Preface

For many years, abstract algebra has been one of my favorite classes to teach at Pomona College, and this text has grown out of that experience. My students, by and large, have been eager second- or third-year undergraduates who have had no prior experience with the material. When they start out, they are more or less comfortable with proofs, have had a solid linear algebra class, and are familiar with the arithmetic (but not the theory) of complex numbers. The goal is to give the students a rigorous and motivated introduction to groups, rings, and fields, and to go deep enough into each subject to see the power of abstract thinking and to be convinced that the subject is full of unexpected results. There is more than enough material here for a one-year course, but appropriate selections can be made for a one-semester course as well. While the text is unmistakably for undergraduates and assumes no prior familiarity with the subject, it hopes to nudge students toward thinking like mathematicians by putting a premium on building intuition and by expecting the students to be actively involved in the learning of the material. It has been my experience that after going through this material, the students are amply prepared for graduate level courses in algebra.

Algebraic structures abound in contemporary mathematics, and abstract algebra provides the language for studying them. Consider the following problems:

- You have 47 colors and you want to color a cube by assigning a (not necessarily distinct) color to each face. How many “different” colorings are possible? If you can get from one coloring to another by rotating the cube, then count the two colorings as the same. (Page 158)
- Can you find the solutions to $x^5 - 10x + 5 = 0$ precisely? (Example 28.18)
- Can you describe all the integer solutions to $x^2 - 3y^2 = z^2$? Or to $y^3 = x^2 + 4$? (Problem 28.4.10 and Proposition 20.18)
- Each of 100 briefcases contains the name of one contestant. Each of the 100 contestants gets to privately examine the contents of 50 of the 100 briefcases. The contestant will be successful if she/he finds the briefcase that has her/his
own name in it. There is no communication between the contestants, but they can agree on a common strategy before the contest. Is there a strategy that with a probability of more than 30% assures the success of every one of the contestants? (Problem 3.3.5)

- Can you double a cube? More precisely, given one edge of a cube, can you construct—using a straightedge and compass—another line segment such that a cube with this new line segment as its side will have a volume twice as much as the original cube? (Corollary 23.18)

- Does there exist a real number $\alpha$ such that rational linear combinations of $1, \alpha, \alpha^2, \ldots$ include every one of $\sqrt[4]{47}, \sqrt[5]{17},$ and $18 - 2\sqrt[3]{19}$? (Question 25.48)

- Let $X$ be a set with 47 elements. Choose 169 one-to-one and onto functions from $X$ to $X$ such that if you compose any two of the functions, you get another function in the set. I predict the following:
  (a) If $f$ and $g$ are two of your functions, then $f(g(x)) = g(f(x))$ for all $x \in X$.
  (b) The identity function is the function $i$ such that $i(x) = x$ for all $x \in X$.
  If $f$ is any of your functions and if you start composing $f$ with itself, then after 169 iterations you will always get $i$.
  Can you prove either prediction? (Problem 6.3.1)

Each of these questions will be answered in this text, but what is somewhat surprising is that abstract algebra provides a common framework for answering them. While it is possible to attack many (but not all) of these problems individually and without recourse to deep theory, an abstract axiomatic development of the properties of algebraic structures will give us the tools and the language necessary to think about them conceptually. The result is a far-reaching, powerful, and—dare I say—beautiful theory.

