CHAPTER 1

Algebraic Persistence

As we saw in the general introduction, the algebraic theory of persistence deals with certain types of diagrams of vector spaces and linear maps, called *persistence modules.* The simplest instances look like this, where the spaces $V_1, \cdots, V_n$ and the maps $v_1, \cdots, v_{n-1}$ are arbitrary:

$$V_1 \xrightarrow{v_1} V_2 \xrightarrow{v_2} \cdots \xrightarrow{v_{n-1}} V_n.$$

Diagrams such as this one are representations of the so-called *linear quiver* $\mathtt{L}_n$. More generally, all persistence modules are representations of certain types of quivers, possibly with relations. Quiver theory provides us not only with a convenient terminology to define persistence modules and describe their properties, but also with a set of powerful structure theorems to decompose them into 'atomic' representations called *interval modules.* Thus, in its algebraic formulation, persistence owes a lot to the theory of quiver representations.

Yet, algebraic persistence cannot be quite reduced to a subset of quiver theory. Shifting the focus from quiver representations to signatures derived from their interval decompositions, it provides very general stability theorems for these signatures and efficient algorithms to compute them. Over time, these signatures—known as the *persistence diagrams*—have become its main object of study and its primary tool for applications. This is the story told in this part of the book: first, how persistence develops as an offspring of quiver theory by focusing on certain types of quivers, and second, how it departs from it by shifting the focus from quiver representations to their signatures.

The first part of the chapter introduces persistence modules using the language of quiver theory. It begins naturally with an overview of the required background material from quiver theory (Section 1), followed by a formal introduction to persistence modules and a review of the conditions under which they can be decomposed (Section 2). The emphasis is on the legacy of quiver theory to persistence.

The second part of the chapter introduces persistence diagrams as signatures for decomposable persistence modules (Sections 3). It then shows how these signatures can be generalized to a class of (possibly indecomposable) representations called the q-*tame* modules (Section 4). The stability properties of these signatures are deferred to Chapter 3.

The chapter closes with a general discussion (Section 5).

Prerequisites. No background on quiver theory is required to read the chapter. However, a reasonable background in abstract and commutative algebra, corresponding roughly to Parts I through III of [111], is needed. Also, some basic notions of category theory, corresponding roughly to Chapters I and VIII of [184], can be helpful although they are not strictly required.

1. A quick walk through the theory of quiver representations

This section gives a brief overview of the concepts and results from quiver theory that will be used afterwards. It also sets up the terminology and notations. The progression is from the more classical aspects of quiver theory to the ones more closely related to persistence. It gives a limited view of the theory of representations, which is a far broader subject.

A more thorough treatment is provided in Appendix A for the interested reader. It includes formal definitions for the concepts introduced here, and a proof outline for Gabriel's theorem, the key result of this section. It also includes further background material, and draws some connections between tools developed independently in persistence and in quiver theory—e.g. *Diamond Principle* of Carlsson and de Silva [49] versus *reflection functors* of Bernstein, Gelfand, and Ponomarev [24].

Quivers. *Quivers* can be thought of as directed graphs, or rather multigraphs, with potentially infinitely many nodes and arrows. Here is a very simple example of quiver:

$$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xleftarrow{b} \underset{3}{\bullet} \xleftarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet} \tag{1.1}$$

and here is a more elaborate example:

$$\begin{tikzcd} & \overset{1}{\bullet} & \\ \underset{2}{\bullet} \arrow[loop left, "b"] \arrow[ur, "a"] \arrow[rr, "c"'] & & \underset{3}{\bullet} \arrow[ul, shift left, "e"'] \arrow[ul, shift right, "d"] \end{tikzcd} \tag{1.2}$$

The quivers we are most interested in are the so-called A_n*-type* quivers, which are finite and linear-shaped, with arbitrary arrow orientations, as in (1.1). Here is a formal general description, where a headless arrow means that the actual arrow orientations can be arbitrary:

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet} \tag{1.3}$$

The special case where all arrows are oriented to the right is called the *linear quiver*, denoted $\mathtt{L}_n$. Not only A_n-type quivers but also their infinite extensions are relevant to persistence theory.

Quiver representations. *Representations* of a quiver $\mathtt{Q}$ over a field $\boldsymbol{k}$ are just realizations of $\mathtt{Q}$ as a (possibly non-commutative) diagram of $\boldsymbol{k}$-vector spaces and $\boldsymbol{k}$-linear maps. For instance, a representation of the quiver (1.1) can be the following:

$$\boldsymbol{k} \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \boldsymbol{k}^2 \xleftarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} \boldsymbol{k} \xleftarrow{\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)} \boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\0 & 1\end{smallmatrix}\right)} \boldsymbol{k}^2 \tag{1.4}$$

or the following:

$$\boldsymbol{k} \xrightarrow{0} 0 \xleftarrow{0} \boldsymbol{k} \xleftarrow{\mathbb{1}} \boldsymbol{k} \xrightarrow{0} 0 \tag{1.5}$$

Here is an example of a representation of the quiver (1.2)—note that none of the triangles commute:

$$
\begin{tikzcd}
 & \boldsymbol{k} & \\
\boldsymbol{k} \arrow[loop left, "\mathbb{1}"] \arrow[ur, "\mathbb{1}"] \arrow[rr, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & & \boldsymbol{k}^2 \arrow[ul, shift right, "(0\ 1)"'] \arrow[ul, shift left, "0"]
\end{tikzcd}
\tag{1.6}
$$

A *morphism* $\phi : \mathbb{V} \to \mathbb{W}$ between two representations of $\mathtt{Q}$ is a collection of linear maps $\phi_i : V_i \to W_i$ at the nodes $\underset{i}{\bullet}$ of $\mathtt{Q}$, such that the following diagram commutes for every arrow $\underset{i}{\bullet} \xrightarrow{a} \underset{j}{\bullet}$ of $\mathtt{Q}$:

$$
\begin{tikzcd}
V_i \arrow[r, "v_a"] \arrow[d, "\phi_i"'] & V_j \arrow[d, "\phi_j"] \\
W_i \arrow[r, "w_a"] & W_j
\end{tikzcd}
\tag{1.7}
$$

For instance, a morphism ϕ from (1.4) to (1.5) can be the following collection of vertical maps making every quadrangle commute:

$$
\begin{tikzcd}
\boldsymbol{k} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, "\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[d, "0"'] & \boldsymbol{k} \arrow[l, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"'] \arrow[d, "-\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[l, "(0\ 1)"'] \arrow[r, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] \arrow[d, "(0\ -1)"'] & \boldsymbol{k}^2 \arrow[d, "0"'] \\
\boldsymbol{k} \arrow[r, "0"] & 0 & \boldsymbol{k} \arrow[l, "0"'] & \boldsymbol{k} \arrow[l, "\mathbb{1}"'] \arrow[r, "0"] & 0
\end{tikzcd}
\tag{1.8}
$$

ϕ is called an *isomorphism* between representations when all its linear maps ϕ_i are ismormophisms between vector spaces. The commutativity condition in (1.7) ensures then that $\mathbb{V}$ and $\mathbb{W}$ have the same algebraic structure.

