CHAPTER 1

Introduction

The concept of a partial dynamical system has been an essential part of Mathematics since at least the late 1800s when, thanks to the work of Cauchy, Lindelöf, Lipschitz and Picard, we know that given a Lipschitz vector field $X$ on an open subset $U \subseteq \mathbb{R}^n$, the initial value problem

$$f'(t) = X(f(t)), \quad f(0) = x_0,$$

admits a unique solution for every $x_0$ in $U$, defined on some open interval about zero.

Assuming that we extend the above solution $f$ to the maximal possible interval, and if we write $\phi_t(x_0)$ for $f(t)$, then each $\phi_t$ is a diffeomorphism between open subsets of $U$. Moreover, if $x$ is in the domain of $\phi_t$, and if $\phi_t(x)$ is in the domain of $\phi_s$, it is easy to see that $x$ lies in the domain of $\phi_{s+t}$, and that

$$\phi_{s+t}(x) = \phi_s(\phi_t(x)).$$

This is to say that, defining the composition $\phi_s \circ \phi_t$ on the largest possible domain where it makes sense, one has that

$$\phi_s \circ \phi_t \subseteq \phi_{s+t},$$

meaning that $\phi_{s+t}$ extends $\phi_s \circ \phi_t$. This extension property is the central piece in defining the notion of a partial dynamical system, the main object of study in the present book.

Since dynamical systems permeate virtually all of mankind’s most important scientific advances, a wide variety of methods have been used in their study. Here we shall adopt an algebraic point of view to study partial dynamical systems, occasionally veering towards a functional analytic perspective.

According to this approach we will extend our reach in order to encompass partial actions on several categories, notably sets, topological spaces, algebras and $C^*$-algebras.

One of our main goals is to study graded $C^*$-algebras from the point of view of partial actions. The fundamental connection between these concepts is established via the notion of crossed product (known to algebraists as skew-group algebra) in the sense that, given a partial action of a group $G$ on an algebra $A$, the crossed product of $A$ by $G$ is a graded algebra. While there are many graded algebras which cannot be built out of a partial action, as above, the number of those that can is surprisingly large. Firstly, there is a vast quantity of graded algebras which, when looked at with the appropriate bias, simply happen to be a partial crossed product. Secondly, and more importantly, we will see that any given graded algebra satisfying quite general hypotheses is necessarily a partial crossed product!

Once a graded algebra is described as a partial crossed product, we offer various tools to study it, but we also dedicate a large part of our attention to the
study of graded algebras per se, mainly through a very clever device introduced by J. M. G. Fell under the name of \textit{C*-algebraic bundles}, but which is now more commonly known as \textit{Fell bundles}. A Fell bundle may be seen essentially as a graded algebra which has been disassembled in such a way that we are left only with the scattered resulting parts.

Our study of Fell bundles consists of two essentially disjoint disciplines. On the one hand we study its internal structure and, on the other, we discuss the various ways in which a Fell bundle may be reassembled to form a C*-algebra. The main structural result we present is that every separable Fell bundle with stable unit fiber algebra must necessarily arise as the semi-direct product bundle for a partial action of the base group on its unit fiber algebra. The study of reassembly, on the other hand, is done via the notions of cross-sectional algebras and amenability.

A number of applications are presented to the study of C*-algebras, notably C*-algebras generated by semigroups of isometries, and the now standard class of graph C*-algebras.

Although not discussed here, the reader may find several other situations where well known C*-algebras may be described as partial crossed products. Among these we mention:

- Bunce-Deddens algebras [46],
- AF-algebras [47],
- the Bost-Connes algebra [15],
- Exel-Laca algebras [57],
- C*-algebras associated to right-angled Artin groups [28],
- Hecke algebras [54],
- algebras associated with integral domains [14], and
- algebras associated to dynamical systems of type $(m,n)$ [7].

Besides, there are numerous other developments involving partial actions whose absence in this book should be acknowledged. First and foremost we should mention that we have chosen to restrict ourselves to discrete groups (i.e., groups without any topology), completely avoiding partial actions of topological groups, even though the latter is a well studied theory. See, for example, [48] and [2].

Although the computation of the K-theory groups of partial crossed product algebras is one of the main focus of the first two papers on the subject, namely [45] and [81], there is no mention here of these developments.

Twisted partial actions and projective partial representations are also absent, even though they form an important part of the theory. See, e.g., [48], [102], [35], [11], [36], [87], [37], [22], [23] and [38].

Even though we briefly discuss the relationship between partial actions and inverse semigroups, many interesting developments have been left out, such as [101], [50], [102], [55], [61], [22], [23] and [24]. The absence of any mention of the close relationship between partial actions and groupoids [3], must also be pointed out.

The study of KMS states for gauge actions on partial crossed products studied in [58] is also missing here.

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