Chapter 1

Tropical Hypersurfaces

The tropical semiring is the triplet \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\) with
\[
x \oplus y := \min(x, y) \quad \text{and} \quad x \odot y := x + y
\]
as the tropical addition and tropical multiplication, respectively. The algebraic properties of the tropical semiring are somewhat modest, if one compares it with rings and fields, which are more common in most areas of mathematics. Still the distributive laws hold, and as an extra catch the new addition is idempotent; i.e., \(x \oplus x = x\) for all \(x \in \mathbb{R}\). The special element \(\infty\) is neutral with respect to the tropical addition. Notice that there are no additive inverses. We abbreviate \(\mathbb{T} = \mathbb{R} \cup \{\infty\}\). We will follow an elementary approach to tropical geometry, namely by studying multivariate polynomials with coefficients in \(\mathbb{T}\) and their evaluation maps. This leads to the definition of tropical hypersurfaces. As their key feature they can be described in terms of polyhedral geometry.

1.1. Tropical Polynomials

To get started, let us try the tropical arithmetic. The equality
\[
(3 \oplus 4) \odot 5 = \min(3, 4) + 5 = 8 = \min(3 + 5, 4 + 5) = (3 \odot 5) \oplus (4 \odot 5)
\]
is an instance of the distributive law, which is a crucial property of a semiring. It is intriguing to look into tropical analogues of classical arithmetic. For instance, a classical theorem from elementary number theory, proved by Fermat, characterizes those natural numbers that can be represented as a sum of two squares. The precise statement and a beautiful proof can be found, e.g., in “The Book” [AZ18, Chapter 4]. If we consider the analogous
question in tropical arithmetic, the simple computation

$$(x \odot x) \oplus (y \odot y) = \min(2x, 2y)$$

reveals that a natural number is a sum of (arbitrarily many) squares tropically if and only if it is even. The purpose of this somewhat underwhelming observation is to illustrate that tropical mathematics should be approached differently in order to get at interesting questions and meaningful solutions. In this book we will let polyhedral geometry lead the way.

When we replace the ordinary arithmetic operations by their tropical counterparts we can create tropical polynomials, such as

$$(1.1) \quad (3 \odot X^3) \oplus (1 \odot X^2) \oplus (2 \odot X) \oplus 4 = \min(3 + 3X, 1 + 2X, 2 + X, 4).$$

This gives rise to a map from $\mathbb{R}$ to $\mathbb{R}$ by substituting the indeterminate $X$ by a real number, and we adopt the notational convention

$$u^a = u \odot u \odot \cdots \odot u = u + u + \cdots + u = a \cdot u$$

for $u \in \mathbb{R}$ and $a \in \mathbb{N}$. The four affine linear functions $3 + 3X, 1 + 2X, 2 + X,$ and (the constant function which is identically) $4$ from equation (1.1) are sketched in Figure 1.1 together with their pointwise minimum. The evaluation yields a piecewise-linear function, where the number of linear pieces is bounded above by the number of monomials. Moreover, this piecewise-linear function is concave; i.e., the set of points below the graph of the function is convex. Notice that tropical polynomials with negative exponents (i.e., tropical Laurent polynomials) make sense, too, by letting $u^{-a} = -a \cdot u$.

Sometimes, just as for classical polynomials over a field of characteristic zero, it is legitimate to identify a tropical polynomial with the function defined by its evaluation. Yet, in set-theoretic terms, a classical or tropical polynomial is defined as a map which assigns a coefficient to each exponent. So terms that do not contribute to the evaluation are kept nonetheless, and this is occasionally important. Therefore, the concept we adopt for a tropical polynomial is the more formal one. However, we often abuse the same notion also for the function obtained from substituting indeterminates by real numbers.

There is a systematic way of turning an ordinary polynomial (with coefficients in a suitable field) into a tropical one. This will justify the seemingly naïve approach above, but we will postpone this discussion until Chapter 2.

The tropical addition can be extended to vectors of real numbers by taking the minimum componentwise. This way we obtain a map

$$\bigoplus : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{T}^d, \ (x, y) \mapsto x \oplus y := \left( \begin{array}{c} x_1 \oplus y_1 \\ x_2 \oplus y_2 \\ \vdots \\ x_d \oplus y_d \end{array} \right).$$
Defining the *tropical scalar multiplication* as

\[
\ast : \mathbb{T} \times \mathbb{T}^d \to \mathbb{T}^d, \ (\lambda, x) \mapsto \lambda \ast x := \begin{pmatrix}
\lambda \circ_{x_1} \\
\lambda \circ_{x_2} \\
\vdots \\
\lambda \circ_{x_d}
\end{pmatrix} = \begin{pmatrix}
\lambda + x_1 \\
\lambda + x_2 \\
\vdots \\
\lambda + x_d
\end{pmatrix} = \lambda \mathbb{1} + x,
\]

we obtain the *tropical semimodule* \((\mathbb{T}^d, \oplus, \ast)\); here \(\mathbb{1}\) denotes the all-ones vector. Clearly, it also makes sense to consider \( \max \) instead of \( \min \) as the tropical addition. In fact, later we will look into this more closely. For now, however, we stick to our choice \( \oplus = \min \).

