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Introduction

The Gelfand-Naimark theorem says that a commutative $C^*$-algebra $A$ with identity is determined up to isomorphism by its spectrum $T$, in the very strong sense that $A$ is isomorphic to the algebra $C(T)$ of continuous functions on $T$. There are also non-commutative $C^*$-algebras with spectrum $T$: for example, the algebra $C(T, M_n(\mathbb{C}))$ of continuous functions from $T$ into the matrix algebra $M_n(\mathbb{C})$. In this book we study theorems which classify algebras with spectrum $T$ and automorphisms of these algebras in terms of topological invariants of the space $T$.

One of the goals of algebraic topology is to associate to a space $T$ algebraic objects whose properties reflect the topological structure of $T$. The ones of interest to us are cohomology groups $H^n(T; \mathbb{Z})$: very roughly speaking, $H^n(T; \mathbb{Z})$ counts $n$-dimensional holes in the space $T$ by chalking up a copy of $\mathbb{Z}$ for each such hole. Our $C^*$-algebras with spectrum $T$ are classified by classes in the 3-dimensional cohomology group $H^3(T; \mathbb{Z})$, and their automorphisms by classes in $H^2(T; \mathbb{Z})$; these theorems were first proved, respectively, by Dixmier and Douady in 1963 and by J. Phillips and Raeburn in 1980. They have been used extensively in recent years: in the analysis of $C^*$-dynamical systems and their crossed products, in the $K$-theory of $C^*$-algebras [150], in differential geometry [14], and in mathematical physics [16]. We shall discuss all the background material needed to prove these classification theorems and apply them to $C^*$-dynamical systems, including the definitions and properties of the cohomology groups themselves.

There are several versions of the Dixmier-Douady Classification Theorem, involving different families of $C^*$-algebras and classifying up to different equivalence relations. The simplest concerns locally trivial bundles over $T$ with fibres isomorphic to the algebra of compact operators on an infinite-dimensional Hilbert space, and identifies them up to bundle isomorphism; we shall prove this one first. For applications to operator algebra, it is more satisfactory to work with continuous-trace $C^*$-algebras, as Dixmier and Douady did. They used continuous fields of $C^*$-algebras, which had been developed by Fell to prove structure theorems for important classes of $C^*$-algebras. However, Fell’s theory never became standard $C^*$-algebraic equipment, so anybody wanting to use the Dixmier-Douady classification faced a heavy start-up cost. We have used instead Rieffel’s theory of Morita equivalence for $C^*$-algebras, which is now recognized to be a fundamental tool for $C^*$-algebraists. It has been folklore for the past 20 years that the Dixmier-Douady classification can be couched in terms of Morita equivalence (see, for example, [5] or [140, 20]), but this version has remained logically dependent on the special structure
theories developed by Fell and Dixmier-Douady. We aim to give a direct treatment, starting from the basic theory of $C^*$-algebras.

Morita equivalence was adapted to $C^*$-algebras by Rieffel in the 1970’s in connection with a $C^*$-algebraic version of the Mackey machine [144, 147], and it has since become a standard tool for analyzing group $C^*$-algebras and crossed products [148, 62]. Although Rieffel’s papers were carefully motivated and expounded, their primary concern was representation theory, and the general theory of Morita equivalence they contained only gradually emerged. We have therefore decided to give a detailed introduction to this general theory. To do this in modern language, we have to discuss Hilbert modules, and we have preferred to be self-contained: the recent expositions of Hilbert modules by Lance [94] and Wegge-Olsen [167] are tilted towards quantum groups and $K$-theory rather than Morita equivalence, and hence their motivating examples are different.

The classification theorems necessarily involve sheaf cohomology. It is slightly nonstandard sheaf cohomology: one has to deal with sheaves of nonabelian groups, and hence the techniques of homological algebra do not apply directly. (And many functional analysts would probably prefer to avoid these techniques anyway.) So we have chosen to give a concrete introduction to sheaf cohomology and fibre bundles based on cocycles. Thus our proof of the bundle-theoretic Dixmier-Douady classification is entirely self-contained, and should be accessible to readers from other areas as well as operator algebraists.

Our treatment of continuous-trace $C^*$-algebras reflects our emphasis on Morita equivalence: for us, a continuous-trace $C^*$-algebra $A$ with spectrum $T$ is almost by definition a $C^*$-algebra which is locally Morita equivalent to the commutative $C^*$-algebra $C(T)$, and the Dixmier-Douady class $\delta(A) \in H^3(T; \mathbb{Z})$ is the obstruction to the existence of a global Morita equivalence. To make all this precise, we need background material on the spectrum and primitive ideal space of a $C^*$-algebra, and we have included an introduction to the subject in Appendix A.

