CHAPTER 2

Eigenvalues of Graphs

In this chapter we demonstrate how certain linear algebraic properties of the adjacency matrix of a graph can be used to obtain information about structural properties of a graph.

2.1. Some Basic Properties

Let $G$ be a graph of order $n$. Thus $G$ has a vertex set $V$ of $n$ vertices which we take to be $1, 2, \ldots, n$ and an edge set $E$ consisting of pairs $\{i, j\}$ of distinct vertices. Sometimes we write these pairs as $ij$ (or as $ji$). With this labeling of vertices, the adjacency matrix of $G$ is the $n \times n$ matrix $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, & \text{if } ij \text{ is an edge,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus $A$ is a symmetric, nonnegative matrix with 0s on its main diagonal, and both the Perron-Frobenius theory and the theory of real symmetric matrices apply.

If we reorder the vertices (attach the labels $1, 2, \ldots, n$ in a different way), the results is a matrix $P^{-1}AP$ for some permutation matrix $P$. The eigenvalues of the graph $G$ are the eigenvalues of the adjacency matrix $A$ and they do not depend on the particular labeling chosen. Since $A$ is a symmetric matrix with zero trace, we order these eigenvalues as

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1,$$

where $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$, and hence $\lambda_n \leq 0$. Here $\lambda_1$ is what we have also called $\lambda_{\text{max}}$. We use $\lambda_1$ and $\lambda_{\text{max}}$ interchangeably and write $\lambda_{\text{max}}(G)$ when it is necessary to avoid ambiguity.

So what does $A$ tell us about $G$? On the one hand, everything since $A$ is just another way to specify $G$. But $G$, and so $A$, have “secrets” that have to be coaxed out of them.

Example 2.1. The adjacency matrix of the complete graph $K_n$ of order $n$ is the matrix $A = J_n - I_n$ where $J_n$ is the $n \times n$ matrix of all 1s. This graph has eigenvalues $-1, \ldots, -1, n - 1$. (For the eigenvalue $n - 1$, $A$ has eigenvector $j_n = (1, 1, \ldots, 1)^t$. The matrix $A - (-1)I_n$ has rank 1 and thus $-1$ is an eigenvalue of multiplicity $n - 1$.)
The adjacency matrix of the complete bipartite graph $K_{m,n}$ is
\[
A = \begin{bmatrix}
O_{m,m} & J_{m,n} \\
J_{n,m} & O_{n,n}
\end{bmatrix},
\]
where $J_{k,l}$ denotes, in general, a $k \times l$ matrix of all 1s. We have
\[
A^2 = \begin{bmatrix}
nJ_m & O_{m,n} \\
O_{n,m} & mJ_n
\end{bmatrix},
\]
a matrix of rank 2. Since the trace of $A$ equals 0, the eigenvalues of $K_{m,n}$ are $-\sqrt{mn}, 0, \ldots, 0, \sqrt{mn}$.

A graph $G$ with adjacency matrix $A$ is connected if and only if there does not exist a permutation matrix $P$ such that
\[
P^{-1}AP = \begin{bmatrix}
A_1 & O \\
O & A_2
\end{bmatrix},
\]
where $A_1$ and $A_2$ are nonvacuous square matrices, that is, if and only if $A$ is irreducible. Since the eigenvalues of $A$ are those of $A_1$ together with those of $A_2$, it follows that the eigenvalues of a graph $G$ are obtained by putting together the eigenvalues of each of its connected components. As a result, in dealing with eigenvalues of graphs, there is usually no loss in generality in assuming that $G$ is connected. In this case, properties PF1 to PF7 and S1 to S5 apply to the irreducible adjacency matrix $A$.

We now consider a number of elementary relations between a connected graph $G$ and its irreducible adjacency matrix $A$ whose eigenvalues are $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$.

E1. The $(i,j)$-entry of $A^k$ is the number of walks of length $k$ between vertex $i$ and vertex $j$. In particular, the trace of $A^2$ is twice the number of edges of $G$. Since the eigenvalues of $A^k$ are $\lambda_n^k, \lambda_{n-1}^k, \ldots, \lambda_1^k$, the number of closed walks of length $k$ in $G$ equals
\[
\sum_{i=1}^{n} \lambda_i^k, \text{ the } k\text{th spectral moment of } A.
\]

**Proof.** Property E1 is easily established. \(\square\)

E2. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ be, respectively, the smallest and largest of the degrees $d_1, d_2, \ldots, d_n$ of the vertices of $G$. Then
\[
\delta \leq \lambda_1 \leq \Delta.
\]
In fact,
\[
\frac{d_1 + d_2 + \cdots + d_n}{n} \leq \lambda_1,
\]
the average degree of a vertex of $G$.