Everything that we said about univariate tropical polynomials extends to the multivariate case. Again each multivariate monomial defines one linear function via evaluation. Formally, a *d-variate tropical polynomial* \( F \) is a map from a finite set \( S \subset \mathbb{Z}^d \) of *exponents* to \( \mathbb{T} \), the set of *coefficients*. If each exponent \( u \in S \) is mapped to the coefficient \( a_u \), then \( F \) induces the
evaluation function which sends $x \in \mathbb{R}^d$ to the real number

$$F(x) = \bigoplus_{u \in S} a_u \odot x_1^{u_1} x_2^{u_2} \ldots x_d^{u_d}$$

$$= \min \{ a_u + u_1 x_1 + u_2 x_2 + \cdots + u_d x_d \mid u \in S \}$$

$$= \min \{ a_u + \langle u, x \rangle \mid u \in S \} .$$

That is, formally tropical polynomials are very similar to classical polynomials with real coefficients, and the key distinction is that we use tropical arithmetic for the evaluation. The support $\text{supp}(F)$ is the set of exponents $u$ for which $a_u \neq \infty$. We will always assume that $F$ has nonempty support; i.e., its evaluation function takes real values only.

The (total) degree of the monomial $x_1^{u_1} x_2^{u_2} \ldots x_d^{u_d}$ is the sum $u_1 + u_2 + \cdots + u_d$ of the exponents, and the degree of $F$ is the maximal degree of a monomial in the support. Recall that, by admitting arbitrary integer exponents, our tropical polynomials are, in fact, tropical analogues of Laurent polynomials.

A (convex) polyhedron in $\mathbb{R}^d$ is the set of feasible solutions to finitely many linear inequalities; or, in more geometric terms, it is the intersection of finitely many affine half-spaces. If it is bounded, then it is a polytope.

Throughout this book we will heavily use notions from polyhedral geometry; see also Section A.1 in Appendix A. A (proper) face of a polyhedron $P$ is the intersection of $P$ with one of its supporting hyperplanes; additionally we view $\emptyset$ and $P$ as trivial faces of $P$. Equivalently, a face is the set of optimal solutions of some linear program with $P$ as the feasible domain; see Section 7.1 for more on linear optimization. A maximal proper face, i.e., a face of codimension 1, is called a facet.

For a $d$-variate tropical polynomial $F$, we call the set

$$\mathcal{D}(F) := \{ (x, s) \in \mathbb{R}^{d+1} \mid x \in \mathbb{R}^d, s \in \mathbb{R}, s \leq F(x) \}$$

the dome of $F$.

**Proposition 1.1.** The dome $\mathcal{D}(F)$ of the $d$-variate tropical polynomial $F$ is an unbounded polyhedron of dimension $d + 1$.

**Proof.** By construction, the set $\mathcal{D}(F)$ is the intersection of finitely many affine half-spaces in $\mathbb{R}^{d+1}$ and hence is a polyhedron. Choose some $x \in \mathbb{R}^d$. By definition $\text{supp}(F) \neq \emptyset$, and hence $F(x)$ is a real number. For any $s < F(x)$ the point $(x, s)$ lies in $\mathcal{D}(F)$, which shows that $\mathcal{D}(F)$ is not empty. Further, since $s < F(x)$, the point $(x, s)$ has a positive distance from each facet, which is defined by some monomial of $F$. Hence for some $\rho > 0$ the closed ball of radius $\rho$ around $(x, s)$ is entirely contained in $\mathcal{D}(F)$. This proves that the polyhedron is full-dimensional. As $s$ can be chosen arbitrarily small, we also see that $\mathcal{D}(F)$ cannot be bounded. \qed
1.1. Tropical Polynomials

That $\mathcal{D}(F)$ is a full-dimensional unbounded convex polyhedron is a consequence of the fact that its outer facet normal vectors do not positively span the entire space $\mathbb{R}^{d+1}$. Indeed, all of them point upward, which we take as a synonym for the positive $e_{d+1}$-direction. The proof above just spells out the details in an elementary way.

Notation 1.2. Whenever it comes to computing with explicit coordinates, one must choose whether vectors are rows or columns. Throughout this book we take the following approach: Generally, vectors denoting points are columns. This entails that matrices act on the left, and linear forms are given as row vectors. This agrees with the majority of linear algebra textbooks. However, if there is sufficient hope that confusion can be avoided, we take the freedom to also write coordinate vectors describing individual points as rows without explicitly marking that such a vector ought to be transposed.

