Chapter 1

Lebesgue Measure in Euclidean Space

In the first section of this chapter we give an introduction to Lebesgue measure.

In the second section we give the elegant Hadwiger–Ohlmann proof of the Brunn–Minkowski theorem, a geometric version of the arithmetic-geometric mean inequality. As a simple consequence of this we derive the isodiametric inequality which says that among Borel sets in \( \mathbb{R}^n \) of the same diameter, the ball has the greatest volume.

In the third section we derive the still-wonderful covering theorem of Vitali for Vitali families of balls.

We close the chapter with a section entitled Notes and Remarks. This is a catch-all which contains historical comments and some miscellaneous results that we found interesting and/or important—results that add to the message contained in the chapter.

1. An Introduction to Lebesgue Measure

By an interval in \( \mathbb{R}^k \) we mean any set \( I \) of the form

\[
I = I_1 \times \cdots \times I_k,
\]

where \( I_1, \ldots, I_k \) are finite intervals in \( \mathbb{R} \). We do not ask that \( I_1, \ldots, I_k \) all be open, closed, or half-open/half-closed; mixtures are just fine. Each \( I_j \subseteq \mathbb{R} \)
Lebesgue Measure in Euclidean Space

has a length \( l(I_j) \) and with this in mind we define the “volume” of \( I \) by

\[
\text{vol}(I) = \prod_{j \leq k} l(I_j).
\]

**Definition 1.1.** Let \( A \subseteq \mathbb{R}^k \). We define the *outer measure* or *Lebesgue outer measure* of \( A \), \( m^*(A) \) by

\[
m^*(A) = \inf \left\{ \sum_n \text{vol}(I_n) : I_n \text{ is an interval in } \mathbb{R}^k, A \subseteq \bigcup_n I_n \right\}.
\]

**Regarding edges.** There is a great deal of latitude with regard to the nature of the edges of the intervals in the coverings of a set \( A \subseteq \mathbb{R}^k \) that are used to compute \( m^*(A) \). For instance, if we wish we can assume each edge has length less than \( \delta > 0 \). This is plain since any interval \( I \) in \( \mathbb{R}^k \) is the union of nonoverlapping intervals, all of whose edges have length less than \( \delta \) and the sum of whose volume totals \( I \)'s volume.

Further, we can assume we’re covering \( A \) by *open* intervals, that is, all the edges are open. In fact, if \( (I_j) \) is a covering of \( A \) by intervals and \( \epsilon > 0 \), then for each \( j \) we can enlarge \( I_j \) to an open interval \( J_j \), \( I_j \subseteq J_j \), and

\[
\text{vol}(J_j) < \text{vol}(I_j) + \epsilon/2.
\]

It follows that in computing \( m^*(A) \), if \( (I_j) \) is a covering of \( A \) by intervals, then we can find a sum \( \sum \text{vol}(J_j) \) that is as close as we please to \( \sum \text{vol}(I_j) \), where \( (J_j) \) is a covering of \( A \) by open intervals. Hence, we can restrict our attention to finding the infimum of such sums \( \sum \text{vol}(J_j) \), where \( (J_j) \) is an open covering of \( A \) by intervals.

**Theorem 1.2.** Basic results about Lebesgue outer measure:

(i) If \( A \subseteq B \), then

\[
m^*(A) \leq m^*(B).
\]

(ii) If \( A = \bigcup_n A_n \), then

\[
m^*(A) \leq \sum_n m^*(A_n).
\]

(iii) For any interval \( I \),

\[
m^*(I) = \text{vol}(I).
\]
Proof. (ii) We can, and do, assume that $\sum_n m^*(A_n) < \infty$. With this in mind, let $\epsilon > 0$ be given, and choose for each $n$ a sequence $(I_{n,j})$ of intervals that cover $A_n$ and satisfy
\[ \sum_j \text{vol}(I_{n,j}) < m^*(A_n) + \frac{\epsilon}{2^{n+2}}. \]

Since $A = \bigcup_n A_n \subseteq \bigcup_{n,j} I_{n,j}$,
\[ m^*(A) \leq \sum_{n,j} \text{vol}(I_{n,j}) \leq \sum_n \left( m^*(A_n) + \frac{\epsilon}{2^n} \right) \leq \sum_n m^*(A_n) + \epsilon. \]

(iii) Let $\epsilon > 0$ be given, and let $(I_j)$ be a covering of $I$ by open intervals such that
\[ \sum_j \text{vol}(I_j) < m^*(I) + \epsilon. \]

Take any closed subinterval $J$ of $I$. Since $J$ is compact there is a $j_0$ such that $J \subseteq I_1 \cup \cdots \cup I_{j_0}$. Let’s look closely to the intervals $I_1, \ldots, I_{j_0}, J$. Each $(k-1)$-dimensional face of these intervals lies in a $(k-1)$-dimensional hyperplane in $\mathbb{R}^k$; in turn, these hyperplanes divide $I_1, \ldots, I_{j_0}$ into closed intervals $K_1, \ldots, K_{n_{j_0}}$, similarly $J$ is divided into closed intervals $J_1, \ldots, J_{m_{j_0}}$ by the same hyperplane. (Think of the case $n = 3$.) Since $J \subseteq \bigcup_{j \leq j_0} I_j$, each $J_m$ is one of the $K_n$’s so that
\[ \text{vol}(J) = \sum_{m \leq m_{j_0}} \text{vol}(J_m) \leq \sum_{n \leq n_{j_0}} \text{vol}(K_n) \leq \sum_{j \leq j_0} \text{vol}(I_j) \leq m^*(I) + \epsilon. \]

This is so for every closed subinterval $J$ of $I$ so
\[ \text{vol}(I) \leq m^*(I) + \epsilon; \]
epsilonlonics soon tell us that
\[ \text{vol}(I) \leq m^*(I). \]
The reverse is plain. \hfill $\square$

Some ground work is needed to prepare the way for measurable sets.
Theorem 1.3. More basic results about Lebesgue outer measure:

