ \le\ 0 \ .$$

We first dispose of some easy cases. If $y \in \mathbb{H}_{\mathrm{Berk}}$, then $g_\varphi(x, y)$ is bounded as a function of x, and (10.53) holds trivially. Let $\mathcal{A}$ be the attracting basin in $\mathbb{P}^1_{\mathrm{Berk}}$ of $E_\varphi(K)$. By Lemma 10.41, $\mathcal{A}$ is an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$. If $x \in \mathcal{A}$ and $y \in \mathbb{P}^1(K)\backslash\mathcal{A}$, then $\varphi^{(n)}(x)$ accumulates at points of $E_\varphi(K)$ for large n. This means $g_\varphi(\varphi^{(n)}(x), y)$ is close to $g_\varphi(z, y)$ for some $z \in E_\varphi(K)$, so (10.53) holds. Likewise, for $x \in \mathbb{H}_{\mathrm{Berk}}\backslash\mathcal{A}$ and $y \in \mathbb{P}^1(K) \cap \mathcal{A}$, the iterates $\varphi^{(n)}(x)$ remain bounded away from y, and again (10.53) holds.

Thus it suffices to prove (10.53) for $x \in \mathbb{H}_{\mathrm{Berk}}\backslash\mathcal{A}$ and $y \in \mathbb{P}^1(K)\backslash\mathcal{A}$. For such x, y, a potential problem arises when the sequence $\{\varphi^{(n)}(x)\}$ has y as an accumulation point.

In order to deal with this, we first make a reduction. Note that if k is any positive integer and the analogue of (10.53) holds with φ replaced by $\varphi^{(k)}$ and d replaced by d^k, then (10.53) holds for φ. (This follows easily from Lemma 10.16.) Suppose $z \in \mathbb{P}^1(K)$ is not exceptional. Consider the points $z_0 = z$, $z_1 = \varphi(z)$, and $z_2 = \varphi^{(2)}(z)$. We claim that $m_\varphi(z_i) < d$ for some i. Otherwise, by Lemma 10.40, two of them would have to coincide. If $z_0 = z_1$ or $z_0 = z_2$, then z would be exceptional. If $z_1 = z_2$, then z_1 would be exceptional, and then z, which belongs to $GO(z_1)$, would also be exceptional. Hence the formula $m_{\varphi\circ\psi}(z) = m_\varphi(\psi(z)) \cdot m_\psi(z)$ (see Proposition 9.28(C)) shows that if $z \in \mathbb{P}^1(K)$ is not an exceptional point for φ, then $m_{\varphi^{(3)}}(z) \le d^3 - 1$. So in proving (10.53), after replacing φ by $\varphi^{(3)}$ if necessary, we can assume that $m_\varphi(z) \le d - 1$ whenever $z \in \mathbb{P}^1(K)$ is not exceptional.

We now require the following estimate, whose proof is given after Corollary 10.44 below:

Lemma 10.42. *Suppose $m_\varphi(z) \le d-1$ for all $z \in \mathbb{P}^1(K)\backslash E_\varphi(K)$. Let $\mathcal{A}$ be the attracting basin of $E_\varphi(K)$. Then there is a constant $C > 0$ such that for all $x \in \mathbb{P}^1_{\mathrm{Berk}}\backslash\mathcal{A}$ and all $y \in \mathbb{P}^1(K)\backslash\mathcal{A}$,*

$$(10.54)\qquad g_\varphi(\varphi(x), y) \ \le\ C + (d-1) \max_{\varphi(z)=y} g_\varphi(x, z) \ .$$

For $x \in \mathbb{H}_{\mathrm{Berk}}\backslash\mathcal{A}$ and $y \in \mathbb{P}^1(K)\backslash\mathcal{A}$, it follows from (10.54) that

$$g_\varphi(\varphi^{(n)}(x), y) \ \le\ C(n) + (d-1)^n \max_{\varphi^n(z)=y} g_\varphi(x, z)$$

for all $n \ge 1$, where

$$\begin{aligned} C(n) &= C + (d-1)C + (d-1)^2 C + \cdots + (d-1)^{n-1} C \\ &= O\big(\max(n, (d-1)^n)\big) \ . \end{aligned}$$

Furthermore, the function $G(z) := g_\varphi(x,z)$ is bounded on $\mathbb{P}^1_{\mathrm{Berk}}$, since $x \in \mathbb{H}_{\mathrm{Berk}}$. Hence

$$\limsup \frac{1}{d^n} g_\varphi(\varphi^{(n)}(x), y) \leq \limsup \frac{1}{d^n}\left(C(n) + (d-1)^n \max_{z \in \mathbb{P}^1_{\mathrm{Berk}}} g_\varphi(x,z)\right) \leq 0$$

as desired. □

To prove Lemma 10.42, we will need the following result about power series:

LEMMA 10.43 (Non-Archimedean Rolle's theorem). *Let K be a complete and algebraically closed non-Archimedean field of characteristic 0. Put $\rho_K = 1$ if K has residue characteristic 0, and put $\rho_K = |p|^{1/(p-1)} < 1$ if K has residue characteristic $p > 0$.*

Suppose $f(T) \in K[[T]]$ converges in $D(0,r)^-$. If $f(T)$ has two or more zeros in $D(0,\rho_K r)^-$, then $f'(T)$ has at least one zero in $D(0,r)^-$.

This is proved in [**85**, p. 316] when $K = \mathbb{C}_p$. The proof is carried over to the general case in Proposition A.20. The hypothesis that $\mathrm{char}(K) = 0$ is used to assure that $|p| \neq 0$ when K has residue characteristic $p > 0$.

Let $\|x,y\|$ denote the spherical distance on $\mathbb{P}^1(K)$ (see §4.3). Recall that for $a \in \mathbb{P}^1(K)$ and $r > 0$, we write

$$B(a,r)^- = \{z \in \mathbb{P}^1(K) : \|z,a\| < r\} .$$

COROLLARY 10.44 (Uniform Injectivity Criterion). *Let K be a complete and algebraically closed non-Archimedean field of characteristic 0, and let $\varphi(T) \in K(T)$ be a nonconstant rational function. Then there is an $r_0 > 0$ (depending only on φ) such that if $a \in \mathbb{P}^1(K)$, $0 < r \leq r_0$, and $B(a,r)^-$ does not contain any critical points of φ, then φ is injective on $B(a,\rho_K r)^-$.*

PROOF. Let X be the set of all zeros and poles of $\varphi(T)$ in $\mathbb{P}^1(K)$. Put

$$r_0 = \min\left(1, \min_{\substack{\alpha,\beta \in X \\ \alpha \neq \beta}} (\|\alpha,\beta\|)\right) .$$

Let $a \in \mathbb{P}^1(K)$ and $0 < r \leq r_0$ be given. By replacing $\varphi(T)$ with $1/\varphi(T)$ if necessary, we can assume that $B(a,r)^-$ does not contain any poles of φ. If we identify $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$, then after changing coordinates by an appropriate element of $\mathrm{GL}_2(\mathcal{O}_K)$ (where $\mathcal{O}_K$ is the valuation ring of K), we can assume that $a = 0$. Observe that such a change of coordinates preserves the spherical distance. In this situation $B(a,r)^- = D(0,r)^- \subset D(0,1) \subset K$. Note that $D(0,r)^-$ does not contain any poles of $\varphi(T)$ and that $\|x,y\| = |x-y|$ for $x,y \in D(0,r)^-$.

Expand $\varphi(T)$ as a power series $f(T) \in K[[T]]$. Then $f(T)$ converges in $D(0,r)^-$. By Lemma 10.43, if $D(0,r)^-$ does not contain any critical points of $\varphi(T)$, then $\varphi(T)$ is injective on $D(0,\rho_K r)^-$. □

Proof of Lemma 10.42. Let $x \in \mathbb{P}^1_{\text{Berk}} \backslash \mathcal{A}$ and $y \in \mathbb{P}^1(K) \backslash \mathcal{A}$, and assume that $m_\varphi(z) \le d-1$ for all $z \in \mathbb{P}^1(K) \backslash E_\varphi(K)$.

By Theorem 10.18, proving (10.54) is equivalent to showing that

$$g_\varphi(x, \varphi^*(y)) \ \le \ C + (d-1) \max_{\varphi(z)=y} g_\varphi(x,z) \ . \tag{10.55}$$

Note that for each fixed $y \in \mathbb{P}^1(K) \backslash \mathcal{A}$, both sides of (10.55) are continuous in x, as functions to the extended reals. The left side is a finite sum of continuous functions, and the right side concerns the maximum of those same functions. Since $\mathbb{P}^1(K) \backslash \mathcal{A}$ is dense in $\mathbb{P}^1_{\text{Berk}} \backslash \mathcal{A}$, it therefore suffices to prove (10.55) for $x \in \mathbb{P}^1(K) \backslash \mathcal{A}$.

If $z_1, \ldots, z_d$ are the points with $\varphi(z_i) = y$ (listed with multiplicities), then

$$g_\varphi(x, \varphi^*(y)) \ = \ \sum_{i=1}^d g_\varphi(x, z_i) \ .$$

Hence to prove (10.55) it is enough to show that there is a constant C such that for each $x, y \in \mathbb{P}^1(K) \backslash \mathcal{A}$, there is some z_i with $\varphi(z_i) = y$ for which $g_\varphi(x, z_i) \le C$.

By Lemma 10.39, the Arakelov-Green's function $g_\varphi(x,y)$ differs from $-\log_v(\delta(x,y)_{\zeta_{\text{Gauss}}})$ by a bounded function on $\mathbb{P}^1_{\text{Berk}} \times \mathbb{P}^1_{\text{Berk}}$. Recall that for $x, y \in \mathbb{P}^1(K)$, $\delta(x,y)_{\zeta_{\text{Gauss}}}$ coincides with the spherical distance $\|x,y\|$. It therefore suffices to show that there is an $\varepsilon > 0$ such that for each $x, y \in \mathbb{P}^1(K) \backslash \mathcal{A}$, there is some z_i with $\varphi(z_i) = y$ for which

$$\|x, z_i\| \ \ge \ \varepsilon \ . \tag{10.56}$$

For each fixed $w \in \mathbb{P}^1(K) \backslash E_\varphi(K)$, Corollary 9.17 and the fact that $m_\varphi(z) \le d-1$ for all $z \in \varphi^{-1}(w)$ imply that one can find an open neighborhood U_w of w in $\mathbb{P}^1_{\text{Berk}}$ and disjoint connected components V, V' of $\varphi^{-1}(U_w)$. After shrinking U_w if necessary, we can assume that V and V' have disjoint closures. Hence, there is an $\varepsilon_w > 0$ such that $\|z, z'\| \ge \varepsilon_w$ for all $z \in \mathbb{P}^1(K) \cap V$, $z' \in \mathbb{P}^1(K) \cap V'$. For any $x \in \mathbb{P}^1(K)$, either $x \notin V$ or $x \notin V'$; it follows from this and the ultrametric inequality $\max(\|x,z\|, \|x,z'\|) \ge \|z,z'\|$ that for each $x \in \mathbb{P}^1(K) \backslash \mathcal{A}$ and each $y \in \mathbb{P}^1(K) \cap U_w$, there is some z_i with $\varphi(z_i) = y$ for which $\|x, z_i\| \ge \varepsilon_w$.

For each of the finitely many nonexceptional critical points $y_1, \ldots, y_t$ of φ, put $w_i = \varphi(y_i)$, choose a neighborhood $U_i = U_{w_i}$ as above, and let $\varepsilon_i := \varepsilon_{w_i}$ be the corresponding bound. For each $y \in \mathbb{P}^1(K) \cap (U_1 \cup \cdots \cup U_r)$, (10.56) holds with $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_t)$.

Finally, put

$$\mathcal{B} \ = \ \mathcal{A} \cup U_1 \cup \cdots \cup U_t \ .$$

By construction, $\varphi^{-1}(\mathcal{B})$ contains all the critical points of φ. Since there are finitely many critical points, there is an $r > 0$ such that for each critical y, the ball $B(y,r)^-$ is contained in $\varphi^{-1}(\mathcal{B})$. After reducing r if necessary, we can assume that $r \le r_0$, where r_0 is the number from Corollary 10.44. By the ultrametric inequality for the spherical distance, for each $z \in \mathbb{P}^1(K)\backslash\varphi^{-1}(\mathcal{B})$, the ball $B(z,r)^-$ does not contain any critical points. Hence Corollary 10.44 shows that φ is injective on $B(z,\rho_K r)^-$.

In particular, each $z \in \mathbb{P}^1(K)\backslash\varphi^{-1}(\mathcal{B})$ has multiplicity 1. Thus, if $y \in \mathbb{P}^1(K)\backslash\mathcal{B}$, then y has d distinct preimages $z_1,\ldots,z_d \in \mathbb{P}^1(K)\backslash\varphi^{-1}(\mathcal{B})$ under φ, and since $\varphi(z_i) = \varphi(z_j) = y$, necessarily $\|z_i,z_j\| \ge \rho_K r$ for all $i \ne j$. By the ultrametric inequality for the spherical distance, for each $x \in \mathbb{P}^1(K)\backslash\mathcal{A}$ there is at least one z_i with $\|x,z_i\| \ge \rho_K r$. Hence, if we take

$$\varepsilon \;=\; \min(\rho_K r, \varepsilon_1, \ldots, \varepsilon_t)\ ,$$

then (10.56) holds for all $x, y \in \mathbb{P}^1(K)\backslash\mathcal{A}$. □

As an application of Theorem 10.36, we give a characterization of the exceptional locus due to Rivera-Letelier.[8]

Proposition 10.45. *Let K be a complete and algebraically closed non-Archimedean field of characteristic* 0*, and let $\varphi(T) \in K(T)$ have degree $d \ge 2$.*

(A) *The exceptional locus E_φ of φ contains at most* 3 *points. At most two exceptional points are in $\mathbb{P}^1(K)$, and at most one is in $\mathbb{H}_{\mathrm{Berk}}$.*

(B) *There is an exceptional point in $\mathbb{H}_{\mathrm{Berk}}$ (necessarily of type* II*) if and only if φ has simple reduction.*

Proof. For (A), note that since $\varphi : \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{P}^1_{\mathrm{Berk}}$ is surjective by Corollary 9.9 (see also [**81**, Lemma 2.5]), φ must map each grand orbit onto itself. So φ maps each finite grand orbit bijectively onto itself, and thus each finite grand orbit is a periodic cycle. The fact that there are at most 2 exceptional points in $\mathbb{P}^1(K)$ is well known and follows from Lemma 10.40, as was noted earlier.

The fact that there is at most one exceptional point in $\mathbb{H}_{\mathrm{Berk}}$ follows from Theorem 10.36. Indeed, suppose $y \in \mathbb{H}_{\mathrm{Berk}}$ is exceptional. After replacing φ by $\varphi^{(k)}$ for an appropriate k, we can assume that y is a fixed point of φ. (Note that $\mu_{\varphi^{(k)}} = \mu_\varphi$ by Lemma 10.16.) For each $n \ge 1$, it follows that

$$\mu^y_{\varphi^{(n)}} \;=\; \frac{1}{d^n}(d^n \delta_y) \;=\; \delta_y\ .$$

Since the $\mu^y_{\varphi^{(n)}}$ converge weakly to μ_φ, we must have $\mu_\varphi = \delta_y$. Thus y is unique.

For assertion (B), suppose first that $y \in \mathbb{H}_{\mathrm{Berk}}$ is exceptional. We have just proved that $\mu_\varphi = \delta_y$ and that y is a fixed point of $\varphi^{(k)}$ for some $k \ge 1$. Moreover, $m_{\varphi^{(k)}}(y) = d^k$ by Theorem 10.2. By Lemma 10.80 below (whose proof uses different ideas than the ones at hand, but only depends on results from Chapter 9), y must be a point of type II. By Corollary 2.13, there is a Möbius transformation $M \in \mathrm{PGL}(2,K)$ such that $M(y) = \zeta_{\mathrm{Gauss}}$. Let $\psi = M \circ \varphi \circ M^{-1}$. Then $\psi^{(k)}(\zeta_{\mathrm{Gauss}}) = \zeta_{\mathrm{Gauss}}$, and $\mu_{\psi^{(k)}} = \delta_{\zeta_{\mathrm{Gauss}}}$ by

[8] The proof of [**81**, Theorem 3], written with $K = \mathbb{C}_p$, remains valid for arbitrary K.

Corollary 10.15. Thus $\psi^{(k)}$ has good reduction by Proposition 10.5. By Corollary 10.17, ψ itself has good reduction. Hence φ has simple reduction.

Conversely, suppose φ has simple reduction. Then by definition, there is a Möbius transformation $M \in \mathrm{PGL}(2, K)$ such that $\psi = M \circ \varphi \circ M^{-1}$ has good reduction. Thus $\mu_\psi = \delta_{\zeta_{\mathrm{Gauss}}}$, so by Corollary 10.15 we have $\mu_\varphi = \delta_y$, where $y = M^{-1}(\zeta_{\mathrm{Gauss}}) \in \mathbb{H}_{\mathrm{Berk}}$ is a point of type II. It follows from Theorem 10.2 that y is exceptional for φ. □

Remark 10.46. An alternate proof of Proposition 10.45(B) can be found in [**5**, Corollary 3.26]. The proof there is also based on properties of the canonical measure μ_φ and its associated Arakelov-Green's function $g_\varphi(x, y)$, especially Theorem 10.18.

As a second application of Theorem 10.36, we show that the canonical measure has point masses only when its support is a single point. This fact is also due to Favre and Rivera-Letelier [**48**]. With their version of Theorem 10.36, the proof remains valid in arbitrary characteristic.

Corollary 10.47. *Let $\varphi(T) \in K(T)$ have degree $d \geq 2$. Then the canonical measure μ_φ never charges points of $\mathbb{P}^1(K)$, and unless φ has simple reduction* (*so that J_φ consists of a single point, necessarily of type* II), *it does not charge points of $\mathbb{H}_{\mathrm{Berk}}$.*

Proof. Since μ_φ has continuous potentials (Proposition 10.7), it cannot charge points of $\mathbb{P}^1(K)$.

Suppose that it charges some $\zeta \in \mathbb{H}_{\mathrm{Berk}}$. Put $\xi = \varphi(\zeta)$, and let $\zeta_1, \ldots, \zeta_n$ be the points in $\varphi^{-1}(\{\xi\})$, labeled in such a way that $\zeta = \zeta_1$. Let their multiplicities be $m_1, \ldots, m_n$ respectively, where $m_1 + \cdots + m_n = d$. By Proposition 9.52,

$$\varphi^*(\delta_\xi) \;=\; \sum_{i=1}^{n} m_i \delta_{\zeta_i} \; .$$

On the other hand, by Theorem 10.2, $\mu_\varphi = (1/d)\varphi^*(\mu_\varphi)$, so $\mu_\varphi(\{\zeta\}) = (m_1/d) \cdot \mu_\varphi(\{\xi\})$; that is

$$\mu_\varphi(\{\varphi(\zeta)\}) \;=\; \frac{d}{m_1} \mu_\varphi(\{\zeta\}) \;\geq\; \mu_\varphi(\{\zeta\}) \; ,$$

with strict inequality unless $m_1 = d$.

Inductively, we see that $\mu_\varphi(\{\varphi^{(k)}(\zeta)\}) \geq \mu_\varphi(\{\zeta\})$ for each $k \geq 0$. Since μ_φ has finite total mass, this is impossible unless there are integers $k_2 > k_1$ for which $\varphi^{(k_2)}(\zeta) = \varphi^{(k_1)}(\zeta)$. In this case, there must be a $\beta > 0$ such that $\mu_\varphi(\{\varphi^{(k)}(\zeta)\}) = \beta$ for each $k \geq k_1$. It follows from the discussion above that $\varphi^{(k)}(\zeta)$ has multiplicity d for each $k \geq k_1$. In particular, $\varphi^{(k_2-1)}(\zeta)$ is the only preimage of $\varphi^{(k_1)}(\zeta)$; hence $\varphi^{(k_1-1)}(\zeta) = \varphi^{(k_2-1)}(\zeta)$. Proceeding inductively backwards, we see that the grand orbit of ζ is finite. Thus ζ is an exceptional point.

By Proposition 10.45, we conclude that ζ is of type II and that φ has simple reduction. □

We conclude this section with a third application of Theorem 10.36: for each $y \in \mathbb{H}_{\text{Berk}}$, the Arakelov-Green's function $g_\varphi(x, y)$ is not only continuous, but Hölder continuous. The Hölder continuity of dynamical Green's functions is well known in the classical case and is due to Favre and Rivera-Letelier in the Berkovich case (see [**47**, Proposition 6.5]). With their version of Theorem 10.36, the result holds in arbitrary characteristic.

We begin by recalling some definitions. For $x, y \in \mathbb{P}^1_{\text{Berk}}$, let $d(x, y) = 2\,\text{diam}(x \vee y) - \text{diam}(x) - \text{diam}(y)$ be the metric introduced in §2.7, which induces the strong topology on $\mathbb{P}^1_{\text{Berk}}$. Here, $x \vee y$ denotes the point where the paths $[x, \zeta_{\text{Gauss}}]$ and $[y, \zeta_{\text{Gauss}}]$ first meet, and $\text{diam}(x)$ is the diameter relative to ζ_{Gauss}.

We will say that a function $G : \mathbb{P}^1_{\text{Berk}} \to \mathbb{R}$ is *Lipschitz continuous* if there is a constant $C > 0$ such that for all $x, y \in \mathbb{P}^1_{\text{Berk}}$,

$$|G(x) - G(y)| \ \leq \ C \cdot d(x, y) \ . \tag{10.57}$$

We will say that G is *Hölder continuous* if there are constants $C > 0$ and $0 < \kappa \leq 1$ such that

$$|G(x) - G(y)| \ \leq \ C \cdot d(x, y)^\kappa \ . \tag{10.58}$$

If κ is given and (10.58) holds, we will say that G is κ-Hölder continuous. In particular, G is Lipschitz continuous iff it is 1-Hölder continuous.

Example 10.48. For any pair of points $w, \zeta \in \mathbb{H}_{\text{Berk}}$, there is a continuous function $G \in \text{BDV}(\mathbb{P}^1_{\text{Berk}})$, uniquely determined up to an additive constant, which satisfies $\Delta G = \delta_w - \delta_\zeta$. Explicitly, $G(z) = -\log_v(\delta(z, w)_\zeta) + C$ for each $z \in \mathbb{P}^1_{\text{Berk}}$ (see Example 5.19 and Lemma 5.24). We claim that G is Lipschitz continuous.

To show this, it suffices to establish (10.57) when x, y are points with $y \in [x, \zeta_{\text{Gauss}}]$. Indeed, if there is a constant C such that (10.57) holds in this case and if arbitrary $x, y \in \mathbb{P}^1_{\text{Berk}}$ are given, put $z = x \vee y$. Then

$$\begin{aligned} |G(x) - G(y)| \ &\leq \ |G(x) - G(z)| + |G(z) - G(y)| \\ &\leq \ C \cdot d(x, z) + C \cdot d(z, y) \ = \ C \cdot d(x, y) \ . \end{aligned}$$

Let $r = \min(\text{diam}(w), \text{diam}(\zeta)) > 0$. Note that G is constant on branches off $\Gamma = [\zeta, w]$, that its restriction to Γ has constant slope 1 (relative to the path distance metric), and that $G(\zeta) \leq G(z) \leq G(w)$ for each $z \in \mathbb{P}^1_{\text{Berk}}$. Assuming $y \in [x, \zeta_{\text{Gauss}}]$, we now consider three possibilities:

First, if $\text{diam}(x), \text{diam}(y) \leq r$, then x and y belong to a single branch off Γ, so $G(x) = G(y)$ and (10.57) holds trivially for any C. Next, if $\text{diam}(x) \leq r/2$ but $\text{diam}(y) \geq r$, let $M = G(w) - G(\zeta)$; then (10.57) holds trivially for any $C \geq 2M/r$. Finally, if $\text{diam}(x), \text{diam}(y) \geq r/2$, let $C \geq 2/r$ be any Lipschitz constant for $\log_v(t)$ on the interval $[r/2, 1] \subset \mathbb{R}$. Then since $G|_{[x,y]}$ is piecewise differentiable and $|G'(z)| \leq 1$ for each $z \in [x, y]$, we have

$$\begin{aligned} |G(x) - G(y)| \ &\leq \ \rho(x, y) \ = \ \log_v(\text{diam}(y)) - \log_v(\text{diam}(x)) \\ &\leq \ C \cdot (\text{diam}(y) - \text{diam}(x)) \ = \ C \cdot d(x, y) \ . \qquad \square \end{aligned}$$

Let μ be a probability measure on $\mathbb{P}^1_{\text{Berk}}$ with continuous potentials, and let $g_\mu(x, y)$ be its normalized Arakelov-Green's function. We will say that μ has *Hölder continuous potentials* if $g_\mu(x, \zeta_{\text{Gauss}})$ is Hölder continuous.

To justify this terminology, we will show that if $G(x) := g_\mu(x, \zeta_{\text{Gauss}})$ is Hölder continuous, with Hölder exponent κ, then for each $y \in \mathbb{H}_{\text{Berk}}$ the function $G_y(x) = g_\mu(x, y)$ is κ-Hölder continuous as well. To see this, note that by (8.62), there is a constant B such that for all $x, y \in \mathbb{P}^1_{\text{Berk}}$

$$g_\mu(x, y) \;=\; -\log_v(\|x, y\|) + G(x) + G(y) + B \;. \tag{10.59}$$

Fix $y \in \mathbb{H}_{\text{Berk}}$. By Example 10.48, $-\log_v(\|x, y\|) = -\log_v(\delta(x, y)_{\zeta_{\text{Gauss}}})$ is Lipschitz continuous, which means that $G_y(x) = g_\mu(x, y)$ is κ-Hölder continuous. (However, the Hölder constant C for G_y may depend on y.)

THEOREM 10.49 (Favre, Rivera-Letelier). *Let $\varphi \in K(T)$ be a rational function with* $\deg(\varphi) \geq 2$. *Then the canonical measure μ_φ has Hölder continuous potentials.*

PROOF. Put $D = \deg(\varphi)$, write $\zeta = \zeta_{\text{Gauss}}$, and let $x_1, \ldots, x_D$ be the preimages of ζ under φ, listed with multiplicities. Let $g \in \text{BDV}(\mathbb{P}^1_{\text{Berk}})$ be a continuous function satisfying

$$\Delta g \;=\; \delta_\zeta - \frac{1}{D}\varphi^*(\delta_\zeta) \;=\; \sum_{k=1}^{D} \frac{1}{D}(\delta_\zeta - \delta_{x_k}).$$

By Example 10.48, such a function exists and is bounded and Lipschitz continuous. Let C_1 be the Lipschitz constant of g, and let C_2 be a bound such that $|g(x)| \leq C_2$ for all $x \in \mathbb{P}^1_{\text{Berk}}$.