Classical algebraic geometry studies (affine) algebraic varieties, which are sets of points where a given collection of polynomials vanishes, i.e., sets where the polynomials attain the value zero. For the tropical evaluation of a polynomial, the value zero is not of any particular relevance (nor any other specific value). Instead the following definition turns out to be fruitful.

Definition 1.3. A $d$-variate tropical polynomial $F$ vanishes at $x \in \mathbb{R}^d$ if the minimum

$$F(x) = \bigoplus_{u \in S} a_u \odot x_1^{u_1} x_2^{u_2} \ldots x_d^{u_d} = \min \{ a_u + \langle u, x \rangle \mid u \in S \}$$

is attained at least twice. The vanishing locus

$$\mathcal{T}(F) := \left\{ x \in \mathbb{R}^d \mid F \text{ vanishes at } x \right\}$$

is the tropical hypersurface defined by $F$.

Notice that only our earlier decision to view tropical polynomials formally, i.e., with all their terms explicitly written down, allows for this definition. The justification that this, indeed, is the proper notion of vanishing is deferred to Chapter 2.

Example 1.4. Definition 1.3 requires (at least) two terms with distinct exponents to attain the minimum. That is, e.g., the univariate tropical polynomial $X \oplus X = \min (X, X) = X$ does not vanish anywhere.

Observation 1.5. Two tropical polynomials with the same support and whose coefficients differ by a constant share the same tropical hypersurface. That is, $F(X) = \bigoplus_{u \in S} a_u \odot X_1^{u_1} \ldots X_d^{u_d}$ and $G(X) = \bigoplus_{u \in S} (a_u \odot c) \odot X_1^{u_1} \ldots X_d^{u_d}$, where $c$ is a real constant, yield $\mathcal{T}(F) = \mathcal{T}(G)$.
Our next step is to reveal the polyhedral structure of a tropical hypersurface. A polyhedral complex is a finite set of polyhedra, called cells of the polyhedral complex, closed with respect to taking faces, such that the intersection of any two polyhedra is a common face, which may be empty. The cells of a polyhedral complex are partially ordered by inclusion, and the empty cell is the unique minimal element. The $k$-skeleton of a polyhedral complex is the polyhedral subcomplex of cells of dimension at most $k$. If $\Pi$ is a polyhedral complex of dimension $d$, then its $(d-k)$-skeleton is also called the codimension-$k$-skeleton of $\Pi$.

**Corollary 1.6.** The tropical hypersurface $T(F)$ coincides with the image of the codimension-2-skeleton of its dome $D(F)$ in $\mathbb{R}^d$ under the orthogonal projection that omits the last coordinate.

**Proof.** Each facet of the polyhedron $D(F)$ from Proposition 1.1 corresponds to one of the tropical monomials $X_1^{i_1}X_2^{i_2}\ldots X_d^{i_d}$. The tropical polynomial $F$ vanishes at $x$ if there are two distinct monomials at which the minimum $F(x)$ is attained. This is the case if and only if the point $(x, F(x))$ in the boundary $\partial D(F)$ is contained in at least two facets. □

This says that a tropical hypersurface of a $d$-variate tropical polynomial is a $(d-1)$-dimensional polyhedral complex, which is pure; i.e., all maximal cells share the same dimension. Observe that $F$ vanishes at $p$ if and only if the piecewise-linear hypersurface $\partial D(F)$ is not differentiable at $(x, F(x))$.

**Example 1.7.** For the univariate tropical polynomial $F(X) = (3 \circ X^3) \oplus (1 \circ X^2) \oplus (2 \circ X) \oplus 4$ from the beginning of this section the dome $D(F)$ is 2-dimensional. Its 0-skeleton consists of the three vertices, $(-2, -3)$, $(1, 3)$, and $(2, 4)$. The tropical hypersurface $T(F)$ is the set $\{-2, 1, 2\}$. See Figure 1.1.

Another way to read Corollary 1.6 is to say that the relative interiors of the facets of the dome $D(F)$ vertically project onto the connected components of the complement of the tropical hypersurface $T(F)$ in $\mathbb{R}^d$. We call the topological closure of one such connected component a region (of linearity) of the tropical hypersurface. The regions generate an ordinary polyhedral decomposition of $\mathbb{R}^d$, called the normal complex, $NC(F)$, whose codimension-1-skeleton is precisely the tropical hypersurface. In the example above, the normal complex consists of the four regions $(-\infty, -2]$, $[-2, 1]$, $[1, 2]$, and $[2, \infty)$.

