Zeta and $L$-functions of complexes

8.1. The building attached to $PGL_n(F)$

Throughout this lecture $F$ denotes a nonarchimedean local field with $q$ elements in its residue field. Such a field is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((x))$, where $q$ is a power of $p$. Denote by $\mathcal{O}_F$ the ring of integers of $F$ and by $\pi$ a uniformizer of $\mathcal{O}_F$, thus the residue field $\mathcal{O}_F/\pi\mathcal{O}_F$ is isomorphic to $\mathbb{F}_q$. Similar to $PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)$ discussed in §4, Lecture 6, each coset $gPGL_n(\mathcal{O}_F)$ in $PGL_n(F)/PGL_n(\mathcal{O}_F)$ can be interpreted as an equivalence class of rank-$n$ lattices (i.e., $\mathcal{O}_F$-modules) $[L_q]$ in $F^n$. They are the vertices of the building $B_n(F)$ attached to $PGL_n(F)$. Two vertices $v_1$ and $v_2$ are adjacent if they can be represented by lattices $L_1$ and $L_2$ such that $L_1 \supseteq L_2 \supseteq \pi L_1 \supseteq \pi L_2$, as before. Note that $L_1/\pi L_1$ is isomorphic to $(\mathbb{F}_q)^n$ and $L_2/\pi L_1$ is a proper nonzero subspace of $\mathbb{F}_q^n$. Thus a maximal set of mutually adjacent vertices has cardinality $n$; they form an $(n - 1)$-dimensional simplicial complex, called a chamber. In other words, vertices $v_1, ..., v_n$ form a chamber if they can be represented by lattices $L_1, ..., L_n$ such that

$$L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n \supseteq \pi L_1,$$

or equivalently,

$$0 \subsetneq L_n/\pi L_1 \subsetneq L_{n-1}/\pi L_1 \subsetneq \cdots \subsetneq L_2/\pi L_1 \subsetneq L_1/\pi L_1 = (\mathbb{F}_q)^n$$

is a complete flag of $\mathbb{F}_q$-subspaces in $(\mathbb{F}_q)^n$. For example, the standard chamber has vertices $[L_{q_i}]$ for $0 \leq i \leq n - 1$, where $g_i$ is the diagonal matrix $\text{diag}(1, ..., 1, \pi, ..., \pi)$ of $\det g_i = \pi^i$. The chambers sharing a fixed vertex correspond bijectively to the complete flags in $(\mathbb{F}_q)^n$. The building $B_n(F)$, as the union of the chambers, is a contractible $(n - 1)$-dimensional simplicial complex. For $0 \leq j \leq n - 1$, the $j$-dimensional facets of the building $B_n$ consist of the $j$-dimensional facets of chambers. The group $PGL_n(F)$ acts on $B_n(F)$ by left translation, preserving $j$-dimensional facets for all $0 \leq j \leq n - 1$. Hence we may take discrete torsion-free cocompact subgroups $\Gamma$ of $PGL_n(F)$ with $\text{ord}_p \det \Gamma \subset n\mathbb{Z}$ and obtain finite $(n - 1)$-dimensional complexes $X_\Gamma = \Gamma \backslash B_n(F)$ as before.

Each vertex $[L_q]$ has a type, given by $\text{ord}_\pi \det g \mod n$. Adjacent vertices have different types. In particular, the vertices in a chamber exhaust all possible types exactly once. The 1-skeleton of $B_n(F)$ is the graph consisting of the vertices and edges in the building; the vertices are partitioned according to their types and the adjacency relation makes it an $n$-partite graph. There are $n - 1$ Hecke operators $A_i, i = 1, ..., n - 1$, based on the double cosets $PGL_n(\mathcal{O}_F)g_iPGL_n(\mathcal{O}_F)$; the action of $A_i$ on $f \in L^2(PGL_n(F)/PGL_n(\mathcal{O}_F))$ is given by

$$A_i f(x) = \sum_{y \text{ adjacent to } x, \text{ type } y = (\text{type } x) + i \mod n} f(y).$$
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Equivalently, we may express the double coset $PGL_n(\mathcal{O}_F)g_iPGL_n(\mathcal{O}_F)$ as a disjoint union $\cup_{j\in I}g_{i,j}PGL_n(\mathcal{O}_F)$ of right $PGL_n(\mathcal{O}_F)$-cosets indexed by a finite set $I_i$, then

$$A_if(gPGL_n(\mathcal{O}_F)) = \sum_{j\in I_i} f(gg_{i,j}PGL_n(\mathcal{O}_F)),$$

similar to what happened for the case $n = 2$ discussed in §4, Lecture 6. The operators $A_i$ and $A_{n-i}$ are transpose of each other, and the $A_i$’s commute. Consequently the induced operators $A_i$ on a finite quotient $X_F$ are simultaneously diagonalizable.

Each $(n-2)$-dimensional simplex is contained in $q+1$ chambers. Each vertex $x$ of $B_n(F)$ has

$$\sigma_{n,i} = \frac{(q^n - 1) \cdots (q - 1)}{(q^{i} - 1) \cdots (q - 1)(q^{n-i} - 1) \cdots (q - 1)}$$

neighbors of type equal to $(\text{type } x) + i \mod n$. We say that $B_n(F)$ is $(q+1)$-regular. Topologically the building $B_n(F)$ is simply connected, hence it serves as the universal cover of its finite left quotients, which are finite simplicial complexes of dimension $n - 1$.

8.2. Spectral theory of regular complexes from $B_n(F)$

MacDonald [70, 71] proved that, for $1 \leq \ell \leq n-1$, the spectrum of $A_{\ell}$ on $B_n(F)$ is $q^{\ell(n-\ell)}/2\Omega_{n,\ell}$, where

$$\Omega_{n,\ell} = \{ \sigma_{\ell}(z_1, ..., z_n) : z_1, ..., z_n \in S^1, z_1 \cdots z_n = 1 \}$$

and $\sigma_{\ell}(z_1, ..., z_n) = \sum_{1 \leq i_1 < ... < i_\ell \leq n} z_{i_1} \cdots z_{i_\ell}$ is the $\ell$th elementary symmetric polynomial in $n$-variables. Note that $\Omega_{2,1}$ is the interval $[-2, 2]$. For $n \geq 3$, the spectrum $\Omega_{n,\ell}$ of $A_{\ell}$ is no longer real; it is a region in the complex plane invariant under multiplication by the $n^{th}$ roots of unity.

The following Alon-Boppana type theorem holds for finite (left) quotients of the building $B_n(F)$. Let $\{X_j\}$ be a family of $(q+1)$-regular finite quotients of $B_n(F)$ with $|X_j| \to \infty$ as $j \to \infty$. The operators $A_{\ell}$ for $1 \leq \ell \leq n-1$ defined above act on functions on vertices of $X_j$. The trivial eigenvalues of $A_{\ell}$ are $q_{n,\ell}^{2\pi i r/n}$ for $r = 1, ..., n$.

**Theorem 8.1** (Li [62]). Assume that each $X_j$ contains a ball isomorphic to a ball in $B_n(F)$ with radius approaching $\infty$ as $j \to \infty$. Then, for each $1 \leq \ell \leq n-1$, the closure of the collection of eigenvalues of $A_{\ell}(X_j)$ for all $j \geq 1$ contains the region $q^{\ell(n-\ell)/2}\Omega_{n,\ell}$.

When $n = 2$, this reduces to the Alon-Boppana Theorem 5.1 and its counterpart Theorem 5.4 for graphs.