(i) If $F_1$ and $F_2$ are disjoint closed bounded sets, then

$$m^*(F_1 \cup F_2) = m^*(F_1) + m^*(F_2).$$

(ii) If $G$ is a bounded open set, then for each $\epsilon > 0$ there is a closed set $F \subseteq G$ such that

$$m^*(F) > m^*(G) - \epsilon.$$

(iii) If $F$ is a closed subset of an open bounded set $G$, then

$$m^*(G \setminus F) = m^*(G) - m^*(F).$$

Proof. (i) Let $\delta > 0$ be chosen so that no interval of diameter less than $\delta$ meets both $F_1$ and $F_2$ (e.g., $\delta < \frac{1}{2} d(F_1, F_2)$, where $d(F_1, F_2)$ is the distance between the disjoint closed sets $F_1$ and $F_2$). Let $\epsilon > 0$ be given. Pick a sequence $(I_i)$ of intervals of diameter less than $\delta$ such that

$$F_1 \cup F_2 \subseteq \bigcup_i I_i, \quad \text{and} \quad \sum_i \operatorname{vol}(I_i) < m^*(F_1 \cup F_2) + \epsilon.$$ 

Denote by $(I^{(1)}_k)$ those intervals among the $I_i$’s that meet $F_1$ and by $(I^{(2)}_j)$ those that meet $F_2$. Then $F_1 \subseteq \bigcup_j I^{(1)}_j$ and $F_2 \subseteq \bigcup_j I^{(2)}_j$ by our judicious concerns over $\delta$. Alas,

$$m^*(F_1) + m^*(F_2) \leq \sum_j \operatorname{vol}(I^{(1)}_j) + \sum_j \operatorname{vol}(I^{(2)}_j) \leq \sum_i \operatorname{vol}(I_i) \leq m^*(F_1 \cup F_2) + \epsilon.$$ 

Epsilonics to the rescue: $m^*(F_1) + m^*(F_2) \leq m^*(F_1 \cup F_2)$.

(ii) Represent $G = \bigcup_i I_i$ where $I_i$’s are nonoverlapping open intervals, and let $\epsilon > 0$ be given; of course $m^*(G) \leq \sum_i \operatorname{vol}(I_i)$, and so there is an $n_0$ such that

$$\sum_{i \leq n_0} \operatorname{vol}(I_i) > m^*(G) - \frac{\epsilon}{2}.$$ 

For each $i \leq n_0$, let $J_i$ be a closed subinterval of the interior of $I_i$ with

$$\operatorname{vol}(J_i) > \operatorname{vol}(I_i) - \frac{\epsilon}{2i}.$$
Then \( F = \bigcup_{i \leq n_0} J_i \) is a closed subset of \( G \) and

\[
m^*(F) = m^*\left( \bigcup_{i \leq n_0} J_i \right)
= \sum_{i \leq n_0} m^*(J_i) \quad \text{(by A)}
= \sum_{i \leq n_0} \text{vol}(J_i)
> \sum_{i \leq n_0} \left( \text{vol}(I_i) - \frac{\epsilon}{2^i} \right)
\geq \sum_{i \leq n_0} \text{vol}(I_i) - \frac{\epsilon}{2}
> m^*(G) - \epsilon.
\]

Notice that the openness of \( G \) was used to represent \( G \) in an appropriate way.

(iii) Let \( \epsilon > 0 \) be given. Using (ii), choose a closed set \( F_1 \subseteq G \setminus F \) so that

\[
m^*(F_1) > m^*(G \setminus F) - \epsilon.
\]

Notice that

\[
m^*(F) + m^*(G \setminus F) \leq m^*(F) + (m^*(F_1) + \epsilon)
= m^*(F \cup F_1) + \epsilon
\leq m^*(G) + \epsilon.
\]

Epsilons take over to say that

\[
m^*(F) + m^*(G \setminus F) \leq m^*(G).
\]

Of course, Theorem 1.2(ii) takes care of the reverse inequality, and with it (iii).

\[\square\]

**Definition 1.4.** A subset \( E \) of \( \mathbb{R}^k \) is **Lebesgue measurable** if given an \( \epsilon > 0 \) there is a closed set \( F \) and open set \( G \) such that

\[ F \subseteq A \subseteq G \quad \text{and} \quad m^*(G \setminus F) < \epsilon. \]

By the complementary nature of open and closed sets, \( E \) is **Lebesgue measurable** if and only if \( E^c \) is

\[ F \subseteq E \subseteq G \iff G^c \subseteq E^c \subseteq F^c \quad \text{and} \quad F^c \setminus G^c = G \setminus F. \]

**Theorem 1.5.** Some basic results on measurable sets:

(i) If \( A \) and \( B \) are measurable, then so is \( A \cap B \).
(ii) A bounded set $B$ is measurable if for each $\epsilon > 0$ there is a compact set $K \subseteq B$ such that 

$$m^*(K) > m^*(B) - \epsilon.$$ 

(iii) Intervals are measurable.

(iv) Sets of outer measure zero are measurable.

(v) If $(A_n)$ is a sequence of disjoint measurable subsets of the interval $I$, then $\bigcup_n A_n$ is measurable also and

$$m^* \left( \bigcup_n A_n \right) = \sum_n m^*(A_n).$$

(vi) If $(A_n)$ is any sequence of disjoint measurable subsets of $\mathbb{R}^k$, then $\bigcup_n A_n$ is measurable and

$$m^* \left( \bigcup_n A_n \right) = \sum_n m^*(A_n).$$

**Proof.** (i) Pick $F_A, F_B$ closed and $G_A, G_B$ open such that 

$$F_A \subseteq A \subseteq G_A \quad \text{and} \quad m^*(G_A \setminus F_A) \leq \frac{\epsilon}{2},$$

$$F_B \subseteq B \subseteq G_B \quad \text{and} \quad m^*(G_B \setminus F_B) \leq \frac{\epsilon}{2}.$$ 