We originally intended to phrase all our results and calculations in terms of Morita equivalence. However, as we progressed we found this attitude restrictive, and noticed in particular that it is often easier to compute $\delta(A)$ using local rank-one projections and partial isometries, as in [127, §2]. We have come to believe that the power and enduring interest of the Dixmier-Douady theory lies in its many formulations, and we hope that our treatment will prove useful from several viewpoints. We have therefore been careful to tie up our main theorem with the bundle-theoretic version by including a discussion of the Stabilization Theorems of Kasparov and Brown-Green-Rieffel.

One punchline of our book is a version of the Dixmier-Douady Theorem in which the map $A \mapsto \delta(A)$ induces an isomorphism of a Brauer group of continuous-trace algebras with fixed spectrum $T$ onto the cohomology group $H^3(T; \mathbb{Z})$. This lovely theorem is part of the folklore we are aiming to tidy up: it was mentioned almost in passing by Taylor in 1975 [161], and discussed in seminar notes by Green around 1978 [61]. However, even to formulate it, we need nontrivial facts about the spectra of tensor product $C^*$-algebras; again we have developed the necessary material in an Appendix.

When we started this book, one goal was to give a readable and complete intro-
Introduction to the research area which has been described by Rosenberg as the study of the “fine structure of the Mackey machine” [151]. Loosely speaking, this concerns actions of groups on continuous-trace algebras and the structure of the associated crossed products. We are still several books short of this goal, so we have compromised by providing an overview of the area and of the remaining background. This is the content of Chapter 7. We have used as a unifying theme the equivariant Brauer group introduced in [20]. The structure theory of this group involves a variety of cohomological invariants and principal bundles, and describes many fascinating relations between them; there are many more, we suspect, waiting to be found.

We have made a serious effort to keep required background to a minimum. There is by now a fairly standard first course in $C^*$-algebras, covering the Gelfand-Naimark Theorem, the continuous functional calculus, positivity, and the GNS-construction. We assume that readers have seen this; everything we need is in the first 3 chapters of Murphy’s book [111], for example. We make the usual conventions of the subject: for example, homomorphisms of $C^*$-algebras are always $*$-preserving, and ideals are closed and two-sided unless otherwise stated. We have had to use some general topology and functional analysis, but we have tried not to overdo this, and to give detailed references. (Our preferred authority here is Pedersen’s book [125], which covers both topology and functional analysis, is elegantly written, and has a good and practical selection of topics.) On the other hand, we have been careful to develop all the algebraic topology we need, and have arranged our material so that parts involving locally compact groups and integration can be avoided without loss of continuity.

We hope that students who have seen the basics of $C^*$-algebras at the Honours level in the UK or Australia, the Diploma level in Europe, or the beginning graduate level in North America can read this book and enjoy doing so. We have been careful to make the different sections of the book as self-contained as possible, so that those who want to know about Hilbert modules and Morita equivalence, the spectrum of a $C^*$-algebra, tensor products of $C^*$-algebras or basic sheaf cohomology should be able to turn to the relevant section and learn something useful without absurd difficulty.

Reader’s Guide

Chapter 1 is about the algebra of compact operators and its properties. This algebra is fundamental both in the structure theory of nice $C^*$-algebras and as motivation for the theory of Morita equivalence. More experienced readers could probably skip or skim this chapter, but others may find it helpful to see our notation and viewpoint in a relatively elementary setting.

Chapter 2 is a self-contained introduction to Hilbert modules, multiplier algebras, and Rieffel’s general theory of induced representations. Induced representations are defined using tensor products of Hilbert modules, but these are technically much easier than the tensor products of $C^*$-algebras discussed in Appendix B, and
we do not assume previous familiarity with tensor products of any sort. Rieffel's motivation came from the unitary representation theory of locally compact groups, and we have discussed this application in outline at the end of each section. However, this material can be safely and logically skipped by those who are not interested or not familiar with integration on locally compact groups. For those who are, we have included a detailed discussion in Appendix C.

Chapter 3 concerns imprimitivity bimodules and Morita equivalence of $C^*$-algebras, and builds on our previous treatment of Hilbert modules. We discuss in detail the Rieffel correspondence between ideals and quotients of Morita-equivalent $C^*$-algebras. From this and the definition of the topology on the primitive ideal space, we obtain the existence of the Rieffel homeomorphism between the primitive ideal spaces of Morita-equivalent $C^*$-algebras. Section 3.4, on the external tensor product of imprimitivity bimodules, uses the basic properties of the spatial tensor product discussed in Appendix B.1. Since this material is not used until Chapter 6, Section 3.4 can be skipped at this stage.

Chapter 4 is a self-contained introduction to sheaf theory covering the material necessary for basic applications to $C^*$-algebras, which should be accessible even to homologically-challenged functional analysts. The only slightly nonstandard algebraic background required is the direct-limit construction discussed in Appendix D.1. Those seeking instant gratification in the form of applications to $C^*$-algebras should see Proposition 4.27, which will be much more informative to those who have read Chapter 1. All the sheaf cohomology needed for Chapter 5 is contained in Section 4.1. In Section 4.2 we discuss fibre bundles and their classification in terms of sheaf cohomology: these ideas are used freely in the recent literature on crossed products of continuous-trace $C^*$-algebras, as outlined in Chapter 7. We can then give the Dixmier-Douady classification for locally trivial fibre bundles with fibre the algebra $K(H)$ of compact operators. This version of the Dixmier-Douady classification theorem uses only the material in Chapters 1 and 4. In the last few pages of Chapter 4, we explain why this version of the theorem is not ideal for those working in $C^*$-algebras.

