Preface

Most students of mathematics have a general perception of number systems that they encounter during their studies. They can identify the five basic number systems of mathematics: natural numbers, integers, rational numbers, real numbers, and complex numbers. However, when it comes to defining various kinds of numbers, not to mention proving their properties, many students appear to be unprepared. The aim of this book is the successive rigorous construction and development of the five basic number systems of mathematics. Accordingly, the book is organized into five chapters.

Peano arithmetic of natural numbers is rightly regarded as a springboard into the development of other number systems and eventually into algebra and analysis. The first section of Chapter 1 introduces a Peano system as an algebraic structure satisfying three axioms and postulates that such a structure exists. In the next three sections, operations of addition and multiplication and an order relation on the underlying set of a Peano system are defined and their properties are established. In the rest of Chapter 1 some aspects of the theory of natural numbers are discussed. Specifically, they include isomorphism of Peano systems (Section 1.5), a set-theoretic model of natural numbers (Section 1.6), and recursion and induction (Sections 1.7 and 1.8). Several examples of algebraic structures that are of importance in abstract algebra are briefly introduced in Section 1.9.

The number system $\mathbb{Z}$ of integers is the subject of study in the first four sections of Chapter 2. The set of integers endowed with addition and multiplication operations is an archetypal example of an algebraic structure called “ring”. Basic properties of rings and their special instances called “integral domains”, together with examples, are covered in the last section of Chapter 2.

Rational numbers are defined as classes of “equivalent fractions” in Chapter 3. As in the previous chapters, the first four sections are devoted to rigorous development of the arithmetic of rational numbers. The set $\mathbb{Q}$ of rational numbers endowed with arithmetic operations and a natural order relation on it is an example of an “ordered field”. Fields and ordered fields are covered in Section 3.5 where examples of a finite field and an ordered field different from $\mathbb{Q}$ are also presented. In
Section 3.6 an important “analytical” concept of convergence appears for the first time in the book. Cauchy sequences (also known as fundamental sequences) with terms in $\mathbb{Q}$ are important tools in the development of the system of real numbers in Chapter 4. Two limitations of the number system $\mathbb{Q}$ are the subject of discussion in Section 3.7. First, some simple quadratic equations such as $x^2 = 2$ do not have a solution in the field $\mathbb{Q}$. Second, there are Cauchy sequences in $\mathbb{Q}$ that are not convergent. An example of such a sequence is presented in the last section of Chapter 3. This example also demonstrates that the equation $x^2 = 2$ does not have a solution in the set of rational numbers.

The real numbers are introduced in Chapter 4 as equivalence classes of Cauchy sequences in $\mathbb{Q}$. In Sections 4.2–4.4, the set of real numbers $\mathbb{R}$ is described as an ordered field. Important completeness properties of the field $\mathbb{R}$ are established in the next two sections, hence resolving incompleteness of the field $\mathbb{Q}$. Some well-known properties of real continuous functions on $\mathbb{R}$ are found in the last section of this chapter.

The last chapter of the book presents a brief development of complex numbers. In Sections 5.1 and 5.2, the field $\mathbb{C}$ of complex numbers is defined and its basic properties are discussed. The highlight of this chapter is the proof of the Fundamental Theorem of Algebra in Section 5.8. The material covered in Sections 5.3–5.7 presents properties of the field $\mathbb{C}$ that are essential for the proof of this celebrated theorem.

The Peano arithmetic presented in Chapter 1 is the basis for all other theories of number systems developed in this book. However, the development of the system of natural numbers requires instructions stated in a language outside the system. This language is provided by set theory. It was not my intention to include set theory, not to mention elementary logic, into the main body of the book. Put differently, it is assumed that the reader is familiar with mathematical proofs and properties of sets. However, a primer on “naive” set theory is included in Appendix A.

Although the proofs in the book do not require any knowledge of mathematics beyond elementary set theory, this material is not meant for the college freshman with minimum preparation from high school. A more likely readership would include upper-undergraduate level students with some “hands-on” experience with higher” mathematics. I believe that in fact the book will find a much wider audience.

In my opinion, the most (and perhaps the only) effective way of learning mathematics is by “doing it”. There are 220 exercises in the book. Every chapter including the appendix has its own set of exercises.

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