Index

$2^{11} - 1 = 2047 = 23 \cdot 89$
and binary Golay codes, 268
and perfect numbers, 267
$2^{11} - 1 = 2047 = 23 \cdot 89$
and Hudalricus Regius, 267
and L. E. Dickson, 267
(7, 3, 1)
block design, 29, 105, 137, 165,
209, 210, 214
complex and quaternion subal-
gebras of the octonions, 225
difference set, 107, 182, 214
doubly regular tournament, 214
Fano plane, 137, 165
Hamming $H(7, 4)$ code, 172
Heawood graph, 210
Heawood’s map on the torus, 210
multiplication rule for octonion
units, 224
orthogonal Latin squares of or-
der 2, 212
P-positions for three-heap Nim, 29
Rick’s Tricky Six Puzzle, 299–302
skew-Hadamard matrix, 215
Steiner triple system, 105
the $\mathcal{F}_7$ matroid, 229
three-circle Venn diagram, 172
(11, 5, 2) biplane, 257–273
and difference sets, 259
and error-correcting codes, 266–
270
and Galois, 265
and Mathieu groups, 274–277
and Steiner systems, 270–272
affine planes, 140–142
$AG(2, F_5)$, 141
Alberti, Leon Battisti
and perspective in art, 123
Alekseev, V. E., 89
and partitioning problems
$x + y = z$, 89
algebraic number fields, 192–198
quadratic (degree-2) fields, 194,
195
biquadratic (degree-4) fields, 194,
195
octic (degree-8) fields, 194, 217
Anderson, Ian, 311
arithmetic progressions, 91
Aschbacher, Michael, 122
and symmetric (79, 13, 2) designs,
122
associahedrons, 55
automorphism, 147
of a graph, 147
<table>
<thead>
<tr>
<th>Concept</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>outer automorphism</td>
<td>279, 287</td>
</tr>
<tr>
<td>automorphism group</td>
<td>147, 152, 153</td>
</tr>
<tr>
<td>of a block design</td>
<td>147</td>
</tr>
<tr>
<td>of a Steiner system</td>
<td>274</td>
</tr>
<tr>
<td>of Rick's Tricky Six Puzzle</td>
<td>279, 286–293</td>
</tr>
<tr>
<td>of the (7, 3, 1) block design</td>
<td>148–153, 265</td>
</tr>
<tr>
<td>on the octahedron</td>
<td>150–152</td>
</tr>
<tr>
<td>on the regular heptagon</td>
<td>150</td>
</tr>
<tr>
<td>of the Fano plane</td>
<td>147, 148</td>
</tr>
<tr>
<td>of the Golay code $G_{12}$</td>
<td>309</td>
</tr>
<tr>
<td>of the Hamming code $H(7, 4)$</td>
<td>253</td>
</tr>
<tr>
<td>of the Heawood graph</td>
<td>147</td>
</tr>
<tr>
<td>of the Hoffman–Singleton graph</td>
<td>303</td>
</tr>
<tr>
<td>of the quadratic residue code $C_{17}$</td>
<td>255</td>
</tr>
<tr>
<td>of $S(5, 8, 24)$</td>
<td>311</td>
</tr>
<tr>
<td>of the (11, 5, 2) biplane</td>
<td>261–265</td>
</tr>
<tr>
<td>Babylonic cuneiform</td>
<td>51</td>
</tr>
<tr>
<td>Beatty sequences</td>
<td>8–18, 21, 34, 41, 78</td>
</tr>
<tr>
<td>complementary</td>
<td>9–11, 21, 22, 34</td>
</tr>
<tr>
<td>Beatty’s Theorem</td>
<td>9</td>
</tr>
<tr>
<td>Beatty, Samuel</td>
<td>10</td>
</tr>
<tr>
<td>Berlekamp, Elwyn R.</td>
<td>31</td>
</tr>
<tr>
<td>Binet J. P. M.</td>
<td></td>
</tr>
<tr>
<td>and the Fibonacci numbers</td>
<td>41</td>
</tr>
<tr>
<td>binomial coefficients</td>
<td>45–47</td>
</tr>
<tr>
<td>biplanes</td>
<td></td>
</tr>
<tr>
<td>symmetric $(v, k, 2)$ designs</td>
<td>261</td>
</tr>
<tr>
<td>block designs</td>
<td>117–123, 132–136</td>
</tr>
<tr>
<td>$(7, 3, 1)$</td>
<td>120, 138, 139, 142, 146–152, 194, 209</td>
</tr>
<tr>
<td>Bruck–Ryser–Chowla Theorem</td>
<td>119</td>
</tr>
<tr>
<td>dual designs</td>
<td>139</td>
</tr>
<tr>
<td>elementary relations</td>
<td>119</td>
</tr>
<tr>
<td>resolvable designs</td>
<td>187, 189, 190, 196, 203</td>
</tr>
<tr>
<td>symmetric designs</td>
<td>119–122, 133–138, 147, 185, 194, 196–198</td>
</tr>
<tr>
<td>Bose’s theorem</td>
<td></td>
</tr>
<tr>
<td>for projective planes</td>
<td>135</td>
</tr>
<tr>
<td>on Singer designs</td>
<td>133, 135</td>
</tr>
<tr>
<td>Bouton’s Theorem</td>
<td>28, 34, 244</td>
</tr>
<tr>
<td>Bouton, Charles L.</td>
<td>28</td>
</tr>
<tr>
<td>bow ties</td>
<td>13–19</td>
</tr>
<tr>
<td>bricks</td>
<td></td>
</tr>
<tr>
<td>and enumeration problems</td>