8.3. Ramanujan complexes as finite quotients of $B_n(F)$

In view of the definition of Ramanujan graphs for regular and bi-regular bipartite graphs as discussed in Lectures 5 and 6, a finite quotient $X$ of $B_n(F)$ is called a Ramanujan complex if, for $1 \leq \ell \leq n-1$, all nontrivial eigenvalues of $A_{\ell}$ on $X$ fall in $q^{\ell(n-\ell)/2}\Omega_{n,\ell}$, the spectrum of $A_{\ell}$ on $B_n(F)$. Theorem 8.1 says that a Ramanujan complex is spectrally extremal. Like graphs, there are explicitly constructed families of Ramanujan complexes.
Theorem 8.2. For \( q \) equal to a prime power and \( n \geq 2 \), there exist explicitly constructed infinite families of \((q+1)\)-regular \((n-1)\)-dimensional Ramanujan complexes arising as quotients of \( B_n(F) \).

The explicit constructions rely on the Ramanujan conjecture established for certain representations of \( GL_n \) or the multiplicative group of division algebras over function fields. We sketch the basic approach, which is similar to the explicit construction for Ramanujan graphs by Lubotzky-Phillips-Sarnak explained in §5, Lecture 6. Let \( K = \mathbb{F}_q(x) \) be a rational function field such that \( F = K_v \) is the completion of \( K \) at a degree-1 place \( v \) not equal to the place at infinity. Let \( H \) be a central simple division algebra of dimension \( n^2 \) over \( K \), totally ramified at \( \infty \) and unramified at \( v \). Put \( D = H^\times/\text{center} \). Then \( D(K_v) \cong PGL_n(F) \) so that the building \( B_n(F) \) is isomorphic to \( D(K_v)/D(O_v) \). By strong approximation theorem, for a compact open subgroup \( \mathcal{K} \) of \( \prod_{w \notin v, \infty} D(O_w) \), the double coset space

\[
X_{\mathcal{K}} = D(K) \backslash D(A_K)/D(K_\infty)D(O_v)\mathcal{K}
\]

can be expressed locally at \( v \) as

\[
X_{\mathcal{K}} = \Gamma_{\mathcal{K}} \backslash D(K_v)/D(O_v) = \Gamma_{\mathcal{K}} \backslash B_n(F).
\]

Here \( \Gamma_{\mathcal{K}} = D(K) \cap \mathcal{K} \) is a discrete cocompact subgroup of \( B_n(F) \). Shrinking \( \mathcal{K} \) if necessary we may also assume \( \Gamma_{\mathcal{K}} \) is torsion-free and \( \text{ord}_{\mathcal{K}} \det \Gamma_{\mathcal{K}} \equiv 0 \mod n \) so that it preserves the type of each vertex. As such, \( X_{\mathcal{K}} \) is a finite \((q+1)\)-regular \((n-1)\)-dimensional simplicial complex.

The space of functions on vertices of \( X_{\mathcal{K}} \) are automorphic forms on \( D(A_K) \) right invariant by \( D(K_v)D(O_v)\mathcal{K} \). It contains the constant functions and their twists by \( n \)th roots of unity. They are common eigenfunctions of \( A_\ell \) for \( 1 \leq \ell \leq n-1 \) with trivial eigenvalues \( q_n \cdot e^{2\pi ir/n} \) for \( r = 1, \ldots, n \). By Jacquet-Langlands correspondence the orthogonal complement \( V_\mathcal{K} \) of these functions can be interpreted as certain automorphic forms on \( PGL_n(A_K) \) with the same eigenvalues for the Hecke operators \( A_\ell \) for \( 1 \leq \ell \leq n-1 \). As Lafforgue [55] proved the Ramanujan conjecture for cuspidal representations for \( GL_n(A_K) \), so we just have to choose appropriate \( \mathcal{K} \) so that only cuspidal automorphic forms occur in \( V_\mathcal{K} \) in order that \( X_{\mathcal{K}} \) is Ramanujan. By varying the congruence subgroups \( \mathcal{K} \), an infinite family of \((n-1)\)-dimensional Ramanujan complexes is obtained.

It should be pointed out that, when \( n = 2 \), the building \( B_2(F) \) is a \((q+1)\)-regular tree and any choice of a compact open subgroup \( \mathcal{K} \) of \( \prod_{w \notin v, \infty} D(O_w) \) such that \( \Gamma_{\mathcal{K}} \) is torsion-free gives rise to a Ramanujan graph. For \( n \geq 3 \), Lubotzky, Samuels and Vishne [69] showed that the same method yields Ramanujan complexes if \( n \) is a prime. When \( n \) is a composite, more care is required in order to avoid the residual spectrum of \( PGL_n \) to occur in the space \( V_\mathcal{K} \) since such representations do not satisfy the Ramanujan conjecture. They showed that there are indeed infinitely many \( \mathcal{K} \) giving rise to complexes that are not Ramanujan.

There are three explicit constructions of Ramanujan complexes in the literature, by Li [62], Lubotzky-Samuels-Vishne [68], and Sarveniazi [86] from 2004 to 2007. At the time when their works were done, the Jacquet-Langlands correspondence for the multiplicative groups of division algebras of dimension \( n^2 \) over function fields was established only for primes \( n \). The proof of the correspondence for all \( n \) appeared only in 2016 by Badulescu and Roche [7]. Li’s construction [62] avoided this problem by using the results of Laumon-Rapoport-Stuhler [59],
where the Ramanujan conjecture was shown to hold for automorphic representations of \( D(k_K) \) having a local component at a place \( w \neq v, \infty \) being Steinberg. Her construction did not use Lafforgue’s result [55] either, but required the division algebras to ramify at least at 4 places. The other two constructions assumed the validity of the Jacquet-Langlands correspondence and used Lafforgue’s result, but could be applied to division algebras totally ramified only at two places. The method in [68] closely parallels that in [67] by finding a group to represent the vertices of \( B_n(F) \). So did [86], which established this fact in a different and elegant way. To show variations of viewpoints, we sketch below Sarveniazi’s approach [86].

Denote by \( H \) the simple central division algebra over \( K = \mathbb{F}_q(t) \) of dimension \( n^2 \) so that

\[
H(K) = \mathbb{F}_{q^n}(t) + \mathbb{F}_{q^n}(t)\tau + \cdots + \mathbb{F}_{q^n}(t)\tau^{n-1},
\]

where \( \tau^n = t \) and \( \tau\alpha = \alpha^n\tau \) for \( \alpha \in \mathbb{F}_{q^n} \). Then \( H \) is totally ramified at \( t = \infty \) with invariant \(-1/n\) and at \( t = 0 \) with invariant \( 1/n \), and unramified elsewhere.

The group of norm 1 elements in \( \mathbb{F}_{q^n} \) is

\[
N_1 = \{ \alpha \in \mathbb{F}_{q^n} : N_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\alpha) = 1 \} = \{ \beta^{q^{-1}} : \beta \in \mathbb{F}_{q^n}^\times \}. \]

Here the second expression follows from the Hilbert Theorem 90. It has cardinality

\[
|N_1| = \frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + \cdots + q + 1.
\]

Observe that, up to multiplication by elements in \( N_1 \), \( 1 - \alpha \tau \) with \( \alpha \in N_1 \) are the elements in \( H(K) \) of reduced norm \( 1 - t \).