Then $F = F_A \cap F_B$ is closed, $G = G_A \cap G_B$ is open, $F \subseteq A \cap B \subseteq G$, and $G \setminus F \subseteq (G_A \setminus F_A) \cup (G_B \setminus F_B)$ so that

$$m^*(G \setminus F) \leq m^*(G_A \setminus F_A) + m^*(G_B \setminus F_B) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

(ii) Suppose the bounded set $B$ satisfies the conditions set forth, and let $\epsilon > 0$ be given. We can find a compact set $K \subseteq B$ such that 

$$m^*(K) \geq m^*(B) - \frac{\epsilon}{2}.$$ 

But $m^*(B) < \infty$. (Why is that?) So we can cover $B$ by a sequence $(I_j)$ of open intervals each of diameter less than $\frac{1}{3\omega}$ such that

$$\sum_j \text{vol}(I_j) < m^*(B) + \frac{\epsilon}{2}.$$
Let $G$ be the union of all those $I_j$'s that meet $B$. $K \subseteq B \subseteq G$ and $G$ is bounded. So our preparatory work in Theorem 1.3(iii) tells us that
\[
m^*(G\setminus K) = m^*(G) - m^*(K) \\
\leq \sum_j m^*(I_j) - m^*(K) \\
= \sum_j \text{vol}(I_j) - m^*(K) \\
\leq m^*(B) - \frac{\epsilon}{2} - m^*(K) \\
= m^*(B) - m^*(K) + \frac{\epsilon}{2} \\
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

(iii) After all, if $I$ is an interval, then $m^*(I) = \text{vol}(I)$, and so we can plainly approximate $I$ from the inside by compact intervals.

(iv) If $m^*(N) = 0$ and $\epsilon > 0$ is given, there must be a sequence $(I_j)$ of open intervals so $N \subseteq \bigcup_j I_j$ and
\[
\sum_j \text{vol}(I_j) \leq m^*(N) + \epsilon = \epsilon.
\]

Then $G = \bigcup_j I_j$ and $F = \emptyset$ soon show the way to $N$’s measurability.

(v) Let $\epsilon > 0$ be given. Choose compact sets $F_n \subseteq A_n$ so
\[
m^*(F_n) > m^*(A_n) - \frac{\epsilon}{2^{n+1}}.
\]

Since
\[
m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n),
\]
there is an $n_0 \in \mathbb{N}$ so that
\[
\sum_{n \leq n_0} m^*(A_n) > m^*\left(\bigcup_n A_n\right) - \frac{\epsilon}{2}.
\]

If $F = \bigcup_{n \leq n_0} F_n$, then $F$ is compact (it’s closed, and being a subset of $I$, bounded). Hence by Theorem 1.3(i)
\[
m^*(F) = \sum_{n \leq n_0} m^*(F_n) > \sum_{n \leq n_0} m^*(A_n) - \frac{\epsilon}{2} > m^*\left(\bigcup_n A_n\right) - \epsilon.
\]

We’ve just taken the bounded set $\bigcup_n A_n$ and, for each $\epsilon > 0$, found a compact set $F$, contained in $\bigcup_n A_n$ so that
\[
m^*(F) > m^*\left(\bigcup_n A_n\right) - \epsilon.
\]
Therefore \( \bigcup_n A_n \) is measurable thanks to (ii). Let’s check the sums: for any \( n_0 \in \mathbb{N} \)

\[
\sum_{n \leq n_0} m^*(A_n) < \sum_{n \leq n_0} \left( m^*(F_n) + \frac{\epsilon}{2^{n+1}} \right)
\leq \sum_{n \leq n_0} m^*(F_n) + \frac{\epsilon}{2}
= m^*\left( \bigcup_{n \leq n_0} F_n \right) + \frac{\epsilon}{2}
\leq m^*\left( \bigcup_n A_n \right) + \epsilon.
\]

Epsilonics assure us that

\[
\sum_{n \leq n_0} m^*(A_n) \leq m^*\left( \bigcup_n A_n \right),
\]

and this is so for each \( n_0 \). It follows that

\[
\sum_n m^*(A_n) \leq m^*\left( \bigcup_n A_n \right).
\]

We’ve already seen the reverse, so we have (v).

(vi) We’ll bootstrap our way from (v) to (vi). To start, let \((I_m)\) be a sequence of disjoint intervals whose union is \( \mathbb{R}^k \) and such that any bounded set in \( \mathbb{R}^k \) is covered by finitely many \( I_m \)’s.

For each \( m, n \in \mathbb{N} \), let

\[
A_{m,n} = I_m \cap A_n
\]

be that part of \( A_n \) inside \( I_m \). Each \( A_{m,n} \) is measurable ((iii) and (i)) and the \( A_{m,n} \)’s are pairwise disjoint. Look to

\[
\tilde{A}_m = \bigcup_n A_{m,n},
\]

the part of \( \bigcup_n A_n \) in \( I_m \). By (v), \( \tilde{A}_m \) is measurable. Further, the \( \tilde{A}_m \)’s are pairwise disjoint and \( \bigcup_n \tilde{A}_m = \bigcup_n A_n \).

Let \( \epsilon > 0 \) be given. For each \( m \), choose a closed set \( F_m \subseteq \tilde{A}_m \) and an open set \( G_m \), which is a bounded open set, \( \tilde{A}_m \subseteq G_m \) such that

\[
m^*(G_m \setminus F_m) < \frac{\epsilon}{2^m}.
\]

Look at \( F = \bigcup_m F_m \) and \( G = \bigcup_m G_m \). \( F \) is closed (if \((x_n)\) is a convergent sequence of points of \( F \), then \((x_n)\) is bounded, and so for some \( m_0 \), \((x_n)\) is
a sequence in $\bigcup_{m \leq m_0} F_m$, hence visits one of $F_1, \ldots, F_{m_0}$ infinitely often—whichever $F_j$ it visits so often contains its limit) and $G$ is open.

$$F = \bigcup_m F_m \subseteq \bigcup_m \overline{A}_m = \bigcup_m A_m = \bigcup_m \overline{A}_m \subseteq \bigcup G_m = G.$$ 

Further,

$$G \setminus F = \bigcup_m (G_m \setminus F_m) \subseteq \bigcup_m (G_m \setminus F_m),$$

so

$$m^*(G \setminus F) \leq \sum_m m^*(G_m \setminus F_m) \leq \sum_m \frac{\epsilon}{2m} = \epsilon,$$

and $\bigcup_n A_n$ is measurable.