<td>77</td>
</tr>
<tr>
<td>and the Fibonacci numbers</td>
<td>41</td>
</tr>
<tr>
<td>and the MOG</td>
<td>319–324</td>
</tr>
<tr>
<td>Brooks, R. L.</td>
<td>94</td>
</tr>
<tr>
<td>and perfect squared squares</td>
<td>94</td>
</tr>
<tr>
<td>Brown, Ezra</td>
<td>155, 187, 209, 217, 257</td>
</tr>
<tr>
<td>$(7, 3, 1)$ and combinatorics</td>
<td>209–215</td>
</tr>
<tr>
<td>$(7, 3, 1)$ and normed algebras</td>
<td>217–227</td>
</tr>
<tr>
<td>Kirkman’s Schoolgirls</td>
<td></td>
</tr>
<tr>
<td>fields, spreads, and hats</td>
<td>187–207</td>
</tr>
</tbody>
</table>
the (11, 5, 2) biplane, codes, designs, and groups, 257–273
the groups $PSL(2, 7)$ and $GL(3, 2)$
and why they are isomorphic, 155–163
Bruce, J. W., 49
and the Lucas–Lehmer primality test, 49
Bruck–Ryser–Chowla Theorem, 119
Brunelleschi, Filippo
and perspective in art, 123
Catalan numbers, 39, 54–65, 69, 71, 74
André’s Reflection Principle, 57
associahedrons, 56
Dyck paths, 58–60
formula for, 57
grouping of products, 55
in Catalan, 60
strings of $L$’s and $R$’s, 57
triangulations of regular polygons, 59
walks, 60
Cayley, Arthur
and Kirkman’s Schoolgirls Problem, 206
and partitioning polygons, 64
and the Cayley–Dickson construction, 223–226
and the octonions, 181, 223
codewords, 167–173, 203, 205
coil diagrams, 91
coin-turning games, 243–255
Mock Turtles, 250–254
Moebius, 254–255
Mogul, 255
Turning Corners, 247–249
Turning Turtles, 245–247
combinations, 46
combinatorial games, 21–38, 243–255
complete graphs, 90
genus of $K_n$, 92
$K_{25}$, 103
$K_5$, 100, 101, 294
$K_n$, 90–92
complex numbers, 220
and the Two-Squares Identity, 220
continued fraction, 43
Conway worms, 15
Conway, John Horton, 12–19, 35
*On Numbers And Games*, 36
and Nim multiplication, 247
and Rick’s Tricky Six Puzzle, 283
and the “alias-alibi” problem, 284
and the hexacode $H_6$, 326
and the Hoffman–Singleton graph, 304
Conway worms, 15
kites and darts, 15
monads, duads, and synthemes, 291
short and long bow ties, 15
the Ace, 15
current graphs, 93–97
Dürer, Albrecht
and perspective in art, 124
darts, 13–15
Dawson’s Chess, 31
de Bruijn cycles, 115–116
de Bruijn graph, 116
de Moivre, Abraham
and the Fibonacci numbers, 41
Denniston, R. H. F., 206
and Kirkman parades, 206
Desargues, Girard
and Desargues’ Theorem, 124
and projective geometry, 124
Dickson, Leonard Eugene, 155
and $2^{11} - 1 = 2047 = 23 \cdot 89$, 268
and $PSL(2, 7)$, 155
and the Cayley–Dickson construction, 223–226
difference sets, 105, 107–113, 115–117, 120–122
$(7, 3, 1)$, 107, 110, 113–115, 117
$(11, 5, 2)$, 108, 110, 113, 114, 120, 260
$(13, 4, 1)$, 110
$(15, 7, 3)$, 115
$(16, 6, 2)$, 38, 111, 112, 183
$(19, 9, 4)$, 110
$(21, 5, 1)$, 115, 294
$(23, 11, 5)$, 110
$(31, 15, 7)$, 110
$(37, 9, 2)$, 115
$(40, 13, 4)$, 115
$(43, 21, 10)$, 110
Hadamard, 111, 183
Paley, 109, 183, 214
planar, 108, 109
Singer, 116, 135, 136
Whiteman, 112
Diophantus, 218
Two-Squares Identity, 220
directed Euler circuit, 116
dodecads, 314
and $S(5, 8, 24)$, 314
Dots and Boxes, 31
duads, 287–297
dual graphs, 96, 97, 144
Duijvestijn, A. J. W., see perfect squared square, 94
Dyck paths, 58–60, 77, 78
eigenvalues and eigenvectors
and the Fibonacci numbers, 41
Emmy Noether Boarding School, 188
error-correcting codes, 167–174, 266–270
Euclid
and the Elements, 42
and the GCD algorithm, 42
Euler, Leonard
Euler’s polyhedral formula, 97
Euler, Leonhard
and Latin squares, 212
Euler characteristic, 98
Euler circuit, 116
Euler’s conjecture disproved, 212
Euler’s misattribution, 51
Euler’s polyhedral formula, 97, 99, 99
for nonplanar graphs, 99
sketch of a proof, 98
evil numbers, 250–252
face-centered-cubic lattice, 176–179
Fano plane, 132, 137, 138, 144–148, 152, 155, 156, 165, 172, 182, 183, 238, 300
Fano, Gino, 132
Fermat, Pierre
2^{11} - 1 = 2047 = 23 \cdot 89
and binary Golay codes, 268
and perfect numbers, 267