Think of \( \tau \) as the Frobenius automorphism on \( \mathbb{F}_{q^n} \) sending \( x \) to \( x^q \) and regard a polynomial \( \sum_{0 \leq i \leq n-1} a_i \tau^i \) with \( a_i \in \mathbb{F}_{q^n} \) as an \( \mathbb{F}_q \)-linear endomorphism on \( \mathbb{F}_{q^n} \) which sends \( x \) to \( \sum_{0 \leq i \leq n-1} a_i x^q \). View \( \mathbb{F}_{q^n} \) as an \( n \)-dimensional vector space over \( \mathbb{F}_q \). Given a complete flag of \( \mathbb{F}_q \)-vector spaces

\[
0 \subset <v_1> \subset <v_1, v_2> \subset \cdots <v_1, v_2, ..., v_n> = \mathbb{F}_{q^n},
\]

we shall find unique elements \( 1 - \alpha_1 \tau, 1 - \alpha_2 \tau, ..., 1 - \alpha_n \tau \) with \( \alpha_i \in N_1 \) such that for \( 1 \leq i \leq n \),

\[
<v_1, ..., v_i> \text{ is the kernel of } (1 - \alpha_i \tau) \cdots (1 - \alpha_1 \tau) \text{ on } \mathbb{F}_{q^n}.
\]

First we note that for any \( \beta^{1-q} \in N_1 \), the kernel of \( 1 - \beta^{1-q} \tau \) is 1-dimensional, equal to \( <\beta> \). This is because \( x \in \mathbb{F}_{q^n}^\times \) is such that \( (1 - \beta^{1-q} \tau)(x) = x - \beta^{1-q} x^q = 0 \) if and only if \( x^{q-1} = \beta^{-1} \), which means that \( x \) and \( \beta \) differ by a multiple in \( \mathbb{F}_{q^n}^\times \). Hence we choose \( \alpha_1 = v_1^{-1} \) so that \( <v_1> \) is the kernel of \( 1 - \alpha_1 \tau \). The image of \( v_2 \) under \( 1 - \alpha_1 \tau \) is \( v_2 - \alpha_1 v_2 = \beta_2 \) a nonzero element in \( \mathbb{F}_{q^n} \). Letting \( \alpha_2 = \beta_2^{1-q} \), we see that \( v_2 \) is annihilated by \( (1 - \alpha_2 \tau)(1 - \alpha_1 \tau) \) so that the kernel of \( (1 - \alpha_2 \tau)(1 - \alpha_1 \tau) \) is precisely \( <v_1, v_2> \). Inductively, suppose we have found \( \alpha_1, ..., \alpha_i \) in \( N_1 \) so that the kernel of \( (1 - \alpha_i \tau) \cdots (1 - \alpha_1 \tau) \) is \( <v_1, ..., v_i> \) for \( 1 \leq i < N \), then \( \alpha_{i+1} = \beta_{i+1}^{1-q} \) with \( \beta_{i+1} = (1 - \alpha_i \tau) \cdots (1 - \alpha_1 \tau)(v_{i+1}) \) has the desired property that the kernel of \( (1 - \alpha_{i+1} \tau) \cdots (1 - \alpha_1 \tau) \) is \( <v_1, ..., v_{i+1}> \). Note that the \( \alpha_i \)'s are uniquely determined by the complete flag.

Since the product \( (1 - \alpha_n \tau) \cdots (1 - \alpha_1 \tau) = 1 + \cdots + a_n \tau^n \) is a polynomial in \( \tau \) of degree \( n \) which is the zero map on \( \mathbb{F}_{q^n} \), and so is \( 1 - \tau^n = 1 - t \). The difference of the two, if nonzero, would yield a nontrivial polynomial relation of degree \( \leq n - 1 \)
in the Frobenius of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, contradicting the linear independence of $1, \tau, \ldots, \tau^{n-1}$. Hence we have
\[
(1 - \alpha_n \tau) \cdots (1 - \alpha_1 \tau) = 1 - t.
\]
This shows that a complete flag of $\mathbb{F}_{q^n}$ gives rise to a unique factorization of $1 - t$ as a product of $n$ terms of the form $1 - \alpha \tau$ with $\alpha \in N_1$. Conversely, any such factorization yields a complete flag with $<v_1, \ldots, v_i>$ being the kernel of $(1 - \alpha_i \tau) \cdots (1 - \alpha_1 \tau)$. Hence there is a bijection between the complete flags of $\mathbb{F}_{q^n}$ and the factorizations of $1 - t$ as a product of $n$ terms of the form $1 - \alpha \tau$ with $\alpha \in N_1$.

Now let $F$ be the completion of $K = \mathbb{F}_q(t)$ at the place $t = 1$. Put $D = H^\times/center$. As $H$ is unramified at $t = 1$, we have $D(F) \cong PGL_n(F)$ and the building $B_n(F) = D(F)/D(O_F)$. Let $S$ consist of $1 - \alpha_1 \tau, (1 - \alpha_2 \tau)(1 - \alpha_1 \tau), \ldots, (1 - \alpha_{n-1} \tau) \cdots (1 - \alpha_1 \tau)$, where all $\alpha_i$ are in $N_1$, and each element of $S$ divides $1 - t$.

To show that $S$ is symmetric, it suffices to prove that given $u = (1 - \alpha_i \tau) \cdots (1 - \alpha_1 \tau)$ in $S$ there is an element $v = (1 - \alpha_n \tau) \cdots (1 - \alpha_{i+1} \tau)$ in $S$ such that the product $vu$ is $1 - t$. Since $1 - t$ lies in the center of $H$, this would imply $v$ and $u$ are inverse to each other in $D(F)$. To find $v$, write $<v_1 >= ker(1 - \alpha_1 \tau), <v_1, v_2 >= ker(1 - \alpha_2 \tau)(1 - \alpha_1 \tau), \ldots, <v_1, \ldots, v_i >= ker u$ as we did before. This gives a partial flag $0 < v_1 < v_1, v_2 < v_1, \ldots, v_i > \in \mathbb{F}_{q^n}$, which we complete by adding a choice of subspaces
\[
<v_1, \ldots, v_i, v_{i+1} > < v_1, \ldots, v_{i+1}, v_{i+2} > < \cdots < v_1, \ldots, v_n > \subseteq \mathbb{F}_{q^n}.
\]
As shown before, there are unique $\alpha_{i+1}, \ldots, \alpha_n$ in $N_1$ such that this complete flag corresponds to the factorization
\[
(1 - \alpha_n \tau) \cdots (1 - \alpha_{i+1} \tau)(1 - \alpha_i \tau) \cdots (1 - \alpha_1 \tau) = 1 - t.
\]
Then $v = (1 - \alpha_n \tau) \cdots (1 - \alpha_{i+1} \tau)$ lies in $S$ and $vu = 1 - t$, as desired.

Let $\Lambda$ be the group generated by $S$. Sarveniazi showed that the 1-skeleton of $B_n(F)$ is the Cayley graph $Cay(\Lambda, S)$. To see this, recall that all chambers containing a fixed vertex $x$ correspond bijectively to the complete flags in $\mathbb{F}_{q^n}$. By viewing $\mathbb{F}_{q^n}$ as an $n$-dimensional vector space over $\mathbb{F}_q$, these chambers correspond bijectively to the factorizations of $1 - t$ as a product of $n$ factors of the form $1 - \alpha \tau$ with $\alpha \in N_1$, as observed before. In other words, given a factorization
\[
(1 - \alpha_n \tau)(1 - \alpha_{n-1} \tau) \cdots (1 - \alpha_1 \tau) = 1 - t,
\]
we obtain a chamber with vertices $x_1 = x, x_2 = (1 - \alpha_1 \tau)x_1, x_3 = (1 - \alpha_2 \tau)(1 - \alpha_1 \tau)x_1 = (1 - \alpha_2 \tau)x_2, \ldots, x_n = (1 - \alpha_{n-1} \tau)x_{n-1}, \text{and } x_1 = (1 - \alpha_n \tau)x_n$.

