Preface

The discrete Painlevé equations are nonlinear difference equations that arise as compatibility conditions of linear systems. The deceptive simplicity of this statement hides deep layers of mathematical properties, which are outlined in these lectures.

These properties are discrete versions of the ones that characterise integrable partial differential equations, such as the Korteweg-de Vries equation, and integrable ordinary differential equations, such as the Painlevé equations. These integrable PDEs and ODEs all arise as compatibility conditions of associated linear problems, which are called Lax pairs. The discovery of solitons [ZK65] and the inverse scattering transform method [GGKM67, AKN74] for solving integrable PDEs provides the beginnings of this theory.

The Painlevé equations were discovered more than a century ago [Cla18, Pai06], but modern interest was sparked by the surprising discovery that they arise as symmetry reductions of integrable PDEs [ARS78, ARS80a, ARS80b]. They appear widely in applications, ranging from fluid dynamics, plasmas, optics and general relativity to random matrix theory [TW94]. The first identification of a difference equation as a discrete Painlevé equation came from the theory of orthogonal polynomials [Sho39, FIK91]. Now more than twenty classes of such integrable equations are known.

The original naming of discrete Painlevé equations came from their continuum limits, but these limits are perhaps their least interesting property. They have a separate rich spectrum of properties, independent of continuum limits, that make them worthy of attention. They share a deep geometric property characterising their initial value (or phase) spaces. They arise when we walk from one tile to another on a lattice defined by reflections associated with an affine Coxeter or Weyl group. They are dynamical systems with zero algebraic entropy [BV99]. Their general solutions provide new higher transcendental functions. Like the differential case, there is reason to expect that these new functions may arise universally as models in many contexts.

Discrete Painlevé equations are second-order nonlinear, nonautonomous difference equations governing functions $w$ of a discrete variable $n$, say. Nonautonomous means they have coefficients that are explicit functions of $n$. There are three different types of equations, according to whether the coefficients are linear, exponential or elliptic functions of $n$. These respective cases are labelled as additive, multiplicative, or elliptic difference equations in the literature, denoted with prefixes $d$-, $q$-, or $ell$- in front of the equation’s name.

In the autonomous limit, their solutions parametrise elliptic curves, with the discrete motion on each curve provided by addition theorems. In the general case, the solutions are still associated with curves, but instead of moving along one curve,
the solutions move from curve to curve in a way that is explicitly described through algebro-geometric operations.

In limiting cases of parameters, explicit solutions of discrete Painlevé equations are known to include rational functions of their coefficients and special functions, such as basic and generalized hypergeometric functions. Like the latter functions, all the solutions satisfy recurrence relations and transformations, which help to characterise their properties.

This book collects the material I presented as principal lecturer at a Conference Board of Mathematical Sciences and National Science Foundation conference in Texas in 2016. Because the audience varied widely in their background knowledge, the lectures necessarily started with elementary introductory material. The present book aims to develop the theory at a higher level, but no attempt is made here to state detailed, general theorems or to provide complete proofs. Instead, the book relies on providing essential points of many arguments through explicit examples that I hope will be useful for applied mathematicians.

The book is oriented towards a reader with a graduate level of knowledge of mathematics, covering complex analysis and differential equations theory to projective geometry. But because it ranges from asymptotics and methods of applied mathematics to reflection groups, foliations and similar abstract theory, appendices are provided to cover further background material.

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