For each $n \geq 1$, put

$$g_n \;=\; \sum_{k=0}^{n-1} \frac{1}{D^k}\, g \circ \varphi^{(k)} \;.$$

Then $g_n \in \text{BDV}(\mathbb{P}^1_{\text{Berk}})$, and $\Delta g_n = \delta_\zeta - (1/D^n)(\varphi^{(n)})^*(\delta_\zeta)$. The functions g_n converge uniformly on $\mathbb{P}^1_{\text{Berk}}$ to

$$g_\infty \;=\; \sum_{k=0}^{\infty} \frac{1}{D^k}\, g \circ \varphi^{(k)} \;. \tag{10.60}$$

By Theorem 10.36, the measures $(1/D^n)(\varphi^{(n)})^*(\delta_\zeta)$ converge weakly to μ_φ. Hence Proposition 5.32 shows that $g_\infty \in \text{BDV}(\mathbb{P}^1_{\text{Berk}})$ and that

$$\Delta(g_\infty) \;=\; \delta_\zeta - \mu_\varphi \;.$$

However, $\Delta(g_\mu(x, \zeta)) = \delta_\zeta - \mu_\varphi$ as well, and both $g_{\mu_\varphi}(x, \zeta)$ and $g_\infty(x)$ are continuous functions of x. By Lemma 5.24, there is a constant C_3 such that for each $x \in \mathbb{P}^1_{\text{Berk}}$,

$$g_{\mu_\varphi}(x, \zeta) \;=\; g_\infty(x) + C_3 \;.$$

We will now use the representation (10.60) to show that $g_{\mu_\varphi}(x,\zeta)$ is Hölder continuous. Let $N \ge 0$ be an integer to be specified later. By (10.60), for any $x, y \in \mathbb{P}^1_{\rm Berk}$,

$$(10.61)\quad \big|g_\infty(x) - g_\infty(y)\big| \ \le\ \sum_{k=0}^{N-1} \frac{1}{D^k}\left|g(\varphi^{(k)}(x)) - g(\varphi^{(k)}(y))\right| + \sum_{k=N}^{\infty} \frac{2C_2}{D^k}\ .$$

By Proposition 9.37, there is a constant M such that for all $z, w \in \mathbb{P}^1_{\rm Berk}$,

$$(10.62)\quad d(\varphi(z),\varphi(w)) \ \le\ M \cdot d(z,w)\ .$$

Iterating (10.62), we see that

$$(10.63)\quad \sum_{k=0}^{N-1} \frac{1}{D_k}\left|g(\varphi^{(k)}(x)) - g(\varphi^{(k)}(y))\right| \ \le\ \left(\sum_{k=0}^{N-1} \frac{C_1 M^k}{D^k}\right) d(x,y)\ .$$

Without loss of generality, we can assume that $M > D$ and that $d(x,y) < 1$. It follows from (10.61) and (10.63) that

$$(10.64)\quad \big|g_\infty(x) - g_\infty(y)\big| \ \le\ \frac{C_1}{\frac{M}{D}-1}\cdot\Big(\frac{M}{D}\Big)^N \cdot d(x,y) + \frac{2C_2}{D-1}\cdot\Big(\frac{1}{D}\Big)^N\ .$$

If we take $N = \lceil -\log(d(x,y))/\log(M)\rceil$, the two terms on the right side of (10.64) are of the same order of magnitude, and simple algebraic manipulations show that with $\kappa = \log(D)/\log(M)$, there is a constant C independent of x, y such that

$$(10.65)\quad \big|g_\infty(x) - g_\infty(y)\big| \ \le\ C \cdot d(x,y)^\kappa\ .$$

Thus $g_{\mu_\varphi}(x,\zeta)$ is Hölder continuous. □

10.5. The Berkovich Fatou and Julia sets

In this section, we define the Berkovich Fatou and Julia sets, give some characterizations of those sets, and note some of their structural properties. As in §10.4, we assume throughout this section that $\mathrm{char}(K) = 0$.

Definition 10.50. Let $\varphi \in K(T)$ be a rational map of degree $d \ge 2$. The *Berkovich Julia set* J_φ of φ is the support of the canonical measure μ_φ, and the *Berkovich Fatou set* F_φ is the complement in $\mathbb{P}^1_{\rm Berk}$ of J_φ.

Note that since the Berkovich Julia set of φ is the support of the probability measure μ_φ, it is by definition always nonempty. We will eventually give four characterizations of J_φ:

(1) J_φ is the support of the canonical measure μ_φ (Definition 10.50).
(2) J_φ is the locus of points $x \in \mathbb{P}^1_{\rm Berk}$ having the property that for every Berkovich neighborhood V of x, the union of the forward images $\varphi^{(n)}(V)$ contains $\mathbb{P}^1_{\rm Berk}\backslash E_\varphi(K)$ (and in particular contains all of $\mathbb{H}_{\rm Berk}$ and omits at most two points of $\mathbb{P}^1(K)$) (Theorem 10.56).

(3) J_φ is the smallest nonempty closed set in $\mathbb{P}^1_{\mathrm{Berk}}$, disjoint from the classical exceptional locus $E_\varphi(K)$, which is both forward and backward invariant under φ (Corollary 10.57).
(4) J_φ is the closure of the set of repelling periodic points in $\mathbb{P}^1_{\mathrm{Berk}}$ (Theorem 10.88).

The equivalence between these properties is due to Rivera-Letelier. For $K = \mathbb{C}_p$, he discussed (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) with us during a visit in 2005, and the general case is contained in [**80**]. The equivalence (1) $\Leftrightarrow$ (3) is stated in his note with Favre [**47**] and proved in [**48**, Theorem A]. The equivalence (2) $\Leftrightarrow$ (3) is stated in [**48**, Proposition 2.7]. In §10.6, we will discuss the relation between (1) and a more 'classical' definition of J_φ in terms of equicontinuity which holds when $K = \mathbb{C}_p$ but not in general; this fact is also due to Rivera-Letelier (unpublished).

Our first observation is that the Berkovich Julia sets (resp. Berkovich Fatou sets) of φ and $\varphi^{(n)}$ coincide for any integer $n \geq 1$:

LEMMA 10.51. *For any $n \geq 1$, we have $J_\varphi = J_{\varphi^{(n)}}$ and $F_\varphi = F_{\varphi^{(n)}}$.*

PROOF. This is an immediate consequence of Lemma 10.16. □

A subset $S \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ is called *completely invariant* for φ if $z \in S$ if and only if $\varphi(z) \in S$. We next observe:

LEMMA 10.52. *The Berkovich Julia and Fatou sets J_φ, F_φ are completely invariant for φ.*

PROOF. Write $\mu = \mu_\varphi$. If $\varphi(x) \in F_\varphi$, then there is an open neighborhood V of $\varphi(x)$ such that $\mu(V) = 0$. But then $\mu(\varphi^{-1}(V)) = (\varphi_*\mu)(V) = \mu(V) = 0$. As x belongs to the open set $\varphi^{-1}(V)$, it follows that $x \notin \mathrm{supp}(\mu)$, that is, $x \in F_\varphi$. Thus F_φ is backward invariant for φ.

Conversely, if $x \in F_\varphi = \mathrm{supp}(\mu)^c$, then there are an open neighborhood U of x and a continuous function f on $\mathbb{P}^1_{\mathrm{Berk}}$ such that $f \equiv 1$ on U and $f \equiv 0$ on $\mathrm{supp}(\mu)$. Let $V = \varphi(U)$, which is an open neighborhood of $\varphi(x)$ by the Open Mapping Theorem. Then $(\varphi_*f)(z) \geq 1$ for all $z \in V$, and $(\varphi_*f)(z) \geq 0$ for all $z \in \mathbb{P}^1_{\mathrm{Berk}}$. Thus

$$0 \;\leq\; \mu(V) \;\leq\; \int_{\mathbb{P}^1_{\mathrm{Berk}}} (\varphi_*f)\, d\mu = \int_{\mathbb{P}^1_{\mathrm{Berk}}} f\, d\varphi^*\mu = \deg(\varphi)\cdot \int_{\mathbb{P}^1_{\mathrm{Berk}}} f\, d\mu \;=\; 0$$

since $f \equiv 0$ on $\mathrm{supp}(\mu)$ and $\varphi^*\mu = \deg(\varphi)\cdot\mu$. It follows that $\mu(V) = 0$, and therefore $\varphi(x) \notin \mathrm{supp}(\mu)$, that is, $\varphi(x) \in F_\varphi$. □

We now use Proposition 10.45 to characterize maps with simple reduction in terms of their Julia sets. Recall that φ has *simple reduction* if, after a change of coordinates, it has good reduction.

LEMMA 10.53. *The Julia set J_φ consists of a single point* (*which is necessarily a point of $\mathbb{H}_{\mathrm{Berk}}$ of type* II) *if and only if φ has simple reduction.*

Proof. This is an immediate consequence of Corollary 10.47, but the following self-contained argument is easier.

If $J_\varphi = \{\zeta\}$ consists of a single point, then the result follows from Proposition 10.45, since ζ is then exceptional by the complete invariance of the Berkovich Julia set.

On the other hand, suppose φ has simple reduction. If φ has *good* reduction, then Example 10.4 shows that μ_φ is a point mass supported at the Gauss point. For the general case, if φ has good reduction after conjugating by a Möbius transformation $M \in \mathrm{PGL}(2, K)$, then by Corollary 10.15 the measure μ_φ is a point mass supported at the type II point $M(\zeta_{\mathrm{Gauss}})$. □

The following result and its corollary will be used several times below:

Lemma 10.54. *Every attracting periodic point $x \in \mathbb{P}^1(K)$ for φ is contained in the Berkovich Fatou set.*

Proof. Let $x \in \mathbb{P}^1(K)$ be an attracting periodic point of period n. Replacing φ by $\varphi^{(n)}$, we may assume that x is a fixed point of φ. Let U be a small open disc in $\mathbb{P}^1(K)$ centered at x which is contained in the (classical) attracting basin of x, and let $y \in U$ be a nonexceptional point of $\mathbb{P}^1(K)$. By Theorem 10.36, J_φ is contained in the closure of the set of inverse iterates of y. Since $\varphi(U) \subseteq U$, none of the inverse iterates of y is contained in U, and therefore U is disjoint from the closure of the set of inverse iterates of y. In particular, it follows that $x \notin J_\varphi$. □

Corollary 10.55. *Every point in $E_\varphi(K)$ is contained in F_φ.*

Proof. Every exceptional point in $\mathbb{P}^1(K)$ is a (super)attracting fixed point, so the result follows from Lemma 10.54. □

The following result is another important consequence of the equidistribution theorem for preimages of nonexceptional points (Theorem 10.36):

Theorem 10.56. *The Berkovich Fatou set F_φ coincides with*

(A) *the set of all points $x \in \mathbb{P}^1_{\mathrm{Berk}}$ having a neighborhood V whose forward iterates under φ omit at least one point of $\mathbb{P}^1_{\mathrm{Berk}} \backslash E_\varphi(K)$,*

(B) *the set of all points $x \in \mathbb{P}^1_{\mathrm{Berk}}$ having a neighborhood V whose forward iterates under φ omit at least three points of $\mathbb{P}^1(K)$.*

Equivalently, the Berkovich Julia set J_φ coincides with

(A′) *the set of all points in $x \in \mathbb{P}^1_{\mathrm{Berk}}$ such that for each neighborhood V of x, the union $\bigcup_{n=1}^\infty \varphi^{(n)}(V)$ contains $\mathbb{P}^1_{\mathrm{Berk}} \backslash E_\varphi(K)$,*

(B′) *the set of all points in $x \in \mathbb{P}^1_{\mathrm{Berk}}$ such that for each neighborhood V of x, the union $\bigcup_{n=1}^\infty \varphi^{(n)}(V)$ omits at most two points of $\mathbb{P}^1(K)$.*

Proof. It suffices to prove the assertions concerning F_φ.

We first prove (A). Let F' be the set of all points in $\mathbb{P}^1_{\mathrm{Berk}}$ having a neighborhood V whose forward iterates under φ omit at least one point of $\mathbb{P}^1_{\mathrm{Berk}} \backslash E_\varphi(K)$. We want to show that $F' = F_\varphi$. First, suppose that $z \in F'$;

then one can find a neighborhood V of z and a point $y \in \mathbb{P}^1_{\mathrm{Berk}} \backslash E_\varphi(K)$ not belonging to $\varphi^{(n)}(V)$ for any $n \geq 1$. If $\mu^y_{\varphi^{(n)}}$ denotes the probability measure $d^{-n}(\varphi^{(n)})^*(\delta_y)$, then $V \cap \mathrm{supp}(\mu^y_{\varphi^{(n)}}) = \emptyset$ for all n, and thus $V \cap \mathrm{supp}(\mu_\varphi) = \emptyset$ by Theorem 10.36. It follows that $F' \subseteq F_\varphi$.

Conversely, let J' be the complement of F'. We claim that J' is the smallest completely invariant nonempty compact subset of $\mathbb{P}^1_{\mathrm{Berk}}$ disjoint from $E_\varphi(K) := E_\varphi \cap \mathbb{P}^1(K)$. Assuming the claim, since $J_\varphi = \mathrm{supp}(\mu_\varphi)$ is completely invariant under φ, we obtain $J' \subseteq J_\varphi$, and hence $F_\varphi \subseteq F'$ as desired. (Note that $J_\varphi \cap E_\varphi(K) = \emptyset$ by Corollary 10.55.)

To prove the claim, we must show that if Z is any nonempty completely invariant compact subset of $\mathbb{P}^1_{\mathrm{Berk}}$ which is disjoint from $E_\varphi(K)$, then Z contains J'. If U denotes the complement of Z, then U is also completely invariant under φ. In particular, $\varphi^{(n)}(U) \subseteq U$ for all $n \geq 1$. If φ does not have simple reduction, then it follows from Proposition 10.45 that $Z = U^c$ contains a nonexceptional point, and thus U is contained in F', which is equivalent to the assertion of the claim.

It remains to prove the claim in the case where φ has simple reduction and $Z = \{\zeta\}$ with $\zeta \in \mathbb{H}_{\mathrm{Berk}}$ an exceptional point of φ. Conjugating by a Möbius transformation, we may assume without loss of generality that φ has *good* reduction and that $Z = \{\zeta_{\mathrm{Gauss}}\}$. In this case, since we already know that $J' \neq \emptyset$, it suffices to prove that every point of $U = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta_{\mathrm{Gauss}}\}$ belongs to F'. Define the *chordal diameter* of an open set $V \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ to be $\sup_{x,y \in V \cap \mathbb{P}^1(K)} \|x, y\|$. If x is any point of U and V is any sufficiently small connected open neighborhood of x contained in U, then the chordal diameter of V is some positive real number $\delta < 1$. Since φ has good reduction, it follows from [**93**, Theorem 2.17(a)] that φ is everywhere nonexpanding with respect to the spherical metric on $\mathbb{P}^1(K)$; this implies that the open set $\varphi^{(n)}(V)$ has chordal diameter at most δ for all $n \geq 1$. If y is any point of $\mathbb{H}_{\mathrm{Berk}}$ such that $0 < \rho(y, \zeta_{\mathrm{Gauss}}) < 1 - \delta$, then since the generalized Hsia kernel $\delta(\cdot, \cdot)_{\zeta_{\mathrm{Gauss}}}$ extends the spherical metric on $\mathbb{P}^1(K)$, one verifies easily that y cannot belong to $\varphi^{(n)}(V)$ for any $n \geq 0$. Since y is automatically nonexceptional, it follows that x belongs to F' as desired. This completes the proof of (A).

The proof of (B) is similar; the only additional thing which needs to be proved is that if φ has good reduction and $x \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta_{\mathrm{Gauss}}\}$, then with V chosen as above, there exists a nonexceptional point in $\mathbb{P}^1(K)$ which does not belong to $\bigcup_{n \geq 0} \varphi^{(n)}(V)$. With y chosen as above, any (nonexceptional) point $z \in \mathbb{P}^1(K)$ whose retraction to the segment $[y, \zeta_{\mathrm{Gauss}}]$ lies in the open interval $(y, \zeta_{\mathrm{Gauss}})$ will work. □

From the proof of Theorem 10.56, we obtain:

COROLLARY 10.57. *The Berkovich Julia set J_φ is the smallest completely invariant nonempty compact subset of $\mathbb{P}^1_{\mathrm{Berk}}$ disjoint from $E_\varphi(K)$.*

The following is an immediate consequence of the definition of J_φ and Theorem 10.36:

COROLLARY 10.58. *If z_0 is any point of $\mathbb{P}^1_{\text{Berk}} \backslash E_\varphi(K)$, the closure of the backward orbit of z_0 contains J_φ. In particular, the backward orbit of any point in J_φ is dense in J_φ.*

The following consequence of Theorem 10.56 is analogous to Theorem 4.7 of [**73**].

THEOREM 10.59 (Transitivity Theorem). *Let $x \in J_\varphi$ and let V be an open neighborhood of x. Let U be the union of the forward images $\varphi^{(n)}(V)$ for $n \geq 0$. Then:*

(A) *U contains $\mathbb{P}^1_{\text{Berk}} \backslash E_\varphi(K)$: in particular, if $\text{char}(K) = 0$, it contains all of $\mathbb{H}_{\text{Berk}}$ and all but at most two points of $\mathbb{P}^1(K)$.*
(B) *U contains all of J_φ.*
(C) *If V is sufficiently small, then $U = \mathbb{P}^1_{\text{Berk}} \backslash E_\varphi(K)$.*

PROOF. (A) We first show that $\mathbb{P}^1(K) \backslash E_\varphi(K) \subset U$. Note that the set $\mathbb{P}^1(K) \backslash U$ can contain at most two points; otherwise, since $\varphi(U) \subseteq U$, it would follow from Theorem 10.56(B) that U is contained in the Fatou set, contradicting the fact that x belongs to both J_φ and U. Let $z \in \mathbb{P}^1(K) \backslash U$. Since $\varphi(U) \subseteq U$, each iterated preimage of z must belong to the finite set $\mathbb{P}^1(K) \backslash U$. It follows that z is exceptional, that is, $z \in E_\varphi(K)$.

The fact that $\mathbb{H}_{\text{Berk}}$ is contained in U follows similarly. If $y \in \mathbb{H}_{\text{Berk}} \backslash U$, then certainly $y \notin E_\varphi(K)$. By Corollary 10.58, the set $\{\varphi^{(n)*}(y)\}_{n \geq 1}$ has J_φ in its closure. On the other hand, $\varphi(U) \subset U$, so these preimages belong to $\mathbb{P}^1_{\text{Berk}} \backslash U$, meaning that $J_\varphi \subset \mathbb{P}^1_{\text{Berk}} \backslash U$. However this is a contradiction, since $x \in J_\varphi$ and $x \in U$.

(B) This is immediate from part (A) and Corollary 10.57.

(C) If $V \subseteq \mathbb{P}^1_{\text{Berk}} \backslash E_\varphi$, it follows from the proof of (A) that $U = \mathbb{P}^1_{\text{Berk}} \backslash E_\varphi$. □

COROLLARY 10.60. *If φ does not have simple reduction, then J_φ is perfect (that is, contains no isolated points). In particular, J_φ either consists of a single point or it is uncountable.*

PROOF. It suffices to prove that J_φ contains a dense set of nonisolated points. By Proposition 10.45, we see that $E_\varphi \cap \mathbb{H}_{\text{Berk}} = \emptyset$. Furthermore, J_φ is infinite, since if it were finite, it would consist of exceptional points in $\mathbb{P}^1(K)$, which are necessarily in the Fatou set by Corollary 10.55. So J_φ contains at least one limit point z_0, and z_0 is nonexceptional. Since the measures $(1/d^n)\varphi^{(n)*}(\delta_{z_0})$ converge weakly to μ_φ, the iterated preimages of z_0 form a dense set of points in J_φ. If $w \in (\varphi^{(n)})^{-1}(\{z_0\})$, then for any neighborhood U of w, the image $\varphi^{(n)}(U)$ is a neighborhood of z_0 (Corollary 9.10). Since J_φ is backwards invariant, U contains points of J_φ distinct from z_0. As U is arbitrary, it follows that w is also a limit point of J_φ. □

COROLLARY 10.61. *The Julia set J_φ is either connected or else has uncountably many connected components.*

PROOF. In the case where J_φ is perfect, the proof is identical to its classical counterpart (see [**73**, Corollary 4.15]). The only other possibility is that J_φ consists of a single point, in which case it is trivially connected. $\square$

10.6. Equicontinuity

In classical non-Archimedean dynamics, one usually defines the Fatou set of a rational map in terms of equicontinuity. Here we explore the relation between our definition and definitions in terms of equicontinuity. We assume throughout that $\text{char}(K) = 0$ and that $\varphi(T) \in K(T)$.

For $x, y \in \mathbb{P}^1(K)$, we write $\|x, y\|$ for the spherical distance (see §4.3), and for $0 < r \in \mathbb{R}$ we write $B(x, r)^- = \{y \in \mathbb{P}^1(K) : \|x, y\| < r\}$.

DEFINITION 10.62 (Equicontinuity on $\mathbb{P}^1(K)$). Let U be an open subset of $\mathbb{P}^1(K)$, and let $\mathcal{F}$ be a collection of functions $f : U \to \mathbb{P}^1(K)$. We say that $\mathcal{F}$ is *equicontinuous* on U if for every $x \in U$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f(B(x,\delta)^-) \subseteq B(f(x),\varepsilon)^-$$

for every $f \in \mathcal{F}$.

A point $x \in \mathbb{P}^1(K)$ is in the *classical Fatou set* for φ iff the collection $\{\varphi^{(n)}\}$ of iterates of φ is equicontinuous on some open neighborhood of x in $\mathbb{P}^1(K)$. The *classical Julia set* is defined to be the complement of the classical Fatou set in $\mathbb{P}^1(K)$.

We now seek to extend the definition of equicontinuity so that it works in the context of $\mathbb{P}^1_{\text{Berk}}$, which need not be metrizable. Fortunately, since $\mathbb{P}^1_{\text{Berk}}$ is a compact Hausdorff space, it has a canonical *uniform structure* (cf. §A.9), and the theory of uniform spaces provides for us a natural notion of equicontinuity. Concretely, this yields the following definition:

DEFINITION 10.63 (Equicontinuity on $\mathbb{P}^1_{\text{Berk}}$). If U is an open subset of $\mathbb{P}^1_{\text{Berk}}$ and $\mathcal{F}$ is a collection of functions $f : U \to \mathbb{P}^1_{\text{Berk}}$, we say that $\mathcal{F}$ is *equicontinuous* on U if for every $x \in U$ and every finite covering $\mathcal{C} = V_1 \cup \cdots \cup V_t$ of $\mathbb{P}^1_{\text{Berk}}$ by simple domains, there exists an open neighborhood U_x of x in $\mathbb{P}^1_{\text{Berk}}$ such that $f(U_x)$ is contained in some V_i for every $f \in \mathcal{F}$.

We define the *Berkovich equicontinuity locus* for φ to be the set of all $x \in \mathbb{P}^1_{\text{Berk}}$ for which the collection $\{\varphi^{(n)}\}$ of iterates of φ is equicontinuous on some Berkovich open neighborhood of x.

THEOREM 10.64. *The Berkovich equicontinuity locus and the classical Fatou sets are both contained in the Berkovich Fatou set.*

PROOF. Let x_0 be a point in either the Berkovich equicontinuity locus or the classical Fatou set. We will show that if U is a sufficiently small open neighborhood of x_0, then μ_φ restricted to U is the zero measure.

First assume that $x_0 \in \mathbb{A}^1_{\mathrm{Berk}}$. By Theorem 10.2, it suffices to show that $\hat{h}_{\varphi,v,(\infty)}$ is harmonic on some neighborhood $U \subset \mathbb{A}^1_{\mathrm{Berk}}$ of x_0. To do this, we follow an argument of Fornaess and Sibony [**52**], in outline. Define

$$\begin{aligned} O &= \{x \in \mathbb{A}^1_{\mathrm{Berk}} \ : \ [T]_x < 2\} \ , \\ E &= \{x \in \mathbb{A}^1_{\mathrm{Berk}} \ : \ [T]_x \le \frac{1}{2}\} \ . \end{aligned}$$

We claim that there exists a neighborhood U of x_0 such that for each $n \ge 1$, either $\varphi^{(n)}(U) \subseteq O$ or $\varphi^{(n)}(U) \subseteq \mathbb{P}^1_{\mathrm{Berk}} \backslash E$. Indeed, if x_0 is in the Berkovich equicontinuity locus, then this follows directly from the fact that $\{O, E^c\}$ is a finite covering of $\mathbb{P}^1_{\mathrm{Berk}}$ by simple domains, while if x_0 is in the classical Fatou set, it follows easily from the ultrametric inequality. Passing to a subsequence $\varphi^{(n_k)}$ and replacing φ by $1/\varphi$ if necessary, we may therefore assume without loss of generality that $\varphi^{(n_k)}(U) \subseteq O$ for all k, or, equivalently, that $[\varphi^{(n_k)}(T)]_x < 2$ for all $x \in U$ and all k. In particular, in the notation of §10.1, $F_1^{(n_k)}(1,T) \ne 0$ on U for all k.

For $x \in U$ we have

$$\begin{aligned} \hat{h}^{(n_k)}_{\varphi,v,(\infty)}(x) &= \frac{1}{d^{n_k}} \max(\log_v([F_1^{(n_k)}(1,T)]_x), \log_v([F_2^{(n_k)}(1,T)]_x)) \\ &= \frac{1}{d^{n_k}} \log_v([F_1^{(n_k)}(1,T)]_x) + \frac{1}{d^{n_k}} \max(1, \log_v([\varphi^{(n_k)}(T)]_x)) \ . \qquad (10.66) \end{aligned}$$

The last term in (10.66) converges uniformly to 0, since the quantity $[\varphi^{(n_k)}(T)]_x$ is uniformly bounded as k varies. Moreover, the term $\frac{1}{d^{n_k}} \log_v([F_1^{(n_k)}(1,T)]_x)$ is harmonic on U for all k. Since $\hat{h}^{(n_k)}_{\varphi,v,(\infty)}$ converges uniformly to $\hat{h}_{\varphi,v,(\infty)}$, it follows that $\frac{1}{d^{n_k}} \log_v([F_1^{(n_k)}(1,T)]_x)$ converges uniformly to $\hat{h}_{\varphi,v,(\infty)}$ on U. Therefore $\hat{h}_{\varphi,v,(\infty)}$ is harmonic on U as desired.

Finally, for the case $x_0 = \infty$, apply a similar argument to the function $\hat{h}_{\varphi,v,(0)}$, using the coordinate function $1/T$ instead of T. □

We now prove that the intersection of F_φ with $\mathbb{P}^1(K)$ is precisely the classical Fatou set. The proof will make use of the following non-Archimedean Montel theorem due to Hsia [**61**]:

THEOREM 10.65 (Hsia's theorem). *Let $D \subset \mathbb{P}^1(K)$ be a closed disc, and let $\mathcal{F}$ be a family of meromorphic functions on D. Suppose that $\bigcup_{f \in \mathcal{F}} f(D)$ omits two distinct points of $\mathbb{P}^1(K)$. Then $\mathcal{F}$ is equicontinuous on D.*

REMARK 10.66. There does not seem to be a satisfactory way to generalize Hsia's theorem to $\mathbb{P}^1_{\mathrm{Berk}}$. This is one of the reasons why we have defined the Berkovich Julia set as the support of μ_φ and not in terms of equicontinuity. For example, let $\{\alpha_n\}$ be a set of coset representatives for the different residue classes in $D(0,1)$ over $\mathbb{C}_p$ and let $f_n = z + \alpha_n$. Then the sequence of functions $f_n = z + \alpha_n$ takes $\mathcal{D}(0,1)$ to itself (so $\bigcup_{n \ge 1} f_n(\mathcal{D}(0,1))$ omits an entire open disc in $\mathbb{P}^1_{\mathrm{Berk}}$) but it is not equicontinuous at the Gauss point of $\mathbb{P}^1_{\mathrm{Berk}}$. See Example 10.70 and Remark 10.71 below for related considerations.

THEOREM 10.67. *If $x \in \mathbb{P}^1(K)$, then x is in the Berkovich Fatou set of φ if and only if x is in the classical Fatou set of φ.*

PROOF. If x is in the classical Fatou set of φ, then we have just seen that x belongs to the Berkovich Fatou set of φ.

