Preface

In this monograph I present a systematic theory of weak solutions in Hilbert-Sobolev spaces of initial-boundary value problems for parabolic systems of partial differential equations. The goal is to provide a practical reference for fundamental results often considered well-known but often difficult to access in the desired generality, as well as to provide instructional material for students preparing for research in applied analysis.

The development begins in Chapter 2 with an abstract theory in which $\mathcal{H}_0$ and $\mathcal{H}_1$ are separable Hilbert spaces with $\mathcal{H}_1$ dense and continuously included in $\mathcal{H}_0$ and $A$ is a mapping of the time axis into the set of bilinear forms on $\mathcal{H}_1 \times \mathcal{H}_1$. The problem is to construct a mapping $u$ for which the equation

$$\langle u(t), v \rangle_{\mathcal{H}_0} = \langle \varsigma_0, v \rangle_{\mathcal{H}_0} + \int_0^t (-A(s)(u(s), v) + F(s) \cdot v) ds$$

holds for all test functions $v \in \mathcal{H}_1$, for all time $t$ and for given $\varsigma_0 \in \mathcal{H}_0$ and $F$ mapping time into the dual $\mathcal{H}_1^*$. This abstract theory is then applied in Chapters 3 and 5 to initial-boundary value problems for linear parabolic systems with $\mathcal{H}_0$ a closed subspace of $[L^2(\Omega)]^N$ for a given open set $\Omega \subset \mathbb{R}^n$ (not assumed to be bounded) and $\mathcal{H}_1$ a closed subspace of $[H^1(\Omega)]^N$ whose elements satisfy essential boundary conditions. The basic theory of Chapter 3 addresses questions of measurability, continuity and energy conservation in the absence of regularity, and includes special considerations for problems with symmetry and representations of solutions in terms of spectral measures. Higher-order regularity of solutions is discussed in Chapter 5 in three related but categorically different formulations: the dynamical systems (Bochner space) formulation in which solutions are mappings from the time axis into various Sobolev spaces, the weak derivative formulation in which solutions are locally integrable on the space-time cross product, and the continuous derivative formulation familiar from calculus. Chapter 5 also includes a discussion of instantaneous regularization of solutions whose data fail to satisfy compatibility conditions. (Chapter 4 is a brief excursion into higher order regularity theory for elliptic systems, required for the application in Chapter 5.)

This linear theory is then applied in chapter 6 to quasilinear systems in which $A$ and $F$ may depend on $u$. A basic theory of solutions with minimal regularity is given, corresponding to the basic theory for linear problems. This includes the theorems of Leray and Hopf for the Navier-Stokes equations as well as results for more general problems in which nonlinearities have specific, controlled growth as $|u| \to \infty$. Results for more highly regular solutions of quasilinear systems are then derived by iteration from the corresponding regularity theory for linear problems. This includes local-in-time existence with large compatible data, global existence for data near an attracting rest point, invariant regions for quasilinear systems, and
a large data, global existence result in two and three space dimensions for certain systems with symmetry.

This book was written in a continuous, linear process in which the material itself dictated formulation, order of topics and level of detail. This choice was made to insure an internal consistency of notation, terminology and perspective that would not have been achieved by collating results from the literature (no claim is made to mathematical originality, however). Maximal generality was an important goal, and this has occasionally resulted in considerable but unavoidable technicality. I have therefore attempted to present proofs in such a way that the experienced reader can easily grasp the flow of the arguments without reading every detail, but at the same time the intrepid student will succeed in checking these details without undue difficulty.

The following are the major theorems of the book, listed here for convenient reference:

- Basic existence and uniqueness: Theorem 3.6 for general linear systems and Theorem 3.7 for systems which are symmetric to leading order.
- Spectral representations: Theorem 3.8.
- Regularity for linear elliptic system: Theorems 4.1 and 4.2.
- Higher order regularity for linear parabolic systems: Theorem 5.5 for compatible data, Theorem 5.8 for incompatible data.
- Theorems of Leray and Hopf for the Navier-Stokes system: Theorem 6.5.
- Local existence of smooth solutions of quasilinear systems: Theorem 6.10.
- Global existence for dissipative quasilinear systems with small initial data, Theorem 6.12.
- Invariant regions for quasilinear systems, Theorem 6.13.
- Global existence for certain symmetric quasilinear systems with large data in two and three space dimensions, Theorem 6.14.

Few readers will read the book cover-to-cover. The following suggestions for reading plans may therefore be considered (the first three should include the basic material in Chapter 1 and sections 2.1., 2.2 and 3.1–3.3):

- Any of the following extensions of the basic theory:
  - systems with symmetry (sections 2.6, 2.7 and 3.4)
  - Navier-Stokes equations (sections 6.1 and 6.2)
  - quasilinear systems with polynomial growth (sections 6.1 and 6.3).
- Higher order regularity for linear parabolic systems (sections 2.2–2.4, 5.1 and 5.2) with the possible addition of initial layer regularization (sections 2.5 and 5.3).
- Higher order regularity for quasilinear parabolic systems (sections 2.2–2.4, 5.1 and 5.2 are prerequisite, 6.4 is essential, and any of 6.5, 6.6 or 6.7 may be included independently).
- Chapter 4 is a self-contained treatment of higher-order regularity for elliptic systems.
• Section A.4 is a self-contained treatment of calculus on Lipschitz hypersurfaces, including the construction of the Radon measure corresponding to surface area and the normal vector field, the trace theorem $W^{1,p}(\Omega) \to L^p(\partial \Omega)$ and the divergence theorem for $W^{1,p}$ integrands.

• Section A.5 is a self-contained treatment of certain spaces $L^{p,q}(\Omega \times I)$ where $I$ is an interval in $\mathbb{R}$ and $\Omega$ is an open set in $\mathbb{R}^n$. The elements of these spaces are in canonical one-to-one correspondence with those of the Bochner spaces $L^p(I; L^q(\Omega))$ but are measurable on the cross product. This correspondence finds application throughout the book in navigating different formulations of regularity.

In all cases students should be familiar with the basic facts of measure theory and functional analysis, including weak derivatives and Sobolev spaces. Spectral measures appear twice in the book but their use can be side-stepped with minimal loss by appeal to the spectral theorem for compact self-adjoint operators.

The Hilbert space framework chosen here has the virtues of relative simplicity and broad applicability. Other approaches, including semigroup theory, Green's functions, and the method of continuation (Schauder theory) have important uses as well, however, but could not be included. For these the reader may consult standard references such as Ladyzhenskaya et al. [26], Friedman [15] and [16] and Evans [12].

Many of the perspectives and formulations in this book reflect the influences of collaborators, colleagues and students, too numerous to mention, with whom I have had the good fortune to interact over many years. My thanks to them all. Of course, responsibility for any shortcomings, deficiencies or errors lies solely with me.

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