The category of representations. The representations of a given quiver $\mathtt{Q}$ over a fixed base field $\boldsymbol{k}$, together with the morphisms connecting them, form a category denoted $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$. This category turns out to be *abelian*, so some of the nice properties of single vector spaces (which can also be viewed as representations of the quiver $\bullet$ having one node and no arrow) carry over to representations of arbitrary quivers. In particular:

- There is a zero object in the category, called the *trivial representation*. It is made up only of zero vector spaces and linear maps. For instance, the trivial representation of the quiver (1.1) is

$$0 \longrightarrow 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow 0$$

- We can form internal and external direct sums of representations. For instance, the external direct sum of (1.4) and (1.5) is the following representation of (1.1), where the spaces and maps are naturally defined as the direct sums of their counterparts in (1.4) and (1.5):

$$\boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)} \boldsymbol{k}^2 \xleftarrow{\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)} \boldsymbol{k}^2 \xleftarrow{\left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)} \boldsymbol{k}^3 \xrightarrow{\left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)} \boldsymbol{k}^2$$

- We can define the *kernel*, *image* and *cokernel* of any morphism $\phi : \mathbb{V} \to \mathbb{W}$. These are defined *pointwise*, with respectively $\ker \phi_i$, $\operatorname{im} \phi_i$ and $\operatorname{coker} \phi_i$ attached to each node $\underset{i}{\bullet}$ of $\mathtt{Q}$, the linear maps between nodes being induced from the ones in $\mathbb{V}$ and $\mathbb{W}$. For instance, the kernel of the morphism in (1.8) is

$$0 \xrightarrow{\ 0\ } \boldsymbol{k}^2 \xleftarrow{\ 0\ } 0 \xleftarrow{\ 0\ } \boldsymbol{k} \xrightarrow{\binom{1}{0}} \boldsymbol{k}^2$$

As expected, ϕ is an isomorphism if and only if both $\ker \phi$ and $\operatorname{coker} \phi$ are trivial.

Not all properties of single vector spaces carry over to representations of arbitrary quivers though. Perhaps the most notable exception is semisimplicity, i.e. the fact that a subspace of a vector space always has a complement: there is no such thing for representations of arbitrary quivers. For instance, $\mathbb{W} = \ 0 \xrightarrow{\ 0\ } \boldsymbol{k}$ is a *subrepresentation* of $\mathbb{V} = \ \boldsymbol{k} \xrightarrow{\ \mathbb{1}\ } \boldsymbol{k}$, i.e. its spaces are subspaces of the ones in $\mathbb{V}$ and its maps are the restrictions of the ones in $\mathbb{V}$, yet $\mathbb{W}$ is not a *summand* of $\mathbb{V}$, i.e. there is no subrepresentation $\mathbb{U}$ such that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$. This obstruction is what makes the classification of quiver representations an essentially more challenging problem than for single vector spaces.

Classification of quiver representations. Given a fixed quiver $\mathtt{Q}$ and a fixed base field $\boldsymbol{k}$, what are the isomorphism classes of representations of $\mathtt{Q}$ over $\boldsymbol{k}$? This central problem in quiver theory has a decisive impact on persistence, as it provides decomposition theorems for persistence modules. Although solving it in full generality is an essentially impossible task, under some restrictions it becomes remarkably simple. For instance, assuming the quiver $\mathtt{Q}$ is finite and every representation of $\mathtt{Q}$ under consideration has finite *dimension* (defined as the sum of the dimensions of its constituent vector spaces), we benefit from the Krull-Remak-Schmidt principle, that is:

THEOREM 1.1 (Krull, Remak, Schmidt). *Let $\mathtt{Q}$ be a finite quiver, and let $\boldsymbol{k}$ be a field. Then, every finite-dimensional representation $\mathbb{V}$ of $\mathtt{Q}$ over $\boldsymbol{k}$ decomposes as a direct sum*

$$\mathbb{V} = \mathbb{V}^1 \oplus \cdots \oplus \mathbb{V}^r \tag{1.9}$$

where each $\mathbb{V}^i$ is itself indecomposable *i.e. cannot be further decomposed into a direct sum of at least two nonzero representations. Moreover, the decomposition (*1.9*) is unique up to isomorphism and permutation of the terms in the direct sum.*

The proof of existence is an easy induction on the dimension of $\mathbb{V}$, while the proof of uniqueness can be viewed as a simple application of Azumaya's theorem [14]. This result turns the classification problem into the one of identifying the isomorphism classes of indecomposable representations. Gabriel [133] settled the question for a small subset of the finite quivers called the *Dynkin* quivers. His result asserts that Dynkin quivers only have finitely many isomorphism classes of indecomposable representations, and it provides a simple way to identify these classes. It also introduces a dichotomy on the finite connected quivers, between the ones that are Dynkin, for which the classification problem is easy, and the rest, for which the problem is significantly harder if at all possible[1]. Luckily for us, A_n-type quivers are Dynkin, so Gabriel's theorem applies and takes the following special form:

THEOREM 1.2 (Gabriel for A_n-type quivers). *Let* $\mathtt{Q}$ *be an* A_n*-type quiver, and let* $\boldsymbol{k}$ *be a field. Then, every indecomposable finite-dimensional representation of* $\mathtt{Q}$ *over* $\boldsymbol{k}$ *is isomorphic to some* interval representation $\mathbb{I}_{\mathtt{Q}}[b,d]$*, described as follows:*

$$\underbrace{0 \overset{0}{\longrightarrow} \cdots \overset{0}{\longrightarrow} 0}_{[1,\,b-1]} \overset{0}{\longrightarrow} \underbrace{\boldsymbol{k} \overset{\mathbb{1}}{\longrightarrow} \cdots \overset{\mathbb{1}}{\longrightarrow} \boldsymbol{k}}_{[b,\,d]} \overset{0}{\longrightarrow} \underbrace{0 \overset{0}{\longrightarrow} \cdots \overset{0}{\longrightarrow} 0}_{[d+1,\,n]}$$

Combined with Theorem 1.1, this result asserts that every finite-dimensional representation of an A_n-type quiver $\mathtt{Q}$ decomposes uniquely (up to isomorphism and permutation of the terms) as a direct sum of interval representations. This not only gives an exhaustive classification of the finite-dimensional representations of $\mathtt{Q}$, but it also provides complete descriptors for their isomorphism classes, as any such class is fully described by the collection of intervals $[b,d]$ involved in its decomposition. This collection of intervals is at the origin of our persistence barcodes.

Unfortunately, Theorems 1.1 and 1.2 are limited in two ways for our purposes. First, by only considering quivers indexed over finite sets, whereas we would like to consider arbitrary subsets of $\mathbb{R}$. Second, by restricting the focus to finite-dimensional representations, whereas in our case we may have to deal with representations including infinitely many nontrivial spaces, or spaces of infinite dimension. The rest of Section 1 is a review of several extensions of the theorems that address these limitations.