In this setting the Hecke operators $A_\ell, 1 \leq \ell \leq n - 1$, can be neatly expressed as sending $f \in L^2(\Lambda)$ to
\[
A_\ell f(x) = \sum f((1 - \alpha_\ell \tau) \cdots (1 - \alpha_1 \tau)x),
\]
where the sum is over all elements $(1 - \alpha_\ell \tau) \cdots (1 - \alpha_1 \tau)$ in $S$ which have reduced norm $(1 - t)^\ell$.

Sarveniazi constructed infinite families of finite $(q + 1)$-regular Ramanujan complexes by taking the building modulo varying monic irreducible polynomials $f(t) \in \mathbb{F}_q[t]$ of degree $d$ and coprime to $t$ and $t - 1$. To simplify our exposition, assume $n$ divides $d$ so that the residue field of $f(t)$ contains the field $\mathbb{F}_{q^n}$. (This resembles the choice of $q \equiv 1 \mod 4$ in the construction of Ramanujan graphs by
Lubotzky-Phillips-Sarnak discussed in §5, Lecture 6.) In other words, the 1-skeleton of the finite \((n-1)\)-dimensional Ramanujan simplicial complex is

\[
Cay(\Lambda \mod f(t), \mod S f(t)).
\]

Like in [67], he further identified the vertex set as \(PGL_n(F_{q^d})\) if \(1-t\) is not an \(r\)th power modulo \(f(t)\) for any \(r\) and \(r > 1\), and \(PSL_n(F_{q^d})\) if \(1-t\) is an \(n\)th power modulo \(f(t)\). If \(1-t\) is congruent to an \(r\)th power modulo \((t)^f\) for some \(r\) and \(r > 1\), then the group is an index-\(r\) subgroup of \(PGL_n(F_{q^d})\) containing \(PSL_n(F_{q^d})\). The element \(1-\alpha \tau\) in \(S\) is identified with the following \(n \times n\) matrix:

\[
\begin{pmatrix}
1 & -\alpha & 1 & -\alpha^q \\
1 & 1 & -\alpha^q & \\
& & \ddots & \\
-\alpha^{q^{n-1}} & & & \\
\end{pmatrix} \in PGL_n(F_{q^d}).
\]

The combinatorial meaning and possible applications of Ramanujan complexes have been active research topics in recent years. For instance, it was discovered that explicit constructions of Ramanujan complexes also lead to optimal logical gates for quantum computers on a single qubit [85].

### 8.4. Zeta and \(L\)-functions of finite quotients of \(B_3(F)\)

Let \(\Gamma\) be a discrete torsion-free cocompact subgroup of \(PGL_3(F)\) such that \(\text{ord}_\pi \det \Gamma \equiv 0 \mod 3\). Then left translation by \(\Gamma\) on the building \(B_3(F)\) preserves the types of vertices. The quotient

\[
X_\Gamma = \Gamma \backslash B_3(F) = \Gamma \backslash PGL_3(F) / PGL_3(O_F)
\]

is a finite \((q+1)\)-regular 2-dimensional simplicial complex. Our goal is to define the zeta function \(Z(X_\Gamma, u)\) of \(X_\Gamma\) such that it has the following nice properties possessed by zeta functions of graphs:

1. It is a rational function with a closed form expression;
2. It provides topological and spectral information of \(X_\Gamma\);
3. It satisfies the Riemann Hypothesis if and only if \(X_\Gamma\) is Ramanujan.

The first task is to decide what kind of “closed geodesics” to count. The 2-dimensional building \(B_3(F)\) is obtained by gluing together its “apartments” along certain chambers. Each apartment is a Euclidean plane tiled by equilateral triangles, which are chambers. The Euclidean metric on each apartment in turn yields the metric on the building \(B_3(F)\) and thus on the quotient \(X_\Gamma\). Recall that adjacent vertices have different types. The type of a directed edge from vertex \(x\) to vertex \(y\) is defined to be \(\text{type}(y) - \text{type}(x)\). A directed path using edges of the same type \(i\) is
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called a path of type $i$; the same path traveled in opposite direction has type $3-i$. Out of a vertex in an apartment there are three directed edges of type 1 (bold in the left figure) and three of type 2 (bold in the right figure) as depicted below:

A geodesic cycle $C$ of $X_\Gamma$ is a closed 1-dimensional geodesic path in $X_\Gamma$. It has a starting vertex and orientation. It lifts to a geodesic path $C'$ starting at a vertex in $B_3(F)$, which can be shown to lie in an apartment $A$ of $B_3(F)$. Hence it is a straight line in $A$ starting at a vertex $v$ and ending at a vertex $\gamma v$ for some non-identity element $\gamma \in \Gamma$. To achieve our goals, it turns out that we should only consider those $C$ contained in the 1-skeleton of $X_\Gamma$ so that $C'$ is a straight line in $A$ using edges of the same type $i \in \{1, 2\}$. Furthermore, a geodesic cycle $C$ is required to remain a geodesic cycle after changing the starting vertex. This is equivalent to $v$, $\gamma v$ and $\gamma^2 v$ being collinear in $A$. Such a $C$ is called a geodesic cycle of type $i$; the same cycle traveled in opposite direction is a geodesic cycle of type $3-i$. As before, $C$ is primitive if it is not obtained by repeating a shorter geodesic cycle more than once. Two geodesic cycles are equivalent if they differ by starting points. Denote by $[C]$ the equivalence class of $C$. As before, the primes of $X_\Gamma$ are equivalence classes of primitive geodesic cycles. The geometric length $\ell(C)$ of $C$ is the number of edges in $C$.

For $n \geq 1$, let $N_n(X_\Gamma)$ denote the number of geodesic cycles in $X_\Gamma$ of type 1 and geometric length $n$; it is also equal to that of type 2. For $i = 1, 2$ the $i$ edge zeta function of $X_\Gamma$ is defined as

$$Z_{1,i}(X_\Gamma, u) = \exp \left( \sum_{n=1}^{\infty} \frac{N_n(X_\Gamma)}{n} u^n \right) = \prod_{[C] \text{ prime of type } i} \frac{1}{1 - u^{\ell(C)}}.$$

Combined, they define the (edge) zeta function for $X_\Gamma$:

$$Z(X_\Gamma, u) = Z_{1,1}(X_\Gamma, u)Z_{1,2}(X_\Gamma, u^2) = \prod_{i=1}^{2} \prod_{[C] \text{ prime of type } i} \frac{1}{1 - u^{\ell(C)i}}.$$

Given a type 1 edge $v_1 \rightarrow v_2$, its type 1 neighbors are defined to be the type 1 edges $v_2 \rightarrow v_3$ in $B_3(F)$ such that the vertices $v_1, v_2, v_3$ do not form a chamber. Similar to Theorem 4.3, $Z_{1,1}(X_\Gamma, u)^{-1}$ is a polynomial in $u$ which can be expressed in terms of the type 1 edge adjacency matrix $T_E$ of $X_\Gamma$:

$$Z_{1,1}(X_\Gamma, u) = \frac{1}{\det(I - T_E u)}.$$
The stabilizer of the type 1 edge from \([L_1] \) to \([L_{\text{diag}(1,1,\pi)}] \) in the standard chamber is the parahoric subgroup

\[
E = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \text{PGL}_3(\mathcal{O}_F) : \pi|g, \pi|h \right\}.
\]