**Proof.** The first inequalities are a consequence of PF7. The second inequality follows from S2 by choosing $x = j_n^t \in \mathbb{R}^n$, the vector of all 1s. \(\square\)
E3. If the diameter of $G$ is $d$, then the number of distinct eigenvalues of $G$ is at least $d + 1$. Put another way, if the number of distinct eigenvalues of $G$ is $p$, then the diameter of $G$ is at most $p - 1$.

**Proof.** The adjacency matrix $A$ is symmetric and so $A$ is similar to a diagonal matrix and the minimum polynomial of $A$ has degree equal to $p$. There exist vertices $k$ and $l$ which are at distance $d$ in $G$; hence the $(k,l)$-entry of $A^d$ is positive but the $(k,l)$-entry of $A^r$ is 0 for $r < d$. Hence $A^d$ is not a linear combination of $I_n, A, A^2, \ldots, A^{d-1}$, implying that the degree $p$ of the minimal polynomial is at least $d + 1$. \hfill $\square$

E4. If $H$ is an induced subgraph of $G$, then the eigenvalues of $H$ interlace those of $G$.

**Proof.** This is a direct consequence of S5 since the adjacency matrix of $H$ is a principal submatrix of the adjacency matrix of $G$. \hfill $\square$

E5. The graph $G$ is bipartite if and only if its collection of eigenvalues is symmetric about 0, in fact, if and only if $\lambda_n = -\lambda_1$.

**Proof.** The graph $G$ is bipartite if and only if its adjacency matrix $A$ can be taken in the form

$$A = \begin{bmatrix} O_p & A_1 \\ A_1^t & O_q \end{bmatrix}$$

for some $p$ and $q$ with $p + q = n$.

Let $x = (x^1, x^2)^t$ be an eigenvector of $A$ for an eigenvalue $\lambda$. Here $x^1$ and $x^2$ have $p$ and $q$ components, respectively. Then

$$Ax = A \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} A_1 x^2 \\ A_1^t x^1 \end{bmatrix} = \lambda \begin{bmatrix} x^1 \\ x^2 \end{bmatrix},$$

and

$$A \begin{bmatrix} -x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} -A_1 x^2 \\ -A_1^t x^1 \end{bmatrix} = \begin{bmatrix} -\lambda x^1 \\ -\lambda x^2 \end{bmatrix} = -\lambda \begin{bmatrix} -x^1 \\ x^2 \end{bmatrix},$$

This implies that $A$ has as many eigenvalues equal to $\lambda$ as it has equal to $-\lambda$.

Now suppose that the eigenvalues of $A$ are symmetric about 0. Then

$$\text{trace}(A^{2k+1}) = \sum_{i=1}^n \lambda_i^{2k+1} = 0 \text{ for all } k \geq 0.$$ 

It follows from E1 that $G$ cannot have any closed walks or cycles of any odd length. Hence $G$ is bipartite.

Now assume only that $\lambda_n = -\lambda_1$. The largest eigenvalue of $A^2$ is $\lambda_1^2$ and it is not a simple eigenvalue. Thus by PF1 for irreducible nonnegative matrices, $A^2$ cannot be irreducible. Thus after
simultaneous permutations of rows and columns, we may assume that
\[ A^2 = \begin{bmatrix} B_1 & O_{k,l} \\ O_{l,k} & B_2 \end{bmatrix} \]
for some positive integers \( k \) and \( l \) with \( k + l = n \). This determines a bipartition of the vertex set into two nonempty parts \( V_1 \) and \( V_2 \) of sizes \( k \) and \( l \), respectively, such that there are no walks of length 2 between \( V_1 \) and \( V_2 \). Suppose \( a \) and \( b \) are distinct vertices in \( V_1 \) that are joined by an edge. Take a shortest path from \( a \) to a vertex in \( V_2 \), say, \( a = w_0, w_1, \ldots, w_p, w_{p+1} \) where \( w_0, w_1, \ldots, w_p \) are in \( V_1 \) and \( w_{p+1} \) is in \( V_2 \). If \( p \geq 1 \), then \( w_{p-1}, w_p, w_{p+1} \) is a walk of length 2 from \( V_1 \) to \( V_2 \). If \( p = 0 \), then \( b, a, w_1 \) is such a walk. Thus in both cases we get a contradiction, proving that there are no edges joining vertices in \( V_1 \). Similarly, there are no edges joining vertices in \( V_2 \), and we conclude that \( G \) is bipartite.