Now

$$A_n = \bigcup A_{m,n},$$

so

$$m^*(A_n) \leq \sum_m m^*(A_{m,n}).$$

It follows that

$$\sum_n m^*(A_n) \leq \sum_n \sum_m m^*(A_{m,n})$$

$$= \sum_m \sum_n m^*(A_{m,n})$$

$$= \sum_m m^*(\overline{A}_m),$$

by (v). Take $m \in \mathbb{N}$. Then

$$\sum_{j \leq m} m^*(\overline{A}_j) \leq \sum_{j \leq m} (m^*(F_j) + m^*(G_j \setminus F_j))$$

$$= m^*\left(\bigcup_{j \leq m} F_j\right) + \sum_{j \leq m} m^*(G_j \setminus F_j) \quad \text{(by A)}$$

$$\leq m^*\left(\bigcup_{j \leq m} F_j\right) + \sum_{j \leq m} \frac{\epsilon}{2^j}$$

$$\leq m^*\left(\bigcup_n A_n\right) + \epsilon.$$ 

The usual epsilonics leads us to conclude that

$$\sum_m m^*(\overline{A}_m) \leq m^*\left(\bigcup_n A_n\right)$$
and, in tandem with what gone on before, we see
\[ \sum_n m^*(A_n) \leq m^*\left( \bigcup_n A_n \right). \]
Again the reverse holds without assumption, so (vi) is proved. \(\Box\)

**Theorem 1.6** (The Fundamental Theorem of Lebesgue Measure). **In summary**

(i) \(m^*\) is a nonnegative, extended real-valued function defined for every subset of \(\mathbb{R}^k\) which assigns
- to each interval, a value equal to its volume,
- to each set, a value common to all its translates,
- to bigger sets, bigger values,
- to compact sets, finite values,
- to nonempty sets, nonnegative values,
and is countably subadditive in doing so.

For any \(A \subseteq \mathbb{R}^k\),
\[ m^*(A) = \inf\{m^*(G) : G \text{ is open } A \subseteq G\}. \]

(ii) The Lebesgue measurable subsets of \(\mathbb{R}^k\) form a \(\sigma\)-field \(\mathcal{M}\) of sets containing every open set, closed set, interval, and set of outer measure zero; \(E \in \mathcal{M}\) if and only if \(E\)'s translates are members of \(\mathcal{M}\).

(iii) \(m^*\) is countably additive on \(\mathcal{M}\) and for \(E \in \mathcal{M}\),
\[ m^*(E) = \sup\{m^*(K) : K \text{ is compact } K \subseteq E\}. \]

2. **The Brunn–Minkowski Theorem**

**Theorem 1.7** (The Brunn–Minkowski Theorem). Let \(n \geq 1\), and let \(\lambda_n\) denote Lebesgue measure on \(\mathbb{R}^n\). If \(A, B\), and \(A + B\) are measurable subsets of \(\mathbb{R}^n\), then
\[ (\lambda_n(A + B))^{1/n} \geq (\lambda_n(A))^{1/n} + (\lambda_n(B))^{1/n}, \]
where
\[ A + B = \{a + b : a \in A, b \in B\}. \]
Notice that (BM) is a geometric generalization of the arithmetic-geometric mean inequality since if $A$ and $B$ are rectangles with sides of length $(a_j)^n_{j=1}$ and $(b_j)^n_{j=1}$, respectively, then (BM) looks like

$$(BM') \quad \left[ \prod_{i}^{n} (a_j + b_j) \right]^{1/n} \geq \left( \prod_{1}^{n} a_j \right)^{1/n} + \left( \prod_{1}^{n} b_j \right)^{1/n}.$$ 

Homogeneity lets us reduce this to the case where $a_j + b_j = 1$ for each $j$. But now the arithmetic-geometric mean inequality assures that

$$\frac{1}{n} \sum_{j=1}^{n} a_j \geq \left( \prod_{1}^{n} a_j \right)^{1/n} \quad \text{and} \quad \frac{1}{n} \sum_{j=1}^{n} b_j \geq \left( \prod_{1}^{n} b_j \right)^{1/n}.$$ 

So (noting that $\sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} (a_j + b_j) = \sum_{j=1}^{n} 1 = n$)

$$1 = \frac{1}{n} \cdot n \geq \left( \prod_{1}^{n} a_j \right)^{1/n} + \left( \prod_{1}^{n} b_j \right)^{1/n},$$

which is $(BM')$. Thus we have proved (BM) for boxes, rectangular parallelepipeds whose sides are parallel to the coordinate hyperplanes.

**Proof.** To prove (BM), suppose that each of $A$ and $B$ is the union of finitely many rectangles whose interiors are disjoint. We proceed by induction on the total number of rectangles in $A$ and $B$. It is important to realize that the inequality is unaffected if we translate $A$ and $B$ independently: in fact, replacing $A$ by $A + h$ and $B$ by $B + k$ replaces $A + B$ by $A + B + h + k$, and the corresponding measures are the same as what we started with. If $R_1$ and $R_2$ are essentially disjoint rectangles in the collection making up $A$, then they can be separated (a translation may be necessary) by a coordinate hyperplane, $\{x_j = 0\}$ say. Thus we may assume that $R_1$ lies in $A^{-} = A \cap \{x_j \leq 0\}$ and $R_2$ lies in $A^{+} = A \cap \{x_j \geq 0\}$. Notice that $A^{+}$ and $A^{-}$ each contain at least one rectangle less than does $A$ and $A = A^{+} \cup A^{-}$.