Thus the type 1 edges in \(B_3(F) \) are parameterized by \(\text{PGL}_3(F)/E \). The operator \(T_E \) can be expressed as a parahoric Hecke operator based on the \(E\)-double coset \(E \text{diag}(1,1,\pi)E \). As type 2 edges are the opposite of type 1 edges, the type 2 adjacency matrix is the transpose \(T_E^t \) of \(T_E \), and

\[
Z_{1,2}(X_\Gamma, u) = \frac{1}{\det(I - T_E^t u)}.
\]

Therefore

\[
Z(X_\Gamma, u) = \frac{1}{\det(I - T_E u) \det(I - T_E^t u^2)}.
\]

It is much more challenging to find an expression for \(Z(X_\Gamma, u) \) analogous to Ihara’s theorem for graphs, providing topological information involving the Euler characteristic

\[
\chi(X_\Gamma) = \#\text{vertices} - \#\text{edges} + \#\text{chambers}
\]

of \(X_\Gamma \) and the spectral information involving the eigenvalues of the Hecke operators \(A_1 \) and \(A_2 \) acting on functions on vertices of \(X_\Gamma \). Since \(X_\Gamma \) is 2-dimensional, this will also involve backtrackless sequences of adjacent chambers (i.e. sharing edges) of \(X_\Gamma \), called \textit{galleries}. We shall only count closed geodesic galleries in \(X_\Gamma \) whose boundaries are geodesic cycles in \(X_\Gamma \) of type 2, as shown in the two examples below.

![Two geodesic galleries in \(B_3(F) \)](image)

Define primitive and equivalent closed geodesic galleries similar to those of cycles. Call the equivalence classes of such primitive closed geodesic galleries 2-dimensional primes of type 2. Note that the same gallery traveled in reverse direction has boundary of type 1, hence the 2-dimensional primes of type 1 are the opposite of those of type 2. The length of a gallery is the number of chambers in the sequence. Denote by \(M_n(X_\Gamma) \) the number of closed geodesic galleries in \(X_\Gamma \) of length \(n \). Define the gallery zeta function of \(X_\Gamma \) to be

\[
Z_{2,2}(X_\Gamma, u) = \exp \left( \sum_{n=1}^{\infty} \frac{M_n(X_\Gamma)}{n} u^n \right) = \prod_{2\text{-dim prin } [C] \text{ of type 2}} \frac{1}{1 - u^{\ell(C)}}.
\]
The Iwahori subgroup
$$B = \{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \text{PGL}_3(\mathcal{O}_F) : \pi|d, \pi|g, \pi|h \}$$
is the stabilizer of the three vertices of the standard chamber, hence it stabilizes a
chamber together with an orientation, called a directed chamber, which is expressed
as an ordered triple of the three vertices of the chamber up to cyclic permutation.
So $\text{PGL}_3(F)/B$ parameterizes all directed chambers in the building $\mathcal{B}_3(F)$. Define
the out-neighbors of a directed chamber $(v_1, v_2, v_3)$ to be the directed chambers
$(v_2, v_3, v_4)$ in $\mathcal{B}_3(F)$ with $v_4 \neq v_1$. Denote by $T_B$ the adjacency matrix of directed
chambers of $X_\Gamma$. It is an Iwahori Hecke operator based on the $B$-double coset
$$B \begin{pmatrix} 0 & 1 & 0 \\ \pi & 0 & 0 \\ 0 & 0 & \pi \end{pmatrix} B.$$ Then, expression like Theorem 4.3 holds for the gallery zeta
function as well:
$$Z_{2,2}(X_\Gamma, u) = \frac{1}{\det(I - T_B u)}.$$
Now we are ready to state our main theorem for the zeta function of a finite quotient
of $\mathcal{B}_3(F)$.

**Theorem 8.3** (Kang-Li [46], Kang-Li-Wang [48], Kang-Li [47]). The edge zeta
function for $X_\Gamma$ is a rational function in $u$ which can be expressed in two ways:

$$Z(X_\Gamma, u) = \frac{1}{\det(I - T_B u) \det(I - T_E u^2)} \quad \text{(2)}$$

$$= \frac{(1 - u^3)^{\chi(X_\Gamma)}}{\det(I - A_1 u + A_2 qu^2 - q^3 u^3) \det(I + T_B u)} \quad \text{(3)}$$

Here $\chi(X_\Gamma)$ is the Euler characteristic of $X_\Gamma$, $A_1$ and $A_2$ are Hecke operators, $T_E$ is
a parahoric Hecke operator, and $T_B$ is an Iwahori Hecke operator introduced above.

This theorem has three proofs, revealing different interpretations of the identity. In [46] the numbers $N_n(X_\Gamma)$ and $M_n(X_\Gamma)$ were computed and compared in terms of contributions from elements in $\Gamma$ up to conjugation. The identity was first discovered from these computations. The second proof [48] is representation-theoretic. The authors computed the eigenvalues of the operators $T_E$, $T_B$, $A_1$ and $A_2$ and compared them. The third proof [47] is cohomological, analogous to that for graphs but more involved.

The zeta function $Z(X_\Gamma, u)$ is said to satisfy the Riemann Hypothesis if all nontrivial zeros of $\det(I - A_1 u + A_2 qu^2 - q^3 u^3)$ from the nontrivial eigenvalues of $A_1$ and $A_2$ have the same absolute value $q^{-1}$. This happens if and only if $X_\Gamma$ is Ramanujan. It was shown in [48, Theorem 2] that the Ramanujan condition has two more equivalent statements in terms of the operators $T_E$ and $T_B$, respectively.

**Theorem 8.4** (Kang-Li-Wang [48]). The following statements are equivalent:

1. $X_\Gamma$ is Ramanujan;
2. The nontrivial zeros of $\det(I - A_1 u + A_2 qu^2 - q^3 u^3)$ have the same absolute value $q^{-1}$;
3. The nontrivial zeros of $\det(I - T_E u)$ have absolute values $q^{-1}$ and $q^{-1/2}$;
4. The nontrivial zeros of $\det(I - T_B u)$ have absolute values $1$, $q^{-1/2}$, and $q^{-1/4}$.
This is reminiscent of the Riemann Hypothesis for the zeta function \( Z(V, u) \) attached to a smooth irreducible projective surface \( V \) defined over a finite field \( \mathbb{F}_q \) discussed in §5 of Lecture 1. Recall that the numerator and denominator of \( Z(V, u) \) are decomposed into products \( P_1(V, u)P_3(V, u) \) and \( P_0(V, u)P_2(V, u)P_4(V, u) \), respectively, where \( P_i(V, u) = \det(I - \Phi^{(i)}u) \) is the determinant of the induced Frobenius operator \( \Phi^{(i)} \) acting on the finite-dimensional vector space, the \( i \)th étale cohomology \( H^i(V) \) of \( V \), and the roots of \( P_i(V, u) \neq 1 \) all have the same absolute value \( q^{-i/2} \). The above theorem is extended to \( PGL_n(F) \) by First in [24].