Property E5 illustrates very clearly how knowledge of eigenvalues can imply structural properties of a graph.

2.2. Eigenvalues and Graph Parameters

In this section we shall see several instances of how graph eigenvalues can give information about some fundamental graph parameters. The graph parameters imply something about the structure of a graph which, in turn, implies something about the structure of its adjacency matrix. This is then used to get information about the eigenvalues.

Let \( G \) be a graph of order \( n \), and let \( n^+, n^0, \) and \( n^- \) denote, respectively, the number of positive, zero, and negative eigenvalues of \( G \). We consider two graph parameters. The independence number of \( G \) is the maximum size \( \alpha(G) \) of a set of vertices no two of which are joined by an edge (the largest order of an induced subgraph with no edges). The chromatic number of \( G \) is the smallest integer \( \chi(G) \) such that the vertices can be partitioned into independent sets (the smallest number of colors that can be used to color the vertices so that no two vertices of the same color are joined by an edge).

We implicitly assume that \( G \) is connected, although it may not always be necessary.

**Theorem 2.2.** \( \alpha(G) \leq \min\{n - n^+, n - n^-\} \).

**Proof.** Since \( G \) has an independent set of vertices of cardinality \( \alpha(G) \), \( G \) has an induced subgraph \( H \) of order \( \alpha(G) \) with no edges, whose adjacency matrix is therefore \( O_{\alpha(G)} \). By the interlacing property S5,
\[ \lambda_{n - \alpha(G) + i} \leq \lambda_i(H) \leq \lambda_i(G) \quad (1 \leq i \leq \alpha(G)). \]
Thus
\[ \lambda_n(G) \leq 0 = \lambda_{\alpha(H)} \leq \lambda_\alpha(G), \]
and so \( n^- \leq n - \alpha(G) \), equivalently, \( \alpha(G) \leq n - n^- \). Using \(-A\) in place of \(A\), we also get \( \alpha(G) \leq n - n^+ \).

Since the eigenvalues of the complete graph \(K_n\) are \(-1, \ldots, -1, n - 1\) with \(n^- = n - 1\), equality holds in Theorem 2.2 for \(K_n\).

Let \(G\) be a graph of order \(n\) with vertex set \(V\). The complement of \(G\) is the graph \(\overline{G}\) on \(V\) in which two distinct vertices are joined by an edge if and only if they are not joined by an edge in \(G\). If \(A\) is the adjacency matrix of \(G\), then \(J_n - I_n - A\) is the adjacency matrix of \(\overline{G}\). A graph is regular of degree \(k\) provided that each vertex has the same degree \(k\). A regular graph of degree \(k\) has \(k\) as its largest eigenvalue with a corresponding eigenvector \(j_n^t = (1, 1, \ldots, 1)^t \in \mathbb{R}^n\). If \(G\) is regular of degree \(k\), then its complement \(\overline{G}\) is regular of degree \(n - 1 - k\). The eigenvalues of the complement of a regular graph can be easily obtained from the eigenvalues of the graph itself.

**Lemma 2.3.** Let \(G\) be a graph of order \(n\) which is regular of degree \(k\), and let the eigenvalues of \(G\) be \(\lambda_1, \ldots, \lambda_2, \lambda_1 = k\). Then the eigenvalues of \(\overline{G}\) are \(n - 1 - k, -1 - \lambda_2, \ldots, -1 - \lambda_n\).