What to do with $B$? Well, slide $B$ over so if $B^{+} = B \cap \{x_j \geq 0\}$, then

$$\frac{\lambda_n(B^{+})}{\lambda_n(B)} = \frac{\lambda_n(A^{+})}{\lambda_n(A)};$$

of course this entails

$$\frac{\lambda_n(B^{-})}{\lambda_n(B)} = \frac{\lambda_n(A^{-})}{\lambda_n(A)}$$

as well. But

$$(A^{+} + B^{+}) \cup (A^{-} + B^{-}) \subseteq A + B,$$

and the union on the left hand side is essentially disjoint. Moreover, the total number of rectangles in either $A^{+}$ and $B^{+}$ or in $A^{-}$ and $B^{-}$ is less than that in $A$ and $B$. Our induction hypothesis applies.
1. Lebesgue Measure in Euclidean Space

The result?
\[
\lambda_n(A + B) \geq \lambda_n(A^+ + B^+) + \lambda_n(A^- + B^-)
\]
\[
\geq (\lambda_n(A^+)^{1/n} + \lambda_n(B^+)^{1/n})^n + ((\lambda_n(A^-)^{1/n} + \lambda_n(B^-)^{1/n})^n
\]

(induction hypothesis)
\[
= \lambda_n(A^+) \left(1 + \left(\frac{\lambda_n(B)}{\lambda_n(A)}\right)^{1/n}\right)^n + \lambda_n(A^-) \left(1 + \left(\frac{\lambda_n(B)}{\lambda_n(A)}\right)^{1/n}\right)^n
\]
\[
= \lambda_n(A) \left(1 + \left(\frac{\lambda_n(B)}{\lambda_n(A)}\right)^{1/n}\right)^n
\]
\[
= (\lambda_n(A)^{1/n} + \lambda_n(B)^{1/n})^n,
\]
and that’s all she wrote. Thus we have (BM) for finite unions of boxes.

If \(A\) and \(B\) are open sets of finite measure, then once given a margin of error, \(\epsilon > 0\), we can find unions \(A\epsilon, B\epsilon\) of essentially disjoint rectangles such that \(A\epsilon \subseteq A, B\epsilon \subseteq B\), and
\[
\lambda_n(A) \leq \lambda_n(A\epsilon) + \epsilon, \ \lambda_n(B) \leq \lambda_n(B\epsilon) + \epsilon.
\]

Once done, \(A\epsilon + B\epsilon \subseteq A + B\), and we can apply the work of the previous paragraphs:
\[
\lambda_n(A + B)^{1/n} \geq \lambda_n(A\epsilon + B\epsilon)^{1/n}
\]
\[
\geq (\lambda_n(A) - \epsilon)^{1/n} + (\lambda_n(B) - \epsilon)^{1/n}.
\]

Let \(\epsilon \to 0\). It follows that (BM) holds for open sets \(A\) and \(B\) of finite measure.

If \(A\) and \(B\) are compact sets, then \(A + B\) is compact as well. Look at \(A\epsilon = [d(x, A) < \epsilon]\). Then \(A\epsilon\) is open, contains \(A\) and \(A \epsilon \subseteq A\). Similarly \(B\epsilon\) is defined analogously. Plainly, \(A\epsilon\) and \(B\epsilon\) are open and bounded. What can we say about \((A + B)\epsilon\)? The same conclusions can be drawn. Further,
\[
A + B \subseteq A\epsilon + B\epsilon \subseteq (A + B)_{2\epsilon}.
\]

Applying (BM) to \(A\epsilon, B\epsilon\), we get
\[
\lambda_n((A + B)_{2\epsilon})^{1/n} \geq \lambda_n(A\epsilon + B\epsilon)^{1/n}
\]
\[
\geq \lambda_n(A\epsilon)^{1/n} + \lambda_n(B\epsilon)^{1/n}
\]
\[
\geq \lambda_n(A)^{1/n} + \lambda_n(B)^{1/n}.
\]

Letting \(\epsilon \to 0\) gives (BM) for \(A, B, A + B\).

The general situation of \(A, B\) and \(A + B\) measurable follows by approximating from within: if \(\tilde{A}, \tilde{B}\) are compact sets with \(\tilde{A} \subseteq A, \tilde{B} \subseteq B\), then \(\tilde{A} + \tilde{B}\)
is a compact set inside $A + B$, and
\[
\lambda_n(A + B) \geq \lambda_n(\tilde{A} + \tilde{B}) \geq (\lambda_n(\tilde{A})^{1/n} + \lambda_n(\tilde{B})^{1/n})^n.
\]
Let $\lambda_n(\tilde{A}) \nearrow \lambda_n(A), \lambda_n(\tilde{B}) \nearrow \lambda_n(B)$, and be done with it.
\[\Box\]

With the Brunn–Minkowski inequality in hand we can easily find our way to the isodiametric inequality.

**Theorem 1.8.** For any Borel set $B \subseteq \mathbb{R}^n$ we have
\[
\lambda_n(B) \leq \left(\frac{\text{diam } B}{2}\right)^n \lambda_n(B_{\ell^2_n}),
\]
where as usual $B_{\ell^2_n}$ denotes the closed unit ball of $\mathbb{R}^n$.

**Proof.** Without loss of sleep we can assume that
\[
d := \text{diam } B < \infty.
\]
This in mind, realize that if $x, y, \in B$, then $||x - y|| \leq d$. Hence
\[
B - B \subseteq dB_{\ell^2_n},
\]
and so
\[
2\lambda_n(B)^{1/n} = \lambda_n(B)^{1/n} + \lambda_n(B)^{1/n} = \lambda_n(B)^{1/n} + \lambda_n(-B)^{1/n} \leq \lambda_n(B - B)^{1/n} (\text{Brunn–Minkowski to the rescue!}) \leq \lambda_n(dB_{\ell^2_n})^{1/n} = d\lambda_n(B_{\ell^2_n})^{1/n}.
\]
It’s easy to deduce the conclusion of the theorem from this inequality:
\[
2\lambda_n(B)^{1/n} \leq \text{diam}(B)\lambda_n(B_{\ell^2_n})^{1/n}.
\]
\[\Box\]

3. **Covering Theorem of Vitali**

**Theorem 1.9** (Vitali). Let $\mathcal{F}$ be a family of closed balls in $\mathbb{R}^n$ that covers a set $E$ in the sense of Vitali, that is, given an $\epsilon > 0$ and $x \in E$, there is a $B \in \mathcal{F}$ such that the diameter of $B$ is less than $\epsilon$ and $x \in B$. Then $\mathcal{F}$ contains a disjoint sequence that covers $\lambda_n—almost all of $E$.