It is worthwhile to explicitly dualize the construction of the polyhedron $D(F)$. The idea is to associate with a tropical polynomial the point configuration given by its support and to interpret the coefficients as a height
function. The convex hull of \( \text{supp}(F) \) is the \textit{Newton polytope} of \( F \), denoted as \( \mathcal{N}(F) \). That is, if \( F(X) = \bigoplus_{u \in S} a_u \odot X_1^{u_1} X_2^{u_2} \ldots X_d^{u_d} \), then
\[
\mathcal{N}(F) = \text{conv} \{ u \in S \mid a_u \neq \infty \} \subset \mathbb{R}^d.
\]
The set
\[
\tilde{\mathcal{N}}(F) = \text{conv} \{(u, r) \in \mathbb{Z}^d \times \mathbb{R} \mid u \in S, r \geq a_u \} \subset \mathbb{R}^{d+1}
\]
is called the \textit{extended Newton polyhedron} \( \tilde{\mathcal{N}}(F) \) of \( F \). Formally, the extended Newton polyhedron is the convex hull of infinitely many points. So we should make sure its name is justified. This requires showing that \( \tilde{\mathcal{N}}(F) \) can be described by finitely many linear inequalities. In view of the Weyl–Minkowski Theorem, Theorem A.2, this amounts to verifying that the extended Newton polyhedron can be written as the sum of a polytope and a finitely generated cone. The latter is accomplished by the following.

\textbf{Observation 1.8.} The extended Newton polyhedron is the Minkowski sum of a polytope and a single ray. More precisely,
\[
\tilde{\mathcal{N}}(F) = \text{conv} \{(u, a_u) \mid u \in \text{supp}(F)\} + \text{pos}\{e_{d+1}\}.
\]
Here \( \text{pos}(M) = \{ \sum \lambda_i m_i \mid \lambda_i \geq 0, m_i \in M \} \) is the \textit{cone} spanned by the set \( M \), and “+” denotes Minkowski addition.

\textbf{Example 1.9.} We continue Example 1.7. The support of the tropical polynomial \( F \) is the four-point set \( \{0, 1, 2, 3\} \) on the real line, and so the Newton

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tropical-polynomial-figure}
\caption{Support and extended Newton polyhedron of a univariate tropical polynomial.}
\end{figure}
polytope $\mathcal{N}(F)$ is the interval $[0, 3]$. In this case the extended Newton polyhedron $\tilde{\mathcal{N}}(F)$ has the four vertices $(0, 4), (1, 2), (2, 1),$ and $(3, 3)$ in $\mathbb{R}^2$. Figure 1.2 shows a picture.

Before we continue with the analysis of the Newton polytopes and the extended Newton polyhedra, we wish to look at the situation in more generality.

1.2. Regular Subdivisions

Consider a finite set of points $A$ in $\mathbb{R}^d$. A (polyhedral) subdivision of $A$ is a finite polytopal complex whose vertices lie in $A$ and that covers the convex hull $\text{conv} A$. If all cells of the subdivision are simplices, it is called a triangulation. A trivial example to keep in mind is the following: Each finite set of points is trivially subdivided by its convex hull. The example below, however, is more interesting.

Example 1.10. Let $A = \{(u, v) \in \mathbb{Z}^2 \mid u \geq 0, v \geq 0, u + v \leq 4\}$ be the set of lattice points in the triangle $\text{conv}\{(0, 0), (4, 0), (0, 4)\}$. Figure 1.3 shows a subdivision of $A$ with ten maximal cells, nine triangles, and one quadrangle. More details on that subdivision will be given in Example 1.12 below.

Now we look at an arbitrary height function $\omega : A \to \mathbb{R}$. This gives rise to the unbounded polyhedron

$$U(A, \omega) = \text{conv}\{(u, \omega(u)) \mid u \in A\} + \text{pos}\{e_{d+1}\} \quad (1.2)$$

in $\mathbb{R}^{d+1}$. A face of $U(A, \omega)$ is called a lower face if it has an outward pointing normal vector $h$ satisfying $\langle h, e_{d+1} \rangle < 0$. That is to say, the normal vector $h$ is pointing downward. In general, a face of an unbounded polyhedron may be unbounded or bounded.

Observation 1.11. The lower faces of $U(A, \omega)$ are precisely those that are bounded.

The lower faces form a subcomplex in the boundary of $U(A, \omega)$, and projecting down yields the regular subdivision $\Sigma(A, \omega)$ of $A$ induced by $\omega$. If the points in $A$ are in convex position, i.e., $A$ is the set of vertices of a convex polytope, then the vertices of any regular subdivision are precisely the points in $A$. If $A \subset \mathbb{R}^d$ is arbitrary and $\omega$ maps each point $p$ to the square of its Euclidean norm $\|p\|^2$, then the regular subdivision of $A$ induced by $\omega$ is known as the Delaunay subdivision of $A$. This is dual to what is known as the Voronoi diagram of $A$. In this sense, regular subdivisions generalize the Delaunay subdivision of a point configuration, which is why they also go by the name “weighted Delaunay subdivisions”.
### 1.2. Regular Subdivisions

Figure 1.3. Regular subdivision of 15 points in $\mathbb{R}^2$ as described in Example 1.12. The four points $(0, 3), (1, 2), (1, 3),$ and $(3, 1)$ do not form vertices of any cell. While $(1, 2)$ occurs in a quadrangular 2-cell, the other three do not occur at all.