Conversely, suppose $x \in \mathbb{P}^1(K)$ belongs to the Berkovich Fatou set of φ. By Theorem 10.56(B), there is a Berkovich disc V containing x such that the union U of the forward orbits of V omits at least three points y_1, y_2, y_3 of $\mathbb{P}^1(K)$. At least one of these, say y_1, must be nonexceptional. Since $U \cap \mathbb{P}^1(K)$ omits the infinitely many preimages of y_1, Hsia's theorem shows x is contained in the classical Fatou set of φ. Thus the intersection of the Berkovich Fatou set with $\mathbb{P}^1(K)$ is contained in the classical Fatou set. □

By a theorem of Benedetto (see [**13**, Corollary 1.3]), the classical Fatou set of φ is always nonempty (in contrast to the situation over the complex numbers). We therefore conclude from Theorem 10.67:

COROLLARY 10.68. *The Berkovich Fatou set F_φ is always nonempty.*

The following corollary is stronger than what one obtains classically over $\mathbb{C}$, due to the absence in the non-Archimedean setting of rational maps with empty Fatou set:

COROLLARY 10.69. *The Berkovich Julia set J_φ always has empty interior.*

PROOF. If J_φ has an interior point z_1, then choosing a neighborhood $V \subset J_\varphi$ of z_1, the union U of forward images of V under φ is both contained in J_φ (by the forward invariance of J) and everywhere dense in $\mathbb{P}^1_{\text{Berk}}$ (since $\mathbb{P}^1_{\text{Berk}} \backslash U$ contains at most 2 points by the Transitivity Theorem). Since J_φ is closed, this implies that $J_\varphi = \mathbb{P}^1_{\text{Berk}}$ and thus that F_φ is empty, contradicting Corollary 10.68. □

Theorem 10.64 asserts that the Berkovich equicontinuity locus is always contained in the Berkovich Fatou set. When $K = \mathbb{C}_p$, we will see in Theorem 10.72 that the reverse inclusion holds as well, so that the Berkovich Fatou set coincides with the Berkovich equicontinuity locus. This relies on deep results of Rivera-Letelier concerning the structure of the Berkovich Fatou set over $\mathbb{C}_p$. However, for a general complete and algebraically closed non-Archimedean field K of characteristic 0, the Berkovich Fatou set need not be contained in the Berkovich equicontinuity locus, as we now show.

EXAMPLE 10.70 (A rational map whose Berkovich Fatou set is not contained in the Berkovich equicontinuity locus). Let p be an odd prime, and let K be a complete and algebraically closed non-Archimedean field of characteristic 0 and residue characteristic $p > 0$, having an element c satisfying $|c| = 1$ whose image in the residue field $\tilde{K}$ is transcendental over the prime field $\mathbb{F}_p$. Consider the polynomial map $\varphi(z) = pz^2 + cz$.

Then φ is conjugate to the map $\varphi'(w) = w^2 + cw$, which has good reduction, so J_φ is a point mass supported at a point $\zeta \in \mathbb{H}_{\mathrm{Berk},K}$ with $\zeta \neq \zeta_{\mathrm{Gauss}}$. In particular, the Gauss point of $\mathbb{P}^1_{\mathrm{Berk},K}$ belongs to the Berkovich Fatou set of φ.

However, we claim that $\{\varphi^{(n)}\}$ is not equicontinuous at the Gauss point. To see this, first note that if $|a|, r < p$, then $\varphi(D(a,r)) = D(pa^2 + ca, r)$ and $\varphi(D(a,r)^-) = D(pa^2 + ca, r)^-$. In particular, $\varphi(D(0,1)) = D(0,1)$, so φ fixes the Gauss point. Moreover, $\varphi(D(0,1)^-) = D(0,1)^-$ and if $|a| = 1$, then $\varphi(D(a,1)^-) = D(ca,1)^-$. In particular, the open disc $V' = D(1,1)^-$ has infinitely many distinct preimages under φ, by our assumption on c. Let $\hat{V}' = \mathcal{D}(1,1)^-$ be the Berkovich open disc corresponding to V'. Let $\hat{V}$ be any simple Berkovich neighborhood of the Gauss point containing the complement of $\hat{V}'$ but not $\hat{V}'$ itself. Then $\{\hat{V}, \hat{V}'\}$ is a finite covering of $\mathbb{P}^1_{\mathrm{Berk}}$ by simple domains. Suppose that $\{\varphi^{(n)}\}$ is equicontinuous at ζ_{Gauss}. By the definition of equicontinuity, since $\zeta_{\mathrm{Gauss}} \notin \hat{V}'$, there must exist a simple domain $\hat{U}$ containing ζ_{Gauss} such that $\varphi^{(n)}(\hat{U}) \subseteq \hat{V}$ for all $n \geq 1$. However, this means that $\hat{U}$ must omit some element of the residue class $\varphi^{-n}(\hat{V}') = \mathcal{D}(c^{-n}a, 1)^-$, for each n. By hypothesis, this means that $\hat{U}$ is missing elements from infinitely many residue classes. On the other hand, a simple domain containing ζ_{Gauss} automatically contains all but finitely many residue classes. This contradiction shows that $\{\varphi^{(n)}\}$ is not equicontinuous at the Gauss point, as claimed.

Remark 10.71. Philosophically, the point ζ_{Gauss} in Example 10.70 should be thought of as belonging to the Fatou set, since the iteration of φ in a small neighborhood of ζ_{Gauss} is quite understandable and predictable, and in some sense nearby points stay close together under iteration. However, there does not seem to be a notion of equicontinuity on $\mathbb{P}^1_{\mathrm{Berk}}$ which captures this philosophical sentiment in a precise way.

For this reason we have defined the Berkovich Fatou and Julia sets without using equicontinuity (contrary to our initial inclination). We would like to thank Rivera-Letelier for bringing examples like Example 10.70 to our attention and for his patient insistence that for arbitrary complete algebraically closed fields K, equicontinuity should not be thought of as the basis for the fundamental dichotomy between order and chaos in the context of iteration on $\mathbb{P}^1_{\mathrm{Berk}}$.

We conclude this section by sketching a proof that when $K = \mathbb{C}_p$, the Berkovich Fatou set coincides with the Berkovich equicontinuity locus.

Theorem 10.72 (Rivera-Letelier). *Let $\varphi \in \mathbb{C}_p(T)$ be a rational function with* $\deg(\varphi) \geq 2$. *Take $x_0 \in \mathbb{P}^1_{\mathrm{Berk}}$, and suppose there exists a neighborhood V of x_0 in $\mathbb{P}^1_{\mathrm{Berk}}$ such that $\bigcup_{n=1}^{\infty} \varphi^{(n)}(V)$ omits at least three points of $\mathbb{P}^1(\mathbb{C}_p)$ or at least one point of $\mathbb{H}_{\mathrm{Berk}}$. Then x is in the Berkovich equicontinuity locus of φ.*

Using Theorem 10.56 and Theorem 10.64, Theorem 10.72 implies:

COROLLARY 10.73. *Suppose* $\varphi \in \mathbb{C}_p(T)$*, and let* $\deg(\varphi) \geq 2$*. Then the Berkovich Fatou set of* φ *equals the Berkovich equicontinuity locus of* φ*.*

The proof of Theorem 10.72 is based on Rivera-Letelier's classification of the periodic components of F_φ, which we now recall[9] (cf. [**81, 84**]). Let $\varphi \in \mathbb{C}_p(T)$ be a rational map of degree $d \geq 2$, and define F_φ^{RL} to be the set of $x \in \mathbb{P}^1_{\mathrm{Berk}}$ for which there exists a neighborhood V of x in $\mathbb{P}^1_{\mathrm{Berk}}$ such that $\bigcup_{n=1}^\infty \varphi^{(n)}(V)$ omits at least three points of $\mathbb{P}^1(\mathbb{C}_p)$. By Theorem 10.56, F_φ^{RL} coincides with F_φ. Rivera-Letelier defines the following two types of components of F_φ^{RL}:

Immediate basins of attraction: If $z_0 \in \mathbb{P}^1(\mathbb{C}_p)$ is an attracting periodic point of period n, its *basin of attraction* is the set

$$\mathcal{A}_{z_0}(\varphi) \;=\; \{z \in \mathbb{P}^1_{\mathrm{Berk}} \;:\; \varphi^{(nk)}(z) \to z_0 \text{ when } k \to \infty\} \;.$$

One easily shows that $\mathcal{A}_{z_0}(\varphi)$ is open and invariant under $\varphi^{(n)}$. The connected component $\mathcal{A}^0_{z_0}(\varphi)$ of $\mathcal{A}_{z_0}(\varphi)$ containing z_0 is called the *immediate basin of attraction* of z_0; it is also open and invariant under $\varphi^{(n)}$. Rivera-Letelier has shown that $\mathcal{A}_{z_0}(\varphi)$ is contained in F_φ^{RL} and that $\mathcal{A}^0_{z_0}(\varphi)$ is a connected component of F_φ^{RL}. He gives the following description of the action of $\varphi(T)$ on $\mathcal{A}^0_{z_0}(\varphi)$ [**81**, pp. 199–200]:

PROPOSITION 10.74. *Let* $\varphi(T) \in \mathbb{C}_p(T)$ *have degree* $d \geq 2$*, and let* $\mathcal{A}^0_{z_0}(\varphi) \subset \mathbb{P}^1_{\mathrm{Berk}}$ *be the immediate basin of attraction of an attracting periodic point* z_0 *of period* n*. Then there is a decreasing set of open neighborhoods* $\{X_k\}_{k\in\mathbb{Z}}$ *of* z_0*, cofinal in the collection of all neighborhoods of* z_0*, such that* X_k *is a disc for all* $k \geq 0$*,* $\bigcup_k X_k = \mathcal{A}^0_{z_0}(\varphi)$*,* $\bigcap_k X_k = \{z_0\}$*, and* $\varphi^{(n)}(X_k) = X_{k+1}$ *for each* $k \in \mathbb{Z}$*.*

The domain of quasi-periodicity: The *domain of quasi-periodicity* of φ, denoted $\mathcal{E}(\varphi)$, is defined to be the interior of the set of points in $\mathbb{P}^1_{\mathrm{Berk}}$ which are recurrent under φ. (A point is called *recurrent* if it lies in the closure of its forward orbit.) By definition, $\mathcal{E}(\varphi)$ is open and invariant under φ and is disjoint from each basin of attraction. Rivera-Letelier has shown that $\mathcal{E}(\varphi)$ is contained in F_φ^{RL} and that each connected component of $\mathcal{E}(\varphi)$ is in fact a connected component of F_φ^{RL}. The action of $\varphi(T)$ on $\mathcal{E}(\varphi)$ has the following property [**81**, Proposition 4.14]; see our Proposition 10.117 below and the discussion preceding it:

PROPOSITION 10.75. *Let* $\varphi(T) \in \mathbb{C}_p(T)$ *have degree* $d \geq 2$*. For each* $x \in \mathcal{E}(\varphi) \cap \mathbb{H}_{\mathrm{Berk}}$*, there is an* $n \geq 1$ *(depending on* x*) such that* $\varphi^{(n)}(x) = x$*. For each connected open affinoid* X *with* $\overline{X} \subset \mathcal{E}(\varphi)$*, there is an* $N \geq 1$ *(depending on* X*) such that* $\varphi^{(N)}(X) = X$*.*

[9]Technically Rivera-Letelier only works with $F_\varphi \cap \mathbb{P}^1(\mathbb{C}_p)$; in passing to F_φ, we are glossing over some subtleties which will be dealt with in §10.9.

Rivera-Letelier's classification of Fatou components says that immediate basins of attraction and components of the domain of quasi-periodicity are the only periodic Fatou components [**83**, Theorem A]:

Theorem 10.76 (Rivera-Letelier). *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree at least 2. Then every periodic connected component of F_φ^{RL} is either an immediate basin of attraction or a connected component of the domain of quasi-periodicity of φ.*

Remark 10.77. The proof of Theorem 10.76 uses the fact that the residue field of $\mathbb{C}_p$ is a union of finite fields, and in particular the proof does not extend to rational functions defined over an arbitrary K.

A component of F_φ^{RL} which is not preperiodic is called a *wandering component*. One deduces from Theorem 10.76 the following result:

Corollary 10.78 (Rivera-Letelier). *For each connected component U of F_φ^{RL}, exactly one of the following holds:*

(1) *$\varphi^{(m)}(U)$ is an immediate basin of attraction for some $m \geq 0$.*
(2) *$\varphi^{(m)}(U)$ is a connected component of the domain of quasi-periodicity for some $m \geq 0$.*
(3) *U is a wandering component.*

We can now prove Theorem 10.72.

Proof of Theorem 10.72. Take $x_0 \in F_\varphi^{RL}$. Then the connected component U of F_φ^{RL} containing x_0 is one of the three types in the statement of Corollary 10.78. We will prove the result separately for each case. Fix an open covering $V_1, \ldots, V_t$ of $\mathbb{P}^1_{\mathrm{Berk}}$ by finitely many simple domains.

First suppose $\varphi^{(m)}(U)$ is the immediate basin of attraction of an attracting periodic point z_0.

We begin with the case where U itself is the immediate basin of attraction. Let n be the period of z_0, and let W be a neighborhood of z_0 small enough so that for each $\ell = 0, \ldots, n-1$, we have $\varphi^{(\ell)}(W) \subset V_i$ for some i.

Take an affinoid neighborhood Z of x_0 whose closure $\overline{Z}$ is contained in U. By Proposition 10.74, there is an exhaustion of U by connected open affinoids $\{X_k\}_{k \in \mathbb{Z}}$, with $X_k \subset W$ for all large k, such that $\varphi^{(n)}(X_k) = X_{k+1}$ for each k. Since $\overline{Z}$ is compact, it is contained in X_{k_0} for some k_0. Hence there is a k_1 such that $\varphi^{(nk_1)}(Z) \subset W$. By our choice of W, for each $k \geq k_1$ there is some V_i with $\varphi^{(k)}(Z) \subset V_i$. After shrinking Z if necessary, we can assume this holds for $0 \leq k < k_1$ as well. Thus the $\varphi^{(k)}(T)$ are equicontinuous at x_0.

More generally, if there is an $m \geq 1$ such that $\varphi^{(m)}(U)$ is an immediate basin of attraction and if Z is the neighborhood constructed above for $\varphi^{(m)}(x_0)$, then we can choose a neighborhood Z_0 of x_0 small enough so that $\varphi^{(m)}(Z_0) \subset Z$, and for each $\ell = 0, 1, \ldots, m-1$ there is some i for which $\varphi^{(\ell)}(Z_0) \subset V_i$. So once again the family $\varphi^{(k)}(T)$ is equicontinuous at x_0.

Next suppose that $\varphi^{(m)}(U)$ is a connected component of $\mathcal{E}(\varphi)$. We begin with the case where U itself is a component of $\mathcal{E}(\varphi)$. Take $x_0 \in U$.

First assume $x_0 \in \mathbb{P}^1(\mathbb{C}_p)$. By Theorem 10.67, the iterates $\varphi^{(k)}(T)$ are equicontinuous at x_0 relative to the spherical distance $\|x, y\|$. Let $\zeta_1, \ldots, \zeta_s$ be the finitely many points in $\partial V_1 \cup \cdots \cup \partial V_t$, and put $\varepsilon = \min_{1 \le j \le s} \operatorname{diam}(\zeta_j)$. (See (2.20) for the definition of diam.) By the definition of equicontinuity in a metric space, there is a $\delta > 0$ such that if $x \in \mathbb{P}^1(K)$ and $\|x, x_0\| < \delta$, then $\|\varphi(x), \varphi(x_0)\| < \varepsilon$. Let $B(x_0, \delta)^- = \{z \in \mathbb{P}^1(K) : \|z, x_0\| < \delta\}$, and let $Z = \mathcal{B}(x_0, \delta)^-$ be the associated Berkovich open ball.

Fix $k \in \mathbb{N}$. By construction, $\varphi^{(k)}(B(x_0, \delta)^-) \subset B(\varphi^{(k)}(x_0), \varepsilon)^-$, so by continuity, $\varphi^{(k)}(Z) \subset \mathcal{B}(\varphi^{(k)}(x_0), \varepsilon)^-$. Since $\operatorname{diam}(z) < \varepsilon$ for each $z \in \mathcal{B}(\varphi^{(k)}(x_0), \varepsilon)^-$, no ζ_j can belong to $\varphi^{(k)}(Z)$, and hence there is some V_i with $\varphi^{(k)}(Z) \subset V_i$.

Next assume $x_0 \in \mathbb{H}_{\mathrm{Berk}}$. By Proposition 10.75, there is an $n \ge 1$ such that $\varphi^{(n)}(x_0) = x_0$. We claim that the neighborhoods Z of x_0 with $\varphi^{(n)}(Z) = Z$ are cofinal in all neighborhoods of x_0. To see this, let V be a neighborhood of x_0. After shrinking V if necessary, we can assume that V is a connected open affinoid with $\overline{V} \subset U$. By Proposition 10.75, there is an $N \ge 1$ such that $\varphi^{(N)}(V) = V$. For each $\ell = 1, \ldots, N$, $\varphi^{(n\ell)}(V)$ is open and $x_0 \in \varphi^{(n\ell)}(V)$. Put

$$Z \;=\; \bigcap_{\ell=1}^{N} \varphi^{(n\ell)}(V) \;.$$

Then Z is open, $\varphi^{(n)}(Z) = Z$, and $x_0 \in Z \subseteq V$.

For each $k = 1, \ldots, n$, choose a neighborhood W_k of x_0 small enough so that $\varphi^{(k)}(W_k) \subset V_i$ for some i depending on k. Put $W = \bigcap_{k=1}^{n} W_k$, and let $Z \subset W$ be a neighborhood of x_0 such that $\varphi^{(n)}(Z) = Z$. Then for each $k \ge 0$, there is some V_i with $\varphi^{(k)}(Z) \subset V_i$. Hence the $\varphi^{(k)}(T)$ are equicontinuous at x_0.

More generally, if $\varphi^{(m)}(U)$ belongs to $\mathcal{E}(\varphi)$ and if $x_0 \in U$, one can show that the $\varphi^{(k)}(T)$ are equicontinuous at x_0 by an argument much like the one in the case of immediate basins of attraction.

Lastly, suppose U is a wandering component. Let $\zeta_1, \ldots, \zeta_s$ be the finitely many points belonging to $\partial V_1 \cup \cdots \cup \partial V_t$. Since U is wandering, each ζ_j belongs to $\varphi^{(n)}(U)$ for at most one n. Thus there is an N such that for each $1 \le j \le s$, we have $\zeta_j \notin \varphi^{(k)}(U)$ for all $k > N$. Now fix $x_0 \in U$, and choose a subdomain $Z \subset U$ containing x_0 small enough so that for each $k \le N$, $\varphi^{(k)}(Z)$ is contained in some V_i. For $k > N$, $\varphi^{(k)}(Z)$ is a domain which does not contain any boundary point of any V_i, so it must belong to some V_i as well. It follows that $\{\varphi^{(k)}(T)\}$ is equicontinuous at x_0. □

Remark 10.79. The proof which we have just given (including equicontinuity on the wandering components) is due to Rivera-Letelier (unpublished). We thank him for allowing us to reproduce his argument here.

10.7. Fixed point theorems and their applications

In this section, we obtain further information about the Berkovich Julia and Fatou sets. We continue our blanket assumption that $\mathrm{char}(K) = 0$, but most of the arguments do not use this.

Let $\varphi(T) \in K(T)$ be nonconstant. Recall that a fixed point of $\varphi(T)$ in $\mathbb{P}^1(K)$ is called *superattracting* if its multiplier $\lambda = 0$, *attracting* if $|\lambda| < 1$, *indifferent* if $|\lambda| = 1$ and *repelling* if $|\lambda| > 1$. Following Rivera-Letelier, we call a fixed point x of $\varphi(T)$ in $\mathbb{H}_{\mathrm{Berk}}$ *indifferent* if it has multiplicity $m_\varphi(x) = 1$, and *repelling* if $m_\varphi(x) > 1$. The reason for this terminology will be explained below. A periodic point for $\varphi(T)$, with period n, will be called indifferent or repelling according to whether the corresponding fixed point of $\varphi^{(n)}(T)$ is indifferent or repelling.

We first show that repelling periodic points are necessarily of type I or II, and that all repelling periodic points belong to the Berkovich Julia set J_φ. We next prove a fixed point theorem for expanding domains and show that if $\deg(\varphi) \geq 2$, then $\varphi(T)$ always has at least one repelling fixed point. We apply this to show that the topological connected components of the Berkovich Fatou set coincide with Rivera-Letelier's Fatou components. We also prove fixed point theorems for sets stabilized by or contracted by $\varphi(T)$. Finally, using a classical argument of Fatou involving homoclinic orbits (see [**73**, §14]), we show that repelling periodic points are dense in J_φ.

Our first lemma, which combines [**82**, Lemmas 5.3 and 5.4], shows that repelling fixed points in $\mathbb{H}_{\mathrm{Berk}}$ must be of type II:

Lemma 10.80 (Rivera-Letelier). *Let $\varphi(T) \in K(T)$ have $\deg(\varphi) \geq 2$. If x is a fixed point of $\varphi(T)$ of type* III *or* IV*, then:*

(A) *x is an indifferent fixed point.*

(B) *$\varphi_*(\vec{v}) = \vec{v}$ for each $\vec{v} \in T_x$.*

Proof. First suppose x is a fixed point of type III, and put $m = m_\varphi(x)$. Let $\vec{v}_1$ and $\vec{v}_2$ be the tangent vectors at x. By Lemma 9.33, $m_\varphi(\vec{v}_1) = m_\varphi(\vec{v}_2) = m$, and there is a segment $[x, c]$ with initial tangent vector $\vec{v}_1$ such that $\rho(x, \varphi(z)) = m \cdot \rho(x, z)$ for each $z \in [x, c]$. Without loss of generality we can assume that c is of type II. Since x is of type III, it follows that $\rho(x, c) \notin \log_v(|K^\times|)$. Note that $\log_v(|K^\times|)$ is a divisible group, since K is algebraically closed. If $\varphi_*(\vec{v}_1) = \vec{v}_2$, then

$$\rho(c, \varphi(c)) \;=\; \rho(c, x) + \rho(x, \varphi(c)) \;=\; (1 + m)\rho(x, c) \ .$$

However $\rho(c, \varphi(c)) \in \log_v(|K^\times|)$ since c and $\varphi(c)$ are of type II. This contradicts $\rho(x, c) \notin \log_v(|K^\times|)$, so it must be that $\varphi_*(\vec{v}_1) = \vec{v}_1$. Hence $[x, c]$ and $\varphi([x, c]) = [x, \varphi(c)]$ share a common initial segment. Let $y \in [x, c]$ be a type II point close enough to x that $\varphi(y) \in [x, c]$, and put $r = \rho(x, y)$. Then

$$mr \;=\; \rho(x, \varphi(y)) \;=\; r + \rho(y, \varphi(y)) \ .$$

Since $\rho(y, \varphi(y))$ belongs to $\log_v(|K^\times|)$ but r does not, this is a contradiction unless $m = 1$.

Next suppose x is of type IV. If we identify $\mathbb{P}^1_{\text{Berk}}$ with $\mathbb{A}^1_{\text{Berk}} \cup \{\infty\}$, then under Berkovich's classification theorem (Theorem 1.2), x corresponds to a nested sequence of closed discs in K with empty intersection:

$$D(a_1, r_1) \supset D(a_2, r_2) \supset D(a_3, r_3) \supset \cdots .$$

Since $\varphi(T)$ has only finitely many fixed points and finitely many poles in $\mathbb{P}^1(K)$, we can assume without loss of generality that none of the $D(a_i, r_i)$ contains a fixed point or a pole. We can also assume that each $r_i \in |K^\times|$. Let $z_i \in \mathbb{P}^1_{\text{Berk}}$ be the point corresponding to $D(a_i, r_i)$. The points z_i all belong to the path $[x, \infty]$.

Put $m = m_\varphi(x)$ and let $\vec{v}$ be the unique tangent vector at x. Clearly $\varphi_*(\vec{v}) = \vec{v}$. By Lemma 9.33, $m_\varphi(\vec{v}) = m$, and there is an initial segment $[x, c] \subset [x, \infty]$ such that $\rho(x, \varphi(z)) = m \cdot \rho(x, z)$ for each $z \in [x, c]$. Since $D(a_i, r_i)$ contains no poles of $\varphi(T)$, by Corollary A.18 its image $\varphi(D(a_i, r_i))$ is a disc $D(a_i', R_i)$. This disc corresponds to $\varphi(z_i)$ under Berkovich's classification theorem. Since $[x, c]$ and $\varphi([x, c]) = [x, \varphi(c)]$ have the same initial tangent vector (the unique $\vec{v} \in T_x$), they share an initial segment. Hence for sufficiently large i, both z_i and $\varphi(z_i)$ belong to $[x, c]$. Suppose $m > 1$. Then

$$R_i := \operatorname{diam}_\infty(\varphi(z_i)) = \operatorname{diam}_\infty(x) + m \cdot \rho(x, z_i) > \operatorname{diam}_\infty(z_i) = r_i .$$

Since the associated Berkovich discs $\mathcal{D}(a_i, r_i)$ and $\mathcal{D}(a_i', R_i)$ both contain x and since $R_i > r_i$, it follows that $D(a_i, r_i)$ is properly contained in $D(a_i', R_i)$.

We claim that $D(a_i, r_i)$ contains a fixed point of $\varphi(T)$ (in K), contradicting our assumption. After changing coordinates if necessary, we can assume that $D(a_i, r_i) = D(0, 1)$ and $D(a_i', R_i) = D(0, R)$ where $R > 1$. Expand $\varphi(T)$ as a power series $f(T) = \sum_{k=0}^\infty a_k T^k$ converging on $D(0, 1)$. Then $\lim_{k\to\infty} |a_k| = 0$, and by the Maximum Principle $\max_k(|a_k|) = R$. By the ultrametric inequality, the coefficients of the power series $g(T) = f(T) - T$ have the same properties, and the theory of Newton polygons shows that $g(T) = 0$ has a solution in $D(0, 1)$ (see Proposition A.16). This point is a fixed point of $\varphi(T)$. □

Let $x_0 \in \mathbb{P}^1_{\text{Berk}}$ be a fixed point of $\varphi(T)$. We now give a description of the local action of $\varphi(T)$ at x_0, which sheds light on the terminology "attracting", "indifferent", or "repelling".

First suppose $x_0 \in \mathbb{P}^1(K)$. After a change of coordinates, we can assume that $x_0 \neq \infty$. In this setting, if $\lambda = \varphi'(x_0)$ is the multiplier of x_0, we can expand $\varphi(T)$ as a power series which converges in a neighborhood of x_0:

$$\varphi(T) = x_0 + \lambda(T - x_0) + \sum_{k=2}^\infty a_k (T - x_0)^k . \tag{10.67}$$

If $\lambda \neq 0$, then there is a disc $D(x_0, R)$ such that $|\varphi(x) - x_0| = |\lambda||x - x_0|$ for each $x \in D(x_0, R)$. Using the theory of Newton polygons, it is easy to show that for each $0 < r \leq R$, we have $\varphi(D(x_0, r)) = D(x_0, |\lambda| r)$ (see Corollary A.17). By continuity, $\varphi(T)$ maps the closed Berkovich disc $\mathcal{D}(x_0, r)$ to $\mathcal{D}(x_0, |\lambda| r)$ and the open Berkovich disc $\mathcal{D}(x_0, r)^-$ to $\mathcal{D}(x_0, |\lambda| r)^-$.

Furthermore, if $x_r \in \mathbb{P}^1_{\text{Berk}}$ is the point corresponding to $D(x_0, r)$ under Berkovich's classification theorem, the fact that $\varphi(D(x_0, r)) = D(x_0, |\lambda|r)$ means that $\varphi(x_r) = x_{|\lambda|r}$. Thus, if $|\lambda| < 1$, the points x_r approach x_0 along the segment $[x_0, x_R]$; if $|\lambda| = 1$, the segment $[x_0, R]$ is fixed; and if $|\lambda| > 1$, the x_r recede from x_0.

If $\lambda = 0$, so that x_0 is superattracting, let a_m be the first nonzero coefficient in the expansion (10.67). Then $m \geq 2$, and Corollary A.17 shows there is a disc $D(x_0, R)$ such that for $0 < r \leq R$, $\varphi(D(x_0, r)) = D(x_0, |a_m|r^m)$. As above, $\varphi(\mathcal{D}(x_0, r)) = \mathcal{D}(x_0, |a_m|r^m)$, $\varphi(\mathcal{D}(x_0, r)^-) = \mathcal{D}(x_0, |a_m|r^m)^-$, and $\varphi(x_r) = x_{|a_m|r^m}$. If r is small enough, then $|a_m|r^m < r$, and the points x_r approach x_0 along $[x_0, x_R]$ as $r \to 0$.