**Proof.** First we suppose $E$ is bounded and contained in the bounded open set $G$. We disregard any members of $\mathcal{F}$ that aren’t contained in $G$ as well as those that don’t intersect $E$. The resulting family, which we will still refer
to as $\mathcal{F}$ covers $E$ in the sense of Vitali. Our proof will be by “exhaustion”. To start, let $R_1$ be

$$R_1 := \sup \{ \text{radius}(B) : B \in \mathcal{F} \}.$$ 

Choose $B_1 \in \mathcal{F}$, say centered at $a_1$, with radius $r_1$ so that

$$\frac{R_1}{2} \leq r_1.$$

Next let $R_2$ be

$$R_2 = \sup \{ \text{radius}(B) : B \in \mathcal{F}, B_1 \cap B = \emptyset \}.$$

Choose $B_2 \in \mathcal{F}$, say, centered at $a_2$ with radius $r_2$ so that $B_2 \cap B_1 = \emptyset$ and

$$\frac{R_2}{2} \leq r_2.$$

Continue down this primrose lane, after $k$ steps, let $R_{k+1}$ be

$$R_{k+1} = \sup \{ \text{radius}(B) : B \in \mathcal{F}, B \cap (B_1 \cup \cdots \cup B_k) = \emptyset \}.$$ 

Choose $B_{k+1} \in \mathcal{F}$ with center $a_{k+1}$, radius $r_{k+1}$ so

$$B_{k+1} \cap (B_1 \cup \cdots \cup B_k) = \emptyset$$

and

$$\frac{R_{k+1}}{2} \leq r_{k+1}.$$

Naturally,

$$R_1 \geq R_2 \geq \cdots.$$

More is so. The $B_k$’s are pairwise disjoint so

$$\sum_k \lambda_n(B_k) = \lambda_n \left( \bigcup_k B_k \right) \leq \lambda_n(G) < \infty;$$

hence

$$\lim_k \lambda_n(B_k) = 0.$$

But

$$\lambda_n(B_k) = c_n r_k^n$$

(while we can tell what $c_n$ is, this precise bit of information is unnecessary at this juncture), so

$$\lim_k r_k = 0.$$

Since $0 \leq R_k \leq 2r_k$, we see

$$\lim_k R_k = 0.$$

The $B_k$’s eat up $E$. How? What does it mean for $x \in E$ not to be in $\bigcup_{k \leq K} B_k$? Well $x \notin \bigcup_{k \leq K} B_k$ means

$$d \left( x, \bigcup_{k \leq K} B_k \right) = \delta > 0.$$
Choose $B \in \mathcal{F}$ so $x \in B$ and $B$ (which is centered at say $a$ with radius $r$) has radius less than $\delta/2$.

Each point of $B$ is within $2r$ of $x$ and $2r < \delta$. Hence $B$ is disjoint from $B_1 \cup \cdots \cup B_K$. $B$ is one of those balls belonging to $\mathcal{F}$ that take part in defining $R_{K+1}$; in particular, $r \leq R_{K+1}$. But

$$\lim_j R_j = 0,$$

and the $R_j$’s are monotone and nonincreasing. Hence there is a $j$ (which is necessarily bigger than $K$) so that

$$R_{j+1} < r \leq R_j.$$

Since $r > R_{j+1}$, $B$ must meet one of the balls $B_1, \ldots, B_j$. Suppose $B \cap B_m \neq \emptyset$. Since $B$ does not meet $B_1 \cup \cdots \cup B_K$, $m > K$. It follows that for any $c \in B \cap B_m$,

$$||x - a_m|| \leq ||x - a|| + ||a - c|| + ||c - a_m||$$

(where $a_m$ is the center of $B_m$ and $a$ is the center of $B$)

$$\leq r + r + r_m$$

$$= 2r + r_m.$$

So

$$||x - a_m|| \leq 2r + r_m,$$

where

$$r \leq R_j \leq R_m \leq 2r_m.$$

Hence

$$||x - a_m|| \leq 2r + r_m \leq 2 \cdot (2r_m) + r_m \leq 5r_m.$$

To summarize: if $x \in E \setminus \bigcup_{k \leq K} B_k$, then $x$ belongs to the union $\bigcup_{k \leq K} \tilde{B}_k$ of those closed balls $\tilde{B}_k$, where $\tilde{B}_k$ shares the center $a_k$ with $B_k$ but $\tilde{B}_k$ has five times the radius of $B_k$.

In other words,

$$\lambda_n \left( E \setminus \bigcup_{k \leq K} B_k \right) \leq \sum_{k > K} \lambda_n(\tilde{B}_k)$$

$$= \sum_{k > K} 5^n \lambda_n(B_k) \to 0,$$

as $K \to \infty$,

since, as we have already noted,

$$\sum_k \lambda_n(B_k) = \lambda_n \left( \bigcup_k B_k \right) \leq \lambda_n(G) < \infty.$$

This finishes the proof for $E$’s that are bounded.
1. Lebesgue Measure in Euclidean Space

For general $E$, look at the sets
$$E_m = \{ x \in E : m - 1 \leq ||x|| < m \}, \ m \in \mathbb{N}.$$ 

Of course
$$\lambda_n(E \setminus \bigcup_m E_m) = 0,$$
and each $E_m$ is as we’ve discussed above. What’s more, the $G$ we chose at the very start of our proof can be chosen to be the open set
$$\{ x \in \mathbb{R}^n : m - 1 < ||x|| < m \},$$
so the disjointness is achieved by passing from one $E_m$ to the next. \hfill \Box

4. Notes and Remarks

Our treatment of Lebesgue measure is an elaboration of Oxtoby’s development as presented in his wonderful volume *Measure and category* [96].