**Example 1.12.** The subdivision in Figure 1.3 is regular. A height function $\omega$ is given by

\[
\begin{align*}
(0, 0) \mapsto & 8, \quad (1, 0) \mapsto 4, \quad (0, 1) \mapsto 2, \quad (2, 0) \mapsto 1, \quad (1, 1) \mapsto 0, \\
(0, 2) \mapsto & 1, \quad (3, 0) \mapsto 2, \quad (2, 1) \mapsto 0, \quad (1, 2) \mapsto 0, \quad (0, 3) \mapsto 4, \\
(4, 0) \mapsto & 8, \quad (3, 1) \mapsto 4, \quad (2, 2) \mapsto 0, \quad (1, 3) \mapsto 2, \quad (0, 4) \mapsto 0.
\end{align*}
\]

The vertices of the quadrangular cell $\text{conv}\{(1, 1), (2, 1), (2, 2), (0, 4)\}$ are lifted to height zero. The point $(1, 3)$ also lies in that quadrangle, but it is lifted higher than all the vertices. The interior point $(1, 2)$, however, is also lifted to height zero. In fact, this is the only point in the set $A$ which is lifted to a point which lies in the relative interior of a (quadrangular) face of the polyhedron $U(A, \omega)$, which is why it receives a special mark in Figure 1.3. Occasionally we say that a point like $(1, 2)$ occurs in the corresponding cell; vertices of cells always occur.

There are two things to keep in mind. First, most point configurations admit subdivisions which are not regular; see Problem 1.32. Second, the set of height functions that define one fixed regular subdivision is a relatively open polyhedral cone, called the secondary cone of the subdivision. In particular, such a height function is never unique. For more details, see Section A.4.

The attentive reader will have noticed that the description of the extended Newton polyhedron of a tropical polynomial $F$ in Observation 1.8 agrees with the definition in (1.2), where the set $A$ is the support of $F$ and
the height function is given by the coefficients. That is to say, studying the extended Newton polyhedra of tropical polynomials is the same as studying height functions and regular subdivisions of finite configurations in \( \mathbb{Z}^d \). The regular subdivision of \( \text{supp}(F) \) induced by the coefficients of \( F \) is denoted as \( \mathcal{S}(F) \). In view of the following result \( \mathcal{S}(F) \) is called the dual subdivision of the tropical hypersurface \( \mathcal{T}(F) \).

**Theorem 1.13.** Let \( F \) be a \( d \)-variate tropical polynomial.

1. There is an inclusion-reversing bijection, \( \beta \), between the faces of the dome \( \mathcal{D}(F) \) and the bounded faces of the extended Newton polyhedron \( \tilde{\mathcal{N}}(F) \).
2. Via orthogonal projection the proper faces \( \tilde{\mathcal{N}}(F) \) correspond to the cells of the dual subdivision \( \mathcal{S}(F) \) of \( \text{supp}(F) \).
3. In this way \( \beta \) induces a bijection between the \( k \)-dimensional cells of the tropical hypersurface \( \mathcal{T}(F) \) to the \( (d-k) \)-dimensional cells of \( \mathcal{S}(F) \) for \( 0 \leq k < d \). The regions of \( \mathcal{T}(F) \) correspond to the vertices of \( \mathcal{S}(F) \).

**Proof.** Let \( F(X) = \bigoplus_{u \in S} a_u \odot X^u \) with \( S = \text{supp}(F) \). Each facet of the polyhedron \( \mathcal{D}(F) \) corresponds to some term of \( F \). The point \( (w, F(w)) \in \mathbb{R}^{d+1} \) is contained in the facet defined by \( a_u \odot X^u \) if and only if the minimum \( F(w) \) equals \( a_u + w_1 \cdot u_1 + \cdots + w_d \cdot u_d \). Letting \( (w, 1) := (w_1, w_2, \ldots, w_d, 1) \in \mathbb{R}^{d+1} \) this is equivalent to saying that the linear form \( (w, 1) \) attains its minimum on the polyhedron \( \tilde{\mathcal{N}}(F) \) at the point \( (u, a_u) \). This means that \( -(w, 1) \) is an element of the (outer) normal cone of the point \( (u, a_u) \) in the boundary of \( \tilde{\mathcal{N}}(F) \). Conversely, for each vector \( -(w, 1) \) in the normal cone of \( (u, a_u) \) the point \( w \) is contained in the facet of \( \mathcal{D}(F) \) defined by \( a_u \odot X^u \).

Now the vectors of the form \( (v, 1) \) and their positive multiples are precisely those vectors in \( \mathbb{R}^{d+1} \) which form an angle of more than \( \pi/2 \) with the direction \( e_{d+1} \) of the lifting. It follows that a face of \( \tilde{\mathcal{N}}(F) \) which admits \( -(v, 1) \) as an outer normal vector is a lower face. By Observation 1.11 the lower faces are precisely the bounded ones. As a poset the normal fan of \( \tilde{\mathcal{N}}(F) \) is anti-isomorphic to the face lattice of \( \tilde{\mathcal{N}}(F) \).