Let \( K = \mathbb{F}_q(T) \) be a rational function field and let \( v \) be a degree-1 place of \( K \) so that the completion \( K_v \) of \( K \) at \( v \) is isomorphic to \( F = \mathbb{F}_q((T)) \). Similar to what we saw for curves, in [59] Laumon, Rapoport and Stuhler proved the existence of certain moduli surfaces \( V \) over \( K \) which have good reduction at \( v \) and the existence of suitably chosen discrete torsion-free cocompact subgroups \( \Gamma(V) \) of \( PGL_3(F) \) arising from certain division algebras of dimension 9 over \( K \) unramified at \( v \) so that in the zeta function \( Z(X_{\Gamma(V)}, u) \) of the complex \( X_{\Gamma(V)} = \Gamma(V)\backslash PGL_3(F)/PGL_3(\mathcal{O}_F) \), the factor

\[
\prod_{j=1}^3(1 - \mu_3^ju(1 - \mu_3^juu_3^jv))
\]

is equal to the factor \( P_2(V, u) \) occurring in the zeta function of \( V \) mod \( v \). Here \( \mu_3 \) is a primitive cubic root of unity and the factor in the denominator comes from trivial eigenvalues of \( A_1 \) and \( A_2 \).

Another connection with number theory is that the factor

\[
\frac{1}{\det(I - A_1(X_{\Gamma})u + A_2(X_{\Gamma})qu^2 - q^3u^3)}
\]

occurring in the zeta function \( Z(X_{\Gamma}, u) \) of \( X_{\Gamma} \) is a Langlands \( L \)-function. To explain this, we begin with the definition of Langlands \( L \)-functions. A reductive group \( G \) over \( F \) has a Langlands dual group \( G^\vee \) over \( \mathbb{C} \). Each irreducible unramified representation \( \sigma \) of \( G(F) \) is induced from an unramified character of the maximal torus of \( G(F) \). To \( \sigma \) we associate a conjugacy class \( \{c(\sigma)\} \) of \( G^\vee(\mathbb{C}) \). The Langlands \( L \)-function of \( \sigma \) with respect to a chosen representation \( r \) of \( G^\vee(\mathbb{C}) \) is defined as

\[
L(\sigma, r, u) = \frac{1}{\det(I - r(c(\sigma))u)}.
\]

Given a discrete torsion-free cocompact subgroup \( \Gamma \) of \( G(F) \), the Langlands \( L \)-function of \( \Gamma \backslash G(F) \) with respect to \( r \) is

\[
L(\Gamma\backslash G(F), r, u) = \prod_{\sigma} L(\sigma, r, u),
\]

where \( \sigma \) runs through all unitary irreducible unramified representations of \( G(F) \) occurring in the space \( L^2(\Gamma\backslash G(F)) \), counting multiplicity.

The Langlands dual group of \( PGL_3(F) \) is \( SL_3(\mathbb{C}) \). We choose \( r \) to be the standard representation \( r_{\text{std}} \) of \( SL_3(\mathbb{C}) \), namely the identity map from \( SL_3(\mathbb{C}) \) to itself, viewed as a 3-dimensional representation of \( SL_3(\mathbb{C}) \). A unitary unramified irreducible representation \( \sigma \) occurring in \( L^2(\Gamma\backslash PGL_3(F)) \) is a principal series representation induced from three unramified characters \( \chi_1, \chi_2, \) and \( \chi_3 \)
8.4. ZETA AND L-FUNCTIONS OF FINITE QUOTIENTS OF $B_3(F)$

One verifies that the Langlands $L$-function for $\Gamma \backslash \text{PGL}_3(F)$ is

$$L(\Gamma \backslash \text{PGL}_3(F), r_{\text{std}}, qu) = \frac{1}{\det(I - A_1(X_\Gamma)u + A_2(X_\Gamma)qu^2 - q^3u^3)}.$$  

The group $\Gamma$ is the fundamental group of $X_\Gamma$. Let $\rho$ be a $d$-dimensional unitary representation of $\Gamma$. The Artin $L$-function for 1-dimensional complexes of $X_\Gamma$ is defined as

$$L_1(X_\Gamma, \rho, u) = \prod_{[C] \text{ prime of type 1}} \frac{1}{\det(I - \rho(Frob_{[C]})u^{\ell([C])})}.$$  

Here, given a prime $[C]$, lift $C$ to a path $C'$ in $B_3(F)$ starting at a vertex $P$, then the end point of $C'$ is $\gamma P$ for some $\gamma \in \Gamma$. The conjugacy class of $\gamma$ is denoted $\text{Frob}_{[C]}$, called the Frobenius conjugacy class at the prime $[C]$. Unlike the case of graphs, there is no good algebraic description of such $\gamma$. More precisely, conjugacy classes of primitive elements in $\Gamma$ do give rise to $\text{Frob}_{[C]}$ at primes $[C]$; on the other hand, there are $\text{Frob}_{[C]}$ which are not conjugacy classes of primitive elements. See [46] for more detail. Similarly one defines the Artin $L$-function $L_2(X_\Gamma, \rho, u)$ for 2-dimensional complexes of $X_\Gamma$ as a product over equivalence classes of primitive closed geodesic galleries whose boundaries are geodesic cycles of type 2. Then each Artin $L$-function is a rational function in $u$ and expressions similar to the zeta functions attached to $X_\Gamma$ hold for Artin $L$-functions. More precisely, we have

**Theorem 8.5 (Kang-Li [47]).** Let $\rho$ be a $d$-dimensional unitary representation of $\Gamma$. Then

1. For $i = 1, 2$, the Artin $L$-function $L_i(X_\Gamma, \rho, u)$ is invariant under induction.
2. There exist operators $T_{E}(\rho)$ and $T_{B}(\rho)$ such that

$$L_1(X_\Gamma, \rho, u) = \frac{1}{\det(I - T_{E}(\rho)u)}, \quad L_2(X_\Gamma, \rho, u) = \frac{1}{\det(I - T_{B}(\rho)u)};$$

and

$$(1 - u^3)^{\chi(X_\Gamma)d}L(\text{Ind}_{\Gamma}^{\text{PGL}_3(F)} \rho, r_{\text{std}}, qu) = \frac{L_1(X_\Gamma, \rho, u)L_1(X_\Gamma, \rho, u^2)}{L_2(X_\Gamma, \rho, -u)}.$$  

Here the Langlands $L$-function is the product of the Langlands $L$-functions of the irreducible representations, counting multiplicity, occurring in the induced representation $\text{Ind}_{\Gamma}^{\text{PGL}_3(F)} \rho$ of $\text{PGL}_3(F)$. This theorem is proved using cohomological method, similar in spirit to the argument in §4, Lecture 4 but a lot more complicated.
8.5. Zeta functions of finite quotients of the building $\Delta(F)$ of $Sp_4(F)$

Let $V$ be a 4-dimensional vector space over $F$ equipped with the nondegenerate alternating bilinear form $\langle \cdot, \cdot \rangle$, whose values on the standard basis $\{e_1, e_2, f_1, f_2\}$ of $V$ are given by

$$\langle e_1, f_2 \rangle = \langle e_2, f_1 \rangle = 1$$

and

$$\langle e_i, f_i \rangle = \langle e_i, e_i \rangle = \langle f_i, f_i \rangle = 0 \quad \text{for } i \in \{1, 2\}.$$ 

The linear transformations on $V$ preserving the bilinear form $\langle \cdot, \cdot \rangle$ constitute the symplectic group $Sp_4(F)$. Its associated building $\Delta(F)$ is a contractible 2-dimensional simplicial complex. Each apartment of $\Delta(F)$ is an Euclidean plane, tiled by isosceles right triangles. Each isosceles right triangle is a chamber, the two vertices on its hypotenuse are special vertices, and the third vertex is nonspecial. Depicted in Figure 4 is the standard apartment $\mathcal{A}$ in which the vertices are described using the lattice model. More precisely, a vertex marked by $[a, b, c, d]$ represents the equivalence class of the rank-4 lattice with a basis $\pi^a e_1, \pi^b e_2, \pi^c f_1, \text{ and } \pi^d f_2$. The special vertices are at the crossing of solid lines and the remaining vertices are nonspecial.