Infinite-dimensional representations. Theorems 1.1 and 1.2 turn out to be still valid if infinite-dimensional representations are considered as well. This is thanks to a powerful result of Auslander [12] and Ringel and Tachikawa [217] dealing with left modules over Artin algebras[2]. The connection with quiver representations is done through the construction of the so-called *path algebra* of a quiver $\mathtt{Q}$, denoted $\boldsymbol{k}\mathtt{Q}$, which in short is the $\boldsymbol{k}$-algebra generated by the finite oriented paths in $\mathtt{Q}$, with the product operator induced by concatenations of paths. The path algebra is an Artin algebra whenever $\mathtt{Q}$ is finite with no oriented cycle, which happens e.g. when $\mathtt{Q}$ is an A_n-type quiver. There is then an equivalence of categories between $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$ and the left modules over $\boldsymbol{k}\mathtt{Q}$. The result of Auslander [12] and Ringel and Tachikawa [217], combined with Gabriel's and Azumaya's theorems, gives the following decomposition theorem for finite- or infinite-dimensional representations of A_n-type quivers:

THEOREM 1.3 (Auslander, Ringel, Tachikawa, Gabriel, Azumaya). *Let* $\mathtt{Q}$ *be an* A_n*-type quiver, and let* $\boldsymbol{k}$ *be a field. Then, every indecomposable representation of* $\mathtt{Q}$ *over* $\boldsymbol{k}$ *is isomorphic to an interval representation* $\mathbb{I}_{\mathtt{Q}}[b,d]$*, and every representation of* $\mathtt{Q}$*, whether finite- or infinite-dimensional, is isomorphic to a (possibly infinite) direct sum of interval representations. Moreover, this decomposition is unique up to isomorphism and permutation of the terms.*

[1]There is in fact a trichotomy: beside the Dynkin quivers are the so-called *tame* quivers, for which the classification problem is still feasible (albeit harder); the rest of the quivers are called *wild* because for them the classification problem is an essentially impossible task.

[2]An Artin algebra is a finitely generated algebra over an Artinian ring, i.e. a commutative ring that satisfies the descending chain condition on ideals: every nested sequence of ideals $I_1 \supseteq I_2 \supseteq \cdots$ stabilizes eventually.

Inifinite extensions of $\mathtt{L}_n$. As a first step toward expanding the index set, consider the following countable extensions of the linear quiver $\mathtt{L}_n$, indexed respectively over $\mathbb{N}$ and $\mathbb{Z}$:

$$\mathtt{N}: \qquad \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots$$

$$\mathtt{Z}: \qquad \cdots \longrightarrow \underset{-2}{\bullet} \longrightarrow \underset{-1}{\bullet} \longrightarrow \underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots$$

Webb [238] has extended Theorems 1.1 and 1.2 to the quiver $\mathtt{Z}$ in the following way, where a representation $\mathbb{V}$ of $\mathtt{Z}$ is called *pointwise finite-dimensional* when each of its constituent vector spaces has finite dimension[3]. The decomposition of representations of $\mathtt{N}$ is obtained as a special case, in which the vector spaces assigned to nodes with negative index are trivial. The uniqueness of the decomposition (for both $\mathtt{Z}$ and $\mathtt{N}$) follows once again from Azumaya's theorem.

THEOREM 1.4 (Webb). *Let* $\boldsymbol{k}$ *be a field. Then, any pointwise finite-dimensional representation of* $\mathtt{Z}$ *over* $\boldsymbol{k}$ *is a direct sum of interval representations.*

As it turns out, there is an equivalence of categories between the representations of $\mathtt{Z}$ and the $\mathbb{Z}$-graded modules over the polynomial ring $\boldsymbol{k}[t]$. Webb's proof works in the latter category. In light of this equivalence, Theorem 1.4 can be viewed as a generalization of the classical structure theorem for finitely generated (graded) modules over a (graded) principal ideal domain [163, §3.8]. The importance of the pointwise finite-dimensionality assumption is illustrated by the following example.

EXAMPLE 1.5 (Webb [238]). For each integer $m \geq 0$, let $\boldsymbol{k}_m$ denote a copy of the field $\boldsymbol{k}$. Let then $\mathbb{V} = (V_i, v_i^j)$ be the representation of $\mathtt{Z}$ defined by:

$$\begin{aligned}
&\forall i \geq 0,\ V_i = \prod_{m \geq 0} \boldsymbol{k}_m, \\
&\forall i < 0,\ V_i = \prod_{m \geq -i} \boldsymbol{k}_m, \\
&\forall i \leq j,\ v_i^j \text{ is the inclusion map} V_i \hookrightarrow V_j.
\end{aligned}$$

For $i \geq 0$, V_i is isomorphic to the space of sequences $(x_0, x_1, x_2, \cdots)$ of elements in $\boldsymbol{k}$, and is therefore uncountably-dimensional. For $i < 0$, V_i is isomorphic to the space of all such sequences satisfying the extra condition that $x_0 = \cdots = x_{-i-1} = 0$. Suppose $\mathbb{V}$ decomposes as a direct sum of interval representations. Since each map v_i^j is injective for $i < 0$ and bijective for $i \geq 0$, all the intervals must be of the form $(-\infty, +\infty)$ or $[i, +\infty)$ for some $i \leq 0$. Since the quotient V_{i+1}/V_i has dimension 1 for $i < 0$, each interval $[i, +\infty)$ occurs with multiplicity 1 in the decomposition. Since $\bigcap_{i<0} V_i = 0$ (i.e. the only sequence $(x_0, x_1, x_2, \cdots)$ such that $0 = x_0 = x_1 = x_2 = \cdots$ is the identically zero sequence), the interval $(-\infty, +\infty)$ does not occur at all in the decomposition. Hence, we have $\mathbb{V} \cong \bigoplus_{i \leq 0} \mathbb{I}[i, +\infty)$, and therefore V_0 is countably-dimensional, a contradiction.

[3]This is a weaker assumption than having $\mathbb{V}$ itself be finite-dimensional when the quiver is infinite.

Representations of arbitrary subposets of $(\mathbb{R}, \leq)$. We now consider extensions of the index set to arbitrary subsets T of $\mathbb{R}$. For this we work with the poset $(T, \leq)$ directly, rather than with some associated quiver. Regarding $(T, \leq)$ as a category in the natural way[4], we let a *representation of* $(T, \leq)$ be a functor to the category of vector spaces. What this means concretely is that the representation defines vector spaces $(V_i)_{i \in T}$ and linear maps $(v_i^j : V_i \to V_j)_{i \leq j \in T}$ satisfying the following constraints induced by functoriality:

$$\begin{array}{rcll} v_i^i & = & \mathbb{1}_{V_i} & \text{for every } i \in T, \text{ and} \\ v_i^k & = & v_j^k \circ v_i^j & \text{for every } i \leq j \leq k \in T. \end{array} \tag{1.10}$$

More generally, one can define representations for any given poset as functors from that poset to the category of vector spaces.

Crawley-Boevey [93] has extended Theorems 1.1 and 1.2 to representations of arbitrary subposets of $(\mathbb{R}, \leq)$. Pointwise finite-dimensionality is understood as in Theorem 1.4. The proof uses a specialized version of the *functorial filtration* method of Ringel [216]. The uniqueness of the decomposition once again follows from Azumaya's theorem.