This allows us to define the map \( \beta \) as follows. If \( a_u \odot X^u \) defines a facet of \( \mathcal{D}(F) \), we can pick an interior point, \( w \in \mathbb{R}^d \), in the corresponding region. Then \( F(w) = a_u + w_1 \cdot u_1 + \cdots + w_d \cdot u_d \). Since \( w \) is generic the linear form \( -(w, 1) \) is contained in a maximal normal cone. Equivalently, \( -(w, 1) \) attains its maximum at a vertex, \( \beta(u) \), of \( \tilde{\mathcal{N}}(F) \). This extends to arbitrary faces of the dome by sending intersections of facets to convex hulls of vertices. So \( \beta \) reverses the inclusion, and this proves the first claim.
The orthogonal projection from $\mathbb{R}^{d+1}$ to $\mathbb{R}^{d}$ which omits the last coordinate maps the poset of lower faces of $\tilde{\mathcal{N}}(F)$ to the polytopal complex $\mathcal{S}(F)$ subdividing the point set $\text{supp}(F)$. The 0-dimensional cells of $\mathcal{S}(F)$ form a subset of $\text{supp}(F)$, and their convex hull coincides with $\mathcal{N}(F)$. Each $k$-dimensional cell of the tropical hypersurface $\mathcal{T}(F)$ corresponds to a $k$-face of $\mathcal{D}(F)$, also via orthogonal projection. This shows the rest. □

**Example 1.14.** The height function in (1.3), which induces the regular subdivision in Figure 1.3, gives rise to the bivariate tropical polynomial

$$F(X,Y) = \min(8, 4 + X, 2 + Y, 1 + 2X, X + Y, 1 + 2Y, 2 + 3X, 2X + Y,$$

$$X + 2Y, 4 + 3Y, 8 + 4X, 4 + 3X + Y, 2X + 2Y, 2 + X + 3Y, 4Y),$$

whose degree equals four. The tropical hypersurface $\mathcal{T}(F)$ is the *tropical plane curve* shown in Figure 1.4. Each region is marked with its *dominant term*, which attains its minimum at all points of that region. For instance, $F(0,0) = 0$, and that minimum is attained at $X + Y$, $2X + Y$, $2X + 2Y$, and $4Y$. These four terms correspond to the vertices of the quadrangle inspected in Example 1.12. The point $(1, 2)$, which is marked yellow in

![Figure 1.4. Tropical plane curve of degree four and its regions, marked with their dominant terms.](image-url)
Figure 1.3, corresponds to the term $X + 2Y$ which also attains zero at $(0, 0)$. Four among the 15 terms of $F$ do not correspond to a region of the tropical plane curve $\mathcal{T}(F)$; these are $X + 2Y$, $4 + 3Y$, $2 + X + 3Y$, and $4 + 3X + Y$. However, in contrast to $X + 2Y$ the minimum $F(X, Y)$ is never attained at, e.g., $4 + 3Y$.

The not necessarily regular subdivisions of a fixed point configuration $A$ are partially ordered in a natural way. Let $\Sigma$ and $\Sigma'$ both be subdivisions of $A$. If each cell of $\Sigma$ is a subpolytope of some cell of $\Sigma'$, then $\Sigma$ is said to refine $\Sigma'$. Conversely, $\Sigma'$ coarsens $\Sigma$. In the refinement partial ordering the trivial subdivision $\text{conv} A$ is the unique coarsest subdivision, albeit not a proper one. The finest subdivisions are the triangulations which use all points in $A$.

1.3. Minimum versus Maximum

Choosing min as our tropical addition is by no means canonical. Let us explore what happens if we exchange min by max. The equality $\min(-x, -y) = -\max(x, y)$ establishes an isomorphism of semirings between $(\mathbb{R}, \min, +)$ and $(\mathbb{R}, \max, +)$. With respect to the maximum operation $-\infty$ is the neutral element. So we arrive at two versions of the tropical semiring, which we write as

$$T_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +) \quad \text{and} \quad T_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$$

if we need to distinguish. To the $d$-variate min-tropical polynomial

$$F(X) = \min \left\{ a_u + \langle u, X \rangle \mid u \in S \right\}$$

we can associate the (likewise $d$-variate) max-tropical polynomial

$$F^*(X) = \max \left\{ -a_u + \langle u, X \rangle \mid u \in S \right\} ,$$

where we replace min by max and take the negatives of the coefficients. Evaluating $F^*$ at the point $-x \in \mathbb{R}^d$ now gives

$$F^*(-x) = \max \left\{ -a_u + \langle u, -x \rangle \mid u \in S \right\} = \max \left\{ -(a_u + \langle u, x \rangle) \mid u \in S \right\} = -\min \left\{ a_u + \langle u, x \rangle \mid u \in S \right\} = -F(x) .$$

From this computation it follows that $F$ vanishes at $x$ if and only if $F^*$ vanishes at $-x$. Therefore, we have the duality relation

$$\mathcal{T}^{\max}(F^*) = -\mathcal{T}^{\min}(F) \quad (1.4)$$

of tropical hypersurfaces. In particular, the image of a min-tropical hypersurface under the reflection at the origin is a max-tropical hypersurface and vice versa. See Figures 1.5(a) and 1.5(b) for a min-tropical line in $\mathbb{R}^3/\mathbb{R}1$. 