Next suppose x_0 is a fixed point of $\varphi(T)$ in $\mathbb{H}_{\text{Berk}}$ of type II. By Corollary 9.25, $m_\varphi(x_0) = \deg(\tilde{\varphi})$, where $\tilde{\varphi}(T) \in k(T)$ is the reduction of $\varphi(T)$ in suitable coordinates with respect to x_0 and $k = \tilde{K}$ is the residue field of K (see the discussion preceding Corollary 9.25). Furthermore, the tangent vectors $\vec{v}_\alpha \in T_{x_0}$ correspond bijectively to points $\alpha \in \mathbb{P}^1(k)$. Under this correspondence $\varphi_*(\vec{v}_\alpha) = \vec{v}_{\tilde{\varphi}(\alpha)}$. By Corollary 9.25, $m_\varphi(x_0, \vec{v}_\alpha) = m_\alpha$ where $m_\alpha = m_{\tilde{\varphi}}(a) \geq 1$ is the usual (algebraic) multiplicity of $\tilde{\varphi}(T)$ at α. Furthermore, by Theorem 9.22, for each $\vec{v}_\alpha \in T_{x_0}$,

$$r_\varphi(x_0, \vec{v}_\alpha) \;:=\; d_{\vec{v}_\alpha}(\rho(\varphi(x), x_0))(x_0) \;=\; m_\alpha \; . \tag{10.68}$$

Loosely speaking, (10.68) says that under the action of $\varphi(T)$, points sufficiently near x_0 in the direction $\vec{v}_\alpha$ recede from x_0 in the direction $\vec{v}_{\tilde{\varphi}(\alpha)}$ at the rate m_α. More precisely, by Lemma 9.33, there is a segment $[x_0, x_\alpha]$ with initial tangent vector $\vec{v}_\alpha$ such that $\varphi([x_0, x_\alpha]) = [x_0, \varphi(x_\alpha)]$ is a segment with initial tangent vector $\vec{v}_{\tilde{\varphi}(\alpha)}$ and for each $x \in [x_0, x_\alpha]$,

$$\rho(\varphi(x), x_0) \;=\; m_\alpha \cdot \rho(x, x_0) \; . \tag{10.69}$$

Furthermore, if $\mathcal{A}_{x_0,x_\alpha}$ is the (generalized Berkovich open) annulus associated to the segment (x_0, x_α), then $\varphi(\mathcal{A}_{x_0,x_\alpha}) = \mathcal{A}_{x_0,\varphi(x_\alpha)}$ is the annulus associated to the segment $(x_0, \varphi(x_\alpha))$, and $\text{Mod}(\mathcal{A}_{x_0,\varphi(x_\alpha)}) = m_\alpha \cdot \text{Mod}(\mathcal{A}_{x_0,x_\alpha})$.

If $m_\varphi(x_0) = 1$, then $\deg(\tilde{\varphi}(T)) = 1$ and $m_{\tilde{\varphi}}(\alpha) = 1$ for each $\alpha \in \mathbb{P}^1(k)$. This means that φ_* permutes the tangent directions at x_0 and $\varphi(T)$ locally preserves path distances at x_0: $\rho(\varphi(x), x_0) = \rho(x, x_0)$ in (10.69). In this sense, the action of $\varphi(T)$ at x_0 is "indifferent".

By contrast, if $m_\varphi(x_0) > 1$, then $\deg(\tilde{\varphi}(T)) \geq 2$, so $\tilde{\varphi}(T)$ has at least one critical point α_0. Since $m_{\tilde{\varphi}}(\alpha_0) \geq 2$, points in the tangent direction $\vec{v}_{\alpha_0}$ on the segment $[x_0, x_{\alpha_0}]$ are "repelled" from x_0 in the direction $\vec{v}_{\tilde{\varphi}(\alpha_0)}$ at a rate $m_{\alpha_0} \geq 2$. In physical terms, one can visualize this action as being like an accretion disc swirling around a quasar, with one or more "jets" emanating from it. In this sense, the action of $\varphi(T)$ at x_0 is "repelling".

We will now show that, as in the classical complex case, repelling periodic points of φ always belong to the Berkovich Julia set J_φ. The proof is the same as [**83**, Proposition 5.1].

THEOREM 10.81 (Rivera-Letelier). *Let* $\varphi(T) \in K(T)$ *have* $\deg(\varphi) \geq 2$. *Then each repelling periodic point for* $\varphi(T)$ *in* $\mathbb{P}^1_{\mathrm{Berk}}$ *belongs to* J_φ.

PROOF. Let x be a repelling periodic point of $\varphi(T)$ of period n. Since $J_{\varphi^{(n)}} = J_\varphi$ by Lemma 10.51, we can assume that x is a repelling fixed point.

First suppose $x \in \mathbb{P}^1(K)$. After a change of coordinates we can assume that $x \in K$ and $|x| \leq 1$. Let λ be the multiplier of x; by hypothesis $|\lambda| > 1$. There is a disc $D(x, r) \subset K$ such that $\|\varphi(z), x\| = |\varphi(z) - x| = |\lambda||z - x| = |\lambda| \cdot \|z, x\|$ for all $z \in D(x, r)$; without loss of generality, we can assume that $r < 1$. If $|z - x| \leq r/|\lambda|^n$, then $\|\varphi^{(n)}(z), x\| = |\lambda|^n \cdot \|z, x\|$. Thus the iterates $\{\varphi^{(n)}\}$ are not equicontinuous at x, and so $x \in J_\varphi$ by Theorem 10.67.

Next suppose $x \in \mathbb{H}_{\mathrm{Berk}}$. By Lemma 10.80, x is of type II. Let k be the residue field of K, and let $\tilde{\varphi}(T) \in k(T)$ be the nonconstant reduction of $\varphi(T)$ at x, in suitable coordinates. By hypothesis, $m_\varphi(x) \geq 2$; by Corollary 9.25, $m_\varphi(x) = \deg(\tilde{\varphi}(T))$.

As noted earlier, after identifying T_x with $\mathbb{P}^1(k)$, we have $\varphi_*(\vec{v}_\alpha) = \vec{v}_{\tilde{\varphi}(\alpha)}$ and $m_\varphi(x, \vec{v}_\alpha) = m_{\tilde{\varphi}}(\alpha)$. Recall that we write $\mathcal{B}_x(\vec{v})^-$ for the component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$ in the direction $\vec{v}$.

Fix a neighborhood U of x. There is a finite subset $S \subset \mathbb{P}^1(k)$ such that for each $\alpha \notin S$, the ball $\mathcal{B}_x(\vec{v}_\alpha)^-$ is contained in U. By Proposition 9.41, $\varphi(\mathcal{B}_x(\vec{v}_\alpha)^-)$ contains $\mathcal{B}_x(\vec{v}_{\tilde{\varphi}(\alpha)})^-$. Let S_0 be the (possibly empty) finite subset of S consisting of the points $\alpha \in S$ which are exceptional for $\tilde{\varphi}(T)$. If $\alpha \notin S_0$, then α has infinitely many preimages under the iterates $\tilde{\varphi}^{(n)}(T)$, so there are an $\alpha_0 \notin S$ and an n_0 such that $\tilde{\varphi}^{(n_0)}(\alpha_0) = \alpha$. It follows that $\mathcal{B}_x(\vec{v}_\alpha)^- \subset \varphi^{(n_0)}(\mathcal{B}_x(\vec{v}_{\alpha_0})^-)$, and so

$$\bigcup_{\alpha \notin S_0} \mathcal{B}_x(\vec{v}_\alpha)^- \ \subset \ \bigcup_{n=0}^{\infty} \varphi^{(n)}(U) \ .$$

Now consider the balls $\mathcal{B}_x(\vec{v}_\alpha)^-$ for $\alpha \in S_0$. After replacing φ by an appropriate iterate $\varphi^{(m)}$, we can assume that $\tilde{\varphi}$ fixes each $\alpha \in S_0$ and hence (since α is exceptional) that

$$m_\varphi(x, \vec{v}_\alpha) \ = \ m_{\tilde{\varphi}}(\alpha) \ = \ \deg(\tilde{\varphi}) \ \geq \ 2 \ .$$

For each $\alpha \in S_0$, consider an annulus $\mathcal{A}_x(\vec{v}_\alpha) := \mathcal{A}_{x,c} \subset U$, where $c \in \mathcal{B}_x(\vec{v}_\alpha)^-$ and $\mathcal{A}_{x,c}$ is the annulus with boundary points x, c and main dendrite (x, c), as in §9.3. After replacing $[x, c]$ by an initial segment of itself if necessary, we can assume that $[x, c] \subset \varphi([x, c])$, so that $\mathcal{A}_x(\vec{v}_\alpha) \subset \varphi(\mathcal{A}_x(\vec{v}_\alpha))$. By Proposition 9.44, for each n the image $\varphi^{(n)}(\mathcal{A}_x(\vec{v}_\alpha))$ is either another annulus, a disc, or all of $\mathbb{P}^1_{\mathrm{Berk}}$. Since $\varphi_*^{(n)}(\vec{v}_\alpha) = \vec{v}_\alpha$, if the image under any iterate is a disc or $\mathbb{P}^1_{\mathrm{Berk}}$, then $\mathcal{B}_x(\vec{v}_\alpha)^- \subset \varphi^{(n)}(U)$. Hence we can assume the image under each iterate is an annulus. By Theorem 9.46, the modulus of $\varphi^{(n)}(\mathcal{A}_x(\vec{v}_\alpha))$ is $\deg(\tilde{\varphi})^n \cdot \mathrm{Mod}(\mathcal{A}_x(\vec{v}_\alpha))$. Since these annuli are increasing and their moduli grow to ∞, their union omits at most one point z_α in $\mathcal{B}_x(\vec{v}_\alpha)^-$, which is necessarily of type I.

Put $\mathcal{E}_0(K) = \{z_\alpha\}_{\alpha \in S_0}$. Then $\mathcal{E}_0(K)$ is a finite subset of $\mathbb{P}^1(K)$ and

$$\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathcal{E}_0(K) \ \subset \ \bigcup_{n=0}^{\infty} \varphi^{(n)}(U) \ .$$

Now take any $z \in \mathcal{E}_0(K) \backslash E_\varphi(K)$. Since z is not exceptional for φ, it has infinitely many distinct preimages under the $\varphi^{(n)}(T)$, so some preimage belongs to $\bigcup_{n=0}^{\infty} \varphi^{(n)}(U)$. It follows that z belongs to that set as well. Thus

$$\mathbb{P}^1_{\mathrm{Berk}} \backslash E_\varphi(K) \ \subset \ \bigcup_{n=0}^{\infty} \varphi^{(n)}(U) \ .$$

Since U is arbitrary, Theorem 10.56 shows that $x \in J_\varphi$. □

We now show that in repelling fixed points always exist.

THEOREM 10.82 (Rivera-Letelier[10]). *Let* $\varphi(T) \in K(T)$ *have* $\deg(\varphi) \geq 2$. *Then* $\varphi(T)$ *has at least one repelling fixed point in* $\mathbb{P}^1_{\mathrm{Berk}}$.

PROOF. This follows immediately from Theorem 10.83 below. □

Recall that a *simple domain* $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ is a connected open set whose boundary ∂U is a finite nonempty set, and each $x_i \in \partial U$ is of type II or III. Recall also that U is called *φ-saturated* if U is a connected component of $\varphi^{-1}(\varphi(U))$, or, equivalently, if $\varphi(\partial U) = \partial\varphi(U)$ (cf. Proposition 9.15).

The following result strengthens a fixed point theorem proved by Rivera-Letelier [**84**, Proposition 9.3] (see also [**82**, Lemma 6.2]).

THEOREM 10.83 (Repelling Fixed Point Criterion). *Let* $\varphi(T) \in K(T)$ *have* $\deg(\varphi) \geq 2$. *If either* $U \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ *is a φ-saturated simple domain with* $\overline{U} \subset \varphi(U)$, *or* $U = \mathbb{P}^1_{\mathrm{Berk}}$, *then* U *contains a repelling fixed point for* φ.

PROOF. Given a point $x \in \mathbb{P}^1_{\mathrm{Berk}}$, we will say that x is *strongly involutive* for $\varphi(T)$ if either

(1) x is fixed by $\varphi(T)$ or
(2) $x \neq \varphi(x)$ and the following condition holds: consider the path $[x, \varphi(x)]$; let $\vec{v}$ be its (inward-directed) tangent vector at x, and let $\vec{w}$ be its (inward-directed) tangent vector at $\varphi(x)$. Then $\varphi_*(\vec{v}) = \vec{w}$, and $\vec{v}$ is the only tangent vector $\vec{v}' \in T_x$ for which $\varphi_*(\vec{v}') = \vec{w}$.

We will consider two cases, according as each $x \in U \cap \mathbb{H}_{\mathrm{Berk}}$ is strongly involutive or not.

First suppose some $x_0 \in U \cap \mathbb{H}_{\mathrm{Berk}}$ is not strongly involutive. By definition, x_0 is not fixed by $\varphi(T)$. Set $x_{-1} = \varphi(x_0)$, and consider the segment $[x_{-1}, x_0]$. Let $\vec{v}_0$ be its tangent vector at x_0, and let $\vec{w}_0$ be its tangent vector at x_{-1}. Recall that the map $\varphi_* : T_{x_0} \to T_{x_{-1}}$ is surjective (Corollary 9.20). Since x_0 is not strongly involutive, there is a tangent vector $\vec{w}_1$ at x_0, with $\vec{w}_1 \neq \vec{v}_0$, for which $\varphi_*(\vec{w}_1) = \vec{w}_0$.

---

[10] When $K = \mathbb{C}_p$, this is proved in [**82**, Theorem B]. The argument carries through in the general case with some modifications; see [**80**]. Here we give a different proof.

By Lemma 9.38, there is a segment $[x_0, x_1] \subset U$ with initial tangent vector $\vec{w}_1$ such that $\varphi(T)$ maps $[x_0, x_1]$ homeomorphically onto $[x_{-1}, x_0]$, with $\varphi(x_0) = x_{-1}$ and $\varphi(x_1) = x_0$. Let $\vec{v}_1$ be the tangent vector to $[x_0, x_1]$ at x_1, noting that $\varphi_*(\vec{v}_1) = \vec{v}_0$. Since φ_* maps the tangent space T_{x_1} surjectively onto T_{x_0}, there is a $\vec{w}_2 \in T_{x_1}$ for which $\varphi_*(\vec{w}_2) = \vec{w}_1$. Clearly $\vec{w}_2 \neq \vec{v}_1$, since $\varphi_*(\vec{v_1}) = \vec{v}_0$.

We now iterate this process, inductively constructing segments $[x_1, x_2]$, $[x_2, x_3], \ldots$ contained in U such that $\varphi(T)$ maps $[x_n, x_{n+1}]$ homeomorphically onto $[x_{n-1}, x_n]$, with $\varphi(x_n) = x_{n-1}$ and $\varphi(x_{n+1}) = x_n$, such that the tangent vector $\vec{w}_{n+1}$ to x_n in $[x_n, x_{n+1}]$ is different from the tangent vector $\vec{v}_n$ to x_n in $[x_{n-1}, x_n]$. This means that

$$P \ = \ \bigcup_{n=0}^{\infty} [x_{n-1}, x_n]$$

is an arc.

If P has finite length, then since $\mathbb{H}_{\text{Berk}}$ is complete under $\rho(x, y)$ (Proposition 2.29), the points $\{x_n\}$ converge to a point $x \in \mathbb{H}_{\text{Berk}}$ which is an endpoint of P. Furthermore,

$$\varphi(x) \ = \ \lim_{n \to \infty} \varphi(x_n) \ = \ \lim_{n \to \infty} x_{n-1} \ = \ x \ ,$$

so x is a fixed point of $\varphi(T)$. Clearly $x \in \overline{U}$. We claim that $x \in U$. If $U = \mathbb{P}^1_{\text{Berk}}$, this is trivial. If $U \neq \mathbb{P}^1_{\text{Berk}}$, then since U is φ-saturated and $\overline{U} \subset \varphi(U)$, the boundary points of U are not fixed by $\varphi(T)$. Thus $x \in U$. Let $\vec{v}$ be the tangent vector to P at x. Since $\rho(x, x_n) < \rho(x, x_{n-1}) = \rho(x, \varphi(x_n))$, necessarily $d_{\vec{v}}(\rho(\varphi(z), x))(x) > 1$, so $m_\varphi(x) > 1$ (Theorem 9.19), and x is a repelling fixed point.

On the other hand, if P has infinite length, then we claim that the x_n converge to a point $x \in \mathbb{P}^1(K)$ and that P is a geodesic ray emanating from x. Indeed, we can choose coordinates so that ζ_{Gauss} lies on P and all but a finite initial segment of P belongs to $\mathcal{D}(0, 1)$. In this situation, if $x_n \in \mathcal{D}(0, 1)$ and $\rho(\zeta_{\text{Gauss}}, x_n) = t_n$, then under Berkovich's classification theorem (Theorem 1.2), x_n corresponds to a disc $D(a_n, r_n)$ with $r_n = q_v^{-t_n}$. These discs are nested and their radii go to 0. Since K is complete, their intersection is a point $x \in K$, which is the limit of the x_n in the strong topology. By the same argument as before, $x \in \overline{U}$ and x is a fixed point of $\varphi(T)$. If $U = \mathbb{P}^1_{\text{Berk}}$, trivially $x \in U$. Otherwise, U is a simple domain, so none of its boundary points belongs to $\mathbb{P}^1(K)$, and again $x \in U$. Under the action of $\varphi(T)$, the points x_n recede from x along P. If λ is the multiplier of $\varphi(T)$ at x, this means that $|\lambda| > 1$, so x is a repelling fixed point.

Next assume that each $x \in U \cap \mathbb{H}_{\text{Berk}}$ is strongly involutive.

If $\varphi(T)$ fixes each $x \in U \cap \mathbb{H}_{\text{Berk}}$, then by continuity it fixes each $z \in U$, and in particular $\varphi(z) = z$ for all $z \in \mathbb{P}^1(K) \cap U$. This means that $\varphi(T)$ has infinitely many fixed points in $\mathbb{P}^1(K)$, so $\varphi(T) = T$, which contradicts $\deg(\varphi) \geq 2$. Hence not every $x \in U \cap \mathbb{H}_{\text{Berk}}$ is a fixed point of $\varphi(T)$.

However, we claim that $\varphi(T)$ has at least one fixed point $c \in U \cap \mathbb{H}_{\mathrm{Berk}}$, which could be an indifferent fixed point. Take an arbitrary $a \in U \cap \mathbb{H}_{\mathrm{Berk}}$. If $\varphi(a) = a$, put $c = a$. If $\varphi(a) \neq a$, then by Lemma 10.84 below, the segment $[a, \varphi(a)]$ contains a fixed point c which belongs to U.

If $m_\varphi(c) > 1$, then c is a repelling fixed point and we are done. If $m_\varphi(c) = 1$, we consider two subcases, according as U is a φ-saturated simple domain with $\overline{U} \subset \varphi(U)$ or $U = \mathbb{P}^1_{\mathrm{Berk}}$.

First, let U be a φ-saturated simple domain with $\overline{U} \subset \varphi(U)$. Write $\partial U = \{x_1, \ldots, x_m\}$, where each x_i is of type II or III. Consider the segments $[c, x_i]$, for $i = 1, \ldots, m$. Since $\overline{U} \subset \varphi(U)$ and $\varphi(\partial U) = \partial\varphi(U)$, we have $\varphi(x_i) \notin \overline{U}$ for each i. Since ∂U is finite, there is an i so that $\rho(c, x_i) \leq \rho(c, x_j)$ for all $1 \leq j \leq m$. For this choice of i we must have $\rho(c, x_i) < \rho(c, \varphi(x_i))$. Since $\varphi(c) = c$, Theorem 9.35(B) gives

$$\rho(c, x_i) \ < \ \rho(c, \varphi(x_i)) \ \leq \ \int_{[c,x_i]} m_\varphi(x)\, dx \ .$$

Hence there is a point $b \in (c, x_i)$ with $m_\varphi(b) > 1$. If $\varphi(b) = b$, then b is a repelling fixed point. If $\varphi(b) \neq b$, then by Lemma 10.84 there is a fixed point $d \in [b, \varphi(b)]$ which belongs to U and satisfies $m_\varphi(d) \geq m_\varphi(b)$. Thus d is a repelling fixed point.

Finally, suppose $U = \mathbb{P}^1_{\mathrm{Berk}}$. We consider two more subcases.

First suppose there is a point $a \in \mathbb{H}_{\mathrm{Berk}}$ with $m_\varphi(a) > 1$. If a is fixed by $\varphi(T)$, then a is a repelling fixed point, and we are done. Otherwise, by Lemma 10.84, there is a fixed point $d \in [a, \varphi(a)]$ with $m_\varphi(d) \geq m_\varphi(a)$, so d is a repelling fixed point.

Next suppose that $m_\varphi(a) = 1$ for each $a \in \mathbb{H}_{\mathrm{Berk}}$. We will see that this leads to a contradiction. As shown above, $\varphi(T)$ has at least one fixed point $c \in \mathbb{H}_{\mathrm{Berk}}$. Since $m_\varphi(c) = 1$ and c has $\deg(\varphi) \geq 2$ preimages under $\varphi(T)$, counting multiplicities (Theorem 9.8(C)), there is a point $a \in \mathbb{H}_{\mathrm{Berk}}$ with $a \neq c$ and $\varphi(a) = c$. By Lemma 10.84, there is a fixed point $d \in [a, c]$, and $\varphi(T)$ maps $[a, d]$ homeomorphically onto $[c, d]$. In particular, if $\vec{v}_1$ is the tangent vector to d in the direction of a and $\vec{v}_2$ is the tangent vector at d in the direction of c, then $\varphi_*(\vec{v}_1) = \vec{v}_2$.

We claim that $\varphi(T)$ fixes the entire segment $[c, d]$. Take any $t \in [c, d]$. Since $m_\varphi(x) = 1$ for each $x \in [c, t]$, Theorem 9.35(B) gives

$$\rho(c, \varphi(t)) \ \leq \ \int_{[c,t]} m_\varphi(x)\, dx \ \leq \ \rho(c, t) \ .$$

Similarly, $\rho(d, \varphi(t)) \leq \rho(d, t)$. Since $t \in [c, d]$, the only point $z \in \mathbb{P}^1_{\mathrm{Berk}}$ satisfying $\rho(c, z) \leq \rho(c, t)$ and $\rho(d, z) \leq \rho(d, t)$ is t itself. Thus $\varphi(t) = t$.

Since $\varphi(T)$ fixes $[c, d]$, it follows that $\varphi_*(\vec{v}_2) = \vec{v}_2$. By Theorem 9.22(C),

$$m_\varphi(d) \ = \sum_{\varphi(\vec{w}) = \vec{v}_2} m_\varphi(d, \vec{w}) \ \geq \ m_\varphi(d, \vec{v}_1) + m_\varphi(d, \vec{v}_2) \ \geq \ 2 \ .$$

Thus d is a repelling fixed point for φ. □

The lemma below is needed to complete the proof of Theorem 10.83.

LEMMA 10.84. *Let $\varphi(T) \in K(T)$ be nonconstant, and let $U \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ be a φ-saturated domain with $U \subseteq \varphi(U)$. Suppose that each $x \in U \cap \mathbb{H}_{\mathrm{Berk}}$ is strongly involutive. Let $a \in U \cap \mathbb{H}_{\mathrm{Berk}}$, and put $b = \varphi(a)$.*

If $a \neq b$, then there is a point $c \in (a,b)$ which is fixed by $\varphi(T)$, and $\varphi(T)$ maps $[a,c]$ homeomorphically onto $[b,c]$. Furthermore, $c \in U$ and $m_\varphi(c) \geq m_\varphi(a)$.

PROOF. The proof is by transfinite induction. For readers unfamiliar with transfinite induction, we recommend [**63**, §IV.7]. Recall that each ordinal represents an isomorphism class of well-ordered sets. Transfinite induction generalizes induction on $\mathbb{N}$ to arbitrary well-ordered sets, by adding an additional case to deal with the fact that each ordinal which is not 0 or a successor ordinal must be a limit ordinal. The idea is to start at a and "push" towards b, with each push consisting of either a single step or passage to a limit of steps, until a fixed point is reached. Transfinite induction is needed since we have no control of the step size, so the fixed point might not be reached in a sequence of steps indexed by the natural numbers. We are assured that the process must eventually stop by cardinality considerations.

Put $a_0 = a$, $b_0 = b$. Suppose that for some ordinal ω, we have constructed points $a_\omega, b_\omega \in [a,b]$ with $a_\omega \in U$, $b_\omega = \varphi(a_\omega)$, and $m_\varphi(a_\omega) \geq m_\varphi(a)$, such that $[a,a_\omega) \cap (b_\omega,b] = \emptyset$ and $\varphi(T)$ maps $[a,a_\omega]$ homeomorphically onto $[b,b_\omega]$. If $a_\omega = b_\omega$, then taking $c = a_\omega$, we are done.

If not, let $\vec{v}_\omega$ (resp. $\vec{w}_\omega$) be the tangent vector to $[a_\omega,b_\omega]$ at a_ω (resp. b_ω). Since a_ω is strongly involutive, $\vec{v}_\omega$ is the only tangent vector at a_ω for which $\varphi_*(\vec{v}_\omega) = \vec{w}_\omega$, and Theorem 9.22 shows that $m_\varphi(a_\omega,\vec{v}_\omega) = m_\varphi(a_\omega)$. Put $m_\omega = m_\varphi(a_\omega)$.

By Lemma 9.33, there is an initial segment $[a_\omega,a_{\omega+1}] \subset [a_\omega,b_\omega]$ such that $\varphi(T)$ maps $[a_\omega,a_{\omega+1}]$ homeomorphically onto a segment $[b_\omega,b_{\omega+1}] \subset [b_\omega,a_\omega]$, with $m_\varphi(x) = m_\omega$ for all $x \in [a_\omega,a_{\omega+1}]$. After moving $a_{\omega+1}$ nearer to a_ω if necessary, we can assume that $a_{\omega+1} \in U$ and $[a_\omega,a_{\omega+1}) \cap (b_{\omega+1},b_\omega] = \emptyset$.

Let $\vec{v}'_{\omega+1}$ be the tangent vector to $[a_\omega,a_{\omega+1}]$ at $a_{\omega+1}$. Since $m_\varphi(x) = m_\omega$ for each $x \in (a_\omega,a_{\omega+1})$, Theorem 9.22 shows that $m_\varphi(a_{\omega+1},\vec{v}'_{\omega+1}) = m_\omega$ and in turn that $m_\varphi(a_{\omega+1}) \geq m_\omega$. Thus $m_\varphi(a_{\omega+1}) \geq m_\varphi(a_\omega) \geq m_\varphi(a)$. By induction, $\varphi(T)$ maps $[a,a_{\omega+1}]$ homeomorphically onto $[b,b_{\omega+1}]$ and $[a,a_{\omega+1}) \cap (b_{\omega+1},b] = \emptyset$.

If λ is a limit ordinal and for each $\omega < \lambda$ we have constructed a_ω, b_ω with the properties above, then $\{a_\omega\}_{\omega<\lambda}$ is a Cauchy net in $\mathbb{H}_{\mathrm{Berk}}$. By Proposition 2.29, it has a unique limit point $a_\lambda \in \mathbb{H}_{\mathrm{Berk}}$. Put $b_\lambda = \varphi(a_\lambda)$.