Theorem 1.6 (Crawley-Boevey). *Let $\boldsymbol{k}$ be a field and let $T \subseteq \mathbb{R}$. Then, any pointwise finite-dimensional representation of $(T, \leq)$ over $\boldsymbol{k}$ is a direct sum of interval representations.*

Note that there is a larger variety of interval representations for the posets $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$ than for the A_n-type quivers. Indeed, some intervals may be left-infinite, or right-infinite, or both. Moreover, since $\mathbb{R}$ has limit points, some intervals for $(\mathbb{R}, \leq)$ may be open or half-open. We will elaborate on this point in the next section.

Remark. The connection to quiver theory is somewhat more subtle in this general setting than in the previous ones. First of all, any quiver can be viewed as a category, with one object per node and one morphism per finite oriented path. Its representations are then interpreted as functors to the category of vector spaces. In the case of the quivers $\mathtt{N}$ and $\mathtt{Z}$, the corresponding categories are equivalent to the posets $(\mathbb{N}, \leq)$ and $(\mathbb{Z}, \leq)$ respectively. However, not every quiver is equivalent (as a category) to a poset, and conversely, not every poset is equivalent to a quiver. The reason for the latter limitation is that paths sharing the same source and the same target are not considered equal in a quiver (recall (1.2) and (1.6)), whereas they are in a poset by transitivity. The workaround is to equip the quivers with *relations* that identify the paths sharing the same source and the same target. The representations of the resulting *quivers with relations* have to reflect these identifications. This way, any poset can be made equivalent (as a category) to some quiver with relations, and its representations can be viewed themselves as quiver representations.

2. Persistence modules and interval decompositions

The background material on quiver theory given in Section 1 provides us with a convenient terminology to introduce persistence modules—see also Section 5 for a historical account. From now on and until the end of the chapter, the field $\boldsymbol{k}$ over which representations are taken is fixed.

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[4]i.e. with one object per element $i \in T$ and a single morphism per couple $i \leq j$.

DEFINITION 1.7. Given $T \subseteq \mathbb{R}$, a *persistence module* over T is a representation of the poset $(T, \leq)$.

This definition follows (1.10) and includes the representations of the quivers $\mathtt{L}_n$, $\mathtt{N}$ and $\mathtt{Z}$ as special cases. However, it does not include the representations of general A_n-type quivers, which are gathered into a different concept called *zigzag module*.

DEFINITION 1.8. Given $n \geq 1$, a *zigzag module* of length n is a representation of an A_n-type quiver.

The term 'zigzag' is justified by the following special situation motivated by applications, where every other arrow is oriented backwards:

$$V_1 \xrightarrow{v_1} V_2 \xleftarrow{v_2} V_3 \xrightarrow{v_3} \cdots \xleftarrow{v_{n-3}} V_{n-2} \xrightarrow{v_{n-2}} V_{n-1} \xleftarrow{v_{n-1}} V_n \tag{1.11}$$

Zigzag modules can also be thought of as poset representations. As a directed acyclic graph, an A_n-type quiver $\mathtt{Q}$ is the Hasse diagram[5] of some partial order relation $\preceq$ on the set $\{1, \cdots, n\}$. Since $\mathtt{Q}$ has at most one oriented path between any pair of nodes, it is equivalent (as a category) to the poset $(\{1, \cdots, n\}, \preceq)$, and its representations are also representations of $(\{1, \cdots, n\}, \preceq)$. Thus, we can rewrite Definition 1.8 as follows, which emphasizes its connection to Definition 1.7:

DEFINITION 1.8 (rephrased). Given $n \geq 1$, a *zigzag module* of length n is a representation of the poset $(\{1, \cdots, n\}, \preceq)$, where $\preceq$ is any partial order relation whose Hasse diagram is of type A_n.

Section 1 provides us with powerful structure theorems to decompose these objects. The basic building blocks are the interval representations, called *interval modules* in the persistence literature. Given an arbitrary index set $T \subseteq \mathbb{R}$, an *interval* of T is a subset $S \subseteq T$ such that for any elements $i \leq j \leq k$ of T, $i, k \in S$ implies $j \in S$. The associated interval module has the field $\boldsymbol{k}$ at every index $i \in S$ and the zero space elsewhere, the maps between copies of $\boldsymbol{k}$ being identities and all other maps being zero. This definition is oblivious to the actual map orientations, which depend on the order relation $\preceq$ that equips the index set T. When this relation is obvious from the context, we simply write $\mathbb{I}S$ for the interval module associated with S; otherwise we write $\mathbb{I}_{\preceq}S$, or even $\mathbb{I}_{\mathtt{Q}}S$ when $\preceq$ is specified through its Hasse diagram $\mathtt{Q}$.

The following theorem summarizes the structural results from Section 1 (Theorems 1.1, 1.2, 1.3 and 1.6) and can be thought of as our main heritage from quiver theory. The conditions under which it guarantees the existence of an interval decomposition are sufficient for our purposes.

THEOREM 1.9 (Interval Decomposition). *Given an index set $T \subseteq \mathbb{R}$ and a partial order relation $\preceq$ on T, a representation $\mathbb{V}$ of the poset $(T, \preceq)$ can be decomposed as a direct sum of interval modules in each of the following situations:*

(i) *T is finite and the Hasse diagram of $\preceq$ is of type A_n (which happens in particular when $\preceq$ is the natural order $\leq$ on T, whose Hasse diagram is $\mathtt{L}_n$),*

(ii) *T is arbitrary, $\preceq$ is the natural order $\leq$, and $\mathbb{V}$ is pointwise finite-dimensional.*

Moreover, the decomposition, when it exists, is unique up to isomorphism and permutation of the terms in the direct sum, and each term is indecomposable.

[5] Defined as the graph having $\{1, \cdots, n\}$ as vertex set, and one edge $i \longrightarrow j$ per couple $i \prec j$ such that there is no k with $i \prec k \prec j$.

So, concretely:

(i) any zigzag module decomposes uniquely as a direct sum of interval modules,

(i)-(ii) any persistence module whose index set is finite or whose vector spaces are finite-dimensional decomposes uniquely as a direct sum of interval modules.

A persistence or zigzag module $\mathbb{V}$ that decomposes as a direct sum of interval modules is called *interval-decomposable*. The converse (called *interval-indecomposable*) means that either $\mathbb{V}$ decomposes into indecomposable representations that are not interval modules, or $\mathbb{V}$ does not decompose at all as a direct sum of indecomposable representations. In principle, interval-decomposability is a stronger concept than the classical notion of decomposability into indecomposables from quiver theory. Nevertheless, the tools introduced by Webb [238] can be used to prove both concepts equivalent for modules over (subsets of) $\mathbb{Z}$ [192], while to our knowledge the question is still not settled for modules over $\mathbb{R}$.