Historically, different strands have come together to create the common language of algebraic structures that is at the core of modern abstract algebra. One thread is the attempt to solve algebraic equations. In fact, the word “algebra” is from the Arabic “al-jabr” and translates to “completion” or “restoration”, referring to moving a negative quantity to the other side of an equation where it becomes positive. In the medieval Islamic world, where algebra started to become a discipline separate from geometry, the central problem was that of solving of equations. The quadratic equation allows us to solve any equation of degree 2, and similar—but more complicated—formulas for cubic and quartic equations were found in sixteenth century Italy. The quest for solving the quintic resulted in deeper studies of permutations and eventually the advent of group theory and the Galois theory of fields. From this beginning, group theory has evolved into mathematicians’ preferred language for the study of symmetries in whatever context. A second thread was the investigation of Diophantine equations—that is, finding integer solutions to equations with many variables such as $x^n + y^n = z^n$. This, together with other problems in number theory, led to the desirability of doing arithmetic and number theory with collections of numbers other than integers. Commutative ring theory is what resulted. Having arisen from old historical roots, the methods and techniques of abstract algebra permeate all of modern mathematics.
This text introduces groups, rings, and fields to a student who is seeing these concepts for the first time and yet wants to gain a somewhat sophisticated taste of the material. The choice of material and the mix of results and problems reflects this pedagogical aim. As such the book is not comprehensive, and the proofs are not necessarily the sleekest proofs available. While the text tells an astounding story—starting from very meager beginnings and building a sophisticated edifice—the main task will be for you the reader to engage the material directly. The large selection of problems will facilitate your endeavor. Since you are assumed to be new to abstract algebra, the writing is somewhat conversational and verbose toward the beginning and becomes more terse as the text progresses. An attempt is made to give you a taste of how mathematicians think about the subject, and so, in addition to the usual definitions and theorems, the text tries to help build your intuition for the material. The proofs of some of the theorems are relegated to the problems. This is because proofs are important, and I want you to figure out some of them for yourself. Since the reader is learning this material for the first time, sometimes topics are repeated. A topic may make its first appearance in an exercise followed by a fuller treatment later. Sometimes a more specialized result is presented before the more general result. To facilitate self-study, many problems have hints, some have short answers, and over 100 problems are solved completely. The hints, short answers, and solutions are all at the back of the book. You are urged to start working on a problem without looking at the back. Only when you are truly frustrated—a necessary part of the learning process—should you look to see if there is a hint, an answer, or a solution. If you have done a problem, but there is a solution at the back, then I urge you to read the solution anyway since it may provide a bit of additional insight. The problems that are important to the development of the subject have boldfaced numbers, while those with a complete solution in the back have italicized numbers.

Groups, rings, and fields have much in common, and an important part of the modern treatment of abstract algebra is an emphasis on the similarities of these and other algebraic structures. However—in the last analysis—to get deep and powerful results, you have to go beyond the commonalities. Groups are mathematicians’ way of thinking about and working with symmetry; commutative rings came about when a need arose to do arithmetic in more general settings than the integers; and field theory originated in the pursuit of solving polynomial equations in one variable. While one of the stories of this book is that all of these things are related to each other, another part of the narrative develops the distinct personalities of each of groups, rings, and fields. As a student of the subject, you need to develop different and separate intuition for each of the structures. It is possible, in an introductory class, to give a survey of these subjects, focusing on their common aspects, and not go too deeply into any of them. My approach has been to go far enough in each topic to showcase some aspect of the deeper theory while constantly bringing out the commonalities.

To the Instructor. The group theory portion of the text has three somewhat unusual features. If you use this text, it will be very hard to avoid the first feature (and you may be better off with a different text if you don’t buy into this approach), but the other two features are quite optional.
First, group actions are introduced very early. In the mathematical world outside group theory, groups appear and show their properties when they act on other objects. Group actions not only get the students to look at groups as “groups of symmetries”, they bring much rhyme and reason to the study of group theory itself. Many important subgroups are stabilizers of actions, the orbits of an action provide a systematic way of partitioning interesting sets, and much of introductory group theory can be organized—as you can see from the table of contents—around various actions of groups on groups. In addition, in Galois theory, studying the action of the Galois groups on the roots of an equation becomes central. For an introductory class, the early introduction of actions may seem as overburdening the students with another level of abstract constructions. This may be true, but what is gained in perspective and intuition is well worth the price. In fact, I believe that actions actually make group theory easier.

Second, Hasse diagrams of posets—lattice diagrams in most cases of interest here—and homomorphism diagrams (instead of exact sequences) are introduced, and students are encouraged to use them to visualize what is going on and to help in arguing proofs. It has been my experience that if you gain facility with these diagrams—which are ubiquitous in notebooks of professional mathematicians—many statements and many proofs turn from abstract and mysterious to straightforward arguments.