Clearly $a_\lambda \in \overline{U}$ and $a_\lambda, b_\lambda \in [a,b]$. Since U is φ-saturated and $[a,b] \subset \varphi(U)$, the same argument as in the proof of Lemma 9.38 shows that $a_\lambda \in U$. By induction, $\varphi(T)$ maps $[a,a_\lambda]$ homeomorphically onto $[b,b_\lambda]$. Since $[a,a_\omega] \cap [b_\omega,b] = \emptyset$ for each $\omega < \lambda$, it follows that $[a,a_\lambda) \cap (b_\lambda,b] = \emptyset$. Since a_λ is the limit of the points a_ω, Theorem 9.22 shows that $m_\varphi(a_\lambda) \geq \limsup_{\omega<\lambda} m_\omega \geq m_\varphi(a)$.

Finally, since there are ordinals of arbitrarily high cardinality and the map $\omega \mapsto a_\omega$ is injective on ordinals with $a_\omega \neq b_\omega$, it must be that $a_\omega = b_\omega$ for some ω. □

Rivera-Letelier [83] has also proved a fixed point theorem for closed, connected sets which are mapped into themselves by $\varphi(T)$.

THEOREM 10.85 (Stable Set Fixed Point Property). *Let $\varphi(T) \in K(T)$ have degree $d \geq 1$. Let $X \subseteq \mathbb{P}^1_{\mathrm{Berk}}$ be a nonempty connected, closed set* (*for either the weak topology or the strong topology*) *which satisfies $\varphi(X) \subseteq X$. Then X contains a point fixed by φ.*

Furthermore, if X is not reduced to a point, then X contains either
(A) *an attracting fixed point of φ of type* I *or*
(B) *a repelling or indifferent fixed point of φ of type* II.

We remark that a set $X \subset \mathbb{P}^1_{\mathrm{Berk}}$ is connected and closed in the weak topology if and only if it is connected and closed in the strong topology. To see this, first note that by Lemma B.18, X is connected in the weak topology if and only if it is connected in the strong topology. If X is closed in the weak topology, then trivially it is closed in the strong topology. Hence it suffices to observe that if X is connected and closed in the strong topology, then its complement is a union of open discs (for the weak topology), so X is closed in the weak topology.

PROOF. If X consists of a single point, that point is a fixed point of φ, but nothing can be said about its nature. Henceforth we will assume that X contains at least two (hence infinitely many) points.

We first show that there is at least one fixed point in X. As in Lemma 10.84, the idea is to start at an arbitrary point and use transfinite induction to "push" towards a fixed point, with each "push" being either a single step or passage to a limit of steps, until a fixed point is finally reached.

We then use this fixed point to deduce the existence of a fixed point with the properties described.

Since X is connected and contains at least two points, it has points in $\mathbb{H}_{\mathrm{Berk}}$. Fix a point $x_0 \in X \cap \mathbb{H}_{\mathrm{Berk}}$. If $\varphi(x_0) = x_0$, we are done. Otherwise, since $\varphi(x_0) \in X$ and X is connected, the path $[x_0, \varphi(x_0)]$ is contained in X. We claim that there is a point $x_1 \in (x_0, \varphi(x_0))$ such that for each $z \in [x_0, x_1)$ the path $[z, \varphi(z)]$ contains x_1 in its interior. By Lemma 9.31, for all $a, c \in \mathbb{H}_{\mathrm{Berk}}$ the path distance satisfies $\rho(\varphi(a), \varphi(c)) \leq d \cdot \rho(a, c)$. Hence if x_1 is the point in $(x_0, \varphi(x_0))$ for which $\rho(x_0, x_1) = \rho(x_0, \varphi(x_0))/(d+1)$, then for each $z \in [x_0, x_1)$

$$\rho(\varphi(z), \varphi(x_0)) \ \leq \ d \cdot \rho(z, x_0) \ < \ \rho(x_1, \varphi(x_0)) \ .$$

Since $\mathbb{H}_{\mathrm{Berk}}$ is uniquely path-connected, the path from z to $\varphi(z)$ must pass through x_1.

Inductively, suppose $\omega \geq 1$ is an ordinal and that we have constructed points $\{x_\kappa\}_{\kappa \leq \omega} \subset X \cap \mathbb{H}_{\mathrm{Berk}}$ such that for each $\kappa < \omega$,

(1) $[x_0, x_\kappa]$ is a proper initial segment of $[x_0, x_{\kappa+1}]$,
(2) $\varphi(x_\kappa) \neq x_\kappa$, and $[x_\kappa, x_{\kappa+1}]$ is a proper initial segment of $[x_\kappa, \varphi(x_\kappa)]$, such that for each $z \in [x_\kappa, x_{\kappa+1}]$ we have $x_{\kappa+1} \in [z, \varphi(z)]$,
(3) if ω is a limit ordinal, then $x_\omega = \lim_{\substack{\kappa\to\omega \\ \kappa<\omega}} x_\kappa$.

If $\varphi(x_\omega) = x_\omega$, we are done. Otherwise, by the same argument as above, there is a point $x_{\omega+1} \in (x_\omega, \varphi(x_\omega))$ such that $[x_\omega, x_{\omega+1}] \subset X$ and for each $z \in [x_\omega, x_{\omega+1})$ the path from z to $\varphi(z)$ passes through $x_{\omega+1}$. We claim that the path $[x_0, x_{\omega+1}]$ extends $[x_0, x_\omega]$.

If ω is a successor ordinal, say $\omega = \kappa+1$, then by hypothesis for each $z \in [x_\kappa, x_\omega)$ the path from z to $\varphi(z)$ contains x_ω in its interior. This means that $\varphi([x_\kappa, x_\omega))$ is contained in a single connected component V of $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{x_\omega\}$. By continuity, $\varphi(x_\omega)$ belongs to the closure of this component, which is $V \cup \{x_\omega\}$. Since $\varphi(x_\omega) \neq x_\omega$, it must be that $\varphi(x_\omega) \in V$. On the other hand, for each $z \in [x_\kappa, x_\omega]$ the path from z to $\varphi(z)$ contains x_ω, so $z \notin V$. It follows that $[x_\kappa, x_\omega] \cap [x_\omega, \varphi(x_\omega)] = \{x_\omega\}$, so $[x_0, x_{\omega+1}]$ extends $[x_0, x_\omega]$.

On the other hand, if ω is a limit ordinal and if $[x_\omega, x_{\omega+1}]$ does not extend $[x_0, x_\omega]$, then $[x_\omega, \varphi(x_\omega)]$ and $[x_\omega, x_0]$ have a common initial segment $[x_\omega, b]$, where $b \in (x_\omega, \varphi(x_\omega))$. There is an η_0 such that if $\eta_0 \leq \eta < \omega$, the point x_η belongs to $[x_\omega, b]$. By continuity, as $\eta \to \omega$, then $\varphi(x_\eta) \to \varphi(x_\omega)$, so there is an $\eta_1 > \eta_0$ such that if $\eta_1 < \eta < \omega$, the path from x_η to $\varphi(x_\eta)$ contains b. By construction, that path also contains $x_{\eta+1}$, which lies in $[x_\eta, x_\omega] \subset [b, x_\omega]$. This is a contradiction. Hence $[x_0, x_{\omega+1}]$ extends $[x_0, x_\omega]$.

Finally, suppose λ is a limit ordinal and that we have constructed points $\{x_\kappa\}_{\kappa<\lambda}$ satisfying properties (1) and (2) above. Then $P_\lambda := \bigcup_{\kappa<\lambda}[x_0, x_\kappa]$ is isomorphic to a half-open segment.

If P_λ has finite path length, then by the completeness of $\mathbb{H}_{\mathrm{Berk}}$ under $\rho(x, y)$ (Proposition 2.29), the points $\{x_\kappa\}_{\kappa<\lambda}$ converge to a point $x_\lambda \in \mathbb{H}_{\mathrm{Berk}}$. Since X is closed and x_λ is also the limit of the x_κ in the weak topology, $x_\lambda \in X$. Clearly $\{x_\kappa\}_{\kappa\leq\lambda}$ satisfies conditions (1), (2), and (3), so the induction can continue.

If P_λ has infinite path length, we claim that $\{x_\kappa\}_{\kappa<\lambda}$ converges to a point $x_\kappa \in X \cap \mathbb{P}^1(K)$ which is fixed by φ. For each $\kappa < \lambda$, let $\vec{v}_\kappa \in T_{x_\kappa}$ be the tangent vector to P_λ at x_κ in the direction of $x_{\kappa+1}$, and put $\mathcal{B}_\kappa^- = \mathcal{B}_{x_\kappa}(\vec{v}_\kappa)^-$. Then the $\mathcal{B}_\kappa^-$ form a nested collection of balls whose radii $r_\kappa = \mathrm{diam}_{x_0}(\mathcal{B}_\kappa^-)$ approach 0. By the completeness of $\mathbb{P}^1_{\mathrm{Berk}}$ in either the weak topology or the strong topology (Proposition 2.29), their intersection is a point

$$x_\lambda \;=\; \lim_{\kappa\to\lambda} x_\kappa \;\in\; \mathbb{P}^1(K)\ .$$

Since X is closed in the weak topology, $x_\lambda \in X$. Moreover, for each κ we have $\varphi(x_\kappa) \in \mathcal{B}_\kappa^-$, so by continuity $\varphi(x_\lambda) = x_\lambda$.

The induction will eventually terminate, since there are ordinals of cardinality greater than the cardinality of $\mathbb{R}$. When it terminates, it has produced a fixed point in either $X \cap \mathbb{P}^1(K)$ or $X \cap \mathbb{H}_{\mathrm{Berk}}$.

We will now show that the existence of the fixed point constructed above implies the existence of a fixed point with the specified properties.

First suppose the fixed point x_ω belongs to $X \cap \mathbb{P}^1(K)$. The path $[x_0, x_\omega]$ contains points x_κ with $\kappa < \omega$ which are arbitrarily near x_ω in the metric $d(x, y)$. By the description of the action of φ near a fixed point in $\mathbb{P}^1(K)$ given at the beginning of this section, if x_ω were repelling, there would be points x_κ arbitrarily near x_ω which are moved away from x_ω along the path $[x_0, x_\omega]$. If x_ω were indifferent, there would be points x_κ arbitrarily near x_ω which are fixed by φ. Neither of these is the case, so x_ω must be attracting.

Next suppose the fixed point x_ω belongs to $X \cap \mathbb{H}_{\text{Berk}}$. If x_ω is of type II, it may be either repelling or indifferent, but it cannot be attracting since there are no attracting fixed points in $\mathbb{H}_{\text{Berk}}$. If x_ω is of type III or IV, then by Lemma 10.80 it is an indifferent fixed point. Let $\vec{v} \in T_{x_\omega}$ be the tangent vector to the path $[x_0, x_\omega]$ at x_ω. Since $m_\varphi(x_\omega, \vec{v}) = 1$, there is a segment $[x_\omega, a]$ with initial tangent vector $\vec{v}$ which is fixed by φ. As the segments $[x_\omega, x_0]$ and $[x_\omega, a]$ have the same initial tangent vector, they must share a nontrivial initial segment $[x_\omega, b]$. This segment contains infinitely many points of type II which are fixed by φ. □

We can now prove the following counterpart to Theorem 10.83:

Theorem 10.86 (Attracting Fixed Point Criterion). *Let $\varphi(T) \in K(T)$ have $\deg(\varphi) \geq 1$. Let $U \subset \mathbb{P}^1_{\text{Berk}}$ be a simple domain for which $\overline{\varphi(U)} \subset U$. Then U contains an attracting fixed point of φ, necessarily of type* I.

Proof. Since $\varphi(\overline{U}) = \overline{\varphi(U)}$, the set $X = \overline{U}$ satisfies the conditions of Theorem 10.85. Thus $\overline{U}$ contains a type I or type II point x fixed by φ.

Since φ maps ∂U into U, the points of ∂U are not fixed, so x belongs to U. We claim that x must be of type I, in which case Theorem 10.85 shows that it is an attracting fixed point.

Suppose to the contrary that x were of type II. Let $x_1, \ldots, x_m$ be the finitely many boundary points of U; each is of type II or III. Put $r = \min_{1 \leq i \leq m} \rho(x, x_i)$. Then the ball

$$\widehat{\mathcal{B}}(x, r)^- \ = \ \{z \in \mathbb{H}_{\text{Berk}} : \rho(x, z) < r\}$$

is contained in U. By the Incompressibility Lemma (Corollary 9.39), $\widehat{\mathcal{B}}(x, r)^-$ is also contained in $\varphi(U)$. Hence the boundary point of U nearest to x, say x_j, is a point of $\overline{\varphi(U)}$. This contradicts the assumption that $\overline{\varphi(U)} \subset U$. □

As an application of Theorem 10.82, we show that each connected component U of the Berkovich Fatou set has the property that $\bigcup_{n=0}^{\infty} \varphi^{(n)}(U)$ omits at least three points of $\mathbb{P}^1(K)$. By Theorem 10.56, U is a maximal connected open set with respect to this property, so $U \cap \mathbb{P}^1(K)$ is a Fatou component of $F_\varphi \cap \mathbb{P}^1(K)$, in the sense of Rivera-Letelier [**83**].

COROLLARY 10.87. *Let $\varphi(T) \in K(T)$ have degree* $\deg(\varphi) \geq 2$. *Then for each connected component U of $F_\varphi = \mathbb{P}^1_{\text{Berk}} \backslash J_\varphi$, the union $\bigcup_{n=0}^{\infty} \varphi^{(n)}(U)$ omits at least three points of $\mathbb{P}^1(K)$.*

PROOF. First note that if U is a component of F_φ, then $\varphi(U)$ is also a component. Indeed, since $\varphi(U)$ is connected and F_φ is completely invariant under $\varphi(T)$ (Lemma 10.52), $\varphi(U)$ is contained in a component V of F_φ. Let U' be the component of $\varphi^{-1}(V)$ containing U. Since $\varphi^{-1}(V) \subset F_\varphi$ and U is a component of F_φ, it must be that $U' = U$. Hence $\varphi(U) = \varphi(U') = V$.

Thus $\varphi(T)$ acts on the set of components of F_φ.

If there is a component W of F_φ which is preperiodic but not periodic for this action, then every component U has the property in the corollary. Indeed, W has infinitely many distinct preimages W_n under the iterates $\varphi^{(n)}(T)$, and the set of forward images $\{\varphi^{(n)}(U)\}$ can contain only finitely many of the W_n. Hence there is some W_N which is disjoint from $\bigcup_{n=0}^{\infty} \varphi^{(n)}(U)$, and $W_N \cap \mathbb{P}^1(K)$ is infinite.

Similarly, if there is a wandering component (that is, a component that is not preperiodic) or if there are at least two periodic components W, W' with disjoint orbits under $\varphi(T)$, then every component U has the property in the corollary.

Thus we need only consider the case where the components form a single, necessarily finite, periodic orbit. By Theorems 10.81 and 10.82, J_φ contains a repelling fixed point x. According to Lemma 10.80, x is either of type I or type II. If x were of type II, then since $\mathbb{P}^1_{\text{Berk}} \backslash \{x\}$ has infinitely many components and J_φ has empty interior (Corollary 10.69), F_φ would have infinitely many components, a contradiction. Hence $x \in \mathbb{P}^1(K)$. Since $E_\varphi(K) \subset F_\varphi$ (Corollary 10.55), x is not exceptional. Thus x has infinitely many distinct preimages under the $\varphi^{(n)}(T)$, all of which belong to J_φ. For any component U of F_φ, the forward images $\varphi^{(n)}(U)$ omit these points. $\square$

Finally, we show that periodic repelling points are dense in the Berkovich Julia set. (This fact is originally due to Rivera-Letelier [**80**].) The proof is a generalization of a classical argument of Fatou.

THEOREM 10.88 (Rivera-Letelier). *Let $\varphi(T) \in K(T)$ have* $\deg(\varphi) \geq 2$. *Then J_φ is the closure of the set of repelling periodic points of φ in $\mathbb{P}^1_{\text{Berk}}$.*

PROOF. By Theorem 10.81, all the repelling periodic points of φ belong to J_φ, so it suffices to show that those points are dense in J_φ.

By Theorem 10.82, there exists a repelling fixed point $z_0 \in \mathbb{P}^1_{\text{Berk}}$ of φ, and by Theorem 10.81, it is in J_φ. If $J_\varphi = \{z_0\}$, the theorem is clearly true.

Otherwise, let $V \subset \mathbb{P}^1_{\text{Berk}}$ be an open set, disjoint from z_0, which meets J_φ. We will construct a repelling periodic point whose orbit meets V, by first constructing a *homoclinic orbit*: a sequence of points $z_1, z_2, \ldots$ with $\varphi(z_k) = z_{k-1}$ for each $k \geq 1$, such that $\lim_{k\to\infty} z_k = z_0$ and some z_ℓ belongs to V. Thus the sequence $\{z_k\}$ abuts at z_0 under the action of $\varphi(T)$, but its "tail" loops back to z_0 and forms a sequence of points repelled by z_0.

By Theorem 10.59, the iterates $\varphi^{(k)}(V)$ cover J_φ, so there is an $\ell \ge 1$ for which $z_0 \in \varphi^{(\ell)}(V)$. Let $z_\ell \in V$ be such that $\varphi^{(\ell)}(z_\ell) = z_0$, and put $z_{\ell-k} = \varphi^{(k)}(z_\ell)$ for $k = 1, \ldots, \ell-1$. We now consider two cases, according to whether $z_0 \in \mathbb{P}^1(K)$ or $z_0 \in \mathbb{H}_{\text{Berk}}$.

First suppose $z_0 \in \mathbb{P}^1(K)$. After a change of coordinates, we can assume that $z_0 \ne \infty$. Let λ be the multiplier for z_0, so $|\lambda| > 1$. By the discussion earlier in this section, there is an $R > 0$ such that $\varphi(\mathcal{D}(z_0, r)^-) = \mathcal{D}(z_0, |\lambda|r)^-$ for each $0 < r \le R$. Put $Y = \mathcal{D}(z_0, R)^-$. After taking a smaller R if necessary, we can assume that $z_\ell \notin Y$. By Theorem 10.59, the iterates $\varphi^{(k)}(Y)$ cover J_φ, so there is an $m \ge 1$ for which $z_\ell \in \varphi^{(m)}(Y)$. Let $z_{\ell+m} \in Y$ be such that $\varphi^{(m)}(z_{\ell+m}) = z_\ell$. Set $z_{\ell+m-k} = \varphi^{(k)}(z_{\ell+m})$ for $k = 1, \ldots, m-1$.

Since $\varphi(\mathcal{D}(z_0, R/|\lambda|^k)^-) = \mathcal{D}(z_0, R/|\lambda|^{k-1})^-$ for each $k \ge 1$, we can inductively find points $z_{\ell+m+k} \in \mathcal{D}(z_0, R/|\lambda|^k)$ such that $\varphi(z_{\ell+m+k}) = z_{\ell+m+k-1}$ for each k. Clearly $\lim_{k\to\infty} z_k = z_0$.

By the Open Mapping Theorem (Corollary 9.10), $\varphi^{(m)}(Y)$ is an open neighborhood of z_ℓ. Since $\varphi^{(\ell)}$-saturated simple domains are cofinal in the neighborhoods of z_ℓ (see the discussion after Proposition 9.15), we can find a $\varphi^{(\ell)}$-saturated simple domain W_ℓ which contains z_ℓ and whose closure is contained in $V \cap \varphi^{(m)}(Y)$. Let $W_{\ell+m}$ be the connected component of $(\varphi^{(m)})^{-1}(W_\ell)$ which contains $z_{\ell+m}$. Clearly $\overline{W}_{\ell+m} \subset Y = \mathcal{D}(x_0, R)^-$. Inductively, for each $k \ge 1$ let $W_{\ell+m+k}$ be the connected component of $\varphi^{-1}(W_{\ell+m+k-1})$ which contains $z_{\ell+m+k}$; then $\overline{W}_{\ell+m+k} \subset \mathcal{D}(z_0, R/|\lambda|^k)^-$. Likewise, put $W_0 = \varphi^{(\ell)}(W_\ell)$; then W_0 is a neighborhood of z_0, and since W_ℓ is $\varphi^{(\ell)}$-saturated, W_ℓ is a connected component of $(\varphi^{(\ell)})^{-1}(W_0)$.

For some sufficiently large n, we will have $\mathcal{D}(z_0, R/|\lambda|^n) \subset W_0$. Put $q = \ell + m + n$, and put $U = W_{\ell+m+n}$. By Lemma 9.12, U is a simple domain. Furthermore, $W_0 = \varphi^{(q)}(U)$ and U is $\varphi^{(q)}$-saturated.

Since

$$\overline{U} \subset \mathcal{D}(z_0, R/|\lambda|^n)^- \subset W_0 = \varphi^{(q)}(U) ,$$

Theorem 10.83 shows that there is a repelling fixed point for $\varphi^{(q)}(T)$ which belongs to U. It is a repelling periodic point for $\varphi(T)$ whose orbit passes through W_ℓ, hence through V.

Next suppose $z_0 \in \mathbb{H}_{\text{Berk}}$. Let Y be an arbitrary neighborhood of z_0 which does not contain z_ℓ. As before, the iterates $\varphi^{(k)}(Y)$ cover J_φ, so there is an $m \ge 1$ for which $z_\ell \in \varphi^{(m)}(Y)$. Let $z_{\ell+m} \in Y$ be such that $\varphi^{(m)}(z_{\ell+m}) = z_\ell$.

Since the $\varphi^{(\ell)}$-saturated simple domains are cofinal in all neighborhoods of z_ℓ, we can find a $\varphi^{(\ell)}$-saturated simple domain W_ℓ containing z_ℓ, such that $\overline{W}_\ell \subset (V \cap \varphi^{(m)}(Y))$. Set $W_0 = \varphi^{(\ell)}(W_\ell)$; it is an open neighborhood of z_0, and W_ℓ is a component of $(\varphi^{(\ell)})^{-1}(W_0)$. Let $W_{\ell+m}$ be the component of $(\varphi^{(m)})^{-1}(W_\ell)$ containing $z_{\ell+m}$.

Put $Z = W_0 \cap Y$. Let $\tilde{\varphi}(T) \in k(T)$ be the reduction of $\varphi(T)$ at z_0 in suitable coordinates, and parametrize the tangent directions in T_{z_0} by

$\mathbb{P}^1(k)$ in the usual way. Let $S \subset \mathbb{P}^1(k)$ be the finite set of points such that $\mathcal{B}_{z_0}(\vec{v}_\alpha)^- \subset Z$ if $\alpha \notin S$. Let $S_0 \subset S$ be the (possibly empty) set of $\alpha \in S$ which are exceptional for $\tilde{\varphi}(T)$. Let $\beta \in \mathbb{P}^1(k)$ be the point for which $z_{\ell+m} \in \mathcal{B}_{z_0}(\vec{v}_\beta)^-$.

We now consider two subcases, according to whether $\beta \in S_0$ or not.

First suppose $\beta \notin S_0$. Then there are an $\alpha \notin S$ and an $n \geq 1$ with $\tilde{\varphi}^{(n)}(\alpha) = \beta$. By hypothesis, $\mathcal{B}_{z_0}(\vec{v}_\alpha)^- \subset Z$. Since $\varphi^{(n)}(\mathcal{B}_{z_0}(\vec{v}_\alpha)^-)$ contains $\mathcal{B}_{z_0}(\vec{v}_\beta)^-$ (Proposition 9.40), there is a point $z_{\ell+m+n} \in \mathcal{B}_{z_0}(\vec{v}_\alpha)^-$ with $\varphi^{(n)}(z_{\ell+m+n}) = z_{\ell+m}$, which means that $\varphi^{(m+n)}(z_{\ell+m+n}) = z_\ell$. Let $W_{\ell+m+n}$ be the component of $(\varphi^{(m+n)})^{-1}(W_\ell)$ which contains $z_{\ell+m+n}$. Since $W_\ell \subset V$ and $\varphi^{(\ell+m)}(z_0) = z_0$ but $z_0 \notin V$, clearly $z_0 \notin W_{\ell+m+n}$. However, $\mathcal{B}_{z_0}(\vec{v}_\alpha)^-$ is a component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{z_0\}$; this implies that $W_{\ell+m+n} \subseteq \mathcal{B}_{z_0}(\vec{v}_\alpha)^-$. Put $q = \ell + m + n$, and put $U = W_{\ell+m+n}$. As before, U is a simple domain, $W_0 = \varphi^{(q)}(U)$, and U is $\varphi^{(q)}$-saturated. Since

$$\overline{U} \subseteq (\mathcal{B}_{z_0}(\vec{v}_\alpha)^- \cup \{z_0\}) \subset Z \subseteq W_0 = \varphi^{(q)}(U) ,$$

we can conclude the proof by using Theorem 10.83.

Next suppose $\beta \in S_0$. Since β is exceptional for $\tilde{\varphi}(T)$, there is an $s \geq 1$ such that $\tilde{\varphi}^{(s)}(\beta) = \beta$. Since β is the only point $\alpha \in \mathbb{P}^1(k)$ for which $\tilde{\varphi}^{(s)}(\alpha) = \beta$, it follows that $\vec{v}_\beta$ is the only tangent vector $\vec{v} \in T_{z_0}$ for which $\varphi_*^{(s)}(\vec{v}) = \vec{v}_\beta$. Thus

$$M := m_{\varphi^{(s)}}(z_0, \vec{v}_\beta) = \deg(\tilde{\varphi})^s > 1 .$$

By Corollary 9.21 and Theorem 9.22, there is a segment $[z_0, t]$ such that $\varphi^{(s)}([z_0, t])$ is the segment $[z_0, \varphi^{(s)}(t)]$, and both segments have the initial tangent vector $\vec{v}_\beta$. Furthermore $\rho(z_0, \varphi^{(s)}(x)) = M \cdot \rho(z, x)$ for each $x \in [z_0, t]$, and if $\mathcal{A} = \mathcal{A}_{z_0,t}$ is the annulus associated to (z_0, t), then $\varphi^{(s)}(\mathcal{A})$ is the annulus associated to $(z_0, \varphi^{(s)}(t))$. Since any two segments with the same initial tangent vector share a common initial segment, after shrinking $[z_0, t]$ if necessary, we can assume that $[z_0, t] \subset [z_0, \varphi^{(s)}(t)]$.

Clearly $\mathcal{A} \subset \mathcal{B}_{z_0}(\vec{v}_\beta)^-$. After further shrinking $[z_0, t]$, we can arrange that $\mathcal{A} \subset W_0$. Put $r = \mathrm{Mod}(\mathcal{A})$, and consider the iterates $\varphi^{(ks)}(\mathcal{A})$ for $k = 1, 2, \ldots$. By induction, $\varphi^{(ks)}(\mathcal{A}) \subset \varphi^{((k+1)s)}(\mathcal{A})$ for each k. By Proposition 9.44, each $\varphi^{(ks)}(\mathcal{A})$ is either an annulus, an open disc, or all of $\mathbb{P}^1_{\mathrm{Berk}}$. If it is an annulus, it is contained in $\mathcal{B}_{z_0}(\vec{v}_\beta)^-$ and has modulus $M^k r$. If it is a disc, it is equal to $\mathcal{B}_{z_0}(\vec{v}_\beta)^-$. If it is $\mathbb{P}^1_{\mathrm{Berk}}$, it clearly contains $\mathcal{B}_{z_0}(\vec{v}_\beta)^-$.

Since $\overline{W}_\ell$ is connected and contained in V and since $\varphi(z_0) = z_0$ but $z_0 \notin V$, it follows that $z_0 \notin (\varphi^{(m)})^{-1}(\overline{W}_\ell)$ and in turn that $z_0 \notin \overline{W}_{\ell+m}$. Since $z_{\ell+m} \in \mathcal{B}_{z_0}(\vec{v}_\beta)^-$, we must have $\overline{W}_{\ell+m} \subset \mathcal{B}_{z_0}(\vec{v}_\beta)^-$.

