# Contents

Preface vii  
To the Student xvii  
Acknowledgement xxii  

1 Rational Numbers and Missing Numbers 1  
1.1 Integers 2  
1.2 Rational Numbers 3  
1.3 Missing Numbers 4  
1.4 Number Lines 7  
1.5 Many Number Lines 9  
1.6 Historical Note: Other Missing Numbers 10  

2 Limit Points and Sequences 13  
2.1 Intervals and Limit Points 15  
2.2 Sequences and Convergence 16  
2.3 Historical Note: Zeno’s Paradox 19  

3 Decimal Representations of Numbers 21  
3.1 Infinite Decimal Representations 22  
3.2 Rational Numbers as Infinite Decimals 24  
3.3 Existence and Uniqueness 27  
3.4 Historical Note: The Hindu-Arabic Numerals 30  

4 Complete Number Lines 35  
4.1 The Completeness Axiom 36  
4.2 The Square Root of 17 37  
4.3 Historical Note: The Archimedean Dilemma 38  

5 Continuity 41  
5.1 Definitions and Examples 42  
5.2 Generating Continuous Functions 46  
5.3 Missing Numbers as Intermediate Values 47  
5.4 Uniform Continuity 49  
5.5 Historical Note: Continuity via Infinitesimals 51
6 Calculus
   6.1 Integrals 54
   6.2 Derivatives 58
   6.3 The Fundamental Connection 61
   6.4 Historical Note: Calculus via Infinitesimals 62

7 Log and Exponential Functions 65
   7.1 Rational Exponents 66
   7.2 Natural Logarithm 67
   7.3 Exponents Reveal More Missing Numbers 69
   7.4 Historical Note: The 19th-Century Transition 72

8 The Real Number Line 75
   8.1 The Non-Negative Real Numbers $\mathbb{R}_{\geq 0}$ 76
   8.2 The Real Number Line $\mathbb{R}$ 85
   8.3 All Number Lines 89
   8.4 Historical Note: The Hyperreal Numbers 90

9 The Price of Completeness 93
   9.1 Cantor's Set Theory 94
   9.2 Lebesgue's Measure Theory 97
   9.3 Historical Note: More Infinities! 98

A Historical Summary 101

B The Reals via Dedekind Cuts 105
   B.1 Rational and Ghost Downsets 106
   B.2 Construction of the Number Line 107
   B.3 Rational and Irrational Numbers 112

C Guidelines for the Instructor 115
   C.1 Chapter Options 115
   C.2 Teaching through Guided Inquiry 116

Bibliography 121

Index 123