Bernstein’s topological proof [10] of the existence of a non-Lebesgue measurable subset of the real line has many points of interest, not least of which the window it opens to the study of descriptive set theory.

A first step is to notice the following:

**Theorem 1.10.** Any uncountable $G_\delta$ subset of $\mathbb{R}$ contains a homeomorphic copy of the Cantor set $\Delta$.

**Proof.** Let $E$ be such a set. Then there exists a descending sequence $(G_n)$ of open sets in $\mathbb{R}$ so that $E = \bigcap_n G_n$. Let $F$ be the set of all $x \in E$ for which given any open set $U$ containing $x$, $U \cap E$ is uncountable. Notice that $F$ is nonempty: after all, if $F = \emptyset$, then the family of open intervals with rational endpoints and having a countable number of points in common with $E$ would cover $F$; it would follow that $E$ is itself countable, which it is not. Similarly, $F$ has no isolated points. In particular, $F$ is infinite.

Now to start the construction of $\Delta$ inside $E$, let $I(0)$ and $I(1)$ be disjoint closed intervals of length at most $\frac{1}{3}$ whose interiors meet $F$ and whose union is contained in $G_1$.

Now the interior of $I(0)$ has an infinite number of points in common with $F$, and so we can find disjoint closed intervals $I(0,0)$ and $I(0,1)$ of length no more than $\frac{1}{3^2}$ whose interiors meet $F$ and whose union is contained in $G_2 \cap I(0)$. 

Likewise the interior of $I(1)$ has an infinite number of points in common with $F$, and so we can find disjoint closed intervals $I(1,0)$ and $I(1,1)$ of length no more than $\frac{1}{3^2}$ whose interiors meet $F$ and whose union is contained in $G_2 \cap I(1)$.

Etc., etc., etc.

The (compact) set
\[
\bigcap_n \bigcup_{(i_1,\ldots,i_n)\in\{0,1\}^n} I(i_1,\ldots,i_n)
\]
is a homeomorphic copy of the Cantor set and lies inside $\bigcap_n G_n = E$. □

A consequence of this is that the collection of uncountable closed subsets of $\mathbb{R}$ has cardinality $c$, the cardinality of the continuum. Since the collection of open intervals with rational endpoints is a countable collection and each open set is the union of some subfamily of this collection, there is at most $c$ open sets (and so at most $c$ closed sets) in $\mathbb{R}$. On the other hand, there are already $c$ many distinct closed intervals.

Now we’re ready to present the still-beautiful result of F. Bernstein.

**Theorem 1.11** (F. Bernstein). There exists a set $B$ of real numbers such that both $B$ and its complement $B^c$ meet every uncountable closed set in $\mathbb{R}$.

**Proof.** We call on the well-ordering principle. Well-order $\mathbb{R}$.

Using facts established above, we can also well-order the collection $\mathcal{F}$ of all uncountable closed subsets of $\mathbb{R}$ (each of which, is, by the way, of cardinality $c$); we can index this well-ordering by the ordinals $\alpha < \omega_c$, where $\omega_c$ is the first ordinal of cardinality $c$, also known as the first ordinal with $c$ predecessors. So $\mathcal{F} = (F_\alpha : \alpha < \omega_c)$.

We put the points of $\mathbb{R}$ into two baskets $B$ and $B'$: look at $F_1$, the first of $\mathcal{F}$’s members, and take two distinct points from $F_1$ (there are lots of points to choose from) and put one in $B$ and the other in $B'$.

Now go to the second member of $\mathcal{F}$, $F_2$. Take two points of $F_2$, neither of which is one of the points from the previous paragraph. Put one of these points in $B$, the other in $B'$.

Continue in this fashion. At each stage (for each ordinal $\alpha < \omega_c$), pick two points from $F_\alpha$ that as yet are not in either $B$ or $B'$ (there are still plenty of choices). Put one in $B$ and the other in $B'$.

So $B$ is in place, $B^c$ contains $B'$, and we have our “Bernstein set” $B$. 
The point to be made here is that any Bernstein set $B$ is non-Lebesgue measurable. After all, if $F$ is a closed subset of $\mathbb{R}$ that is contained in $B$, then $F \cap B^c = \emptyset$, and so $F$ must be countable, and so is of Lebesgue measure zero. The regularity of Lebesgue measure tells us that $B$ must have measure zero if it is measurable. But then the same reasoning applies to $B^c$ if $B$ is measurable. □

Again we have followed Oxtoby’s direction in the presentation of Bernstein’s result from F. Bernstein [10].

4.2. Vitali’s Proof of the Existence of Nonmeasurable Sets. Bernstein’s proof is topological in nature. There is another proof of the existence of a nonmeasurable set, due to G. Vitali, that is more group theoretic. It proceeds as follows.

Let $U = (-1, 1)$, and $V = (-\frac{2}{3}, \frac{2}{3})$. Denote by $\mathbb{Q}$ the subgroup of rational numbers, and look at $\mathbb{R}/\mathbb{Q}$ which consists of cosets $\mathbb{Q} + x_\alpha$ of $\mathbb{R}$ modulo $\mathbb{Q}$. Notice that each coset $\mathbb{Q} + x_\alpha$ is dense in $\mathbb{R}$. From each coset $\mathbb{Q} + x_\alpha$, pick exactly one $y_\alpha$ that is also in $V$. Then the set $E = \{y_\alpha\}$ is nonmeasurable.