Chapter 5 is the heart of the book: the classification of continuous-trace $C^*$-algebras. To define this class, we need to consider $C^*$-algebras with Hausdorff spectrum. To appreciate the material in Section 5.1, one needs a working knowledge of the spectrum and primitive ideal space of a $C^*$-algebra; all the necessary details are in Appendix A. In Section 5.2, we define continuous-trace $C^*$-algebras, discuss the alternative characterizations, and give many nontrivial examples.

Our treatment of the Dixmier-Douady classification theorem and the analogous classification of automorphisms depends on all the earlier chapters. We give two versions of the Dixmier-Douady Theorem: the classification up to Morita equivalence, and the classification of separable stable continuous-trace $C^*$-algebras up to isomorphism. The latter depends on the Brown-Green-Rieffel Theorem, which says that Morita equivalence is the same as stable isomorphism, and which we prove in Section 5.5. This is a deep theorem, and the discussion in Section 5.5 is necessarily at a higher level than most of the main text. We conclude the chapter with a brief discussion of the pros and cons of the different versions of the Dixmier-Douady classification. At this stage we do not complete the classification
of automorphisms, since the most efficient route to this involves more sophisticated use of tensor products of $C^*$-algebras.

Chapter 6 begins with our punchline: the identification of the Brauer group $\text{Br}(T)$ of continuous-trace $C^*$-algebras with $H^3(T; \mathbb{Z})$. The multiplication in $\text{Br}(T)$ is a balanced $C^*$-algebraic tensor product; to understand it, we need the description of the spectrum of a tensor product from Appendices B.1–B.4. Once we have this balanced tensor product, we can quickly finish the classification of automorphisms of continuous-trace algebras. The other sections of Chapter 6 concern two constructions which yield interesting examples of continuous-trace $C^*$-algebras. The pull-back construction of Section 6.2, which is defined using the balanced tensor product, shows how to make the Brauer group into a functor. The inducing construction of Section 6.3 is a $C^*$-algebraic version of Mackey’s original construction of the Hilbert space of an induced representation; it yields a class of nontrivial continuous-trace $C^*$-algebras whose Dixmier-Douady class we can explicitly compute.

The first two Appendices discuss aspects of the general theory of $C^*$-algebras which would not normally be covered in a first course on the subject, but which are required in the book. The first is a relatively gentle introduction to the spectrum of a $C^*$-algebra which highlights the examples of interest to us. The second is a self-contained treatment of tensor products of $C^*$-algebras whose main goal is to prove that the spectrum of the tensor product of two continuous-trace $C^*$-algebras is the product of their spectra. This result is certainly known, but it is hard to point to a detailed proof in the existing literature. Only the first four sections of Appendix B are used in the book proper; the fifth has been included to provide a detailed source for other known facts. The third Appendix contains a proof of Rieffel’s formulation of Mackey’s Imprimitivity Theorem; this material is logically independent of the rest of the book, and is included because we would be giving a lop-sided view of Morita equivalence if some applications to representation theory were not mentioned.

Hooptedoodle. This book occasionally breaks for a Hooptedoodle. These contain comments which go beyond the prevailing scope and level, but which might help put the material in context. The term Hooptedoodle comes from the prologue of John Steinbeck’s Sweet Thursday, in which one of the main characters criticizes the previous book Cannery Row:

> Sometimes I want a book to break loose with a bunch of hooptedoodle. . . .
> But I wish it was set aside so I don’t have to read it. . . . Then I can skip it if I want to, or maybe go back to it after I know how the story comes out.

Steinbeck then dutifully labels his digressions as hooptedoodles.

Material in our hooptedoodles is not used in the main text, and can easily be skipped. Indeed, they should be skipped if they don’t seem to be helping!

Acknowledgments. We have tried to present a detailed account of the theory of continuous-trace $C^*$-algebras, along with the background necessary to read the current literature. Our account is logical rather than historical; we have not made any attempt to give detailed attributions, and we apologize to those we have not
mentioned*. We do acknowledge obvious debts to Rieffel and to Dixmier-Douady; where our subject matter overlaps, we have been unable to improve on Lance’s treatment of Hilbert modules.

We thank those who helped us with this book. In addition to those who taught and introduced us to the subject, we thank the participants in the various seminars we gave at Newcastle based on parts of the book. We specifically thank Neal Fowler, Astrid an Huef, Wojciech Szmański, David Webb, and especially Paul Muhly for reading large parts of drafts and for their helpful suggestions. The remaining mistakes are the other author’s fault.

*We did start out with better intentions, but soon realized we could cause even more offense this way. Our memories are not what we think they once were.