If some $\varphi^{(ns)}(\mathcal{A})$ is $\mathcal{B}_{z_0}(\vec{v}_\beta)^-$ or $\mathbb{P}^1_{\mathrm{Berk}}$, then trivially $\overline{W}_{\ell+m} \subset \varphi^{(ns)}(\mathcal{A})$. On the other hand, if each $\varphi^{(ns)}(\mathcal{A})$ is an annulus, then since the moduli of those annuli grow to ∞, we must have

$$\bigcup_{k=1}^{\infty} \varphi^{(ks)}(\mathcal{A}) \;=\; \mathcal{B}_{z_0}(\vec{v}_\beta)^- \backslash \{x_\beta\}$$

for some $x_\beta \in \mathbb{P}^1(K)$. Here, x_β depends only on $\varphi(T)$, z_0, and β, but not on W_ℓ. Hence we can assume that W_ℓ was chosen so that $\varphi^{(m)}(x_\beta) \notin \overline{W}_\ell$, which means that $x_\beta \notin \overline{W}_{\ell+m}$. By compactness, again there is some $\varphi^{(ns)}(\mathcal{A})$ which contains $\overline{W}_\ell$.

Choose a point $z_{\ell+m+ns} \in \mathcal{A}$ for which $\varphi^{(ns)}(z_{\ell+m+ns}) = z_{\ell+m}$, and let $W_{\ell+m+ns}$ be the component of $(\varphi^{(ns)})^{-1}(W_{\ell+m}) = (\varphi^{(\ell+m+ns)})^{-1}(W_0)$ which contains $z_{\ell+m+ns}$. By the discussion above,

$$\overline{W}_{\ell+m+ns} \;\subset\; \mathcal{A} \;\subset\; W_0 \ .$$

Thus if we take $q = \ell + m + ns$ and put $U = W_{\ell+m+ns}$, then U is a $\varphi^{(q)}$-saturated simple domain with $\overline{U} \subset \varphi^{(q)}(U)$, and we can conclude the proof by using Theorem 10.83 as before. □

Remark 10.89. It is an open problem to prove the "classical" version of Theorem 10.88 for $\mathbb{P}^1(K)$, that is, to prove that the repelling periodic points of φ in $\mathbb{P}^1(K)$ are dense in the classical Julia set $J_\varphi(K)$ of φ. By a theorem of J.-P. Bézivin [**22**, Theorem 3], if there exists at least one repelling periodic point in $J_\varphi(K)$, then the repelling periodic points of φ in $\mathbb{P}^1(K)$ are dense in $J_\varphi(K)$. Hence the problem is equivalent to the *a priori* weaker assertion that if $J_\varphi(K)$ is nonempty, then there exists at least one repelling periodic point for φ in $\mathbb{P}^1(K)$.

10.8. Dynamics of polynomial maps

In this section, we characterize the Berkovich Julia set and the canonical measure in potential-theoretic terms in the case where $\varphi(T) \in K[T]$ is a polynomial. Here the characteristic of K can be arbitrary.

Let $\varphi(T) \in K[T]$ be a polynomial of degree $d \geq 2$.

Definition 10.90. The *Berkovich filled Julia set* K_φ of φ is

$$K_\varphi \;:=\; \bigcup_{M>0} \{x \in \mathbb{A}^1_{\mathrm{Berk}} \;:\; [\varphi^{(n)}(T)]_x \leq M \text{ for all } n \geq 0\} \ .$$

In other words, K_φ is the set of all $x \in \mathbb{P}^1_{\mathrm{Berk}}$ for which the sequence $\{[\varphi^{(n)}(T)]_x\}_{n\geq 0}$ stays bounded as n goes to infinity. It is clear that K_φ is a compact subset of $\mathbb{P}^1_{\mathrm{Berk}}$ not containing ∞. It is nonempty, because every preperiodic point of φ in $\mathbb{A}^1(K)$ is contained in K_φ. It is not hard to see that K_φ is just the closure in $\mathbb{P}^1_{\mathrm{Berk}}$ of the classical filled Julia set

$$\bigcup_{M>0} \{x \in K \;:\; |\varphi^{(n)}(x)| \leq M \text{ for all } n \geq 0\} \ .$$

It is also easy to see that the complement of K_φ coincides with the attracting basin for the attracting fixed point ∞ of φ.

Both K_φ and ∂K_φ are completely invariant under φ. Since the classical exceptional locus $E_\varphi(K)$ consists of attracting periodic points, which are contained in either the complement of K_φ (in the case of ∞) or the interior of K_φ, we know by Corollary 10.57 that

$$J_\varphi \subseteq \partial K_\varphi . \tag{10.70}$$

We will see below that in fact the Berkovich Julia set of φ equals ∂K_φ.

Recall that the Call-Silverman local height function $\hat{h}_{\varphi,\infty} = \hat{h}_{\varphi,v,(\infty)}$ relative to the point ∞ and the dehomogenization $f_1(T) = 1$, $f_2(T) = \varphi(T)$ is defined for $x \in \mathbb{A}^1(K)$ by the formula

$$\hat{h}_{\varphi,\infty}(x) = \lim_{n\to\infty} \frac{1}{d^n} \max(0, \log_v([\varphi^{(n)}]_x)) .$$

It is a continuous subharmonic function on $\mathbb{A}^1_{\mathrm{Berk}}$ which belongs to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and satisfies

$$\Delta \hat{h}_{\varphi,\infty} = \delta_\infty - \mu_\varphi .$$

Moreover, the definition of $\hat{h}_{\varphi,\infty}$ shows that $\hat{h}_{\varphi,\infty}(x) = 0$ for all $x \in K_\varphi$.

By (10.8) and continuity, we have the transformation law

$$\hat{h}_{\varphi,\infty}(\varphi(x)) = d \cdot \hat{h}_{\varphi,\infty}(x) \tag{10.71}$$

for all $x \in \mathbb{A}^1_{\mathrm{Berk}}$. From this and the fact that $\hat{h}_{\varphi,\infty}(x) - \log^+[T]_x$ is a bounded function on $\mathbb{P}^1_{\mathrm{Berk}}$, one deduces easily that $\hat{h}_{\varphi,\infty}(x) > 0$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}} \backslash K_\varphi$. Thus $K_\varphi = \{x \in \mathbb{P}^1_{\mathrm{Berk}} : \hat{h}_{\varphi,\infty}(x) = 0\}$.

Next consider the Green's function $g_\varphi(x,\infty) = g_{\mu_\varphi}(x,\infty)$. By formula (10.21), for $x \neq \infty$ we have

$$g_\varphi(x,\infty) = \hat{h}_{\varphi,\infty}(x) + C \tag{10.72}$$

where $C = \hat{h}_{\varphi,v,(0)}(\infty) + \log_v(R)$. For each $x \in \mathbb{P}^1_{\mathrm{Berk}}$, Theorem 10.18 and the fact that $\varphi^*((\infty)) = d \cdot (\infty)$ give

$$g_\varphi(\varphi(x),\infty) = g_\varphi(x,\varphi^*(\infty)) = d \cdot g_\varphi(x,\infty) . \tag{10.73}$$

If $x \in K_\varphi$, the iterates $\varphi^{(n)}(x)$ all belong to the compact set K_φ, so the values $g_\varphi(\varphi^{(n)}(x),\infty)$ are uniformly bounded, and iterating (10.73) shows that $g_\varphi(x,\infty) = 0$. Hence $C = 0$ and $g_\varphi(z,\infty) = \hat{h}_{\varphi,\infty}(z)$ for all z.

By formula (10.21) and what has just been shown,

$$g_\varphi(x,y) = \begin{cases} -\log_v(\delta(x,y)_\infty) + \hat{h}_{\varphi,\infty}(x) & \\ \qquad + \hat{h}_{\varphi,\infty}(y) + \log_v(R) & \text{if } x, y \neq \infty , \\ \hat{h}_{\varphi,\infty}(x) & \text{if } y = \infty , \\ \hat{h}_{\varphi,\infty}(y) & \text{if } x = \infty . \end{cases} \tag{10.74}$$

From (10.74) and the fact that $\hat{h}_{\varphi,\infty} \equiv 0$ on K_φ, it follows that for any probability measure ν supported on K_φ, we have

$$\iint_{\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}}} g_{\mu_\varphi}(x,y)\, d\nu(x)\, d\nu(y)$$

$$= \log_v(R) + \iint_{\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}}} -\log_v(\delta(x,y)_\infty)\, d\nu(x)\, d\nu(y) \ . \tag{10.75}$$

Note that μ_φ is supported on J_φ, which is contained in K_φ by (10.70). By Theorem 10.8, the left-hand side of (10.75) is always nonnegative and is zero if and only if $\nu = \mu_\varphi$. On the other hand, by the definition of the logarithmic capacity and the equilibrium measure, the right-hand side is minimized by the equilibrium measure $\mu_{K_\varphi,\infty}$ of K_φ relative to ∞. It follows that $\mu_\varphi = \mu_{K_\varphi,\infty}$ and the capacity $\gamma_\infty(K_\varphi)$ is equal to R. Since μ_φ has continuous potentials, the Green's function $G(z,\infty;K_\varphi)$ is continuous. Thus $G(z,\infty;K_\varphi)$ is equal to the Call-Silverman local height $\hat{h}_{\varphi,\infty}(x)$, since both functions are zero on K_φ, have Laplacian equal to $\delta_\infty - \mu_\varphi$, are continuous on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\infty\}$, and have bounded difference from $\log_v([T]_x)$ as $x \to \infty$.

Recall that $R = |\operatorname{Res}(F_1,F_2)|^{-\frac{1}{d(d-1)}}$. If

$$\varphi(T) \ = \ a_d T^d + a_{d-1}T^{d-1} + \cdots + a_1 T + a_0 \ ,$$

then an easy calculation shows that $|\operatorname{Res}(F_1,F_2)| = |a_d|^d$, so $\gamma_\infty(K_\varphi) = R = |a_d|^{-\frac{1}{d-1}}$.

We can now show that $J_\varphi = \partial K_\varphi$. For this, it suffices to note that by Corollary 7.39, the connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash K_\varphi$ containing ∞ is $U_\infty = \mathbb{P}^1_{\mathrm{Berk}} \backslash K_\varphi = \{x \in \mathbb{P}^1_{\mathrm{Berk}} : \hat{h}_{\varphi,\infty}(x) > 0\}$. This is also the attracting basin of ∞. By Corollary 7.39(C),

$$J_\varphi \ = \ \operatorname{supp}(\mu_\varphi) \ = \ \partial U_\infty \ = \ \partial K_\varphi \ .$$

By Proposition 7.37(A)(6), it follows that $\gamma_\infty(J_\varphi) = \gamma_\infty(K_\varphi)$.

Combining these facts, we have proved the following result:

Theorem 10.91. *Let $\varphi \in K[T]$ be a polynomial with degree $d \geq 2$ and leading coefficient a_d. Then:*

(A) *For all $x \in \mathbb{P}^1_{\mathrm{Berk}}$,*

$$g_{\mu_\varphi}(x,\infty) \ = \ G(x,\infty;K_\varphi) \ = \ \hat{h}_{\varphi,v,(\infty)}(x)$$
$$= \ \lim_{n\to\infty} \frac{1}{d^n} \max(0, \log_v([\varphi^{(n)}]_x)) \ .$$

(B) *The canonical measure μ_φ coincides with $\mu_{K_\varphi,\infty}$, the equilibrium measure of K_φ with respect to ∞.*

(C) $J_\varphi = \partial K_\varphi$.

(D) $\gamma_\infty(J_\varphi) = \gamma_\infty(K_\varphi) = |a_d|^{-1/(d-1)}$.

(E) *The connected component of F_φ containing ∞ is equal to $\mathbb{P}^1_{\mathrm{Berk}} \backslash K_\varphi$, the basin of attraction of ∞ for $\varphi(T)$.*

10.9. Rational dynamics over $\mathbb{C}_p$

The field $\mathbb{C}_p$ is (up to isomorphism) the smallest complete and algebraically closed non-Archimedean field of characteristic 0 and positive residue characteristic p. It can be constructed as the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers and is often viewed as the p-adic analogue of the complex numbers. It is thus of great importance in arithmetic geometry. Thanks to the work of Rivera-Letelier, Benedetto, and others, the dynamics of rational functions $\varphi(T) \in \mathbb{C}_p(T)$ are much better understood than those of rational functions over an arbitrary complete and algebraically closed non-Archimedean field K.

In this section, we summarize the main results of Rivera-Letelier from [**81, 82, 83, 84**] concerning rational dynamics on $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$, formulating his results in the terminology of this book. There are many interesting results in those papers we will not touch upon, and we omit the proofs, so we encourage the reader to consult the original papers. For the reader's convenience, we repeat some of the results and definitions from earlier sections.

Let p be a prime number. The field $\mathbb{C}_p$ is special in several ways:

- It has characteristic 0 and residue characteristic $p > 0$.
- It has a countable dense subfield isomorphic to $\overline{\mathbb{Q}}$ (a fixed algebraic closure of $\mathbb{Q}$).
- Its value group $|\mathbb{C}_p^\times|$ is a rank 1 divisible group isomorphic to $\mathbb{Q}$.
- Its residue field is isomorphic to $\overline{\mathbb{F}}_p$, an algebraic closure of the prime field $\mathbb{F}_p$.

These properties have several consequences. In particular, they imply that $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ has points of all four types, I, II, III, and IV, that $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ has countably many points of type II, with countable branching at each such point, and that $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ (equipped with the Berkovich topology) is metrizable. However, most important for rational dynamics are the strong finiteness properties which follow from the fact that $\overline{\mathbb{F}}_p$ is a union of finite fields. For example, if ζ is a type II fixed point of $\varphi(T) \in \mathbb{C}_p(T)$ and $\tilde{\varphi}(T) \in \overline{\mathbb{F}}_p(T)$ is reduction of $\varphi(T)$ in a corresponding local coordinate, then the forward orbits of $\tilde{\varphi}(T)$ in $\mathbb{P}^1(\overline{\mathbb{F}}_p)$ are automatically finite. Moreover, if $\tilde{\varphi}(T)$ has degree 1, then for some $n \geq 1$, $\tilde{\varphi}^{(n)}(T)$ is the identity map on $\mathbb{P}^1(\overline{\mathbb{F}}_p)$.

These finiteness properties are the ultimate source of Rivera-Letelier's description of the Berkovich Fatou set in $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ as a disjoint union of attracting basins, the "quasi-periodicity domain" and its preimages, and wandering domains. Such a decomposition does not hold over an arbitrary complete and algebraically closed non-Archimedean field K.

Rivera-Letelier starts with a rational function $\varphi(T) \in \mathbb{C}_p(T)$ and studies the action of φ on $\mathbb{P}^1(\mathbb{C}_p)$ and on the "p-adic hyperbolic space" $\mathbb{H}_p$, which turns out to be naturally isomorphic to $\mathbb{H}_{\mathrm{Berk}} = \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(\mathbb{C}_p)$ (see [**82**, §9.3]), though at first glance Rivera-Letelier's definition of $\mathbb{H}_p$ looks rather different from the definition of $\mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(\mathbb{C}_p)$. Rivera-Letelier equips $\mathbb{H}_p$ with the "strong topology" induced by the path distance $\rho(x, y)$ as defined in §2.7. This topology is strictly finer than the Berkovich subspace topology coming from the inclusion $\mathbb{H}_p \subset \mathbb{P}^1_{\text{Berk}}$.

In [**81, 82, 83, 84**], Rivera-Letelier takes the point of view that $\mathbb{P}^1(\mathbb{C}_p)$ is the fundamental object of study and that $\mathbb{H}_p$ is auxiliary. He thus states his main results in terms of $\mathbb{P}^1(\mathbb{C}_p)$. In his later works with Favre [**46, 47, 48**], he adopts a point of view closer to ours, that $\mathbb{P}^1_{\text{Berk}}$ is the appropriate domain for studying non-Archimedean dynamics.

The fundamental idea and principal novelty of Rivera-Letelier's work is that one can deduce useful facts about how $\varphi(T)$ acts on $\mathbb{P}^1(\mathbb{C}_p)$ by studying its action on $\mathbb{H}_p$. In particular, fixed point theorems for the action of $\varphi(T)$ on $\mathbb{H}_p$ play a prominent role in Rivera-Letelier's work.

To each $x \in \mathbb{H}_p$, Rivera-Letelier associates a family of *bouts* (literally: ends). In the context of our definition of $\mathbb{H}_{\text{Berk}}$, a bout $\mathcal{P}$ can be thought of as a tangent vector emanating from x, and the collection of bouts at x can therefore be identified with the "tangent space" T_x at x. To each bout $\mathcal{P}$ at x, one can associate a ball $\mathcal{B}_{\mathcal{P}} \subset \mathbb{P}^1(\mathbb{C}_p)$ in a natural way: $\mathcal{B}_{\mathcal{P}}$ is the set of $y \in \mathbb{P}^1(\mathbb{C}_p)$ such that the path (x, y) lies in the equivalence class $\mathcal{P}$. Thus, if $\mathcal{P}$ corresponds to $\vec{v} \in T_x$, then $\mathcal{B}_{\mathcal{P}} = \mathcal{B}_x(\vec{v})^-$ in our notation above.

However, in Rivera-Letelier's construction of $\mathbb{H}_p$, bouts are defined before points. Rivera-Letelier's definition of a bout [**82**, §2] is as follows. Let

$$B_0 \ \subset \ B_1 \ \subset \ B_2 \ \subset \cdots$$

be an increasing sequence of closed balls in $\mathbb{P}^1(\mathbb{C}_p)$ whose union B is either an open ball or all of $\mathbb{P}^1(\mathbb{C}_p)$. The collection of open annuli (or balls, if $B = \mathbb{P}^1(\mathbb{C}_p)$) $\{B \backslash B_i\}_{i \geq 0}$ is called a *vanishing chain*. Two vanishing chains are considered equivalent if each is cofinal in the other, under the relation of containment. A bout $\mathcal{P}$ is an equivalence class of vanishing chains. The ball $B_{\mathcal{P}}$ associated to a bout $\mathcal{P}$ is the (generalized) ball $B = \bigcup B_i$ attached to any of the vanishing chains in $\mathcal{P}$; it is independent of the vanishing chain.

For Rivera-Letelier, a point $x \in \mathbb{H}_p$ is a collection of bouts $\{\mathcal{P}_\alpha\}$ whose associated balls $B_{\mathcal{P}_\alpha}$ are pairwise disjoint and have $\mathbb{P}^1(\mathbb{C}_p)$ as their union. He calls a point x (and the bouts in it) rational, irrational, or singular, according as x consists of infinitely many bouts, two bouts, or one bout, respectively. Under Berkovich's classification, these correspond to points of type II, III, and IV, respectively. If x is of type II, then for each bout $\mathcal{P}$ associated to x, $\mathcal{B}_{\mathcal{P}}$ is an open ball with radius in $|\mathbb{C}_p^*|$; if x is of type III, then $\mathcal{B}_{\mathcal{P}}$ is an irrational ball; and if x is of type IV, then $\mathcal{B}_{\mathcal{P}} = \mathbb{P}^1(\mathbb{C}_p)$.

Fundamental lemmas. Rivera-Letelier bases his theory on two fundamental results concerning the action of a rational function on $\mathbb{P}^1(\mathbb{C}_p)$. The first ([**82**], Proposition 4.1) enables him to define the action of φ on $\mathbb{H}_p$, by first defining its action on bouts (compare with Lemma 9.45):

LEMMA 10.92. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d = \deg(\varphi) \geq 1$. Let $\mathcal{P}$ be a bout of $\mathbb{P}^1(\mathbb{C}_p)$. Then there exists another bout $\mathcal{P}'$ of the same type (rational, irrational, or singular) and an integer $1 \leq m \leq d$ such that for each vanishing chain $\{C_i\}_{i \geq 0}$ defining $\mathcal{P}$, there is an $N \geq 1$ such that*

(1) $\{\varphi(C_i)\}_{i \geq N}$ is a vanishing chain defining $\mathcal{P}'$,

(2) for each $i \geq N$, the map $\varphi : C_i \to \varphi(C_i)$ is of degree m.

Rivera-Letelier calls $m = m(\mathcal{P})$ the multiplicity of the bout $\mathcal{P}$ and writes $\varphi(\mathcal{P})$ for $\mathcal{P}'$. In our notation, if $\mathcal{P}$ corresponds to $\vec{v} \in T_x$, then $m(\mathcal{P}) = m_\varphi(x, \vec{v})$ and $\varphi(\mathcal{P}) = \varphi_*(\vec{v})$. Rivera-Letelier defines the action of $\varphi(T)$ on a point $x = \{\mathcal{P}_\alpha\}$ by setting $\varphi(x) = \{\varphi(\mathcal{P}_\alpha)\}$. He then defines a local degree $\deg_\varphi(x)$ at each $x \in \mathbb{H}_p$ which coincides with our multiplicity $m_\varphi(x)$. For example, for a type II point he defines $\deg_\varphi(x) = \deg(\tilde{\varphi}(T))$, where $\tilde{\varphi}(T)$ is a reduction of $\varphi(T)$ relative to x and $\varphi(x)$. (See [**82**, Proposition 4.4]; for the definition of $\tilde{\varphi}(T)$, see the discussion preceding Corollary 9.25.) He shows that for any bout $\mathcal{Q}$ belonging to y,

$$\deg_\varphi(x) = \sum_{\substack{\text{bouts } \mathcal{P} \text{ in } x \\ \text{with } \varphi(\mathcal{P}) = \mathcal{Q}}} m(\mathcal{P}) .$$

(Compare this with our Theorem 9.22.) We will henceforth write $m_\varphi(x)$ instead of $\deg_\varphi(x)$.

The second fundamental result ([**82**, Lemma 4.2]) describes the action of φ on the ball $\mathcal{B}_\mathcal{P}$ associated to a bout (compare with our Proposition 9.41):

LEMMA 10.93. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d = \deg(\varphi) \geq 1$. Let $\mathcal{P}$ be a bout with multiplicity $m = m(\mathcal{P})$, and let $\varphi(\mathcal{P})$ be its image under φ. Then there is an integer $m \leq N \leq d$ (depending on $\mathcal{P}$), with $N = m$ if and only if $\varphi(\mathcal{B}_\mathcal{P}) = \mathcal{B}_{\varphi(\mathcal{P})}$, such that*

(1) for each $y \in \mathcal{B}_{\varphi(\mathcal{P})}$ there are exactly N points $x \in \mathcal{B}_\mathcal{P}$ with $\varphi(x) = y$ (counting multiplicities),

(2) for each $y \in \mathbb{P}^1(\mathbb{C}_p) \backslash \mathcal{B}_{\varphi(\mathcal{P})}$ there are exactly $N - m$ points $x \in \mathcal{B}_\mathcal{P}$ with $\varphi(x) = y$ (counting multiplicities).

Rivera-Letelier shows that for each noncritical point $x \in \mathbb{P}^1(\mathbb{C}_p)$, there is a maximal open affinoid containing x on which $\varphi(T)$ is injective, the *component of injectivity* of x ([**81**, Proposition 2.9]):

PROPOSITION 10.94. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be nonconstant. Suppose $x \in \mathbb{P}^1(\mathbb{C}_p)$ is not a critical point. Then there is a maximal open affinoid $V \subseteq \mathbb{P}^1_{\text{Berk}}$ containing x on which $\varphi(T)$ is injective: $\varphi(T)$ is injective on V, and if X an open affinoid containing x on which $\varphi(T)$ is injective, then $X \subseteq V$.*

In [**82**, Corollaries 4.7, 4.8], he shows that $\varphi(T)$, acting on $\mathbb{H}_p$, is continuous relative to the path distance topology. In Theorem 9.35, we have established this for arbitrary fields K.

LEMMA 10.95. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a nonconstant rational function of degree d. Then for any $a, c \in \mathbb{H}_p$,*

$$\rho(\varphi(a), \varphi(c)) \ \leq \ \int_{[a,c]} m_\varphi(x)\, dx \ .$$

In particular, $\rho(\varphi(a), \varphi(c)) \leq d \cdot \rho(a, c)$.

Given $x \in \mathbb{H}_p$, write $\widehat{\mathcal{B}}(x, r)^- = \{z \in \mathbb{H}_p : \rho(x, z) < r\}$ for the open ball of radius r about x, relative to the path distance metric. The following result from [**84**], which Rivera-Letelier calls the "Incompressibility Lemma", shows in particular that $\varphi(T)$ is an open map on $\mathbb{H}_p$ relative to the path distance topology (see our Corollary 9.39):

LEMMA 10.96. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be nonconstant, and let $x \in \mathbb{H}_p$. Then for each $r > 0$,*

$$\widehat{\mathcal{B}}(\varphi(x), r)^- \ \subseteq \ \varphi(\widehat{\mathcal{B}}(x, r)^-) \ .$$

The theory of fixed points. If $x \in \mathbb{P}^1(\mathbb{C}_p)$ is fixed by $\varphi(T)$ and if its multiplier is λ, it is called

(1) an *attracting* fixed point, if $|\lambda| < 1$,

(2) an *indifferent* fixed point, if $|\lambda| = 1$; such a fixed point is called *parabolic* if λ is a root of unity and is *rationally indifferent* otherwise,

(3) a *repelling* fixed point, if $|\lambda| > 1$.

If a point $x \in \mathbb{P}^1(\mathbb{C}_p)$ is fixed by $\varphi^{(n)}(T)$ for some n and if n is the smallest integer with this property, then x is called periodic of order n, and $C = \{x, \varphi(x), \ldots, \varphi^{(n-1)}(x)\}$ is called the cycle associated to x. It is well known that the multiplier of $\varphi^{(n)}(t)$ is the same for all points $t \in C$ and the multiplier of C is defined to be that number. A cycle is called *attracting*, *indifferent*, or *repelling*, according as its multiplier λ satisfies $|\lambda| < 1$, $|\lambda| = 1$, or $|\lambda| > 1$. An attracting cycle $C \subset \mathbb{P}^1(\mathbb{C}_p)$ is called *superattracting* if its multiplier is 0. An indifferent cycle C is called *parabolic* if its multiplier is a root of unity.

Rivera-Letelier calls a fixed point of $\varphi(T)$ in $\mathbb{H}_p$ *indifferent* if $m_\varphi(x) = 1$ and *repelling* if $m_\varphi(x) > 1$. (We have explained the motivation for this terminology in §10.7.) He shows that fixed points of type III or IV in $\mathbb{H}_p$ are necessarily indifferent [**84**, Proposition 5.2]; see our Lemma 10.80.

Repelling fixed points in $\mathbb{H}_p$ come in two kinds, *separable* and *inseparable* [**84**]. If $x \in \mathbb{H}_p$ is a repelling fixed point (necessarily of type II) and $\tilde{\varphi}(T) \in \overline{\mathbb{F}}_p(T)$ is a reduction of $\varphi(T)$ relative to x, then x is called separable if the extension $\overline{\mathbb{F}}_p(T)/\overline{\mathbb{F}}_p(\tilde{\varphi}(T))$ is separable, and it is called inseparable if $\tilde{\varphi}(T) = \tilde{g}(T^p)$ for some $\tilde{g}(T) \in \overline{\mathbb{F}}_p(T)$. If x is separable, then there are a finite number of tangent directions $\vec{v}$ at x for which $m_\varphi(x, \vec{v}) > 1$, while if x is inseparable, then $m_\varphi(x, \vec{v}) \geq p$ for every tangent direction. In terms of the local action of $\varphi(T)$ described in §10.7, if x is separable, then it has a finite number of repelling "jets", while if x is inseparable, then *every* direction is a jet. Inseparable fixed points play an important role in the study of wandering components of the Fatou set.

Periodic points of $\varphi(T)$ in $\mathbb{H}_p$ become fixed points of $\varphi^{(n)}(T)$ for some n, and they are called indifferent, repelling, separable, or inseparable if they have those properties as fixed points of $\varphi^{(n)}(T)$. All the points of a given cycle have the same type.