Indeed, suppose $E$ is measurable. Look at the set

$$S = \left( \mathbb{Q} \cap \left( -\frac{2}{3}, \frac{2}{3} \right) \right) + E.$$

It is the countable union

$$S = \bigcup_{q \in \mathbb{Q} \cap \left( -\frac{2}{3}, \frac{2}{3} \right)} (q + E)$$

of pairwise disjoint sets. To see the disjointness, suppose $q_1$ and $q_2$ are distinct rationals from $(-\frac{2}{3}, \frac{2}{3})$ and

$$(q_1 + E) \cap (q_2 + E) \neq \emptyset.$$ 

Then

$$q_1 + y_\alpha = q_2 + y_\alpha'$$

for some $y_\alpha, y_\alpha' \in E$ so

$$y_\alpha - y_\alpha' = q_2 - q_1 \in \mathbb{Q},$$

which means

$$y_\alpha = y_\alpha'$$

by construction of $E$, and that’s a “no-no”.

So supposing that $E$ is measurable, $S$ is also measurable and is the countable union of sets of the form $q + E$, each of which is measurable and of the same
measure. So \( S \) has two choices in life: either it is of measure zero or it has infinite measure.

Each choice is closed to \( S \)! After all,
\[
\left(-\frac{1}{3}, \frac{1}{3}\right) \subseteq S \subseteq (-1, 1).
\]
Therefore \( E \) is nonmeasurable.

The construction is still more subtle than one might at first suspect. Indeed, if \( \Sigma \) is a any \( \sigma \)-field of subsets of \( \mathbb{R} \) which contains all the Lebesgue measurable sets, and if \( \mu \) is a countably additive nonnegative extended real-valued measure extending Lebesgue measure defined on \( \Sigma \), and both \( \Sigma \) and \( \mu \) are translation invariant, then \( E \notin \Sigma \). After all, if \( E \in \Sigma \) and \( \Sigma \) is translation invariant, then \( S \in \Sigma \) as well; but \( \mu(S) \) is faced with the same choices as the Lebesgue measure of \( S \): \( \mu(S) \) is either 0 or \(+\infty\), yet it’s still so that
\[
\left(-\frac{1}{3}, \frac{1}{3}\right) \subseteq S \subseteq (-1, 1).
\]

G. Vitali \[132\] gave the first example of a nonmeasurable set. His use of the \textit{axiom of choice} led to considerable controversy in its day, and no less an enthusiast about Lebesgue measure than Henri Lebesgue himself was dismayed by the consequences.

The attentive student will notice that Vitali’s construction relies on the existence of a Hamel basis for \( \mathbb{R} \) as a vector space over the subspace \( \mathbb{Q} \) of rationals. It is of more than passing interest to notice that \textit{No Hamel basis over \( \mathbb{Q} \) is a Borel set or even analytic.} As Nadkarni and Sunder \[90\] note, a Hamel basis \( \mathbb{R} \) over \( \mathbb{Q} \) can be Lebesgue measurable. Indeed the usual Cantor set \( \Delta \) sitting inside \([0, 1]\) contains a Hamel basis since
\[
\Delta + \Delta = \{x + y : x, y \in \Delta\}
\]
coincides with the interval \([0, 2]\).

\section*{4.3. Isoperimetric Inequality}

Among the many consequences of the Brunn–Minkowski theorem, we find the following classical result.

\textbf{Theorem 1.12} (Isoperimetric Inequality). Among convex bodies of a given volume, Euclidean balls have the least surface area.

\textbf{Proof.} Let \( C \) be a convex, compact set in \( \mathbb{R}^n \) whose \( n \)-dimensional volume is that of
\[
B := B_{\mathbb{R}^n}^2.
\]
The surface area of $C$ can be expressed by

$$
\text{sa}(\partial C) = \lim_{\epsilon \searrow 0} \frac{\text{vol}_n(C + \epsilon B) - \text{vol}_n(C)}{\epsilon}.
$$

The Brunn–Minkowski inequality tells us that

$$
\text{vol}_n(C + \epsilon B)^{\frac{1}{n}} \geq (\text{vol}_n(C)^{\frac{1}{n}} + \epsilon \text{vol}_n(B)^{\frac{1}{n}}),
$$

so

$$
\text{vol}_n(C + \epsilon B) \geq \left(\text{vol}_n(C)^{\frac{1}{n}} + \epsilon \text{vol}_n(B)^{\frac{1}{n}}\right)^n \geq \text{vol}_n(C) + n\epsilon \text{vol}_n(C)^{\frac{n-1}{n}} \text{vol}_n(B)^{\frac{1}{n}}.
$$

Hence

$$
\text{vol}_n(C + \epsilon B) - \text{vol}_n(C) \geq n\epsilon \text{vol}_n(C)^{\frac{n-1}{n}} \text{vol}_n(B)^{\frac{1}{n}},
$$

and so

$$
\frac{\text{vol}_n(C + \epsilon B) - \text{vol}_n(C)}{\epsilon} \geq n\text{vol}_n(C)^{\frac{n-1}{n}} \text{vol}_n(B)^{\frac{1}{n}}.
$$

Taking the expression (1) into account, we see that

$$
\text{sa}(\partial C) \geq n\text{vol}_n(C)^{\frac{n-1}{n}} \text{vol}_n(B)^{\frac{1}{n}}.
$$

Further factoring into account that we’re supposing $C$ and $B$ have the same volume, we have

$$
\text{sa}(\partial C) \geq n\text{vol}_n(B) = \text{sa}(\partial B)!
$$

Isn’t life beautiful? Isn’t life gay? Isn’t life the perfect way to pass the time of day? \hfill \square

The above proof is taken from Keith Ball’s beautiful essay “An Elementary Introduction to Modern Convex Geometry” [3].

Let $X$ be a bounded compact metric space (the closure of any ball is compact), and let $B$ be a family of closed balls in $X$ with

$$
\sup \{ \text{diam}(B) : B \in B \} < \infty.
$$

Then there is a countable subfamily $\mathcal{C}$ of $B$ consisting of pairwise disjoint balls such that the union of all $B \in \mathcal{B}$ is contained in the union of balls with the same centers as those in $\mathcal{C}$ but five times the radii.

The proof of this and similar variations can be found in the book *Geometry of Sets and Measures in Euclidean Spaces* by Pertti Mattila [80].

More general versions can be studied in Federer’s *Geometric Measure Theory* [38, section 2.8].