![Figure 4. The standard apartment $\mathcal{A}$ of $\Delta(F)$](image)

---

3It is Fig. 1 in [22], used by permission of Oxford University Press.
The group of symplectic similitudes \( GSp_4(F) \) preserves the bilinear form \((\ ,\ )\) up to scalar multiples. Modulo its center, the quotient group \( GSp_4(F) \) acts transitively on the special vertices of \( \Delta(F) \) and preserves nonspecial vertices. As the stabilizer of the special vertex \([0,0,0,0]\) is \( GSp_4(O_F) \), the cosets in \( GSp_4(F)/GSp_4(O_F) \) parameterize the special vertices on the building \( \Delta(F) \). Out of a special vertex, there are two kinds of directed edges: those between two special vertices are of type 1 and those between a special and a nonspecial vertex are of type 2. The type 1 edges are parameterized by the cosets of the Siegel congruence subgroup \( \Gamma \) of \( GSp_4(O_F) \) and the type 2 ones by the cosets of the Klingen congruence subgroup \( \Gamma \) of \( GSp_4(O_F) \). The cosets of the Iwahori subgroup \( I = P_1 \cap P_2 \) parameterize the directed chambers of \( \Delta(F) \). For example, on the standard apartment \( \mathcal{A} \), the 4 vectors out of \( v = [0,0,0,0] \) ending at \([0,0,1,1],[1,0,1,0],[1,1,0,0],[0,1,0,1]\) are type 1 edges, and the 4 vectors ending at \([0,1,1,1],[1,0,1,1],[1,1,1,0],[1,1,0,1]\) are type 2 edges.

Two vertex adjacency operators \( A_1 \) and \( A_2 \) act on special vertices: \( A_1 \) is based on the double coset \( GSp_4(O_F) \) \( diag(1,1,\pi,\pi)GSp_4(O_F) \) and \( A_2 \) is \( q^2+1 \) times the identity operator plus the operator based on the double coset \( GSp_4(O_F) \) \( diag(1,\pi,\pi,\pi^2)GSp_4(O_F) \). For \( i = 1,2 \), the type \( i \) edge adjacency operator \( L_i \) are based on the double cosets \( P_1 diag(1,1,\pi,\pi)P_1 \) and \( P_2 diag(1,\pi,\pi,\pi^2)P_2 \), respectively. We shall be concerned with two directed chamber adjacency operators

\[
L_I \text{ based on the double coset } I t I \text{ with } t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\pi & 0 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} \text{ and } L'_I \text{ based on the double coset } It'I \text{ with } t' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi^{-1} \\ \pi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

See [22,49] for more details.

Let \( \Gamma \) be a discrete torsion-free cocompact subgroup of \( GSp_4(F) \) such that \( \text{ord}_y \text{det} \Gamma \equiv 0 \mod 4 \). Then \( \Gamma \) preserves the types of vertices and \( X_\Gamma = \Gamma \backslash \Delta(F) \) is a finite 2-dimensional simplicial complex. Define 1-dimensional primes of type \( i \) and the zeta function of \( X_\Gamma \) counting the closed geodesic cycles contained in the 1-skeleton of \( X_\Gamma \) using edges of the same type as in the previous section:

\[
Z(X_\Gamma,u) = \prod_{\text{dim}'1 \text{ prime } [C] \text{ of type } i} \frac{1}{1-\psi_i(C)u^r}.
\]

Similar to Theorem 8.3 for finite quotients of the building \( B_3(F) \), we have two closed form expressions for the zeta function of \( X_\Gamma \) in terms of the various operators introduced above.

**Theorem 8.6 (Fang-Li-Wang [22]).** The zeta function \( Z(X_\Gamma,u) \) defined above is a rational function in \( u \) with two expressions:

\[
Z(X_\Gamma,u) = \frac{1}{\text{det}(I-L_{P_1}u) \text{det}(I-L_{P_2}u^2)} \frac{1}{(1-\psi^2)(1-\psi^q)(1-\psi^q-q^2)N(\Gamma)} \frac{1}{\text{det}(I-A_1u+qA_2u^2-q^2A_1u^5+q^6t^4u^3) \text{det}(I-L_Iu)}. \]
Here $\chi(X_\Gamma) = \#\text{vertices} - \#\text{edges} + \#\text{chambers}$ in $X_\Gamma$ is the Euler characteristic of $X_\Gamma$, and $2N(\Gamma)$ is the number of special vertices in $X_\Gamma$.

The dual group of $PGSp_4(F)$ is $\text{Spin}_5(\mathbb{C}) \cong Sp_4(\mathbb{C})$, which has a degree 4 spin representation $r_{\text{spin}}$ and a degree 5 standard representation $r_{\text{std}}$. A unitary irreducible representation $\sigma$ of $PGSp_4(F)$ occurring in $L^2(\Gamma \backslash PGSp_4(F))$ is induced from $\chi_1 \times \chi_2 \times \tau$, where $\chi_1, \chi_2,$ and $\tau$ are unramified characters of $F^\times$ satisfying $\chi_1\chi_2\tau^2 = 1$. The Langlands $L$-function of $\sigma$ with respect to $r_{\text{spin}}$ is

$$L(\sigma, r_{\text{spin}}, u) = \frac{1}{(1 - \chi_1\tau(\pi)u)(1 - \chi_2\tau(\pi)u)(1 - \chi_1\chi_2\tau(\pi)u)(1 - \tau(\pi)u)}$$

and that with respect to $r_{\text{std}}$ is

$$L(\sigma, r_{\text{std}}, u) = \frac{1}{(1 - \chi_1(\pi)u)(1 - \chi_2(\pi)u)(1 - \chi_1^{-1}(\pi)u)(1 - \chi_2^{-1}(\pi)u)(1 - u)}.$$

Analogous to the case of $PGL_2(F)$, one verifies that the zeta function $Z(X_\Gamma, u)$ is related to the Langlands $L$-function of $\Gamma \backslash PGSp_4(F)$ with respect to $r_{\text{spin}}$ as follows:

$$L(\Gamma \backslash PGSp_4(F), r_{\text{spin}}, q^3u) = \frac{1}{\det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)}.$$

It is natural to ask whether there is a combinatorial zeta function attached to $X_\Gamma$ which is related to the Langlands $L$-function with respect to the standard representation $r_{\text{std}}$ of the dual group $\text{Spin}_5(\mathbb{C})$. To answer this, introduce another zeta function

$$Z'(X_\Gamma, u) = \prod_{i=1}^{2} \prod_{1 \text{-dim'} \text{ prime } [C] \text{ of type } i} \frac{1}{(1 - u^{\ell(C)})^{2/i}}.$$

In [49] Kang-Li-Wang obtained a closed form expression of this zeta function, which is more complicated than the theorem above. To describe it, let $A_2' = A_2 - 2q^2I$ and $A_2'' = A_2 - q^2I$.

**Theorem 8.7 (Kang-Li-Wang [49]).**

$$Z'(X_\Gamma, u) = \frac{1}{\det(I - L_{P_1}u)^2} \frac{1}{\det(I - L_{P_2}u)} \frac{1}{(1 - u)^2(1 + u)^{2\chi(X_\Gamma)-2}(1 - qu)^{t_1}(1 + qu)^{t_2}} \frac{1}{\det(I - A_2'u + (qA_1^2 - 2q^2A_2'')u^2 - q^4A_2'u^3 + q^8Iu^4)\det(I + L'Iu)}.$$

Here $t_1$ and $t_2$ are integers determined by representations of certain types occurring in $L^2(\Gamma \backslash PGSp_4(F)/I)$. They sum to $-2(q^2 - 1)N(\Gamma)$, where $2N(\Gamma)$ is the number of special vertices in $X_\Gamma$. For detailed expression, see [49, Theorem 6.3].