2^{11} - 1 = 2047 \equiv 23 \cdot 89
and Hudalricus Regius, 267
and L. E. Dickson, 267

(7, 3, 1)
block design, 29, 105, 137, 165, 209, 210, 214
difference set, 107, 182, 214
doubly regular tournament, 214
Fano plane, 137, 165
Hamming H(7, 4) code, 172
Heawood graph, 210
Heawood’s map on the torus, 210
multiplication rule for octonion units, 224
orthogonal Latin squares of order 2, 212
P-positions for three-heap Nim, 29
Rick’s Tricky Six Puzzle, 299–302
skew-Hadamard matrix, 215
Steiner triple system, 105
the F_7 matroid, 229
three-circle Venn diagram, 172

(11, 5, 2) biplane, 257–273
and difference sets, 259
and error-correcting codes, 266–270
and Galois, 265
and Mathieu groups, 274–277
and Steiner systems, 270–272

affine planes, 140–142
AG(2, \mathbb{F}_5), 141
Alberti, Leon Battisti
and perspective in art, 123
Alekseev, V. E., 89
and partitioning problems
x + y = z, 89
algebraic number fields, 192–198
quadratic (degree-2) fields, 194, 195
biquadratic (degree-4) fields, 194, 195
octic (degree-8) fields, 194, 217
Anderson, Ian, 311
arithmetic progressions, 91
Aschbacher, Michael, 122
and symmetric (79, 13, 2) designs, 122
associahedrons, 55
automorphism, 147
of a graph, 147
outer automorphism, 279, 287
automorphism group, 147, 152, 153
of a block design, 147
of a Steiner system, 274
of Rick’s Tricky Six Puzzle, 279, 286–293
of the \((7, 3, 1)\) block design, 148–153, 265
on the octahedron, 150–152
on the regular heptagon, 150
of the Fano plane, 147, 148
of the Golay code \(G_{12}\), 309
of the Hamming code \(H(7, 4)\), 253
of the Heawood graph, 147
of the Hoffman–Singleton graph, 303
of the quadratic residue code \(C_{17}\), 255
of \(S(5, 8, 24)\), 311
of the \((11, 5, 2)\) biplane, 261–265