**Proof.** Let \(A\) be an adjacency matrix of \(G\). There is an orthogonal matrix \(Q\) of eigenvectors of \(A\) with first column \(j_n^t\) such that
\[
Q^{-1}AQ = \text{diag}(k, \lambda_2, \ldots, \lambda_n).
\]
Then
\[
Q^{-1}(J_n - I_n - A)Q = Q^{-1}J_nQ - Q^{-1}I_nQ - Q^{-1}AQ = \text{diag}(n, 0, \ldots, 0) - I_n - \text{diag}(k, \lambda_2, \ldots, \lambda_n) = \text{diag}(n - 1 - k, -1 - \lambda_1, \ldots, -1 - \lambda_n).
\]
and the result follows.

A clique of the graph \(G\) is a subset of its vertices every pair of which are joined by an edge. Thus a clique is a subset of the vertices that induces a complete graph. The clique number of \(G\) is the maximum size \(\beta(G)\) of a clique. For each \(U \subseteq V\), \(U\) is a clique of \(G\) if and only if its complement \(\overline{U} = V \setminus U\) is an independent set of \(\overline{G}\). Thus \(\beta(G) = \alpha(\overline{G})\).

Let \(n^- = 1, n = -1, n^< = -1\) denote, respectively, the number of eigenvalues of \(G\) which are greater than \(-1\), equal to \(-1\), and less than \(-1\).

**Corollary 2.4.** Let \(G\) be a graph of order \(n\) which is regular of degree \(k\). If \(k = n - 1\), then \(G = K_n\) and \(\beta(G) = n\). If \(k \leq n - 2\), then
\[
\beta(G) \leq \min\{n - n^-> - 1, n - n^< - 1 + 1\}.
\]

**Proof.** By Lemma 2.3 the eigenvalues of \(\overline{G}\) are \(n - 1 - k, -1 - \lambda_2, \ldots, -1 - \lambda_n\). We have that \(n - 1 - k > 0\) since \(k \leq n - 2\). Also \(-1 - \lambda_i > 0\) if and only if \(\lambda_i < -1\), and \(-1 - \lambda_i < 0\) if and only if \(\lambda_i > -1\). The conclusion now follows.
An upper bound on the chromatic number in terms of eigenvalues was discovered by Wilf (1967). It is a consequence of the following lemma. Recall that the minimum degree of a vertex of a graph $G$ is denoted by $\delta(G)$.

**Lemma 2.5.** We have $\chi(G) \leq 1 + p$ where

$$p = \max\{\delta(H) : H \text{ an induced subgraph of } G\}.$$ 

**Proof.** We consider the vertices of $G$ in some order and color them sequentially using a greedy algorithm: we color the first vertex using any color, and when we come to the $k$th vertex, we choose a color that has not been used to color any of the first $k - 1$ vertices that are joined to the $k$th vertex by an edge. Thus, when coloring a vertex, what matters is the number of previously colored vertices joined to it. We show that there is an ordering of the vertices for which the greedy algorithm colors the vertices with at most $1 + p$ colors. Such an ordering is determined as follows:

- The graph $H_n = G$ has a vertex $x_n$ of degree at most $p$.
- The graph $H_{n-1} = G - x_n$ has a vertex $x_{n-1}$ of degree at most $p$.
- The graph $H_{n-2} = G - \{x_{n-1}, x_n\}$ has a vertex $x_{n-2}$ of degree at most $p$.
- 
- The graph $H_1 = G - \{x_2, \ldots, x_{n-1}, x_n\}$ has only one vertex $x_1$ and its degree is 0.

In this way we get an ordering $x_1, x_2, \ldots, x_n$ of the vertices of $G$ such that each vertex is joined by an edge to at most $p$ vertices that come before it. For this ordering, the greedy algorithm never needs more than $1 + p$ colors for a coloring of $G$. \hfill $\square$

**Theorem 2.6.** $\chi(G) \leq 1 + \lambda_1(G)$.

**Proof.** By PF7 and S5, we have

$$\delta(H) \leq \lambda_1(H) \leq \lambda_1(G)$$

for all induced subgraphs $H$ of $G$. By Lemma 2.5, $\chi(G) \leq 1 + \lambda_1(G)$. \hfill $\square$

Hoffman (1977) discovered a lower bound on the chromatic number in terms of the smallest and largest eigenvalue.

**Theorem 2.7.** $\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$ for $n \geq 2$.

**Proof.** First note that since $G$ is connected and $n \geq 2$, $G$ has at least one edge and $\lambda_n$ must be negative. Thus $-\lambda_1/\lambda_n$ is a positive number.