3. Persistence barcodes and diagrams

In order to simplify the description of interval modules over arbitrary subsets of $\mathbb{R}$, including subsets with limit points, we need to decide on a simple and unified writing convention for intervals. For this purpose we will use *decorated real numbers*, which are written as ordinary real numbers with an additional superscript $^+$ (plus) or $^-$ (minus). Whenever the decoration of a number is unknown or irrelevant, we will use the superscript $^\pm$. The order on decorated numbers is the obvious one: $b^\pm < d^\pm$ if $b < d$, or if $b = d$ and $b^\pm = b^-$ and $d^\pm = b^+$. The corresponding dictionary for finite intervals of $\mathbb{R}$ is the following one, where $b^\pm < d^\pm$:

$$
\begin{aligned}
\lceil b^-, d^- \rfloor &\quad \text{stands for} \quad [b, d)\,, \\
\lceil b^-, d^+ \rfloor &\quad \text{stands for} \quad [b, d], \\
\lceil b^+, d^- \rfloor &\quad \text{stands for} \quad (b, d), \\
\lceil b^+, d^+ \rfloor &\quad \text{stands for} \quad (b, d].
\end{aligned}
$$

We will also use the symbols $-\infty$ and $+\infty$ for infinite endpoints. Since intervals are always open at infinity, these implicitly carry the superscripts $-\infty^+$ and $+\infty^-$, which we will generally omit in the notations, so for instance $\lceil -\infty, d^- \rfloor$ stands for the open interval $(-\infty, d)$.

Given an arbitrary index set $T \subseteq \mathbb{R}$, we can now rewrite each interval S of T as $\lceil b^\pm, d^\pm \rfloor \cap T$, for some interval $\lceil b^\pm, d^\pm \rfloor$ of $\mathbb{R}$. Note that the choice of $\lceil b^\pm, d^\pm \rfloor$ may not be unique when T is a strict subset of $\mathbb{R}$. For instance, letting $S = \{2, 3\}$ and $T = \{1, 2, 3, 4\}$, we can write S indifferently as $[2, 3] \cap T$, or $(1, 3] \cap T$, or $(1, 4) \cap T$, or $[2, 4) \cap T$, or more generally $\lceil b^\pm, d^\pm \rfloor \cap T$ for any $[2, 3] \subseteq \lceil b^\pm, d^\pm \rfloor \subseteq (1, 4)$. To remove ambiguities, unless otherwise stated we will always pick the interval of $\mathbb{R}$ that is smallest with respect to inclusion, as per the following rule[6]:

RULE 1.10. *Given an interval S of an index set T, among the intervals of $\mathbb{R}$ whose intersection with T is S, pick the one that is smallest with respect to inclusion, e.g.* $[2, 3]$ *in the previous example.*

[6]This rule has been applied implicitly until now. It is arbitrary and not motivated by mathematical considerations. Other rules can be applied as well, leading to different writings of the same interval S of T.

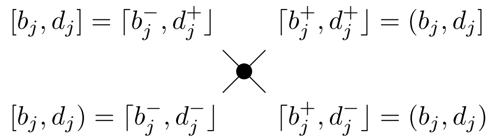

FIGURE 1.1. The four decorated points corresponding to intervals $\lceil b_j^{\pm}, d_j^{\pm} \rfloor$.

For simplicity we will also omit the index set T in the notation when it is irrelevant or obvious from the context. Then, any interval-decomposable module $\mathbb{V}$ can be written uniquely (up to permutation of the terms) as

$$\mathbb{V} \cong \bigoplus_{j \in J} \mathbb{I}\lceil b_j^{\pm}, d_j^{\pm} \rfloor. \tag{1.12}$$

The set of intervals $\lceil b_j^{\pm}, d_j^{\pm} \rfloor$, ordered by the lexicographical order on the decorated coordinates, is called the *persistence barcode* of $\mathbb{V}$. Technically it is a multiset, as an interval may occur more than once. Another representation of the persistence barcode is as a multiset of *decorated points* in the extended plane $\bar{\mathbb{R}}^2 = [-\infty, +\infty]^2$, where each interval $\lceil b_j^{\pm}, d_j^{\pm} \rfloor$ is identified with the point of coordinates (b_j, d_j) decorated with a diagonal tick according to the convention of Figure 1.1. This multiset of decorated points is called the *decorated persistence diagram* of $\mathbb{V}$, noted $\mathsf{Dgm}(\mathbb{V})$. From (1.12),

$$\mathsf{Dgm}(\mathbb{V}) = \{(b_j^{\pm}, d_j^{\pm}) \ \mid \ j \in J\}. \tag{1.13}$$

The *undecorated persistence diagram* of $\mathbb{V}$, noted $\mathsf{dgm}(\mathbb{V})$, is the same multiset without the decorations:

$$\mathsf{dgm}(\mathbb{V}) = \{(b_j, d_j) \ \mid \ j \in J\}. \tag{1.14}$$

Let us give a concrete example taken from our traditional zoo—the background details will be given in Chapter 2.

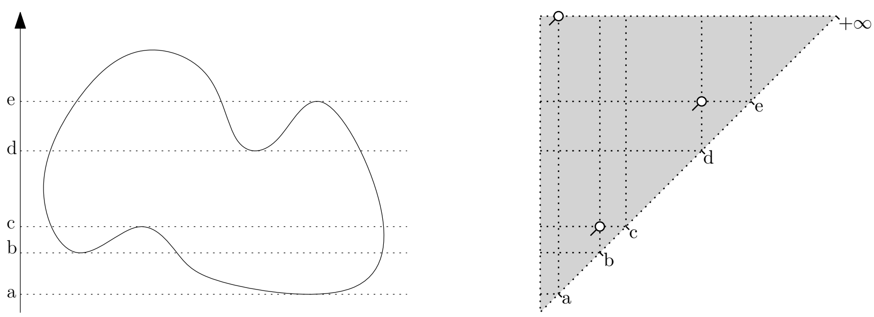

FIGURE 1.2. A classical example in persistence theory. Left: a smooth planar curve X and its y-coordinate or 'height' function $f : X \to \mathbb{R}$. Right: the decorated persistence diagram of $\mathsf{H}_0(\mathcal{F})$.
— *From Chazal et al. [72].*

EXAMPLE 1.11. Consider the curve X in $\mathbb{R}^2$ shown in Figure 1.2, filtered by the height function f. Take the family $\mathcal{F}$ of sublevelsets $F_y = f^{-1}((-\infty, y])$ of f, where parameter y ranges over $\mathbb{R}$. This family is nested, that is, $F_y \subseteq F_{y'}$ whenever $y \leq y'$.

Let us apply the 0-homology functor H_0 to $\mathcal{F}$ and study the resulting persistence module $\mathsf{H}_0(\mathcal{F})$, which encodes the evolution of the connectivity of the sublevel sets F_y as parameter y ranges from $-\infty$ to $+\infty$. This module decomposes as follows:

$$\mathsf{H}_0(\mathcal{F}) \cong \mathbb{I}[a, +\infty) \oplus \mathbb{I}[b, c) \oplus \mathbb{I}[d, e) = \mathbb{I}\lceil a^-, +\infty\rfloor \oplus \mathbb{I}\lceil b^-, c^-\rfloor \oplus \mathbb{I}\lceil d^-, e^-\rfloor,$$

which intuitively means that 3 different independent connected components appear in the sublevel set F_y during the process: the first one at $y = a$, the second one at $y = b$, the third one at $y = d$; while the first one remains until the end, the other two eventually get merged into it, at times $y = c$ and $y = e$ respectively. A pictorial description of this decomposition is provided by the decorated persistence diagram in Figure 1.2.