Babylonian cuneiform, 51
Beatty sequences, 8–18, 21, 34, 41, 78
complementary, 9–11, 21, 22, 34
Beatty’s Theorem, 9
Beatty, Samuel, 10
Berlekamp, Elwyn R., 31
Binet J. P. M.
and the Fibonacci numbers, 41
binomial coefficients, 45–47
biplanes
symmetric \((v, k, 2)\) designs, 261
block designs, 117–123, 132–136
\((7, 3, 1)\), 120, 138, 139, 142, 146–152, 194, 209
\((11, 5, 2)\), 120, 259
\((13, 4, 1)\), 139
\((15, 3, 1)\), 187, 190, 195, 196, 203, 204
\((15, 7, 3)\), 196, 198
\((16, 6, 2)\), 111
\((25, 9, 3)\), 122
\((31, 15, 7)\), 139
Bruck–Ryser–Chowla Theorem, 119
dual designs, 139
elementary relations, 119
resolvable designs, 187, 189, 190, 196, 203, 204
symmetric designs, 119–122, 133–138, 147, 185, 194, 196–198
Bose’s theorem
for projective planes, 135
on Singer designs, 133, 135
Bouton’s Theorem, 28, 34, 244
Bouton, Charles L., 28
bow ties, 13–19
bricks
and enumeration problems, 77
and the Fibonacci numbers, 41
and the MOG, 319–324
Brooks, R. L., 94
and perfect squared squares, 94
Brown, Ezra, 155, 187, 209, 217, 257
\((7, 3, 1)\) and combinatorics, 209–215
\((7, 3, 1)\) and normed algebras, 217–227
Kirkman’s Schoolgirls, fields, spreads, and hats, 187–207
the (11, 5, 2) biplane, codes, designs, and groups, 257–273
the groups $PSL(2, 7)$ and $GL(3, 2)$
and why they are isomorphic, 155–163
Bruce, J. W., 49
and the Lucas–Lehmer primality test, 49
Bruck–Ryser–Chowla Theorem, 119
Brunelleschi, Filippo
and perspective in art, 123
Catalan numbers, 39, 54–65, 69, 71, 74
André’s Reflection Principle, 57
associahedrons, 56
Dyck paths, 58–60
formula for, 57
grouping of products, 55
in Catalan, 60
strings of $L$’s and $R$’s, 57
triangulations of regular polygons, 59
walks, 60
Cayley, Arthur
and Kirkman’s Schoolgirls Problem, 206
and partitioning polygons, 64
and the Cayley–Dickson construction, 223–226
and the octonions, 181, 223
codewords, 167–173, 203, 205
coil diagrams, 91
coin-turning games, 243–255
Mock Turtles, 250–254
Moebius, 254–255
Mogul, 255
Turning Corners, 247–249
Turning Turtles, 245–247
combinations, 46
combinatorial games, 21–38, 243–255
complete graphs, 90
genus of $K_n$, 92
$K_{25}$, 103
$K_5$, 100, 101, 294
$K_n$, 90–92
complex numbers, 220
and the Two-Squares Identity, 220
continued fraction, 43
Conway worms, 15
Conway, John Horton, 12–19, 35
*On Numbers And Games*, 36
and Nim multiplication, 247
and Rick’s Tricky Six Puzzle, 283
and the “alias-alibi” problem, 284
and the hexacode $\mathcal{H}$, 326
and the Hoffman–Singleton graph, 304
Conway worms, 15
kites and darts, 15
monads, duads, and synthemes, 291
short and long bow ties, 15
the Ace, 15
current graphs, 93–97
Dürer, Albrecht
and perspective in art, 124
darts, 13–15
Dawson’s Chess, 31
de Bruijn cycles, 115–116
de Bruijn graph, 116
de Moivre, Abraham
and the Fibonacci numbers, 41
Denniston, R. H. F., 206
and Kirkman parades, 206
Desargues, Girard
and Desargues’ Theorem, 124
and projective geometry, 124
Dickson, Leonard Eugene, 155
and $2^{11} - 1 = 2047 = 23 \cdot 89$, 268
and $PSL(2, 7)$, 155
and the Cayley–Dickson construction, 223–226
difference sets, 105, 107–113, 115–117, 120–122
$(7, 3, 1)$, 107, 110, 113–115, 117
$(11, 5, 2)$, 108, 110, 113, 114, 120, 260
$(13, 4, 1)$, 110
$(15, 7, 3)$, 115
$(16, 6, 2)$, 38, 111, 112, 183
$(19, 9, 4)$, 110
$(21, 5, 1)$, 115, 294
$(23, 11, 5)$, 110
$(31, 15, 7)$, 110
$(37, 9, 2)$, 115
$(40, 13, 4)$, 115
$(43, 21, 10)$, 110
Hadamard, 111, 183
Paley, 109, 183, 214
planar, 108, 109
Singer, 116, 135, 136
Whiteman, 112
Diophantus, 218
Two-Squares Identity, 220
directed Euler circuit, 116
dodecads, 314
and $S(5, 8, 24)$, 314
Dots and Boxes, 31
duads, 287–297
dual graphs, 96, 97, 144
Duijvestijn, A. J. W., see perfect squared square, 94
Dyck paths, 58–60, 77, 78
eigenvalues and eigenvectors
and the Fibonacci numbers, 41
Emmy Noether Boarding School, 188
error-correcting codes, 167–174, 266–270
Euclid
and the Elements, 42
and the GCD algorithm, 42
Euler, Leonard
Euler’s polyhedral formula, 97
Euler, Leonhard
and Latin squares, 212
Euler characteristic, 98
Euler circuit, 116
Euler’s conjecture disproved, 212
Euler’s misattribution, 51
Euler’s polyhedral formula, 97, 99, 99
for nonplanar graphs, 99
sketch of a proof, 98
evil numbers, 250–252
face-centered-cubic lattice, 176–179
Fano plane, 132, 137, 138, 144–148, 152, 155, 156, 165, 172, 182, 183, 238, 300
Fano, Gino, 132
Fermat, Pierre
Index