It is related to the Langlands $L$-function by

$$(1 - q^2u)^{2N(\Gamma)} L(\Gamma \backslash PGSp_4(F), r_{\text{std}}, q^2u) = \frac{1}{\det(I - A_2'u + (qA_1^2 - 2q^2A_2'')u^2 - q^4A_2'u^3 + q^8Iu^4)}.$$

It should be pointed out that, like what we saw in §8.4 for the zeta functions attached to finite quotients of $B_3$, the factors $\frac{1}{\det(I - L'Iu)}$ occurring in $Z(X_\Gamma, u)$ in
Theorem 8.5 and \( \frac{1}{\det(I-L'u)} \) in Theorem 8.6 each arise from counting closed geodesic galleries of spin type and of standard type, respectively. More precisely, a chamber of spin type is a “square” in an apartment obtained by gluing together two chambers which share a type 1 edge. Note that the boundaries of this square are four type 2 edges. A sequence of adjacent chambers of spin type with boundary a geodesic path of type 2 edges is called a geodesic gallery of chambers of spin type. Then \( \frac{1}{\det(I-L'u)} \) counts closed geodesic galleries of spin type in \( X_\Gamma \). A chamber of standard type is a “square” in an apartment obtained by gluing together four chambers along type 2 edges which they share so that the boundaries of the square consist of four type 1 edges. Similarly \( \frac{1}{\det(I-L'u)} \) counts closed geodesic galleries of standard type in \( X_\Gamma \).

The figures below depict a chamber of spin/standard type and 2 geodesic galleries of chambers of the same type containing it:

\begin{align*}
\text{a spin type chamber} & \quad \text{2 spin type geodesic galleries} \\
\text{a standard type chamber} & \quad \text{2 standard type geodesic galleries}
\end{align*}

As before, call a 2-dimensional prime of spin/standard type an equivalence class of primitive closed geodesic galleries of spin/standard type. Then \( \frac{1}{\det(I-L'u)} \) (resp. \( \frac{1}{\det(I-L'u)} \)) is a product of \( \frac{1}{1-u^{\ell(C)}} \) over 2-dimensional primes \( [C] \) of spin (resp. standard) type. Here the length \( \ell(C) \) of \( [C] \) is the number of chambers of spin (resp. standard) type contained in \( C \). The reader is referred to [49] for details.

8.6. Distribution of primes in finite quotients of \( B_3(F) \) and \( \Delta(F) \)

For a 2-dimensional complex \( X \) obtained as a finite quotient of the building \( B_3(F) \) or \( \Delta(F) \), we defined 1-dimensional (resp. 2-dimensional) primes which are equivalence classes of primitive closed geodesic cycles (resp. galleries) in \( X \) of a
given type. In each case there is a zeta function defined as a product over the primes in question, and this zeta function is the reciprocal of a polynomial, of the form $\frac{1}{\det(I-Tu)}$ for a suitable adjacency operator $T$. Parallel to the case of graphs, very recently Li and Matias [64] obtain the ”Prime Geodesic Theorem” for 1- and 2-dimensional primes of the given type by studying the analytic behavior of the associated zeta function, or equivalently, the eigenvalues of $T$. The key is to prove that given two directed edges/chambers of given type, there is a geodesic path of the same type and dimension in $X$ connecting them. This result leads to the determination of the greatest common divisor $\delta$ of the length of primes of given type. Then using the Perron-Frobenius theorem, they determine the largest eigenvalue in absolute value, $\lambda$, of $T$, show that there are $\delta$ distinct eigenvalues of $T$ with absolute value $\lambda$, and each occurs with multiplicity $\kappa$. Their result is summarized below.

**Theorem 8.8 (Prime geodesic theorem for 2-dimensional complexes [64]).** Let $X$ be a finite quotient of the building $B_3(F)$ of $PGL_3(F)$ or $\Delta(F)$ of $PGSp_4(F)$ by a discrete torsion-free cocompact subgroup as before. Given $i \in \{1, 2\}$ and a type, let $\delta$, $\lambda$ and $\kappa$ be as described above. Then for integers $n$ large, we have

$$\#\{i - \dim' \text{ prime } [C] \text{ of } X \text{ of given type } : \ell(C) = n\delta\} \sim \frac{\kappa\lambda^n\delta}{n},$$

and

$$\#\{i - \dim' \text{ prime } [C] \text{ of } X \text{ of given type } : \ell(C) < n\delta\} \sim \frac{\kappa\lambda^n\delta}{(\lambda^\delta - 1)n},$$

where the values of $\lambda$, $\delta$ and $\kappa$ are as follows:

<table>
<thead>
<tr>
<th>building</th>
<th>$B_3(F)$</th>
<th>$B_3(F)$</th>
<th>$\Delta(F)$</th>
<th>$\Delta(F)$</th>
<th>$\Delta(F)$</th>
<th>$\Delta(F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim $i$</td>
<td>1</td>
<td>2</td>
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<tr>
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<td>2</td>
<td>spin</td>
<td>standard</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$q^2$</td>
<td>$q$</td>
<td>$q^2$</td>
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<td>$q^2$</td>
<td>$q^3$</td>
</tr>
<tr>
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<td>2</td>
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</tr>
<tr>
<td>$\kappa$</td>
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The Prime Geodesic Theorem provides an estimate of the number of primes of $X$ of given dimension and type with given length. In the case that $X$ is a Ramanujan complex as discussed in §3, this estimate has the best possible error term $O(\frac{\lambda^{n\delta/2}}{n})$, following [48, Theorem 2] reviewed in §4.

Given a finite unramified Galois cover $Y \to X$ with Galois group $G$, to each $i$-dimensional prime $[C]$ of $X$ of a given type, where $i = 1$ or 2, we associate a conjugacy class in $G$, called $\text{Frob}_{[C]}$, the same way as we did for graphs. Define, for each conjugacy class $C$ of $G$, the set

$$S_C = \{i - \text{dimensional prime } [C] \text{ of } X \text{ of given type } : \text{Frob}_{[C]} = C\}.$$ 

In other words, using the conjugacy classes of $G$ we partition the $i$-dimensional primes of $X$ of a given type, and we may ask a finer distribution question about these primes, namely whether each set $S_C$ has a density. By studying the analytic behavior of the Artin $L$-functions $L(X, \rho, u)$ attached to finite-dimensional unitary irreducible representations $\rho$ of $G$, Li and Matias proved in [64] that $L(X, \rho, u)$ is holomorphic on the closed disk $|u| \leq 1/\lambda$ for all nontrivial $\rho$. Here $\lambda$ is as in
Theorem 8.8. From this they conclude that the Chebotarev density theorem in natural density holds for the cover $Y \to X$.

**Theorem 8.9 (Chebotarev density theorem for 2-dimensional complexes [64]).** Let $X$ be as in Theorem 8.8. Let $Y \to X$ be a finite unramified Galois cover with Galois group $G$. Then the Chebotarev density theorem holds in natural density for $i$-dimensional primes ($i \in \{1, 2\}$) of $X$ of a given type. Specifically, for each conjugacy class $C$ of $G$, the set $S_C$ defined above has natural density $\frac{|C|}{|G|}$.