Suppose we have a coloring of $G$ that uses exactly $q$ colors. Then the vertex set $V$ can be partitioned into $q$ independent sets $V_1, V_2, \ldots, V_q$. Thus the adjacency matrix of $G$ can be taken in the form

$$A = \begin{bmatrix}
O_{m_1} & * & \cdots & * \\
* & O_{m_2} & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & O_{m_q}
\end{bmatrix}.$$
Let

\[ P = \begin{bmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\
\vdots & & \ddots & \ddots & & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 1 \\
\end{bmatrix} \]

be the \( q \times n \) characteristic matrix for this partition; thus for \( i = 1, 2, \ldots, q \), row \( i \) has \( m_i \) consecutive 1s in the positions as shown. Let \( z = (z_1, z_2, \ldots, z_n)^t \) with \( z^t z = 1 \) be a positive eigenvector of \( A \) for its largest eigenvalue \( \lambda_1 \). Let

\[ Q = P \text{ diag}(z_1, z_2, \ldots, z_n), \]

where \( \text{diag}(z_1, z_2, \ldots, z_n) \) is the \( n \times n \) diagonal matrix with diagonal entries \( z_1, z_2, \ldots, z_n \). The rows of \( P \) and \( Q \) are orthogonal and nonzero. Hence there is a diagonal matrix \( D = \text{diag}(a_1, a_2, \ldots, a_q) \) such that \( R = DQ \) has orthonormal rows. By the interlacing property S4,

\[ \lambda_{\max}(RAR^t) \leq \lambda_1. \]

Let \( y = (a_1, a_2, \ldots, a_q)^t \). Then \( y^t R = z^t \) and \( 1 = z^t z = (y^t R)(y^t R)^t = y^t R R^t y = y^t y \). Then \( y^t RAR^t y = z^t A z = \lambda_1 \), and so by property S2, \( \lambda_1 \leq \lambda_{\max}(RAR^t) \). Therefore

\[ \lambda_1 = \lambda_{\max}(RAR^t). \]

Thus

\[ (q-1)\lambda_{\min}(RAR^t) + \lambda_1 = (q-1)\lambda_{\min}(RAR^t) + \lambda_{\max}(RAR^t) \leq \text{trace}(RAR^t) = 0. \]

By the interlacing property S5,

\[ (q-1)\lambda_{\min}(RAR^t) + \lambda_1 \geq (q-1)\lambda_n + \lambda_1, \]

and we get that \( (q-1)\lambda_n + \lambda_1 \leq 0 \), and thus \( (\chi(G) - 1)\lambda_n + \lambda_1 \leq 0 \), as desired. \( \square \)

### 2.3. Graphs with small \( \lambda_{\max} \)

Since the largest eigenvalue of a connected graph is bounded from below by its average degree, one would expect that graphs with a small \( \lambda_{\max} \) are rare and have simple structure.

Consider a connected graph of order \( n \),

(1) \( \lambda_{\max} = 0 \) if and only if \( G = K_1 \) (the trivial graph with one vertex and no edges).

To see this one could use the average degree remark above, or note that if \( G \) had an edge, then

\[ \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix} \]

would be a principal submatrix of \( G \) with 1 as an eigenvalue and then use the interlacing property S5.
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(2) \( \lambda_{\text{max}} \leq 1 \) with equality if and only if \( G = K_2 \) (in which case \( \lambda_{\text{max}} = 1 \)).

If \( n \geq 3 \), then since \( G \) is connected, it has a path of length 2 and thus has
\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]
as a principal submatrix. For this matrix the largest eigenvalue is \( \sqrt{2} > 1 \). Now use again the interlacing property S5.

(3) \( 1 < \lambda_{\text{max}} \leq \sqrt{2} \) if and only if \( G \) is the complete bipartite graph \( K_{1,2} \), for which the largest eigenvalue is \( \sqrt{2} \).

See the argument for (2) above and use the interlacing property S5 again.