and Fermat’s Little Theorem, 50
and the Fermat powers $2^{2^n}$, 36, 247
and the Pell Equation, 51
Fibonacci
Liber abaci, 39
numbers, 12, 39–44, 47
Beatty sequences, 41
counting problems, 40–41
de Moivre–Binet Formula, 41
Euclidean algorithm, 42
Virahanka, 40
origin of the name, 40
rabbits, 39
field, 126, 127, 129, 131, 133, 145
finite field, 129, 130, 133, 146
construction, 129
finite projective planes, 132–140, 144–152, 183, 211, 230, 238, 261, 283, 294
finite projective spaces, 127–152, 198–201, 238, 239
PG(2, $\mathbb{F}_2$), 132
PG(3, $\mathbb{F}_2$) sits inside PG(2, $\mathbb{F}_4$), 300
PG(3, $\mathbb{F}_2$), 183, 185, 189, 199–201, 205
PG(n, $\mathbb{F}_q$), 137
Fink, Alex, 279
Rick’s Tricky Six Puzzle, 279–309
Fisher, R. A., 118
Fisher’s Inequality, 119, 139, 166
Four Color Theorem, 91
fractals, 53
and Pascal’s triangle, 53
Fundamental Theorem of Matroids and Greedy Algorithms, 241
Galois, Évariste, 194, 217
Fundamental Theorem of Galois Theory, 207
Galois group, 207, 217
Galois theory, 194
generalized Fibonacci numbers, 48–53
Brahmagupta–Pell numbers, 50
Chebyshev polynomials, 52, 53
Jacobsthal numbers, 50
Mersenne numbers, 49
genus, 91
of a graph, 91
of the complete graph $K_n$, 92
Golay codes
$G_{11}$, 170, 259, 266, 269, 270, 308
$G_{12}$, 170, 259, 266, 269, 270, 283, 307, 308
and Rick’s Tricky Six Puzzle, 307
Golay, Marcel, 167
see also Golay codes, 167
golden section $\phi$, 8, 12, 14, 43
graph coloring, 91–93
proper, 91–93
graph covering, 90, 99–103
graph decomposition, 99–105
of $K_{25}$ into 100 triangles, 103
of $K_5$ into two 5-cycles, 101
of $K_7$ into seven triangles, 105
graph packing, 90, 99–104
graphs, 56
associahedron graph, 55
and Fermat’s Little Theorem, 50
and the Fermat powers $2^{2^n}$, 36, 247
and the Pell Equation, 51

Fibonacci
- Liber abaci, 39
- numbers, 12, 39–44, 47
  - Beatty sequences, 41
counting problems, 40–41
de Moivre–Binet Formula, 41
Euclidean algorithm, 42
Virahanka, 40
origin of the name, 40
rabbits, 39
field, 126, 127, 129, 131, 133, 145
finite field, 129, 130, 133, 146
construction, 129
finite projective planes, 132–140, 144–152, 183, 211, 230, 238, 261, 283, 294
finite projective spaces, 127–152, 198–201, 238, 239
$PG(2, \mathbb{F}_2)$, 132
$PG(3, \mathbb{F}_2)$ sits inside $PG(2, \mathbb{F}_4)$, 300
$PG(3, \mathbb{F}_2)$, 183, 185, 189, 199–201, 205
$PG(n, \mathbb{F}_q)$, 137

Galois, Évariste, 194, 217
- Fundamental Theorem of Galois Theory, 207
- Galois group, 207, 217
- Galois theory, 194
generalized Fibonacci numbers, 48–53
Brahmagupta–Pell numbers, 50
Chebyshev polynomials, 52, 53
Jacobsthal numbers, 50
Mersenne numbers, 49
genus, 91
- of a graph, 91
- of the complete graph $K_n$, 92