In Theorem 10.81, we have seen that all repelling periodic points belong to the Berkovich Julia set J_φ. In Lemma 10.54, we have seen that all attracting periodic points belong to the Berkovich Fatou set F_φ. Over $\mathbb{C}_p$, in contrast to the situation over $\mathbb{C}$, all indifferent periodic points belong to F_φ:

Proposition 10.97. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $\deg(\varphi) \geq 2$. Then each indifferent periodic point of $\varphi(T)$, in $\mathbb{P}^1(\mathbb{C}_p)$ or in $\mathbb{H}_p$, belongs to F_φ.*

Proof. Let $x_0 \in \mathbb{P}^1_{\mathrm{Berk}}$ be an indifferent periodic point of period n. We claim that there are an $N \geq 1$ and a simple domain V containing x such that V omits a disc in $\mathbb{P}^1_{\mathrm{Berk}}$ and $\varphi^{(nN)}(V) = V$. If this is granted, then x belongs to the Berkovich Fatou set of $\varphi^{(nN)}(T)$, since its forward images omit all points of $\mathbb{P}^1(\mathbb{C}_p)\backslash V$. We finish by noting that $F_\varphi = F_{\varphi^{(nN)}}$ by Lemma 10.51.

We now establish the claim. After replacing $\varphi(T)$ by $\varphi^{(n)}(T)$, we can assume that x_0 is a fixed point of $\varphi(T)$.

If x_0 is of type I, choose coordinates so that $x_0 \neq \infty$ and expand $\varphi(T)$ as a power series as in (10.67). Then because the multiplier λ of x_0 satisfies $|\lambda| = 1$, the theory of Newton polygons shows there is a disc $D(x_0, r)^-$ such that $\varphi(D(x_0, r)^-) = D(x_0, r)^-$ (see Corollary A.17). Let $V = \mathcal{D}(x_0, r)^-$ be the corresponding Berkovich disc. By continuity, $\varphi(V) = V$.

Next suppose x_0 is of type II. Let $\tilde{\varphi}(T) \in \overline{\mathbb{F}}_p(T)$ be a reduction of $\varphi(T)$ at x_0. Since $m_\varphi(x_0) = 1$, Corollary 9.25 shows that $\deg(\tilde{\varphi}) = 1$. Since $\overline{\mathbb{F}}_p$ is a union of finite fields, there is some ℓ such that $\tilde{\varphi}(T) \in \mathrm{PGL}_2(\mathbb{F}_{p^\ell})$. This is a finite group, so there is an $N \geq 1$ for which $\tilde{\varphi}^{(N)}(T) = \mathrm{id}$. This means $(\varphi^{(N)})_*$ fixes each $\vec{v} \in T_{x_0}$.

For all but finitely many tangent directions, we have $\varphi^{(N)}(\mathcal{B}_{x_0}(\vec{v})^-) = \mathcal{B}_{x_0}(\vec{v})^-$. (For example, this holds for each $\mathcal{B}_{x_0}(\vec{v})^-$ not containing a pole of $\varphi^{(N)}(T)$.) Let S be an index set for the remaining tangent directions (we may assume that $S \neq \emptyset$). For each $\vec{v}_\alpha$ with $\alpha \in S$, Lemma 9.33 shows that there is a segment $[x_0, c_\alpha]$ with initial tangent vector $\vec{v}_\alpha$ such that $\varphi^{(N)}([x_0, c_\alpha]) = [x_0, \varphi(c_\alpha)]$ is another segment with the same initial tangent vector $\vec{v}_\alpha = (\varphi^{(N)})_*(\vec{v}_\alpha)$. Furthermore, since $m_{\varphi^{(N)}}(x_0, \vec{v}_\alpha) = m_{\varphi^{(N)}}(x_0) = m_\varphi(x_0)^N = 1$, for each $y \in [x_0, c_\alpha]$ we have $\rho(x_0, y) = \rho(x_0, \varphi^{(N)}(y))$. Since any two segments with the same initial tangent vector have a common initial segment, after moving c_α closer to x_0 if necessary, we can assume that $\varphi^{(N)}(c_\alpha) = c_\alpha$ and that $\varphi^{(N)}(T)$ is the identity on $[x_0, c_\alpha]$. Let $\mathcal{A}_{x_0,c_\alpha}$ be the annulus associated to the segment (x_0, c_α). By Lemma 9.33, we have $\varphi^{(N)}(\mathcal{A}_{x_0,c_\alpha}) = \mathcal{A}_{x_0,c_\alpha}$. Thus, if V is the component of $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{c_\alpha\}_{\alpha\in S}$ containing x_0, it follows that $\varphi^{(N)}(V) = V$.

If x_0 is of type III, then T_{x_0} consists of two tangent vectors and by Lemma 10.80, φ_* fixes them both. If x_0 is of type IV, then T_{x_0} consists of a single tangent vector, and φ_* clearly fixes it. By an argument like the one above, in either case we can construct a simple domain V containing x_0 with $\varphi(V) = V$. □

The basic existence theorem for fixed points of $\varphi(T)$ in $\mathbb{P}^1(\mathbb{C}_p)$ is due to Benedetto [**13**, Proposition 1.2]:

PROPOSITION 10.98. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a nonconstant rational function. Then $\varphi(T)$ has at least two distinct fixed points in $\mathbb{P}^1(\mathbb{C}_p)$, and at least one of these is nonrepelling.*

PROOF. (Sketch) If $\varphi(T)$ has degree d, then φ has exactly $d+1$ fixed points $z_0, z_1, \ldots, z_d$ in $\mathbb{P}^1(\mathbb{C}_p)$, counting multiplicities. Since each z_i can have multiplicity at most d, at least two of the z_i must be distinct. If some z_i has multiplier $\lambda_i = 1$, it is clearly nonrepelling. Otherwise, transferring the Index Formula (see [**73**, Theorem 12.4]) from $\mathbb{C}$ to $\mathbb{C}_p$ using the fact that $\mathbb{C}_p \cong \mathbb{C}$ as fields, Benedetto concludes that

$$\sum_{i=0}^{d} \frac{1}{1-\lambda_i} \ = \ 1 \ .$$

By the ultrametric inequality, this cannot hold if each $|\lambda_i| > 1$. Hence $|\lambda_i| \le 1$ for some i. □

Rivera-Letelier [**82**, Theorem B] proves an existence theorem for repelling fixed points in $\mathbb{P}^1_{\text{Berk}}$, complementing the one above:

PROPOSITION 10.99. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree* $\deg(\varphi) \ge 2$. *Then $\varphi(T)$ has a repelling fixed point in $\mathbb{P}^1_{\text{Berk}}$ (which could lie in either $\mathbb{P}^1(\mathbb{C}_p)$ or $\mathbb{H}_p$).*

The existence of certain kinds of periodic points implies the existence of others. Concerning classical fixed points and cycles, Rivera-Letelier proves the following result [**81**, Corollary 3.17, p. 189]. While the statement is purely about $\mathbb{P}^1(\mathbb{C}_p)$, the proof makes crucial use of the action of φ on $\mathbb{H}_p$:

PROPOSITION 10.100. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d \ge 2$. If $\varphi(T)$ has more than $3d-3$ attracting cycles in $\mathbb{P}^1(\mathbb{C}_p)$, then it has infinitely many.*

For example, $\varphi(T) = T^p$ has infinitely many attracting cycles, given by the roots of unity in $\mathbb{C}_p$ of order coprime to p.

In [**82**, Theorems A, A'], Rivera-Letelier shows:

PROPOSITION 10.101. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree* $\deg(\varphi) \ge 2$. *If $\varphi(T)$ has at least two nonrepelling periodic points in $\mathbb{P}^1(\mathbb{C}_p)$ (counted with multiplicities), then it has a repelling periodic point in $\mathbb{H}_p$ and infinitely many nonrepelling periodic points in $\mathbb{P}^1(\mathbb{C}_p)$.*

Concerning indifferent periodic points, he shows ([**84**, Proposition 5.1], [**81**, Corollary 4.9]):

PROPOSITION 10.102. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree* $\deg(\varphi) \geq 2$. *If φ has an indifferent periodic point in either $\mathbb{H}_p$ or $\mathbb{P}^1(\mathbb{C}_p)$, then it has infinitely many indifferent periodic points in both $\mathbb{H}_p$ and $\mathbb{P}^1(\mathbb{C}_p)$. It also has at least one repelling periodic point in $\mathbb{H}_p$. Each indifferent fixed point in $\mathbb{P}^1(\mathbb{C}_p)$ is isolated.*

Concerning inseparable periodic points in $\mathbb{H}_p$, he proves [**84**, Principal Lemma]:

PROPOSITION 10.103. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d \geq 2$. Then $\varphi(T)$ has either* 0, 1, *or infinitely many inseparable periodic points in $\mathbb{H}_p$.*

He gives $\varphi(T) = T^p + pT^d$, with $d > p$, as an example of a polynomial with infinitely many inseparable periodic points.

Rivera-Letelier also proves theorems limiting the number of periodic cycles. A classical result of Fatou asserts that for a rational function $\varphi(T) \in \mathbb{C}(T)$ of degree $d \geq 2$, each attracting or parabolic cycle attracts at least one critical point of $\varphi(T)$, so there are at most $2d-2$ such cycles. Any isomorphism between $\mathbb{C}_p$ and $\mathbb{C}$ takes 0 to 0 and roots of unity to roots of unity. Using this, he obtains [**81**, Theorem 1, p. 194]:

PROPOSITION 10.104. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d \geq 2$. Then the number of superattracting and parabolic cycles of $\varphi(T)$ in $\mathbb{P}^1(\mathbb{C}_p)$ is at most* $2d-2$.

A well-known result of Shishikura improves Fatou's theorem to say that over $\mathbb{C}$, there are at most $2d-2$ nonrepelling cycles; however the example $\varphi(T) = T^d$, with d coprime to p, shows that this can fail over $\mathbb{C}_p$: the roots of unity contain infinitely many indifferent cycles.

Recall that a point $x \in \mathbb{P}^1_{\mathrm{Berk}}$ is called *exceptional* if the union of its forward and backwards orbits is finite. The exceptional locus in $\mathbb{P}^1(\mathbb{C}_p)$ consists of at most two points. The following result (see [**84**, Theorem 4], or our Proposition 10.45) describes the exceptional locus for $\mathbb{H}_p$:

THEOREM 10.105. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d \geq 2$. Then the exceptional set of $\varphi(T)$ in $\mathbb{H}_p$ consists of at most one point. It is nonempty if and only if the unique exceptional point is a repelling fixed point and after a change of coordinates, $\varphi(T)$ has good reduction at that point. In that case, the exceptional point is the only repelling periodic point of $\varphi(T)$ in $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$.*

Finally, Rivera-Letelier classifies the cases with extreme behavior. The following result combines Theorems 1, 2, and 3 from [**84**].

THEOREM 10.106. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d \geq 2$. Then the number of periodic points of $\varphi(T)$ in $\mathbb{H}_p$ is* 0, 1, *or* ∞. *Moreover:*

(A) *$\varphi(T)$ has no periodic points in $\mathbb{H}_p$ if and only if $\varphi(T)$ has finitely many nonrepelling periodic points in $\mathbb{P}^1(\mathbb{C}_p)$. In that case, $\varphi(T)$ has a unique attracting fixed point $z_0 \in \mathbb{P}^1(\mathbb{C}_p)$, and all other periodic points of $\varphi(T)$ in $\mathbb{P}^1(\mathbb{C}_p)$ are repelling.*

(B) *$\varphi(T)$ has exactly one periodic point in $\mathbb{H}_p$ if and only if, after a change of coordinates, $\varphi(T)$ has inseparable good reduction. In that case, the unique periodic point is an inseparable exceptional fixed point, and all periodic points of $\varphi(T)$ in $\mathbb{P}^1(\mathbb{C}_p)$ are attracting.*

An example of a function satisfying (A) is $\varphi(T) = (T^p - T)/p$ (see Example 10.120 below). A function satisfying (B) is $\varphi(T) = T^p$.

The proofs of the preceding fixed point theorems make use of two general results concerning subsets of $\mathbb{H}_p$ which are contracted (resp. expanded) by $\varphi(T)$. In [**83**, §8], Rivera-Letelier establishes the following fixed point property for subsets of $\mathbb{H}_p$ contracted by $\varphi(T)$ (see our Theorem 10.85):

PROPOSITION 10.107. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be nonconstant, and let $X \subset \mathbb{H}_p$ be a connected set containing at least two points, with $X \supseteq \varphi(X)$. Then X contains either a type* II *fixed point of $\varphi(T)$ in $\mathbb{H}_p$ or a geodesic ray emanating from an attracting fixed point of $\varphi(T)$ in $\mathbb{P}^1(\mathbb{C}_p)$.*

In [**84**, Proposition 9.3], Rivera-Letelier establishes a fixed point property for subsets of $\mathbb{H}_p$ expanded by $\varphi(T)$ (compare with our Theorem 10.83). Given $V \subset \mathbb{H}_p$, denote its closure in the metric space $(\mathbb{H}_p, \rho)$ by $\mathrm{cl}_{\mathbb{H}}(V)$ and denote its boundary by $\partial_{\mathbb{H}} V$.

PROPOSITION 10.108. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be nonconstant, and let $V \subset \mathbb{H}_p$ be a connected open set for which $\mathrm{cl}_{\mathbb{H}}(V) \subset \varphi(V)$ and $\varphi(\partial_{\mathbb{H}} V) = \partial_{\mathbb{H}}(\varphi(V))$. Then V contains either a type* II *fixed point of $\varphi(T)$ in $\mathbb{H}_p$ or a geodesic ray emanating from a repelling fixed point of $\varphi(T)$ in $\mathbb{P}^1(\mathbb{C}_p)$.*

The Fatou set and its components. For a subset $X \subset \mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$, we will write $X(\mathbb{C}_p)$ for $X \cap \mathbb{P}^1(\mathbb{C}_p)$. In particular, if $X = F_\varphi$ is the Berkovich Fatou set, we have seen in Theorem 10.67 that $F_\varphi(\mathbb{C}_p)$ coincides with the classical Fatou set, defined as the set of all points $x \in \mathbb{P}^1(\mathbb{C}_p)$ for which the iterates $\varphi^{(n)}(T)$ are equicontinuous on a neighborhood $U(\mathbb{C}_p)$ of x (relative to the chordal metric $\|x, y\|$).

Recall that Rivera-Letelier works primarily with $\mathbb{P}^1(\mathbb{C}_p)$, not with $\mathbb{P}^1_{\mathrm{Berk}}$. In [**81**], following Benedetto [**14**], he partitions $F_\varphi(\mathbb{C}_p)$ into subsets called *analytic components*. Since we are interested in $\mathbb{P}^1_{\mathrm{Berk}}$, we will define analytic components slightly differently than Rivera-Letelier and Benedetto. Given a set $F \subset \mathbb{P}^1(\mathbb{C}_p)$ and a point $x \in F$, Rivera-Letelier defines the analytic component of x in F to be the union of the sets $X(\mathbb{C}_p)$, as X ranges over all connected open affinoids $X \subset \mathbb{P}^1_{\mathrm{Berk}}$ for which $X(\mathbb{C}_p) \subset F$. However, we define the analytic component of x to be the open set $V \subset \mathbb{P}^1_{\mathrm{Berk}}$ which is the union of the affinoids X above. Thus Rivera-Letelier's analytic component is our $V(\mathbb{C}_p)$.

Analytic components usually have dynamical significance, but this is not always the case. For example, if $\varphi(T)$ is a rational function with good reduction, its Julia set is $\{\zeta_{\text{Gauss}}\}$ and its Fatou set is $F_\varphi = \mathbb{P}^1_{\text{Berk}}\backslash\{\zeta_{\text{Gauss}}\}$, so $F_\varphi(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p)$ and the analytic component of $F_\varphi(\mathbb{C}_p)$ containing any given $x \in \mathbb{P}^1(\mathbb{C}_p)$ is $\mathbb{P}^1_{\text{Berk}}$, which contains ζ_{Gauss}.

In [**83**], Rivera-Letelier introduces a more subtle notion of components of $F_\varphi(\mathbb{C}_p)$, which he calls *Fatou components*. Given a point $x \in F_\varphi(\mathbb{C}_p)$, he defines the Fatou component of x to be the union of all sets $X(\mathbb{C}_p)$, where X is a connected open affinoid with $X(\mathbb{C}_p) \subset F_\varphi(\mathbb{C}_p)$, satisfying the condition that

$$\bigcup_{n=0}^{\infty} \varphi^{(n)}(X(\mathbb{C}_p)) \text{ omits at least three points of } \mathbb{P}^1(\mathbb{C}_p)\ . \tag{10.76}$$

Since we are interested in $\mathbb{P}^1_{\text{Berk}}$, we will define the Fatou component of x to be the union $U^{RL} \subset \mathbb{P}^1_{\text{Berk}}$ of the corresponding Berkovich open affinoids X. Thus, Rivera-Letelier's Fatou component is our $U^{RL}(\mathbb{C}_p)$. The fact that U^{RL} is dynamically meaningful follows from [**83**, Proposition 7.1], which says:

Proposition 10.109. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a rational function, and suppose X_1, X_2 are connected open affinoids satisfying* (10.76)*. If $X_1 \cap X_2$ is nonempty, then $X_1 \cup X_2$ also satisfies* (10.76).

We will now show that the Rivera-Letelier Fatou components U^{RL} coincide with the topological connected components of the Berkovich Fatou set F_φ. To see this, fix $x \in F_\varphi$, let U^{RL} be the Fatou component of x, and let U be the connected component of x in F_φ. Since each point of U^{RL} has a connected open affinoid neighborhood X satisfying (10.76), it follows from Theorem 10.56 that $U^{RL} \subseteq F_\varphi$. Since U^{RL} is connected, we have $U^{RL} \subseteq U$. Conversely, let $X \subset U$ be a connected open affinoid containing x. By Corollary 10.87, X satisfies (10.76), so $X \subset U^{RL}$. Since such sets X exhaust U, it follows that $U \subseteq U^{RL}$.

Rivera-Letelier identifies three kinds of Fatou components of particular interest: immediate basins of attraction, components of the domain of quasi-periodicity, and wandering components.

If $C \subset \mathbb{P}^1(\mathbb{C}_p)$ is an attracting periodic cycle, its *basin of attraction* $\mathcal{A}_C(\varphi)$ is the collection of all points in $\mathbb{P}^1_{\text{Berk}}$ which are topologically attracted by C. It is easy to see that each basin of attraction is an open set (see Lemma 10.41). The *immediate basin of attraction* $\mathcal{A}^0_{z_0}(\varphi)$ of a periodic point $z_0 \in C$ is the analytic component of the basin of attraction which contains z_0. It follows from [**83**, Proposition 6.1] that each immediate basin of attraction is a Fatou component (and therefore a connected component of F_φ).

A point $x \in \mathbb{P}^1(\mathbb{C}_p)$ belongs to the *domain of quasi-periodicity for* φ if there are a neighborhood $U(\mathbb{C}_p)$ of x and a sequence $n_j \to \infty$ such that $\varphi^{(n_j)}(z)$ converges uniformly to the identity on $U(\mathbb{C}_p)$. Rivera-Letelier defines the domain of quasi-periodicity to be the collection of all such x. However, for us the domain of quasi-periodicity $\mathcal{E}(\varphi)$ will be the union of the analytic components associated to that set, so Rivera-Letelier's domain of quasi-periodicity is our $\mathcal{E}(\varphi)(\mathbb{C}_p)$. By [**83**, Proposition 6.1], each analytic component of $\mathcal{E}(\varphi)$ is a Fatou component.

Note that by definition, $\mathcal{E}(\varphi)$ is open and $\varphi(\mathcal{E}(\varphi)) = \mathcal{E}(\varphi)$. It is easy to see that $\mathcal{E}(\varphi^{(n)}) = \mathcal{E}(\varphi)$ for each $n \geq 1$ [**81**, Proposition 3.9].

A point $x \in \mathbb{P}^1_{\mathrm{Berk}}$ is *recurrent* if it belongs to the closure of its forward orbit $\{\varphi^{(n)}(x)\}_{n \geq 0}$. By definition, each point of $\mathcal{E}(\varphi)(\mathbb{C}_p)$ is recurrent. Rivera-Letelier shows that $\mathcal{E}(\varphi)(\mathbb{C}_p)$ is the *interior* of the set of recurrent points in $\mathbb{P}^1(\mathbb{C}_p)$ [**81**, Corollary 4.27].

Let $\mathcal{R}$ be the set of all recurrent points in $\mathbb{P}^1_{\mathrm{Berk}}$, and let $\mathcal{R}^0$ be its interior. Note that by Theorem 10.59, the Berkovich Julia set J_φ is contained in $\mathcal{R}$. It follows from Proposition 10.117 below that $\mathcal{E}(\varphi) \subset \mathcal{R}^0$.

We claim that $\mathcal{E}(\varphi) = \mathcal{R}^0$, that is, $\mathcal{E}(\varphi)$ is the interior of the set of recurrent points in $\mathbb{P}^1_{\mathrm{Berk}}$. Let $x \in \mathcal{R}^0$. Then there is a connected open affinoid neighborhood X of x with $X \subset \mathcal{R}^0$. In particular, $X(\mathbb{C}_p) \subset \mathcal{R}^0(\mathbb{C}_p) = \mathcal{E}(\varphi)(\mathbb{C}_p)$. By definition, this means X is contained in some analytic component of $\mathcal{E}(\varphi)$, so $x \in \mathcal{E}(\varphi)$. Thus $\mathcal{R}^0 \subset \mathcal{E}(\varphi)$.

A domain $\mathcal{D} \subset \mathbb{P}^1_{\mathrm{Berk}}$ is called *wandering* if
(1) its forward images under φ are pairwise disjoint and
(2) it is not contained in the basin of attraction of an attracting cycle.

The following result, which mirrors Fatou's famous classification theorem in complex dynamics, is Rivera-Letelier's description of the classical Fatou set over $\mathbb{C}_p$ [**81**, p. 205]:

THEOREM 10.110 (Classification theorem). *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a rational function of degree $d \geq 2$. Then $F_\varphi(\mathbb{C}_p)$ is the disjoint union of the following three sets:*

(A) *immediate basins of attraction and their preimages,*
(B) *the domain of quasi-periodicity and its preimages,*
(C) *the union of all wandering discs.*

If $\varphi(T)$ has simple reduction, then φ has no wandering discs [**81**, Corollary 4.33]. Benedetto has shown [**12**, Theorem 1.2] that if φ is defined over a field L which is a finite extension of $\mathbb{Q}_p$ and if φ has no "wild recurrent Julia critical points" (recurrent critical points belonging to $J_\varphi(\mathbb{C}_p)$, whose multiplicity is divisible by p), then φ has no wandering discs. In particular, this holds if $\deg(\varphi) \leq p$. On the other hand, Benedetto [**15**] has also given an example of a polynomial in $\mathbb{C}_p[T]$ which has a wandering disc.

Benedetto has conjectured that if $L/\mathbb{Q}_p$ is a finite extension and if $\varphi(T) \in L(T)$ has degree at least 2, then the Fatou set of $\varphi(T)$ has no wandering components. If true, this would be a non-Archimedean analogue of Sullivan's famous "No Wandering Domains" theorem in complex dynamics.

Let $W(\mathbb{C}_p)$ be the union of the wandering discs in $F_\varphi(\mathbb{C}_p)$. Not much is known about the structure of $W(\mathbb{C}_p)$ or the Fatou components corresponding to it. However, in [**81**, Lemma 4.29] Rivera-Letelier shows that the lim inf of the chordal diameters of the forward images of a wandering disc is 0.

In [**83**, Theorem A], Rivera-Letelier shows that each Fatou component associated to $W(\mathbb{C}_p)$ is a wandering domain:

Proposition 10.111 (Classification of Periodic Components). *Suppose $\varphi(T) \in \mathbb{C}_p(T)$ has degree at least 2. Then any periodic Fatou component is either an immediate basin of attraction of an attracting periodic point $x_0 \in \mathbb{P}^1(\mathbb{C}_p)$ or an analytic component of the domain of quasi-periodicity.*

Basins of attraction. Rivera-Letelier gives the following description of the immediate basin of attraction of an attracting periodic point [**81**, Theorem 2, p. 196].

Theorem 10.112 (Description of Immediate Basins of Attraction). *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a rational function of degree $d \geq 2$. Let $\mathcal{A}^0_{z_0}(\varphi)$ be the immediate basin of attraction of an attracting periodic point $z_0 \in \mathbb{P}^1(\mathbb{C}_p)$. Then either*

(A) *$\mathcal{A}^0_{z_0}(\varphi)$ is an open disc $\mathcal{D}$ or*
(B) *$\mathcal{A}^0_{z_0}(\varphi)$ is a domain of* Cantor type*, meaning that its boundary $\partial\mathcal{A}^0_{z_0}(\varphi)$ is a Cantor set.*

Furthermore, the number of attracting periodic cycles whose immediate basin of attraction is of Cantor type is bounded [**81**, Proposition 4.8, p. 197]:

Proposition 10.113. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a rational function of degree $d \geq 2$. Then φ has at most $d-1$ attracting cycles such that the points belonging to them have an immediate basin of attraction of Cantor type.*

Rivera-Letelier gives an explicit description of the action of $\varphi(T)$ on each immediate basin of attraction [**81**, pp. 199–200]:

Proposition 10.114. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d \geq 2$, and let $X \subset \mathbb{P}^1_{\mathrm{Berk}}$ be the immediate basin of attraction of an attracting fixed point x_0. Then there is a decreasing sequence of neighborhoods $\{X_n\}_{n\in\mathbb{Z}}$ of x_0, which is cofinal in the collection of all neighborhoods of x_0, such that X_0 is a disc, $\bigcup_n X_n = X$, $\bigcap_n X_n = \{x_0\}$, and $\varphi(X_n) = X_{n+1}$ for each n.*

The domain of quasi-periodicity. Rivera-Letelier also gives an explicit description of components of the domain of quasi-periodicity [**81**, Theorem 3, p. 211].

THEOREM 10.115 (Description of the Domain of Quasi-periodicity). *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a rational function of degree $d \geq 2$. Then each analytic component U of $\mathcal{E}(\varphi)$ is a strict open affinoid domain, that is, in suitable coordinates,*

$$U = \mathcal{D}(a,r)^- \backslash \bigcup_{j=1}^{n} \mathcal{D}(a_j, r_j) \ , \tag{10.77}$$

where $a, a_1, \ldots, a_n \in \mathbb{C}_p$ and $r, r_1, \ldots, r_n \in |\mathbb{C}_p^\times|$, with pairwise disjoint discs $\mathcal{D}(a_j, r_j)$. Moreover, each boundary point of U is a repelling periodic point.

If $n = 0$ in (10.77), then U is called a *Siegel disc*. If $n \geq 1$, then U is called an *n-Hermann ring*. Rivera-Letelier gives examples of rational functions $\varphi(T)$ with an n-Hermann ring, for each $n \geq 1$ ([**81**, Proposition 6.4, p. 225 and Proposition 6.7, p. 227]).

Since any preimage under φ of a strict open affinoid is itself a strict open affinoid (see Lemma 9.12 and its proof), each connected component of $\bigcup_{n\geq 0} \varphi^{-n}(\mathcal{E}(\varphi))$ is a strict open affinoid.