(4) There are many graphs with \( \lambda_{\text{max}} = 2 \):
   (a) The cycle \( C_n \) for \( n \geq 3 \).
   (b) The complete bipartite graph \( K_{1,4} \).
   (c) The path on 5 vertices with a path of length 2 attached to its middle vertex.
   (d) The path on 7 vertices with an edge attached to its middle vertex.
   (e) The path on 8 vertices with an edge attached to a vertex at distance 2 from one of its end vertices.
   (f) The graph obtained from two paths of length 2 (two copies of \( K_{1,2} \)) by joining their middle vertices by a path of length \( n - 5 \) (\( n \geq 6 \)).

The graphs in (a)-(f) are called Smith graphs. It is straightforward to check that all these graphs have \( \lambda_{\text{max}} = 2 \). As shown in the next theorem, they contain as induced subgraphs all graphs with \( \lambda_{\text{max}} \leq 2 \).

**Theorem 2.8.** The connected graphs with \( \lambda_{\text{max}} \leq 2 \) are precisely the connected, induced subgraphs of the Smith graphs.

**Proof.** We only give a rough outline of the proof. A connected graph \( G \) of order \( n \) can by iteratively constructed by starting with \( K_1 \) and adding a new vertex and one or more edges connected to old vertices. By interlacing S5, each new vertex will have to increase the \( \lambda_{\text{max}} \). Since \( \lambda_{\text{max}} \) is not to exceed 2, \( G \) is either a cycle \( C_n \) (so \( \lambda_{\text{max}} = 2 \)) or a tree. Since \( \lambda_{\text{max}}(K_{1,4}) = 2 \), no other tree can have a vertex of degree 4 or more. If the largest degree of a vertex is 2, then \( G \) is a path and therefore an induced subgraph of a cycle \( C_n \). Thus the only remaining case to handle is a tree with largest degree equal to 3. If \( G \) has more than 1 vertex of degree 3, then \( G \) must be the graph in item (4)(f) with \( \lambda_{\text{max}} = 2 \). Otherwise \( G \) is a tree with a unique vertex of degree 3, and thus consists of three paths at a common vertex. If all these paths have length 2 or more, then \( G \) must be the graph in item...
Thus one of the paths has length 1. If the other two paths have length 3 or more, then $G$ must be the graph in item (4)(d). Otherwise, one of the paths has length less than 3, and $G$ is an induced subgraph of the graph in items (4)(d) and (4)(e).

2.4. Laplacian Matrix of a Graph

A graph can also be described by another matrix whose linear algebraic properties hold combinatorial information about the graph. This matrix has also been extensively investigated.

Again let $G$ be a graph of order $n$ with vertices taken to be $1, 2, \ldots, n$. Let the number of edges of $G$ be $m$ and assume the edges have been labeled as $e_1, e_2, \ldots, e_m$. The vertex-edge incidence matrix of $G$ is the $n \times m$ $(0,1)$-matrix $B$ whose $(i,j)$-entry is 1 if vertex $i$ is a vertex of the edge $e_j$ $(1 \leq i \leq n, 1 \leq j \leq m)$. Thus the number of 1s in each column of $B$ is 2, and the number of 1s in a row is the degree of the corresponding vertex. A signed vertex-edge incidence matrix of $G$ is the matrix $C$ obtained from $B$ by replacing one of the 1s in each column with a $-1$. (Obviously $C$ is not unique in general. One can think of orienting each edge and then assigning a $+1$ to the initial vertex of the edge and a $-1$ to the terminal vertex.) Both $B$ and $C$ determine $G$ up to the labeling of vertices and edges.

The Laplacian matrix of $G$ is the $n \times n$ symmetric matrix $L = D - A$ where $A$ is the adjacency matrix of $G$ and $D$ is the diagonal matrix $\text{diag}(d_1, d_2, \ldots, d_n)$ whose diagonal entries are the degrees of the corresponding vertices. Since all row sums of $L$ equal 0, 0 is an eigenvalue of $L$ with corresponding eigenvector $j_n^t = (1, 1, \ldots, 1)$. We have

$$L = CC^t,$$

and thus $L$ is a singular, positive semi-definite, symmetric matrix. Let the eigenvalues of $L$ be

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0.$$

The following is a classical result.

**Theorem 2.9.** The determinant of each $n-1 \times n-1$ submatrix of the Laplacian matrix of $G$ has absolute value equal to the number $\tau(G)$ of spanning trees of $G$; in fact, the determinant of the $n-1 \times n-1$ submatrix obtained from $L$ by deleting row $i$ and column $j$ equals $(-1)^{i+j}\tau(G)$.