Golay codes
- $G_{11}$, 170, 259, 266, 269, 270, 308
- $G_{12}$, 170, 259, 266, 269, 270, 283, 307, 308
- and Rick’s Tricky Six Puzzle, 307
- Golay, Marcel, 167
- see also Golay codes, 167
golden section $\phi$, 8, 12, 14, 43
graph coloring, 91–93
- proper, 91–93
graph covering, 90, 99–103
graph decomposition, 99–105
- of $K_{25}$ into 100 triangles, 103
- of $K_5$ into two 5-cycles, 101
- of $K_7$ into seven triangles, 105
graph packing, 90, 99–104
graphs, 56
- associahedron graph, 55
connected, 71, 97
de Bruijn, 116
genus of a graph, 91
Moore graphs, 303
paths, 58
planar, 56, 97–99
walks, 78
Graves, John T.
and the Eight-Squares Identity, 223
and the octonions, 181, 222
greedy algorithms
Dijkstra and minimal connectors, 240
Euler and the Königsberg Bridges, 240
Ford–Fulkerson and max-flow min-cut, 240
Hall and maximal matchings, 240
Hungarian Assignment algorithm, Grundy, Patrick M., 31
Kruskal and minimal spanning trees, 240
Menger’s theorem, 240
Prim and minimal spanning trees, 240
groups
$A_8$, 156
$GL(3, 2)$, 147, 148, 152, 153, 155–158, 161, 163, 253
$GL(\mathbb{F}_8)$, 157, 161, 163
$k$-transitive groups, 275
$M_{11}$, 258, 275, 277
$M_{12}$, 258, 275, 277
$M_{24}$, 255, 258, 275, 277
Mathieu groups, 258, 274–277
Multiple Transitivity Theorem, 275
multiply transitive, 275
$PGL(2, 5)$, 279
$PSL(2, 7)$, 147, 148, 153, 155, 156, 159–161, 163, 275
$PSL(2, 11)$, 257
$PSL(2, 17)$, 254
$PSL(3, 4)$, 156
$S_5$, 279
$S_5$ sits specially in $S_6$, 279
$S_6$, 279
simple, 275, 276
$SLF(7)$, 159–163
why $PSL(2, 7)$ and $GL(3, 2)$ are isomorphic, 152–163
Grundy number, 31, 33–35, 244, 245, 247–252, 254, 255
Grundy’s Theorem (see Sprague–Grundy Theorem), 32
Guy, Richard K., 61, 77, 279
Catwalks, Sandsteps and Pascal Pyramids, 61–75
Rick’s Tricky Six Puzzle, 279–309
unique rook circuits, 77–87
Hadamard difference sets, 111
Hadamard matrix, 111, 181–184, 215
normalized, 182
regular, 183
skew-Hadamard matrix, 215
Hadamard, Jacques, 182
Hamilton, William R.
and the quaternions, 221
could not multiply triples, 221
Hamming
distance, 168, 179, 307
sphere, 168, 169, 180
weight, 168, 180
Hamming codes, 170–174, 179, 201, 203–205, 252, 253
(3, 1) triplication code, 170
(7, 4) code, 170
(15, 11) code, 203
extended Hamming code, 179, 252, 253
Hamming, Richard, 167
Hats Games, 202, 203
The Three Hats Game, 202
The Fifteen Hats Game, 203
Heawood graph, 93, 99, 142, 143
Heawood’s conjecture, 91
Heawood, Percy J., 91
hexacode, 326–327
hexads
and $S(5, 6, 12)$, 305, 307
Hoffman–Singleton graph, 303–305
homomorphisms, 152–163
Hudalricus Regius, 268
and $2^{11} − 1 = 2047 = 23 \cdot 89$, 268
hyperplanes, 133, 134
incidence graph, 210
incidence matrix, 265, 266, 270
of $(7, 3, 1)$, 166
of a block design, 166, 190
of a finite projective geometry, 165
of the (15, 11) Hamming code, 205
of the Fano plane, 165, 173, 182, 183
of the Kirkman design, 190, 203, 204
of $PG(3, \mathbb{F}_2)$, 183
inner automorphism, 287
Isaacs’s game, 32
Isaacs, Rufus, 21
isomorphic
block designs, 139, 205
finite fields, 129
games, 21
graphs, 99
groups, 265, 275
$PGL(2, 5)$ and $S_5$, 279
$PSL(2, 7)$ and $GL(3, 2)$, 152
the alternating group $A_8$ and the linear group $L_4(2)$, 325
Hadamard matrices, 182
matroids, 236
subalgebras
of the octonions, 225
of the sedenions, 226
isomorphism
between $GL(3, 2)$ and Aut$(7, 3, 1)$, 153
between $PGL(2, 5)$ and $S_5$, 279
between $PSL(2, 7)$ and $GL(3, 2)$, 155–163
between graphs, 99
between projective spaces and Kirkman designs, 201
exceptional: $A_8 \cong L_4(2)$, 324
of combinatorial games, 22
the Galois correspondence, 207
Kepler conjecture, 176, 178
proof by Hales et al., 178
Kirchhoff’s current law, 93, 95
Kirchhoff, Gustav, 93
Kirkman parades, 206
Kirkman’s Schoolgirls Problem, 187–206
and algebraic number fields, 194–198
and difference sets, 191
and Hamming codes, 203, 204
and hats, 201–205
and Rick’s Tricky Six Puzzle, 299–303
and spreads, 199, 201
and the group $(\mathbb{Z}/2\mathbb{Z})^4$, 207
solutions, 188, 192, 196, 200, 203, 207
Kirkman, Thomas P., 133, 185, 187–191, 196, 200, 201, 205
and Kirkman designs, 196, 200, 201, 205
kites, 13–15
Langford sequences, 6, 78, 90
Langford, Dudley, 5
son playing with blocks, 5, 89, 327
Laplace transform coefficients, 63
Latin squares, 141
and finite projective planes, 211
Euler’s conjecture, 212
Euler’s conjecture disproved, 212
of order $n$, 211
orthogonal, 141, 211
Leech lattice