1.4. The Tropical Projective Torus

and its mirror image; notice that the dual $a^*$ of the linear tropical polynomial $a$ equals $-a$, which is why the apex of the max-tropical hyperplane $T^\text{max}(a^*)$ is $a$.

Remark 1.15. Whenever we replace min by max we also need to redefine the notion of a regular subdivision if we want to make use of results like Theorem 1.13. In fact, it suffices to replace (1.2) by $\text{conv}\{(u, \omega(u)) \mid u \in A\} - \text{pos}\{e_{d+1}\}$ and to focus on the upper facets instead of the lower ones.

We conclude that it is just a matter of taste if one prefers min or max. However, later we will encounter the situation where it is convenient to consider both additive structures at the same time. For now we stick to min for our tropical addition.

1.4. The Tropical Projective Torus

In classical geometry it is often convenient to study projective instead of affine varieties. This means dealing with homogeneous polynomials rather than arbitrary ones. A $d$-variate tropical polynomial $F$ is homogeneous of degree $\delta$ if for all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ we have

\[ F(\lambda \odot x) = F(\lambda \cdot 1 + x) = \lambda^\delta \odot F(x) = \delta \cdot \lambda + F(x), \]

where $1$ denotes the all-ones vector of length $d$. Suppose that $F$ is homogeneous of degree $\delta$, which is the case if and only if each monomial has degree $\delta$. Then $F$ vanishes at $x \in \mathbb{R}^d$ if and only if the minimum $F(x)$ is attained at least twice, or equivalently, the minimum $F(\lambda \odot x) = \delta \cdot \lambda + F(x)$ is attained at least twice for all $\lambda \in \mathbb{R}$. Hence it makes sense to consider the tropical hypersurfaces of homogeneous tropical polynomials in the quotient of $\mathbb{R}^d$ obtained by factoring out the tropical scalar multiplication.

Definition 1.16. The direct product $(\mathbb{C}^\times)^d$ of $d$ copies of the multiplicative group of the complex numbers is called an algebraic $d$-torus. The quotient $\mathbb{R}^d/\mathbb{R}^1$ is called the tropical projective $(d-1)$-torus.

The name “torus” for $(\mathbb{C}^\times)^d$ comes about since for $d = 2$, as a topological space, this is homotopy equivalent to the usual torus $S^1 \times S^1$, which is the oriented closed surface of genus one. In the tropical setting $\mathbb{C}$ is replaced by $\mathbb{R}$, and $0$ is replaced by $\infty$, the neutral element with respect to $\oplus$. The word “projective” refers to taking the quotient by one copy of $\mathbb{R}$, hence the name for $\mathbb{R}^d/\mathbb{R}^1$. The quotient topology lets $\mathbb{R}^d/\mathbb{R}^1$ inherit a topology from the natural topology on $\mathbb{R}^d$. By virtue of the bijective map

\[ (x_1, x_2, \ldots, x_d) + \mathbb{R}^1 = (0, x_2 - x_1, \ldots, x_d - x_1) + \mathbb{R}^1 \]

\[ \mapsto (x_2 - x_1, \ldots, x_d - x_1) \]
the tropical projective torus $\mathbb{R}^d/\mathbb{R}1$ is homeomorphic to $\mathbb{R}^{d-1}$. Often, we will identify $\mathbb{R}^d/\mathbb{R}1$ with $\mathbb{R}^{d-1}$, and if we do so then always with respect to the map (1.6). For a more thorough discussion of the topological situation see Section 5.1. Specializing Corollary 1.6 to the homogeneous case yields:

**Corollary 1.17.** The tropical hypersurface $\mathcal{T}(F)$ of a homogeneous $d$-variate tropical polynomial $F$ of degree $\delta \geq 1$ is either empty or a pure and connected $(d - 2)$-dimensional polyhedral complex in $\mathbb{R}^d/\mathbb{R}1 = \mathbb{R}^{d-1}$.

Often we will treat coordinate vectors of points in $\mathbb{R}^d/\mathbb{R}1$ as homogeneous coordinates; i.e., we identify $(x_1, x_2, \ldots, x_d)$ with $\lambda \odot (x_1, x_2, \ldots, x_d)$ for $\lambda \in \mathbb{R}$.