Third, normal subgroups, quotient groups, and homomorphisms are introduced somewhat late. Homomorphisms are defined early, but their serious treatment waits until Chapter 11. As I will explain, I have reasons for doing so, but, if you prefer, you can easily change course. One could go to normal subgroups and Chapter 10 right after Chapter 6. Normal subgroups and homomorphisms are very important and their study is at the core of group theory. However, my experience has been that they are also difficult concepts for the first time learner, and much is to be gained if the student develops a variety of intuitions about groups before tackling these concepts. When I teach abstract algebra at Pomona College, I follow the order of this book. Hence, the students will see alternating groups and Sylow theorems early. These give the students a feel for finite group theory and allow the construction of many examples. In addition, the students will have worked with orbits of actions extensively. By the time they are asked to consider quotient groups, the construction will almost seem natural.

Because of these features, the instructor has to be careful not to get bogged down in the first few chapters. There is much material here but you can move briskly. In fact, the writing is meant to be read by the students, and this should help the instructor move through the introductory material more quickly. In my own teaching, I often have students read a section and do some of the more computational problems before I discuss the topic in class, and, in fact, I leave the development of some topics entirely to the students. Reading a mathematics text is an important skill, and my hope is that the many remarks and expository discussions will be helpful in this regard.

To give you a sense of what I do with the material, in the first semester of the course I cover group theory (skipping Sections 5.3, 7.3, 11.6, and 12.4 and Chapters 13 and 14) and ring theory (through Chapter 18 proving ED ⇒ PID ⇒ UFD, but
skipping Sections 16.4, 17.2, and 18.5). In the second semester I start with Chapter 19 on polynomial rings and go through Galois theory (skipping Section 19.6 and and Chapter 20, and the discussion of algebraic closures in Section 24.2).

I have kept the main part of the text to material that I want the students to read—any section or chapter that can be skipped in a first reading is marked by an asterisk—but there are many extra problems and mini-projects that can be used to explore topics not covered in the text. Using these and the references provided, the students should be able to design many independent mini-projects. The website for this book (www.ams.org/bookpages/amstext-27) maintained by the publisher has a detailed syllabus for a year-long course based on this text and other supplementary material.

The three parts—groups, rings, and fields—presented here constitute the first volume of an eventual two-volume text. The second volume will cover modules over a PID, an introduction to algebraic geometry via Gröbner bases, and representation theory.

Acknowledgments. I have collected the material for this book over the course of many years, and, as a result I am indebted to many mathematicians and many books. However, I learned algebra primarily from Marty Isaacs at the University of Wisconsin–Madison. I went to Madison with no particular interest in algebra, but Marty’s graduate course in algebra (which years later became Isaacs [Isa94]) was a revelation. All of a sudden, not only could I follow the individual steps in the arguments, but the questions, the techniques, and the whole enterprise made sense. A small part of the attraction was Marty’s emphasis on group actions and on lattice diagrams. Marty’s indirect and direct influence can be seen on every page of this book. If I have been able to transmit even a small part of the excitement that I felt when taking his class, then I will claim the book a success. In addition to Marty’s classes and books, over the years I have relied on the many wonderful texts on abstract algebra. Some of my favorites are Herstein [Her75], Hadlock [Had78], Hartley and Hawkes [HH76], Stewart [Ste15], Dummit and Foote [DF04], Bhattacharya, Jain, and Nagpaul [BJN94], and Goodman [Goo98]. I also want to thank my many students. Not only have they constantly alerted me to typos and mistakes, but their enthusiasm, engagement, and positive feedback convinced me to write the text. It is a cliche to say that the book would not have been possible without the support of my family. But it is true. The book is dedicated to my partner Nanaz and our sons Kiavash and Neema who heard the excuse “I am writing a book” way too often. Finally, I acknowledge my late father Parviz Shahriari, who, as my high school algebra teacher in 10th grade, got me interested in mathematics and whose many books, such as “Raveshhaye Jabr” [Sha70], made high school algebra actually fun.