$E_8$ in $\mathbb{R}^8$, 179, 180
$\Lambda$ in $\mathbb{R}^{24}$, 179, 315, 327
Leech, John, 179
Leonardo of Pisa, see Fibonacci
Loehr, Nicholas A., 155
the groups $PSL(2, 7)$ and $GL(3, 2)$
and why they are isomorphic, 155–163
Lucas numbers, 12
$M_{11}$, 258, 275, 277
$M_{12}$, 258, 275, 277
$M_{24}$, 255, 258, 275, 277
magic squares, 140–142, 191
$3 \times 3$, 140, 191
$5 \times 5$, 141
pandiagonal, 141
map, 71
Hamiltonian, 71
rooted, 71
matroids, 229–242
and bases, 234
and fields, 237
and greedy algorithms, 239
and independence, 236
and independent sets, 231, 232
and optimal spanning trees, 230
and projective planes, 238
and spanning trees, 233
and the Fano plane, 230, 238
basis axioms, 234
cycle matroid, 235
graph matroid, 235
origin of the name, 235
uniform matroid, 235
vector matroid, 235
Meeker, Darcy
and perfect squared squares, 94
Mellinger, Keith A., 187
Kirkman’s Schoolgirls, fields, spreads, and hats, 187–207
Mersenne numbers, 49
Lucas–Lehmer primality test, 49
Mersenne primes, 49
mex, 31, 32, 35, 247, 250
Miracle Octad Generator (MOG) and parity, 319
Miracle Octad Generator (MOG) octads of $S(5, 8, 24)$, 317
Miracle Octad Generator (MOG), 317–327
a picture of $S(5, 8, 24)$, 317, 319
and the Mathieu group $M_{24}$, 317
and $S(3, 4, 16)$, 319
and apples in buckets, 319
and parity, 324
and the exceptional isomorphism $A_8 \cong L_4(2)$, 324
and the hexacode, 326
and the Leech lattice in $\mathbb{R}^{24}$, 327
bricks (eight-element subsets), 319–324
elementary construction, 319
octads of $S(5, 8, 24)$, 326
tetrad, 319–324
the fundamental correspondence, 319
Mock Turtles, 250–254
Moebius transformations, 159, 254, 265, 279, 285, 286, 289
MOG, see Miracle Octad Generator monads, 287–306
Moore graphs, 303
multipliers, 113–115
The Multiplier Theorem, 114
Mystical Hexagram Theorem, 124, 125
N-positions, 25, 26, 28, 29, 31, 34, 244, 248, 249, 251
Nim, 23, 24, 27, 28, 34, 37, 144–146, 243–253
addition, see Nim sum
multiplication, 35–38, 247, 249
product, 247, 248
sum, 27, 34–36, 144–146, 201, 203, 204, 244, 246–248, 251, 252
numbers, 27, 28, 31, 144–146, 203, 244, 247–249
nonexistence of three-dimensional real normed algebras, 226
nonisomorphic simple groups of the same order, 156
normed algebra, 219
octads, 312
and $S(5, 8, 24)$, 312–315
nonobvious properties, 312–315
octonions, 181, 222, 223
and $(7, 3, 1)$, 224
and sums of eight squares, 223
odious numbers, 250–255
OEIS, 61–63, 71
Online Encyclopedia of Integer Sequences, see OEIS
Orbit-Stabilizer Theorem, 148–152, 254, 265
orbits, 114, 115, 148–152
planetary, 176
P-positions, 25, 26, 28–33, 35, 244–255
packings, see graph packings, sphere packings
in $PG(3, \mathbb{F}_2)$, 189
partitioning problems, 63, 64, 89, 91, 99
partitioning polygons, 63, 64
\(x + y = z\), 90, 91, 99
\(x + y = 2z\), 90
\(x + y = 3z\), 91
Pascal's triangle, 39, 44–54, 60
and the Catalan numbers, 55
mod 2 triangle, 53
origins, 47
quarter-pyramid, 69, 70
semi-pyramid, 66, 67, 74
semi-triangle, 63
Pascal, Blaise, 124
and projective geometry, 124
and the Mystical Hexagram Theorem, 124
Paulhus, Mark M., 77
unique rook circuits, 77–87
Penrose tiling, 13–15, 17, 18, 20
ace, 15, 17
aperiodic set, 13
bow ties, 12–17
Conway worms, 17, 18, 20
darts, 13–15
kites, 13–15
rhombus, 14
perfect squared squares, 94–96
permutations, 46
on an error-correcting code, 253
and multipliers, 114–115
on \(\mathbb{Z}/v\mathbb{Z}\), 113
on a set, 46, 114
on block designs, 139
on Rick's Tricky Six Puzzle, 282–285
on sliding block puzzles, 280
on the (11, 5, 2) biplane, 261–265
on the (7, 3, 1) block design, 147–153
that preserve adjacency in a graph, 147
perspective art
and projective geometry, 125
perspective drawing, 123–125
\(PG(2, \mathbb{F}_2)\), 132, 144
\(PG(2, \mathbb{F}_3)\), 139, 140
\(PG(2, \mathbb{F}_4)\), 294
\(PG(2, \mathbb{F}_5)\), 141
\(PG(3, \mathbb{F}_2)\), 183, 185, 189, 199–201, 205
\(PG(3, \mathbb{F}_q)\), 199, 200
\(PG(n, \mathbb{F})\), 126–129
\(PG(n, \mathbb{F}_q)\), 130–137
Piero della Francesca
and perspective in art, 124
Plücker, Julius
and plane cubic curves, 117
and the (9, 12, 4, 3, 1) block design, 118, 120, 122
planar graph, 92, 97–99
point classes, 126, 129
Poncelet, Jean-Victor
and projective geometry, 125
positions, see P-positions, N-positions, terminal positions
projective geometry, 131
projective geometry, 123
and perspective art, 125
projective planes, 86, 126, 132, 135, 137–141, 147, 152, 153
axioms, 126