The persistence measure. Let $\mathbb{V}$ be an interval-decomposable module. From its decorated persistence diagram $\mathsf{Dgm}(\mathbb{V})$ we derive the following measure on rectangles $R = [p, q] \times [r, s]$ in the extended plane with $-\infty \leq p < q \leq r < s \leq +\infty$:

$$\mu_{\mathbb{V}}(R) = \operatorname{card}\left(\mathsf{Dgm}(\mathbb{V})|_R\right), \tag{1.15}$$

where the membership relation for a decorated point $(b^\pm, d^\pm) \in \mathsf{Dgm}(\mathbb{V})$ is defined by:

$$(b^\pm, d^\pm) \in R \iff [q, r] \subseteq \lceil b^\pm, d^\pm \rfloor \subseteq [p, s]. \tag{1.16}$$

The pictorial view of (1.16) is that point (b, d) and its decoration tick belong to the closed rectangle R, as illustrated in Figure 1.3. Then, (1.15) merely defines $\mu_{\mathbb{V}}$ as the counting measure over the restrictions of $\mathsf{Dgm}(\mathbb{V})$ to rectangles, which takes values in $\{0, 1, 2, \cdots, +\infty\}$ (we do not distinguish between infinite values). The

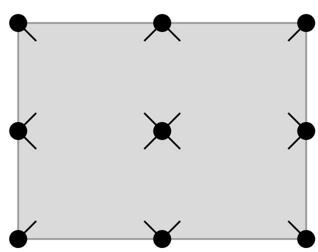

FIGURE 1.3. A decorated point $(b^\pm, d^\pm)$ belongs to a rectangle R if (b, d) belongs to the interior of R, or if it belongs to the boundary of R with its tick pointing towards the interior of R.
— *From Chazal et al. [72].*

term *measure* is motivated by the fact that $\mu_{\mathbb{V}}$ is additive with respect to splitting a rectangle into two rectangles, either vertically or horizontally, with the convention that $x + \infty = +\infty + x = +\infty$:

$$\forall p < x < q \leq r < y < s,$$

$$\left\{ \begin{array}{l} \mu_{\mathbb{V}}([p, q] \times [r, s]) = \mu_{\mathbb{V}}([p, x] \times [r, s]) + \mu_{\mathbb{V}}([x, q] \times [r, s]) \\ \mu_{\mathbb{V}}([p, q] \times [r, s]) = \mu_{\mathbb{V}}([p, q] \times [r, y]) + \mu_{\mathbb{V}}([p, q] \times [y, s]) \end{array} \right. \tag{1.17}$$

This additivity property is illustrated in Figure 1.4, where the claim is that $\mu_{\mathbb{V}}(R) = \mu_{\mathbb{V}}(A) + \mu_{\mathbb{V}}(B) = \mu_{\mathbb{V}}(C) + \mu_{\mathbb{V}}(D)$. Notice how the decoration of a given point on the border between two subrectangles assigns this point uniquely to one of them.

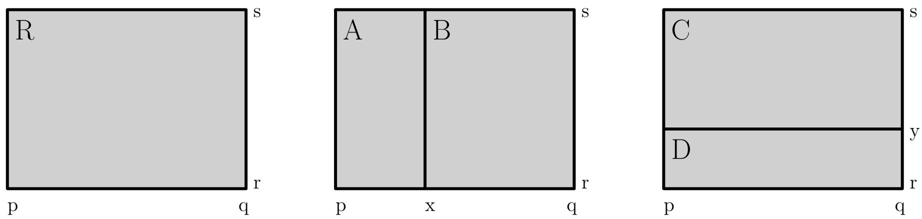

FIGURE 1.4. Additivity of $\mu_{\mathbb{V}}$ under vertical / horizontal splitting.
— *From Chazal et al. [72].*

When $\mathbb{V} = (V_i, v_i^j)$ is a persistence module over $\mathbb{R}$, we can also relate its persistence measure $\mu_{\mathbb{V}}$ more directly to $\mathbb{V}$ through the following well-known inclusion-exclusion formulas, which hold provided that all the ranks are finite or, less stringently, that all but $\operatorname{rank} v_q^r$ are finite—in which case the values of the alternating sums are $+\infty$:

$$\forall p < q \leq r < s \in \mathbb{R},$$

$$\begin{cases} \mu_{\mathbb{V}}([-\infty, q] \times [r, +\infty]) &= \operatorname{rank} v_q^r, \\ \mu_{\mathbb{V}}([-\infty, q] \times [r, s]) &= \operatorname{rank} v_q^r - \operatorname{rank} v_q^s, \\ \mu_{\mathbb{V}}([p, q] \times [r, +\infty]) &= \operatorname{rank} v_q^r - \operatorname{rank} v_p^r, \\ \mu_{\mathbb{V}}([p, q] \times [r, s]) &= \operatorname{rank} v_q^r - \operatorname{rank} v_q^s + \operatorname{rank} v_p^s - \operatorname{rank} v_p^r. \end{cases} \tag{1.18}$$

The first formula counts the number of decorated points of $\mathsf{Dgm}(\mathbb{V})$ that lie inside the quadrant $Q = [-\infty, q] \times [r, +\infty]$, the second formula inside the horizontal strip $H = [-\infty, q] \times [r, s]$, the third formula inside the vertical strip $V = [p, q] \times [r, +\infty]$, the fourth formula inside the rectangle $R = [p, q] \times [r, s]$. These patterns are illustrated in Figure 1.5. Notice how the second, third and fourth formulas are obtained from the first one by counting points inside quadrants and by removing multiple counts (hence the term 'inclusion-exclusion formulas'). These formulas are useful to localize points in the persistence diagram from the sole knowledge of the ranks of the linear maps in $\mathbb{V}$. As we will see next, they can be used to generalize the definition of persistence diagram to a certain class of interval-indecomposable persistence modules.

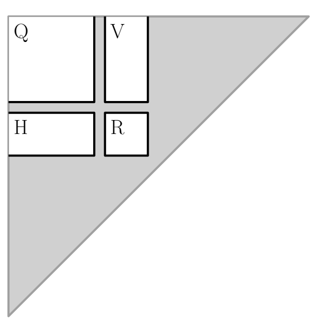

FIGURE 1.5. A quadrant, horizontal strip, vertical strip, and finite rectangle in the half plane above the diagonal.
— *From Chazal et al. [72].*

4. Extension to interval-indecomposable persistence modules

So far we have restricted our attention to modules that are decomposable into interval summands. For these modules we have defined persistence diagrams that encode their algebraic structure in a unique and complete way. These diagrams, together with their derived persistence measures, serve as signatures for the modules, and as we will see in Chapter 3 they can be compared against one another in a natural and theoretically sound way.

