Contents

Preface ix

Chapter 1. Overview 1
  1.1. What is a combinatorial design? 1
  1.2. What is Algebraic Design Theory? 1
  1.3. What is in this book? 2

Chapter 2. Many Kinds of Pairwise Combinatorial Designs 7
  2.1. Orthogonality sets 7
  2.2. Symmetric balanced incomplete block designs 9
  2.3. Hadamard matrices 11
  2.4. Weighing matrices 12
  2.5. Balanced weighing matrices 13
  2.6. Orthogonal designs 14
  2.7. Complex Hadamard matrices 16
  2.8. Complex generalized Hadamard matrices 18
  2.9. Complex generalized weighing matrices 19
  2.10. Generalized Hadamard matrices over groups 19
  2.11. Balanced generalized weighing matrices 22
  2.12. Generalized weighing matrices 23
  2.13. Summary 24

Chapter 3. A Primer for Algebraic Design Theory 27
  3.1. Groups 27
  3.2. Monoids 34
  3.3. Group actions 35
  3.4. Rings 39
  3.5. Matrices 41
  3.6. Linear and related groups 42
  3.7. Representations 44

Chapter 4. Orthogonality 49
  4.1. How many rows can be pairwise A-orthogonal? 49
  4.2. Non-trivial orthogonality sets 50
  4.3. A big picture 51
  4.4. Equivalence 54
  4.5. Matrices, arrays, and designs 60

Chapter 5. Modeling A-Equivalence 63
  5.1. A first look at the automorphism group 63
  5.2. Ambient rings with a model for A-equivalence 65
5.3. Ambient rings for the familiar orthogonality sets 68

Chapter 6. The Grammian 71
6.1. Orthogonality as a Grammian property 71
6.2. Non-degeneracy 72
6.3. Gram completions and composition of orthogonality sets 73
6.4. The Gram Property and Λ-equivalence 74

Chapter 7. Transposability 77
7.1. The main problems 77
7.2. A functional approach to self-duality 78
7.3. Conjugate equivalence operations 80
7.4. A matrix algebra approach to transposability and self-duality 80
7.5. A different kind of transposable orthogonality set 82

Chapter 8. New Designs from Old 85
8.1. Composition 85
8.2. Transference 93

Chapter 9. Automorphism Groups 99
9.1. Automorphism groups of pairwise combinatorial designs 99
9.2. A class of generalized Hadamard matrices 100
9.3. A bound on the size of the automorphism group 103
9.4. Permutation automorphism groups 105
9.5. Automorphism groups of orthogonal designs 106
9.6. Expanded designs 108
9.7. Computing automorphism groups 112
9.8. The associated design 114
9.9. Associated designs and group divisible designs 116
9.10. An isomorphism for weighing matrices 117

Chapter 10. Group Development and Regular Actions on Arrays 119
10.1. Matrix preliminaries 119
10.2. Group-developed arrays 119
10.3. Regular embeddings 121
10.4. Difference sets and relative difference sets 124
10.5. Group ring equations and associates 127
10.6. Finding all associates of an array 129
10.7. An algorithm for solving Problems 10.2.3 and 10.2.4 131
10.8. Composition via associates 132

Chapter 11. Origins of Cocyclic Development 135
11.1. First derivation 135
11.2. Second derivation 140
11.3. Cocycles for cyclic groups 142

Chapter 12. Group Extensions and Cocycles 145
12.1. Central extensions 145
12.2. Cocycles for product groups 150
12.3. Polycyclic presentations 151
12.4. Cocycles from collection in polycyclic groups 153
12.5. Monomial representations and cocycles 157

Chapter 13. Cocyclic Pairwise Combinatorial Designs 161
13.1. The main definitions 161
13.2. Ambient rings with a central group 162
13.3. Some big problems 164
13.4. Central extensions of a design 164
13.5. Approaches to cocyclic designs 165