Index 349

projective spaces, 126–137
  definition, 126–127
dual subspaces, 128
  $PG(n, \mathbb{F})$, 126–129
  $PG(n, \mathbb{F}_q)$, 130
Subspace Correspondence Theorem, 127
projective transformation, 125
  $PSL(2, 5)$, 265
  $PSL(2, 7)$, 265
  $PSL(2, 11)$, 265
  $PSL(2, p)$
    and Galois, 265
and the projective plane of order 4, 294–300
  the six equivalence classes, 284–285
Ringel–Youngs Theorem, 92
rook circuits, 77–87
  4-corner, 80–82
guideposts, 79, 81, 83, 84, 86
  the 4-Corner Principle, 80, 83, 86
  the Cul-de-sac Principle, 80
  the Parity Principle, 79, 81
  the Two Neighbor Principle, 79
rook tour, 78

S(5, 8, 24), 259, 277, 311–315
  and $M_{24}$, 311
  and the Golay code $G_{24}$, 272
  internal structure, 311–315
Restriction Theorem, 312
Sands, Bill
  Sands walks, 63
  Sands’ Theorem, 61
  walks of length $n$, 67
sedenions, 225
  and a $(15, 3, 1)$ block design, 226
Shannon, Claude, 167, 170
  and the first Hamming code, 170
Singer designs, 132, 133, 135, 139
Singer difference sets, 110, 136
Singer, James, 135–140
  Singer’s two theorems on difference sets, 135
Skolem sequences, 7, 8, 78, 89, 90
Skolem, Thoralf, 8
  and Skolem sequences, 7, 8, 78, 89, 90

quaternions, 181, 221, 222
  and $(7, 3, 1)$, 225

Rank-Nullity Theorem, 128
resolvable block designs, see block designs
Restriction Theorem for Steiner Systems, 312
Rick’s Tricky Six Puzzle, 279–309
  and $K_5$, 294
  and $S(5, 6, 12)$, 305
  and five-coloring the edges of $K_6$, 288
  and Kirkman’s Schoolgirls Problem, 299–303
  and outer automorphisms, 287–294
  and Rick Wilson’s theorem, 282
  and the Fano plane, 299–302
  and the Golay code $G_{12}$, 307
  and the Hoffman–Singleton graph, 303–304
and Steiner Triple Systems, 89
Sloane, Neil A. J., 61
and the OEIS, 61
Smith, C. A. B., 94
and perfect squared squares, 94
sphere packing, 174
density, 175
in $\mathbb{R}^2$, 175
in $\mathbb{R}^3$, 176, 178
in $\mathbb{R}^4$, 179
in $\mathbb{R}^8$, 179–181
in $\mathbb{R}^{24}$, 179, 315
kissing number, 178
lattice packing, 177
Sprague, Roland P., 34, 94
Sprague–Grundy Theorem, 34, 37, 244, 251
spreads, 189
in $PG(3, \mathbb{F}_2)$, 198–200
stabilizers, 149, 151, 152
Steiner systems
Restriction Theorem, 271, 274, 312
$S(2, 3, 7)$, 270, 274
$S(3, 4, 8)$, 156, 270
$S(4, 5, 11)$, 271, 273
and $M_{11}$, 273
$S(5, 6, 12)$, 271, 273, 305
and $M_{12}$, 273
and Rick’s Tricky Six Puzzle, 305
$S(5, 7, 28)$, 271
$S(5, 8, 24)$, 259, 271, 272, 274, 277, 311–315
and $G_{24}$, 272
and $M_{24}$, 275
$S(p, q, r)$, 270
with $p > 3$, 271
Steiner triple systems, 90, 103–105, 107, 120, 187, 190, 206
Stirling numbers of the second kind, 64
Stone, A. H., 94
and perfect squared squares, 94
Subspace Correspondence Theorem, 127, 130, 134
sums of squares, 217–226
Two-Squares Identity, 218
Four-Squares Identity, 218
Eight-Squares Identity, 219
no Sixteen-Squares Identity, 226
Sylvester, J. J., 287
monads, duads, and synthemes, 287
symmetric designs, see block designs
symmetry, 13, 16, 33, 67, 70, 103, 146, 152, 176, 190, 257, 258, 261, 262, 274
Taxicab metric, 72
terminal positions, 24–26, 32, 243, 245, 250
three-dimensional real normed algebras, nonexistence of, 226
tiling, 13, 15
aperiodic set, 14
nonperiodic, 13
periodic, 13
prototile, 13, 14
tile, 13
tournaments, 213
doubly regular, 213–215
regular, 213
transitive, 213
transitivity of groups, see groups
Turning Corners, 243
Turning Turtles, 243, 245–249
Tutte, W. T., 94
    and perfect squared squares, 94
vanishing point, 123–125
vector spaces, 126–131, 152, 229, 231,
    232, 235, 236, 238, 241
Venn diagram, 172
Viazofska, Maryna
    and sphere-packing in $\mathbb{R}^8$ and $\mathbb{R}^{24}$, 315
Virahanka, 40
walks, 61–75
    one-dimensional, 63, 69
    two-dimensional, 74
    three-dimensional, 74
    four-dimensional, 74
    on lattice points, 61–75
    using the Taxicab metric, 72
Winning Ways, 31
Wyt Queens, 23
Wythoff’s game, 21–23, 32, 33, 35,
    37
Wythoff, Willem Abraham, 21
Zarankiewicz’s problem, 166