In order to extend the definition of persistence diagram to persistence modules $\mathbb{V}$ that are not interval-decomposable, we proceed backwards compared to Section 3: first, we use the formulas of (1.18) as axioms to define the persistence measure $\mu_{\mathbb{V}}$ of $\mathbb{V}$; then, we show the existence and uniqueness of a multiset of points in the extended plane whose counting measure restricted to rectangles coincides with $\mu_{\mathbb{V}}$. In order to use (1.18), we need to assume that the ranks in the alternating sums are finite, except $\operatorname{rank} v_q^r$, which brings up the following notion of tameness for $\mathbb{V}$:

Definition 1.12. A persistence module $\mathbb{V} = (V_i, v_i^j)$ over $\mathbb{R}$ is *quadrant-tame*, or $\mathfrak{q}$-*tame* for short, if $\operatorname{rank} v_i^j < +\infty$ for all $i < j$.

The reason for this name is obvious from the first formula of (1.18): when $\mathbb{V}$ is interval-decomposable, $\operatorname{rank} v_i^j$ represents the total multiplicity of the diagram inside the quadrant $[-\infty, i] \times [j, +\infty]$. The $\mathfrak{q}$-tameness property allows this multiplicity to be infinite when $i = j$ (i.e. when the lower-right corner of the quadrant touches the diagonal line $y = x$) but not when $i < j$ (i.e. when the quadrant lies strictly above the diagonal line). Note that forcing the multiplicity to be finite even when $i = j$ would bring us back to the concept of pointwise finite-dimensional module.

For a $\mathfrak{q}$-tame persistence module $\mathbb{V}$, we define the persistence measure $\mu_{\mathbb{V}}$ by the formulas of (1.18), which as we saw are well-founded in this case. This measure on rectangles does take values in $\{0, 1, 2, \cdots, +\infty\}$ as before, although the fact that it cannot be negative is not obvious at first sight: it follows from the observation that the formulas of (1.18) actually count the multiplicity of the summand $\mathbb{I}[q, r]$ in the interval-decomposition of the restriction of the module $\mathbb{V}$ to some finite index set J, which is $J = \{q, r\}$ for the first formula, $J = \{q, r, s\}$ for the second, $J = \{p, q, r\}$ for the third, and $J = \{p, q, r, s\}$ for the fourth. In each case, the restriction of $\mathbb{V}$ to this finite index set J is interval-decomposable by Theorem 1.9 (i), so the multiplicity of the summand $\mathbb{I}[q, r]$ is well-defined and non-negative—for further details on this specific point, please refer to [72, §2.1]. In addition to being non-negative, $\mu_{\mathbb{V}}$ is also additive under vertical and horizontal splittings, a straight consequence of its definition (the duplicated terms in the alternating sums cancel out).

Theorem 1.13 ([72, theorem 2.8 and corollary 2.15]). *Given a* $\mathfrak{q}$-*tame module* $\mathbb{V}$, *there is a uniquely defined locally finite decorated multiset* $\mathsf{Dgm}(\mathbb{V})$ *in the open extended upper half-plane* $\{(x, y) \in \bar{\mathbb{R}}^2 \mid x < y\}$ *such that for any rectangle* $R = [p, q] \times [r, s]$ *with* $-\infty \leq p < q < r < s \leq +\infty$,

$$\mu_{\mathbb{V}}(R) = \operatorname{card}\left(\mathsf{Dgm}(\mathbb{V})|_R\right).$$

The proof of this result works by subdividing the rectangle R recursively into subrectangles, and by a limit process it eventually charges the local mass of $\mu_{\mathbb{V}}$ to a finite set of decorated points, whose uniqueness is obtained as an easy consequence of the construction. A preliminary version of this argument appeared in [71] and was based on measures defined on specific families of rectangles. The final version in [72] is more cleanly formalized and considers general rectangle measures, therefore we recommend it to the interested reader.

Theorem 1.13 defines the persistence diagram of a q-tame module $\mathbb{V}$ uniquely as a multiset $\mathsf{Dgm}(\mathbb{V})$ of decorated points lying above the diagonal line $\Delta = \{(x, x) \mid x \in \bar{\mathbb{R}}\}$. Alternately, one can use a persistence barcode representation, in which every diagram point $(b^\pm, d^\pm)$ becomes the interval $\lceil b^\pm, d^\pm \rfloor$. The undecorated version of $\mathsf{Dgm}(\mathbb{V})$, denoted $\mathsf{dgm}(\mathbb{V})$, is obtained as in (1.14) by forgetting the decorations.

These definitions agree with the ones from Section 3 in the sense that, if a persistence module $\mathbb{V}$ over $\mathbb{R}$ is both q-tame and interval-decomposable, then the multiset $\mathsf{Dgm}(\mathbb{V})$ derived from Theorem 1.13 agrees with the one from (1.13) everywhere above the diagonal Δ—see [72, proposition 2.18]. The two multisets may differ along Δ though, since the interval-decomposition of $\mathbb{V}$ may contain summands of type $\mathbb{I}[b, b] = \mathbb{I}\lceil b^-, b^+ \rfloor$, which are not captured by the rectangles not touching Δ. It turns out that either definition of $\mathsf{Dgm}(\mathbb{V})$ can be used in practice, as the natural measures of proximity between persistence modules and between their diagrams, which will be presented in Chapter 3, are oblivious to the restrictions of the diagrams to the diagonal Δ. We will therefore be using both definitions indifferently in the following.

5. Discussion

To conclude the chapter, let us discuss some of its concepts further and put them into perspective.

Persistence modules: a historical account. Several definitions of a persistence module coexist in the persistence literature, following the steady development of the theory towards greater generality and abstraction.

The term *persistence module* was coined originally by Zomorodian and Carlsson [243], but the concept appeared already in [118], where it referred to a finite sequence of finite-dimensional vector spaces connected by linear maps as follows:

$$V_1 \xrightarrow{v_1} V_2 \xrightarrow{v_2} \cdots \xrightarrow{v_{n-1}} V_n \tag{1.19}$$

In other words, a persistence module as per Edelsbrunner, Letscher, and Zomorodian [118] is a finite-dimensional representation of the linear quiver $\mathtt{L}_n$. Zomorodian and Carlsson [243] extended the concept to diagrams indexed over the natural numbers, that is, to representations of the quiver $\mathtt{N}$. Cohen-Steiner, Edelsbrunner, and Harer [87] further extended the concept to work with diagrams indexed over the real line. They defined a persistence module as an indexed family of finite-dimensional vector spaces $\{V_i\}_{i\in\mathbb{R}}$ together with a doubly indexed family of linear maps $\{v_i^j : V_i \to V_j\}_{i\leq j}$ that satisfy the following identity and composition rules:

$$\begin{aligned} \forall i, \quad & v_i^i = \mathbb{1}_{V_i}, \\ \forall i \leq j \leq k, \quad & v_i^k = v_j^k \circ v_i^j. \end{aligned} \tag{1.20}$$

Such objects are pointwise finite-dimensional representations of the poset $(\mathbb{R}, \leq)$, the identity and composition rules (1.20) following from functoriality as in (1.10).

Chazal et al. [71] dropped the finite-dimensionality condition on the vector spaces, and then Chazal et al. [72] replaced $\mathbb{R}$ by any subset $T \subseteq \mathbb{R}$ equipped with the same order relation $\leq$. Hence Definition 1.7.