By calculus, for any function $f : \mathbb{R} \to \mathbb{R}_{>0}$ and any $x \in \mathbb{R}$,

$$\lim_{\delta \to 0} \frac{f(x)^\delta - 1}{\delta} = \ln(f(x)) \ .$$

Rivera-Letelier shows that if an analytic function $f(T) \in \mathbb{C}_p[[T]]$ induces an automorphism of a disc, then its iterates satisfy an analogous limit [**81**, Lemma 3.11, Proposition 3.16]:

LEMMA 10.116. *Let $f(T)$ be an analytic function, defined by a power series with coefficients in $\mathbb{C}_p$, which induces an automorphism of a disc $D(a,R)^-$. Suppose there is a $\gamma < 1$ such that $|f(z) - z|_p < \gamma R$ for all $z \in D(a,R)^-$. Then:*

(A) *There is a bounded analytic function $f_*(T)$ on $D(a,R)^-$, called the* iterated logarithm *of $f(T)$, such that for any sequence of natural numbers $\{n_k\}_{k\geq 0}$ with $|n_k|_p \to 0$, the sequence of functions $(f^{(n_k)}(T) - T)/n_k$ converges uniformly to $f_*(T)$ on $D(a,R)^-$.*

More precisely, there are a constant C and a number $\rho_0 > 0$ depending on γ (but not on f or R), such that for each $n \in \mathbb{N}$ with $|n|_p < \rho_0$ and all $z \in D(a,R)^-$,

$$\left| \frac{f^{(n)}(z) - z}{n} - f_*(z) \right|_p \leq CR\,|n|_p \ . \tag{10.78}$$

(B) *For each $z_0 \in D(a,R)^-$, we have $f_*(z_0) = 0$ iff z_0 is an indifferent periodic point of $f(T)$, and $f_*(z_0) = f_*'(z_0) = 0$ iff z_0 is parabolic. If $f_*(z_0) \neq 0$, then there is an integer $k_0 = k_0(z_0)$ such that on a suitably small neighborhood of z_0, $f^{(k_0)}(T)$ is analytically conjugate to a translation. In particular, the periodic points of f in $D(a,R)^-$ are isolated.*

In [**81**, Lemma 3.11], Rivera-Letelier shows (under the hypotheses of Lemma 10.116) that for each $w \in \mathbb{C}_p$ with $|w| < \rho_0$, there is a canonical automorphism $f^{(w)}$ of $D(a,R)^-$, which coincides with the n-fold iterate $f^{(n)} = f \circ \cdots \circ f$ when $w = n \in \mathbb{N}$. These automorphisms have the property that if $|w_1|, |w_2| < \rho_0$, then $f^{(w_1)} \circ f^{(w_2)} = f^{(w_1+w_2)}$, and for each $z \in D(a,R)^-$

$$\lim_{w \to 0} \frac{f^{(w)}(z) - z}{w} \;=\; f_*(z) \ .$$

He interprets this as saying that the map $w \mapsto f^{(w)}$, which interpolates the map $n \mapsto f^{(n)}$ for $n \in \mathbb{N}$, is a 'flow' attached to the 'vector field' $f_*(z)$.

Let us now consider what Lemma 10.116 says about the induced action of $f(T)$ on the Berkovich disc $\mathcal{D}(a,R)^-$. By [**81**, Lemma 3.11], one has $\sup_{z \in D(a,R)^-} |f_*(z)| \le DR$, where $D = \max_{k \ge 1}(k\gamma^k)$. Put $B = \max(C, D)$; note that B depends only on γ. It follows from (10.78) and standard properties of non-Archimedean power series that if $D(b,r) \subset D(a,R)^-$ is a subdisc with $r \ge BR|n|$ and if $|n| < \rho_0$, then $f^{(n)}(T)$ induces an automorphism of $D(b,r)$. Thus it fixes the point $x \in \mathcal{D}(a,R)^-$ corresponding to $D(b,r)$ under Berkovich's classification theorem. Taking limits over nested sequences of discs, we see that $f^{(n)}(T)$ fixes each $x \in \mathcal{D}(a,R)^-$ with $\mathrm{diam}_\infty(x) \ge BR|n|$. Thus, for any sequence of natural numbers with $\mathrm{ord}_p(n_k) \to \infty$, the iterates $f^{(n_k)}(T)$ "freeze" larger and larger parts of $\mathcal{D}(a,R)^-$, moving the remaining points within subdiscs of radius less than $BR|n_k|$.

We can now describe the action of $\varphi(T)$ on a component of the domain of quasi-periodicity. The following is [**81**, Proposition 4.14, p. 200]:

PROPOSITION 10.117. *Let $\varphi(T) \in \mathbb{C}_p(T)$ be a rational function of degree $d \ge 2$. Let X be a connected closed affinoid contained in a component of the domain of quasi-periodicity $\mathcal{E}(\varphi)$. Choose coordinates on $\mathbb{P}^1$ so that $\infty \notin X$. Then there are a $k \ge 1$ and constants $\gamma < 1$ and $B > 0$ such that:*

- (A) *$\varphi^{(k)}(T)$ induces an automorphism of X, which fixes ∂X and the main dendrite D of X.*
- (B) *If we write $X \backslash (\partial X \cup D)$ as a disjoint union of discs $\mathcal{D}(a, R_a)^-$, then $\varphi^{(k)}(T)$ induces an automorphism of each $\mathcal{D}(a,R_a)^-$, and the action of $f(T) = \varphi^{(k)}(T)$ on $\mathcal{D}(a,R_a)^-$ is the one described in the discussion above, with the indicated γ and B, uniformly for all such discs.*

In particular, this shows that each point of X is recurrent and that each point $x \in X \cap \mathbb{H}_{\mathrm{Berk}}$ is fixed by some iterate $\varphi^{(n)}(T)$. Proposition 10.75, used in the proof that for $\mathbb{P}^1_{\mathrm{Berk},\mathbb{C}_p}$ the Berkovich Fatou set coincides with the Berkovich equicontinuity locus, is an immediate consequence of Proposition 10.117.

We conclude this section by giving Rivera-Letelier's formula for the number of fixed points in $\mathbb{P}^1(\mathbb{C}_p)$ belonging to a component of $\mathcal{E}(\varphi)$.

Let U be a component of $\mathcal{E}(\varphi)$, and let $k \in \mathbb{N}$ be such that $\varphi^{(k)}(U) = U$. Let $\zeta_1, \ldots, \zeta_m$ be the boundary points of U which are fixed by $\varphi^{(k)}(T)$. By Theorem 10.115, each ζ_i is a repelling fixed point of type II. For each ζ_i, we can change coordinates so that it becomes the Gauss point. Let $\tilde{f}_i(T) \in \overline{\mathbb{F}}_p(T)$ be the corresponding reduction of $\varphi^{(k)}(T)$. The tangent direction at ζ_i pointing into U corresponds to a fixed point α_i of $\tilde{f}_i(T)$ in $\mathbb{P}^1(\overline{\mathbb{F}}_p)$; let $n_i(U)$ be the multiplicity of that fixed point (that is, the order of vanishing of $\tilde{f}_i(T) - T$ at α_i). Rivera-Letelier [**81**, Proposition 5.10] shows:

PROPOSITION 10.118. *Let $\varphi(T) \in \mathbb{C}_p(T)$ have degree $d \geq 2$. Let U be a component of the domain of quasi-periodicity $\mathcal{E}(\varphi)$. Suppose that $\varphi^{(k)}(U) = U$, and let $\zeta_1, \ldots, \zeta_m$ be the points of ∂U fixed by $\varphi^{(k)}(T)$. Then each fixed point of $\varphi^{(k)}(T)$ in $U(\mathbb{C}_p)$ is indifferent, and the number of such fixed points* (*counting multiplicities*) *is exactly*

$$2 + \sum_{i=1}^{m} (n_i(U) - 2) \ . \tag{10.79}$$

10.10. Examples

In this section, we provide some examples illustrating the theory developed above. In Examples 10.119 and 10.124–10.126, K is arbitrary; otherwise we take $K = \mathbb{C}_p$.

EXAMPLE 10.119. Consider the polynomial $\varphi(T) = T^2$. The filled Julia set is

$$K_\varphi \ = \ \mathcal{D}(0,1) \ = \ \{z \in \mathbb{A}^1_{\mathrm{Berk}} \ : \ [T]_z \leq 1\} \ .$$

The Julia set of φ is the Gauss point $\zeta_{\mathrm{Gauss}} \in \mathcal{D}(0,1) \subset \mathbb{P}^1_{\mathrm{Berk}}$. The canonical measure μ_φ in this case is a point mass supported at ζ_{Gauss}.

EXAMPLE 10.120. If $\varphi(T) = \frac{T^p - T}{p} \in \mathbb{C}_p[T]$, one can show that $K_\varphi = J_\varphi = \mathbb{Z}_p$, which is also the classical Julia set for φ. By Theorem 10.91, the canonical measure μ_φ coincides with the equilibrium distribution of $\mathbb{Z}_p$ with respect to the point ∞, which is known ([**88**, Example 4.1.24]) to be the normalized Haar measure on $\mathbb{Z}_p$.

EXAMPLE 10.121. The following example is due to Benedetto ([**14**, Example 3]). Let p be an odd prime, and choose $a \in \mathbb{C}_p$ such that $|a| = p^\epsilon > 1$, where $0 < \epsilon \leq \frac{p}{2p+2}$. If $\varphi(T) = T^2(T-a)^p \in \mathbb{C}_p[T]$, then φ does not have simple reduction, but the classical Julia set $J_\varphi(\mathbb{C}_p)$ is nonetheless empty. From Lemma 10.53, Corollary 10.60, and Theorem 10.91, we conclude that J_φ is an uncountable subset of $\mathbb{H}_p$ with the structure of a Cantor set.

Note that for a polynomial $\varphi(T) \in \mathbb{C}_p[T]$ of degree at most $p+1$, Benedetto has shown that $J_\varphi(\mathbb{C}_p)$ is empty if and only if φ has simple reduction (Proposition 4.9 of [**14**]).

EXAMPLE 10.122. This example is due to Rivera-Letelier ([**82**, Example 6.3]). Let p be a prime, and take $\varphi(T) = (T^p - T^{p^2})/p$, with $K = \mathbb{C}_p$.

It is not hard to see that if $|a| \le 1$ and $p^{-1/(p-1)} < r \le 1$, then the preimage under φ of the disc $D(a, r)$ is a disjoint union of p closed discs $D(a_i, r')$ where $r' = (r/p)^{1/p} > p^{-1/(p-1)}$. Put $D = D(0, 1)$, and let $D_1, \ldots, D_p$ be the preimages of D under φ; each D_i has radius $p^{-1/p}$. Inductively, for each $m \ge 2$, $(\varphi^{(m)})^{-1}(D)$ is a disjoint union of p^m closed discs $D_{i_1,\ldots,i_m}$ of radius $r_m = p^{-(1-p^{-m})/(p-1)}$. Here we are indexing the discs by the sequences $\{i_1, \ldots, i_m\}$ with $1 \le i_j \le p$, and i_j is determined by the property that

$$\varphi^{(j-1)}(D_{i_1,\ldots,i_m}) \subset D_{i_j} \quad \text{for } j = 1, \ldots, m .$$

Clearly $D_{i_1,\ldots,i_m} \subset D_{i_1,\ldots,i_{m-1}}$, and $\varphi(D_{i_1,\ldots,i_m}) = D_{i_2,\ldots,i_m}$, that is, the action of φ on iterated preimages of D is conjugate to a left shift on the indices.

One has $D \supset \varphi^{-1}(D) \supset (\varphi^{(2)})^{-1}(D) \supset \cdots$ and

$$J_\varphi = \bigcap_{m=1}^{\infty} (\varphi^{(m)})^{-1}(D) .$$

Under Berkovich's classification theorem, each $x \in J_\varphi$ corresponds to a nested sequence of discs $\{D_{i_1,\ldots,i_m}\}$, so the points of J_φ are in one-to-one correspondence with the sequences $\{i_j\}_{j\ge1}$ with $1 \le i_j \le p$ for each j. Furthermore

$$\mathrm{diam}_\infty(x) = \lim_{m\to\infty} p^{-(1-p^{-m})/(p-1)} = p^{-1/(p-1)} .$$

If the sequence of discs corresponding to x has empty intersection, then x is of type IV, while if it is nonempty, then x is of type II. Furthermore, the action of φ on J_φ is conjugate to a left shift on the index sequences, and it can be seen that x is of type II if and only if the corresponding sequence $\{i_j\}_{j\ge1}$ is periodic. Thus, J_φ is isomorphic to a Cantor set contained in the set of points in $\mathbb{H}_{\mathrm{Berk}}$ with $\mathrm{diam}_\infty(x) = p^{-1/(p-1)}$.

Since $K = \mathbb{C}_p$, there are only countably many points of type II, so J_φ consists of countably many points of type II and uncountably many points of type IV, but no points of type I or III.

Let $X_p = \{1, \ldots, p\}$ and let μ_p be the probability measure on X_p which gives each element mass $1/p$. Equip the space of sequences $X_p^{\mathbb{N}} = \prod_{j=1}^{\infty} X_p$ with the product measure $\mu_p^{\mathbb{N}}$. The canonical measure μ_φ is obtained by transporting $\mu_p^{\mathbb{N}}$ to J_φ.

EXAMPLE 10.123. Let p be an odd prime, take $\varphi(T) = pT^3 + (p-1)T^2$, and let $K = \mathbb{C}_p$.

One checks easily that 0, -1, and $1/p$ are the fixed points of $\varphi(T)$ in $\mathbb{C}_p$, with $|\varphi'(0)| = 0$, $|\varphi'(-1)| = 1$, and $|\varphi'(1/p)| = p$. Thus, 0 is a superattracting fixed point, -1 is an indifferent fixed point, and $1/p$ is a repelling fixed point. It follows that 0 and -1 belong to the Fatou set F_φ, while $1/p$ belongs to the Julia set J_φ.

The reduction of $\varphi(T)$ at $\zeta_{\rm Gauss}$ is $\widetilde{\varphi}(T) = -T^2$, so $\zeta_{\rm Gauss}$ is a repelling fixed point, and it belongs to J_φ. As J_φ contains at least two points, it is infinite, perfect, and has uncountably many connected components. By Proposition 10.45, no point in J_φ is exceptional. This means $1/p$ and $\zeta_{\rm Gauss}$ each have infinitely many preimages under the iterates $\varphi^{(k)}(T)$, and those preimages belong to J_φ. Hence $J_\varphi(\mathbb{C}_p)$ and $J_\varphi \cap \mathbb{H}_{\rm Berk}$ are both infinite. Furthermore, Theorem 10.59 shows that every Berkovich neighborhood of $1/p$ contains preimages of $\zeta_{\rm Gauss}$, and every Berkovich neighborhood of $\zeta_{\rm Gauss}$ contains preimages of $1/p$. In particular, $1/p$ is a limit of points in $J_\varphi \cap \mathbb{H}_{\rm Berk}$.

The next group of examples comes from Favre and Rivera-Letelier [**48**], who describe a class of rational functions whose Berkovich Julia sets and canonical measures can be determined.

Identify $\mathbb{P}^1_{\rm Berk}$ with $\mathbb{A}^1_{\rm Berk} \cup \{\infty\}$, and for each $t \in \mathbb{R}$ let $S(t) \in \mathbb{A}^1_{\rm Berk}$ be the point corresponding to the disc $D(0, q_v^t)$ under Berkovich's classification. Given a collection of numbers $a_1, \ldots, a_n \in K^\times$ satisfying $|a_1| < \cdots < |a_n|$ and positive integers $d_0, d_1, \ldots, d_n$, consider the rational function

$$\varphi(T) \ = \ T^{d_0} \cdot \prod_{k=1}^{n} \left(1 + (T/a_k)^{d_{k-1}+d_k}\right)^{(-1)^k} .$$

Then it is easy to see that $\varphi(T)$ has degree $d = d_0 + d_1 + \cdots + d_n$, and if $z \in K^\times$ and $|z| \neq |a_1|, \ldots, |a_n|$, then

$$|\varphi(z)| \ = \ |z|^{d_0} \cdot \prod_{k=1}^{n} \max\left(1, |z/a_k|^{d_{k-1}+d_k}\right)^{(-1)^k} .$$

Set $a_{n+1} = \infty \in \mathbb{P}^1_{\rm Berk}$, taking $|\infty| = \infty$, and put $c_k = \prod_{j=1}^k a_k^{(-1)^k(d_{k-1}+d_k)}$. It follows from Proposition 2.19 that $\varphi(S(t)) = S(L(t))$ where $L(t)$ is the continuous, piecewise affine map defined by

$$L(t) \ = \begin{cases} d_0 t & \text{if } t \leq \log_v(|a_1|) \ , \\ \log_v(|c_k|) + (-1)^k d_k t & \text{if } \log_v(|a_k|) \leq t \leq \log_v(|a_{k+1}|) \ . \end{cases}$$

If $a_1, \ldots, a_n$ and $d_0, d_1, \ldots, d_n$ are chosen appropriately, then there will be a closed interval $I = [a, b] \subset \mathbb{R}$ and closed subintervals $I_0, \ldots, I_n \subset I$ which are pairwise disjoint except possibly for their endpoints, such that $L(t)$ maps I_j homeomorphically onto I for each $j = 0, \ldots, n$. In this situation $L^{-1}(I) = I_0 \cup \cdots \cup I_n$. Put $J_0 = S(I)$, and for each $m \geq 1$ put $J_m = S(\big(L^{(m)}\big)^{-1}(I))$, so $J_0 \supseteq J_1 \supseteq J_2 \cdots$. Favre and Rivera-Letelier show that $J_m = (\varphi^{(k)})^{-1}(J_0)$ and that

$$J_\varphi \ = \ \bigcap_{m=1}^{\infty} J_m \ .$$

They also show that if $\sum_{k=0}^n 1/d_k = 1$, then $J_\varphi = J_0$; on the other hand, if $\sum_{k=0}^n 1/d_k < 1$, then J_φ is a Cantor set contained in J_0.

Furthermore, if $J_\varphi = J_0$, then μ_φ is the measure on J_0 obtained by transporting the Lebesgue measure on I, normalized to have total mass 1. If J_φ is a Cantor set, then μ_φ is the measure described as follows. For each $m \geq 1$, $\left(L^{(m)}\right)^{-1}(I)$ can be written as a union of closed subintervals $I_{i_1,\ldots,i_m}$, where each i_j satisfies $0 \leq i_j \leq n$ and is determined by the property that

$$L^{(j-1)}(I_{i_1,\ldots,i_m}) \ \subset \ I_{i_j} \ .$$

Then $\mu_\varphi(S(I_{i_1,\ldots,i_m})) = d_{i_1} \cdots d_{i_m}/d^m$.

Example 10.124 (Favre, Rivera-Letelier). Let $n \geq 2$ and take $a \in K$ with $|a| > 1$. Put $a_k = a^{k/n}$ (fixing any choice of the root) for $k = 1, \ldots, n-1$, and let $d_0 = \cdots = d_{n-1} = n$. Consider the ration function of degree n^2

$$\varphi(T) \ = \ T^n \cdot \prod_{k=1}^{n-1} \Big(1 + (T/a_k)^{2n}\Big)^{(-1)^k} \ .$$

Let $I = [0, \log_v(|a|)]$ and let $I_k = [\frac{k}{n}\log_v(|a|), \frac{k+1}{n}\log_v(|a|)]$ for $0 \leq k \leq n-1$. Then $L(I_k) = I$ for each k, and $\sum_{k=0}^{n-1} 1/d_k = 1$.

By the discussion above, $J_\varphi = S(I)$, and μ_φ is the transport of the normalized Lebesgue measure on I.

Example 10.125 (Favre, Rivera-Letelier). Take $n = 2$, and choose $a \in K$ with $|a| > 1$. Put $a_1 = a^{1/2}$, $a_2 = a^{3/4}$ for any choices of the roots, and set $d_0 = 2$, $d_1 = 4$, $d_2 = 4$. Consider

$$\varphi(T) \ = \ \frac{T^2(1 + (T/a_2)^8)}{1 + (T/a_1)^6} \ ,$$

which has degree 10. Let $I = [0, \log_v(|a|)]$ and take

$$I_0 = [0, \frac{1}{2}\log_v(a)] \ , \quad I_1 = [\frac{1}{2}\log_v(a), \frac{3}{4}\log_v(a)] \ , \quad I_2 = [\frac{3}{4}\log_v(a), \log_v(a)] \ .$$

Then $L(I_0) = L(I_1) = L(I_2) = I$, and $1/2+1/4+1/4 = 1$. Again $J_\varphi = S(I)$, and μ_φ is the transport of normalized Lebesgue measure on I.

Example 10.126 (Favre, Rivera-Letelier). Take $n = 1$, and fix $a \in K$ with $|a| > 1$. Let $d \geq 5$ be an integer, take $a_1 = a$, and put $d_0 = d-2, d_1 = 2$. Consider the function

$$\varphi(T) \ = \ \frac{T^{d-2}}{1 + (T/a)^d} \ ,$$

which has degree d. Then

$$L(t) \ = \ \begin{cases} (d-2)t & \text{if } t \leq \log_v(|a|) \ , \\ d\log_v(|a|) - 2t & \text{if } t \geq \log_v(|a|) \ . \end{cases}$$

If $I = [0, \frac{d}{2}\log_v(|a|)]$, $I_0 = [0, \frac{d}{2d-4}\log_v(|a|)]$, and $I_1 = [\frac{d}{4}\log_v(|a|), \frac{d}{2}\log_v(|a|)]$, then I_0 and I_1 are disjoint since $d \geq 5$, and $L(I_0) = L(I_1) = I$. Clearly $I_0 \cup I_1 \subset I$ and $1/d_0 + 1/d_1 < 1$. By the discussion above, J_φ is a Cantor set contained in $S(I)$.

Our final example concerns a rational function originally studied by Benedetto [**11**, p. 14]. It gives an indication of how complicated J_φ can be, in general.

EXAMPLE 10.127. Let p be a prime, take $K = \mathbb{C}_p$, and put

$$\varphi(T) \ = \ \frac{T^3 + pT}{T + p^2} \ .$$

Note that $\varphi(0) = 0$ and $\varphi'(0) = 1/p$ (so that $|\varphi'(0)| > 1$). Thus 0 is a repelling fixed point and it belongs to J_φ. Let $a_0 = 0$. Benedetto shows that for each $m \geq 1$, there is a point $a_m \in \mathbb{C}_p$ with $|a_m| = p^{-1/2^m}$ for which $\varphi(a_m) = a_{m-1}$. Hence, $a_m \in J_\varphi$ as well. The sequence $\{a_m\}_{m \geq 1}$ has no Cauchy subsequences, since if $m > n$, then $|a_m - a_n| = |a_m|$ by the ultrametric inequality, and $\lim_{m\to\infty} |a_m| = 1$. In particular, $\{a_m\}_{m\geq 1}$ has no limit points in $\mathbb{C}_p$, so $J_\varphi(\mathbb{C}_p)$ is not compact.

However, the point ζ_{Gauss}, which corresponds to $D(0,1)$ under Berkovich's classification theorem, is a limit point of $\{a_m\}_{m\geq 1}$ in $\mathbb{P}^1_{\text{Berk}}$. Since J_φ is compact in the Berkovich topology, we must have $\zeta_{\text{Gauss}} \in J_\varphi$. We can see this directly, by considering the reduction of $\varphi(T)$ at ζ_{Gauss}, which is $\widetilde{\varphi}(T) = T^2$. Since $\deg(\widetilde{\varphi}) > 1$, ζ_{Gauss} is a repelling fixed point for $\varphi(T)$ in $\mathbb{H}_{\text{Berk}}$, and it belongs to J_φ by Theorem 10.81.

Since $m_\varphi(\zeta_{\text{Gauss}}) = \deg(\widetilde{\varphi}) = 2$, but $\deg(\varphi) = 3$, the point ζ_{Gauss} must have a preimage under φ distinct from itself. Using Proposition 2.18, one can show that if x_0 is the point corresponding to $D(-p^2, 1/p^3)$ under Berkovich's classification theorem, then $\varphi(x_0) = \zeta_{\text{Gauss}}$. Similarly, for each $k \geq 1$, if x_k is the point corresponding to $D(-p^{2+k}, 1/p^{3+k})$, then $\varphi(x_k) = x_{k-1}$. As $k \to \infty$, the points x_k converge to the point 0 in $\mathbb{P}^1_{\text{Berk}}$. This gives an explicit example of a point in $J_\varphi(\mathbb{C}_p)$ which is a limit of points in $J_\varphi \cap \mathbb{H}_{\text{Berk}}$.

Continuing on, taking preimages of the chain of points $\{x_k\}_{k\geq 0}$, we see that each a_m is a limit of points in $J_\varphi \cap \mathbb{H}_{\text{Berk}}$. However, this is far from a complete description of J_φ. By Corollary 10.60, J_φ is uncountable, and by Theorem 10.88 each point of J_φ is a limit of repelling periodic points. The points $\{a_m\}_{m\geq 1}$ and $\{x_n\}_{n\geq 0}$ are all preperiodic but not periodic, so they are not among the repelling periodic points described in Theorem 10.88, but each one is a limit of such points. According to Bézivin's theorem [**22**, Theorem 3], each a_m is in fact a limit of repelling periodic points in $\mathbb{P}^1(K)$.

However, there are only countably many repelling periodic points in $\mathbb{P}^1(K)$. Furthermore, there are only countably many type II points in $\mathbb{P}^1_{\text{Berk},\mathbb{C}_p}$, and each repelling periodic point in $\mathbb{H}_{\text{Berk}}$ is of type II. Thus, J_φ contains uncountably many other points which we have not yet described.

10.11. Notes and further references

Much of the material in §10.2 also appears in [**5**, Appendix A].

Chambert-Loir [**35**] and Favre and Rivera-Letelier [**46, 47, 48**] have given independent constructions of the canonical measure on $\mathbb{P}^1_{\text{Berk}}$ attached to φ. Also, Szpiro, Tucker, and Piñeiro [**78**] have constructed a sequence of blowups of a rational map on $\mathbb{P}^1/\operatorname{Spec}(\mathcal{O}_v)$, leading to a sequence of discrete measures supported on the special fibers. When suitably interpreted in terms of Berkovich space, the weak limit of these measures gives another way of defining the canonical measure.

The proof we have given of Theorem 10.36 borrows some key ingredients from the work of Favre and Rivera-Letelier, but it is also rather different in several respects. (We make extensive use of the Arakelov-Green's function $g_\varphi(x, y)$ attached to φ, while Favre and Rivera-Letelier's proof does not use this at all.)

As with Theorem 7.52 (see the remarks in §7.10), Arakelov geometry provides another approach to global equidistribution results such as Theorem 10.24 (cf. [**35**],[**94**]).

Our treatment of Fatou and Julia theory on $\mathbb{P}^1_{\text{Berk}}$ owes a great deal to the comments of Juan Rivera-Letelier. We thank him in particular for pointing out Example 10.70, explaining the proof of Theorem 10.72 for wandering domains, and for teaching us about the theory of uniform spaces.

The potential-theoretic approach to Fatou-Julia theory in the classical complex setting began with the work of Brolin [**31**], who proved equidistribution of preimages for polynomial maps and other important facts. It was subsequently used by Lyubich [**72**] and Freire, Lopes, and Mañé [**54**] to prove equidistribution of preimages for rational maps and by Tortrat [**95**] to prove the equidistribution of periodic points. The (pluri)potential approach to complex dynamics is fundamental in higher dimensions, where the most convenient way to define the Julia set is as the support of a suitable current generalizing the canonical measure attached to a rational map in dimension one. For an overview of the pluripotential theoretic approach to complex dynamics in higher dimensions, see the papers [**50, 51, 52, 53**] by Fornaess and Sibony.

We have assumed throughout §10.5–§10.8 that $\operatorname{char}(K) = 0$. As noted in the text (e.g., Remark 10.37), one could give analogues of the results in those sections when $\operatorname{char}(K) = p$, but several of the statements (e.g., that of Proposition 10.45) would have to be modified and the proofs would be correspondingly more involved. Our decision to assume that $\operatorname{char}(K) = 0$ arose partly from our desire to illustrate the close parallels between rational dynamics on $\mathbb{P}^1(\mathbb{C})$ and on $\mathbb{P}^1_{\text{Berk}}$; some of those parallels would be less clear if we tried to deal systematically with the "pathologies" which can arise in characteristic p. As noted in the text, a number of the proofs in §10.5–§10.8 (e.g., the proofs of Corollary 10.47 and Lemma 10.54) extend verbatim to characteristic p if one uses the general form of Theorem 10.36 proved in [**48**].

An earlier version of this chapter served as the basis for §5.10 of Joe Silverman's recent book [**93**]. The treatment given here is a thoroughly revised version of that earlier work, taking into account Example 10.70 and correcting some errors in the original manuscript.