**Proof.** Here is a brief outline of a proof. First suppose that $G$ is not connected. Then $\tau(G) = 0$. Also $L$ is the direct sum of two matrices each of which has 0 as an eigenvalue and is singular. Thus the rank of $L$ is at most $n-2$, and each $n-1 \times n-1$ submatrix has a zero determinant. Now suppose that $G$ is connected. Then the rank of $L$ can be shown to equal $n-1$, and thus the dimension of the nullspace of $L$ is 1. Since $j_n = (1, 1, \ldots, 1)^t$ is in the nullspace, the null space is spanned by $j_n$. Since $L \text{adj}(L) = \det(L)I_n = O$, each column of the adjugate $\text{adj}(L)$ of $L$ is a constant vector. Since $L$ is
symmetric, so is \( \text{adj}(L) \) and hence \( \text{adj}(L) \) is a constant multiple of \( J_n \), the \( n \times n \) matrix of all 1s. It remains to show that this constant is \( \tau(G) \). To do this, we have only to show that one \( n - 1 \times n - 1 \) submatrix of \( L \) has determinant equal to \( \tau(G) \). Applying the Cauchy-Binet theorem for the determinant of the product of two rectangular matrices to a \( n - 1 \times n - 1 \) principal submatrix of \( L = CC^t \), we obtain \( \kappa(G) \).

Examining the coefficient of \( \lambda \) in the characteristic polynomial of \( L \), we see that

\[
\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i.
\]

Let \( \mu(G) = \mu_{n-1} \), the second smallest eigenvalue of the Laplacian matrix of the graph \( G \). Fiedler called \( \mu(G) \) the algebraic connectivity of \( G \). By Theorem 2.9, the algebraic connectivity of \( G \) is positive if and only if \( G \) is connected. The vertex connectivity \( \nu(G) \) of \( G \) is the smallest number of vertices whose removal results in a disconnected graph or a single vertex. Let \( G = K_n \). Then \( \nu(K_n) = n - 1 \) and \( \mu(K_n) = n \) since the eigenvalues of its Laplacian matrix \((n - 1)I_n - (J_n - I_n) = nI_n - J_n \) are \( n, \ldots, n, 0 \).

The next theorem shows that the algebraic connectivity furnishes a lower bound for the vertex connectivity. We omit the proof.

**Theorem 2.10.** Let \( G \) be a graph of order \( n \) different from \( K_n \). Then

\[
\mu(G) \leq \nu(G).
\]

The edge connectivity of graph \( G \) is the smallest number \( \epsilon(G) \) of edges whose removal leaves a disconnected graph. The edge connectivity of a graph with a single vertex is 0. Since the edge connectivity is at least as large as the vertex connectivity, the algebraic connectivity gives a lower bound on the edge connectivity too.

The algebraic connectivity can also be used to give information about the expansive properties of a graph.

**Theorem 2.11.** Let \( G \) be a graph of order \( n \) with vertex set \( V \). Then for \( U \subseteq V \), the number of edges between \( U \) and its complement \( V \setminus U \) is at least

\[
\mu(G) \frac{|U||V \setminus U|}{n}.
\]

**Proof.** If \( U = \emptyset \) or \( V \), then the theorem holds trivially. Otherwise, let \( x = (x_1, x_2, \ldots, x_n)^t \) be defined by

\[
x_i = \begin{cases} 
  n - k & i \in U \\
  -k & i \in V \setminus U.
\end{cases}
\]
Then $x^t j_n = 0$ and $xx^t = kn(n - k)$. Hence

$$x^t L x = \sum_{ij \text{ an edge}} (x_i - x_j)^2$$

$$= (\text{number of edges between } U \text{ and } V \setminus U) \cdot n^2.$$

It follows from property S3 that

$$\mu(G) \leq \frac{x^t L x}{x^t x} \leq \frac{\text{the number of edges between } U \text{ and } V \setminus U n^2}{kn(n - k)}.$$

Thus the number of edges between $U$ and $V \setminus U$ is at least $\mu(G) \frac{k(n-k)}{n}$, $\Box$

Two good general graph theory references are [1, 2]. Most of the results in this chapter can be found in the books [1, 3, 4]. The spectra of a graph is thoroughly investigated in [4].

Bibliography