Carlsson and de Silva [49] extended the concept of persistence module in a different way, by choosing arbitrary orientations for the linear maps connecting the finite-dimensional vector spaces in (1.19). This gave rise to the concept of zigzag module, presented in Definition 1.8 without the finite-dimensionality condition.

More recently, Bubenik and Scott [41] then Bubenik, de Silva, and Scott [40] proposed to generalize the concept of persistence module to representations of arbitrary posets. This generalization reaches far beyond the setting of 1-dimensional persistence, losing some of its fundamental properties along the way, such as the ability to define complete discrete invariants like persistence barcodes in a systematic way. Nevertheless, it still guarantees some 'soft' form of stability, as will be discussed at the end of Chapter 3 and then in Chapter 9.

Interval decompositions in the persistence literature. Although Theorem 1.9 is presented as a byproduct of representation theory in the chapter, it actually took our community some time to realize this connection.

Historically, Zomorodian and Carlsson [243] were the first ones to describe persistence modules in terms of representations. They pointed out the connection between the persistence modules over $\mathbb{N}$ and the graded modules over the polynomial ring $\boldsymbol{k}[t]$ (mentioned after Theorem 1.4), and they used the structure theorem for finitely generated modules over a principal ideal domain as decomposition theorem for finite-dimensional persistence modules over $\mathbb{N}$.

Some time later, Carlsson and de Silva [49] introduced zigzag modules and connected them to finite-dimensional representations of A_n-type quivers. This connection induces a decomposition theorem for finite-dimensional zigzag modules via Gabriel's theorem.

More recently, Chazal et al. [72] pointed out the connection between persistence or zigzag modules over finite sets and representations of finitely generated algebras, and they referred to the work of Auslander [12] and Ringel and Tachikawa [217] to decompose arbitrary persistence or zigzag modules (including infinite-dimensional ones) over finite index sets—Theorem 1.9 (i).

In the meantime, Lesnick [179] introduced our community to the work of Webb [238], which generalizes the decomposition theorem used by Zomorodian and Carlsson [243] to pointwise finite-dimensional modules over $\mathbb{N}$. Crawley-Boevey [93] further extended it to a decomposition theorem for persistence modules over arbitrary subsets of $\mathbb{R}$ under the pointwise finite-dimensionality condition—Theorem 1.9 (ii). His proof turns out to hold under a somewhat weaker (albeit technical) condition [95], which gives hope for tackling the interval decomposability question in greater generality, as will be discussed next.

These results have contributed to shape the theory as we know it today. Among them, let us point out [49] as a key contribution, for bringing the existence of quiver theory and its connection to persistence to the attention of our community, and conversely, for creating an opportunity to advertise persistence among the representation theory community and stimulate interactions. Besides, Carlsson and de Silva [49] proposed a genuinely new constructive proof of Gabriel's theorem in the special case of A_n-type quivers, which is self-contained, requires no prior knowledge of quiver theory, and has eventually led to a practical algorithm for computing decompositions of zigzag modules [50]. For the interested reader, we analyze this proof and establish connections to the so-called *reflection functors* of Bernstein, Gelfand, and Ponomarev [24] in Section 4.4 of Appendix A.

$\mathfrak{q}$*-tameness versus interval-decomposability.* The concepts of interval-decomposable and $\mathfrak{q}$-tame persistence modules over $\mathbb{R}$ are closely related but not identical, and neither of them is a direct generalization of the other. For instance, the module $\bigoplus_{j\in J} \mathbb{I}\lceil b_j^{\pm}, d_j^{\pm}\rfloor$, where the decorated pairs form a dense subset of the half-plane above Δ, is interval-decomposable but not $\mathfrak{q}$-tame—in fact its persistence measure is infinite on every rectangle. By contrast, the module $\prod_{n\geq 1} \mathbb{I}[0, \frac{1}{n}]$ is $\mathfrak{q}$-tame but not interval-decomposable.

Both concepts are related though. As we will see in Chapter 3, $\mathfrak{q}$-tame modules are the limits of pointwise finite-dimensional modules in some metric called the *interleaving distance*. Recall that pointwise finite-dimensional modules are themselves $\mathfrak{q}$-tame by definition and interval-decomposable by Theorem 1.9 (ii), so $\mathfrak{q}$-tame modules are limits of (a subset of) the interval-decomposable modules. Furthermore, Crawley-Boevey [95] proved that any $\mathfrak{q}$-tame persistence module $\mathbb{V}$ admits interval-decomposable submodules $\mathbb{W}$ whose interleaving distance to $\mathbb{V}$ is zero, so $\mathfrak{q}$-tame modules are in fact indistinguishable from interval-decomposable modules in that metric. This result led Bauer and Lesnick [20] to define the undecorated persistence diagram of a $\mathfrak{q}$-tame module $\mathbb{V}$ directly as the diagram of any interval-decomposable submodule $\mathbb{W}$ of $\mathbb{V}$ lying at interleaving distance zero from $\mathbb{V}$. The upcoming Isometry Theorem (Theorem 3.1) implies that this is a sound definition, all such submodules $\mathbb{W}$ having in fact the same undecorated diagram.

One of these submodules stands out: the so-called *radical* $\mathrm{rad}(\mathbb{V})$, defined as follows (where $\mathbb{V} = (V_i, v_i^j)$):

$$\forall j \in \mathbb{R},\ \mathrm{rad}(\mathbb{V})_j = \sum_{i<j} \mathrm{im}\, v_i^j. \tag{1.21}$$

Although not always pointwise finite-dimensional, $\mathrm{rad}(\mathbb{V})$ is interval-decomposable [95], furthermore it makes the quotient module $\mathbb{U} = \mathbb{V}/\,\mathrm{rad}(\mathbb{V})$ *ephemeral*[7], that is:

$$\forall i < j \in \mathbb{R},\ \mathrm{rank}\, u_i^j = 0. \tag{1.22}$$

Thus, every $\mathfrak{q}$-tame module is interval-decomposable 'modulo' some ephemeral module. Chazal, Crawley-Boevey, and de Silva [63] formalized this idea by introducing the so-called *observable category* of persistence modules, defined as the quotient category of $\mathfrak{q}$-tame modules 'modulo' ephemeral modules in the sense of Serre's theory of localization. In this new category, $\mathfrak{q}$-tame modules become interval-decomposable, and (undecorated) persistence diagrams are a complete invariant for them.

The conclusion of this discussion is that $\mathfrak{q}$-tame modules appear as a natural extension of (a subset of) the interval-decomposable modules. In addition, experience shows that $\mathfrak{q}$-tame modules occur rather widely in applications. For instance, as we will see in Chapter 7, the Vietoris-Rips and Čech complexes of a compact metric space have $\mathfrak{q}$-tame persistent homology, whereas they can be very badly behaved when viewed non-persistently. Such examples support the claim that the $\mathfrak{q}$-tame modules are also a natural class of persistence modules to work with in practice.

[7] In fact, $\mathrm{rad}(\mathbb{V})$ is the smallest submodule of $\mathbb{V}$ such that the quotient module is ephemeral.