Chapter 14. Centrally Regular Actions 167
14.1. Cocyclic forms 167
14.2. A lesser expanded design 167
14.3. A pair of lifting homomorphisms 168
14.4. The lift 169
14.5. Translation 170
14.6. Centrally regular embeddings 171
14.7. Finding cocyclic forms 173
14.8. All the cocycles of a design 176

Chapter 15. Cocyclic Associates 177
15.1. Definition of cocyclic associates 177
15.2. The group ring equation for cocyclic associates 178
15.3. The familiar designs 180
15.4. Cocyclic designs and relative difference sets 181
15.5. Normal $p$-complements 182
15.6. Existence conditions for cocyclic Hadamard matrices 183
15.7. Cyclotomic rings and circulant complex Hadamard matrices 185
15.8. Composition of cocyclic associates 190

Chapter 16. Special Classes of Cocyclic Designs 195
16.1. Cocyclic Hadamard matrices 195
16.2. Cocyclic weighing matrices 197
16.3. Cocyclic orthogonal designs 198
16.4. A cocyclic substitution scheme 200
16.5. Cocyclic complex Hadamard matrices 201

Chapter 17. The Paley Matrices 203
17.1. Actions of 2-dimensional linear and semilinear groups 203
17.2. The Paley matrices and their automorphism groups 205
17.3. The regular actions 209

Chapter 18. A Large Family of Cocyclic Hadamard Matrices 215
18.1. On the orders covered 215
18.2. A construction for prime powers congruent to 3 (mod 4) 216
18.3. A construction for prime powers congruent to 1 (mod 4) 218
18.4. Plug-in matrices 220
18.5. Proof of the main theorem and a generalization 221

Chapter 19. Substitution Schemes for Cocyclic Hadamard Matrices 223
19.1. General substitution schemes 224
19.2. Number-theoretic constraints 226
| 19.3. | Further results for group-developed plug-in matrices | 227 |
| 19.4. | Inverting action | 228 |
| 19.5. | Trivial action | 230 |
| 19.6. | Complementary pairs and the Cocyclic Hadamard Conjecture | 232 |
| 19.7. | Existence of group-developed complementary pairs | 233 |

Chapter 20. Calculating Cocyclic Development Rules 239

| 20.1. | Introduction to development tables | 239 |
| 20.2. | Development tables for abelian groups | 240 |
| 20.3. | Development tables revisited | 241 |
| 20.4. | Group cohomology | 242 |
| 20.5. | Constructing a free table | 243 |
| 20.6. | Group homology | 244 |
| 20.7. | Presentations and the Schur multiplier | 246 |
| 20.8. | Constructing a torsion table | 249 |
| 20.9. | Listing the elements of the second cohomology group | 253 |
| 20.10. | Another look at the Cocyclic Hadamard Conjecture | 255 |

Chapter 21. Cocyclic Hadamard Matrices Indexed by Elementary Abelian Groups 257

| 21.1. | Motivation: indexing groups for the Sylvester matrices | 257 |
| 21.2. | The extension problem | 258 |
| 21.3. | Pure Hadamard collection cocycles | 261 |
| 21.4. | Bilinearity and Hadamard cocycles | 262 |
| 21.5. | Solution of the Hadamard cocycle problem | 263 |

Chapter 22. Cocyclic Concordant Systems of Orthogonal Designs 267

| 22.1. | Existence and uniqueness of cocyclic systems of OD(n; 1^k) | 267 |
| 22.2. | A reduction | 268 |
| 22.3. | Solution of the reduced problem | 269 |
| 22.4. | Proof of Theorem 22.1.1 | 270 |
| 22.5. | Removing the zeros | 271 |
| 22.6. | Examples | 272 |

Chapter 23. Asymptotic Existence of Cocyclic Hadamard Matrices 279

| 23.1. | Complex sequences with zero aperiodic autocorrelation | 279 |
| 23.2. | Sets of Hermitian and skew-Hermitian circulant matrices | 281 |
| 23.3. | Sets of cocyclic signed permutation matrices | 282 |
| 23.4. | Existence of cocyclic complex Hadamard matrices | 283 |
| 23.5. | Concluding remarks | 284 |

Bibliography 287

Index 295