**Example 1.18.** For the homogeneous linear tropical polynomial $(a_1 \odot X_1) \oplus (a_2 \odot X_2) \oplus (a_3 \odot X_3)$, with $a_i \in \mathbb{R}$, the tropical hypersurface equals

$$-(a_1, a_2, a_3) + (\mathbb{R}_{\geq 0}(1, 0, 0) \cup \mathbb{R}_{\geq 0}(0, 1, 0) \cup \mathbb{R}_{\geq 0}(0, 0, 1)) \oplus \mathbb{R}1$$

which, as a subset of $\mathbb{R}^3/\mathbb{R}1$, is the same as

$$(0, a_1 - a_2, a_1 - a_3) + (\mathbb{R}_{\geq 0}(0, -1, -1) \cup \mathbb{R}_{\geq 0}(0, 1, 0) \cup \mathbb{R}_{\geq 0}(0, 0, 1)) .$$

By projecting onto the last two coordinates as in (1.6) we obtain the image shown in Figure 1.5(a).

A tropical hyperplane is the tropical hypersurface of a homogeneous linear tropical polynomial

$$\tag{1.7} (a_1 \odot X_1) \oplus (a_2 \odot X_2) \oplus \cdots \oplus (a_d \odot X_d)$$

for some vector $a \in \mathbb{T}^d$, where we assume that the support of $a$ contains at least two elements. This entails that a tropical hyperplane is never empty. We may identify the vector $a$ with the tropical linear form (1.7), and thus

Figure 1.5. Tropical hyperplanes in $\mathbb{R}^3/\mathbb{R}1$ with respect to min and max.
we denote the tropical hyperplane as $\mathcal{T}(a)$. It is generic if $a$ has full support. Adding a constant vector $\lambda \mathbb{1}$ gives the same tropical hyperplane; i.e.,

$$\mathcal{T}(a) = \mathcal{T}(\lambda \mathbb{1} + a) = \mathcal{T}(\lambda \odot a).$$

This means that the set of generic tropical hyperplanes in $\mathbb{R}^d/\mathbb{R}^1$ bijectively corresponds with the points in the tropical projective torus $\mathbb{R}^d/\mathbb{R}^1$. This corresponds to the classical situation where linear hyperplanes in $\mathbb{C}^d$ (or, equivalently, projective hyperplanes in $\mathbb{P}^{d-1}\mathbb{C}$ are parameterized by the points in the projective space $\mathbb{P}^{d-1}\mathbb{C}$. The special point $-a$, where the minimum is attained in all monomials simultaneously, is called the apex of the tropical hyperplane $\mathcal{T}(a)$.

Let us examine the tropical plane curves of degree two. A general homogeneous tropical polynomial of degree two in three indeterminates equals

$$
(a_{200} \odot X_1^2) \oplus (a_{110} \odot X_1 X_2) \oplus (a_{101} \odot X_1 X_3)
\oplus (a_{020} \odot X_2^2) \oplus (a_{011} \odot X_2 X_3) \oplus (a_{002} \odot X_3^2)
$$

(1.8)

for six parameters $a_{ijk} \in \mathbb{T}$. Here we restrict our attention to the case where all exponents are nonnegative; i.e., we are looking at tropical analogues of ordinary rather than Laurent polynomials. The tropical hypersurface defined by (1.8) is a tropical plane conic. An example with the parameter sequence

$$
(a_{200}, a_{110}, a_{101}, a_{020}, a_{011}, a_{002}) = (6, 5, 5, 6, 5, 7)
$$

(1.9)

is shown in Figure 1.6.

Notice that the Newton polytope of the tropical polynomial in (1.8) is contained in the dilated triangle $2 \cdot \Delta_2$, where $\Delta_2$ is the regular triangle spanned by the standard basis vectors $e_1, e_2, e_3$ in $\mathbb{R}^3$. 

![Figure 1.6. A tropical plane conic in $\mathbb{R}^3/\mathbb{R}^1$ (left) and the dual subdivision of the Newton polytope $2\Delta_2$ (right).](image_url)
A tropical polynomial in \(d\) indeterminates which is homogeneous of degree \(\delta \geq 1\) is said to have full support if its support equals \(\delta \Delta_{d-1} \cap \mathbb{Z}^d\), the set of all lattice points in the dilated simplex \(\delta \cdot \Delta_{d-1}\). In particular, in this case the vertices \(\delta e_k\) of \(\delta \cdot \Delta_{d-1}\) are present in the support, and hence the Newton polytope equals \(\delta \Delta_{d-1}\). The following is a direct consequence of Theorem 1.13.

**Corollary 1.19.** Let \(F\) be a homogeneous \(d\)-variate tropical polynomial of degree \(\delta \geq 1\) with full support. Then the tropical hypersurface \(T(F)\) is dual to the 1-coskeleton of the regular subdivision of the dilated simplex \(\delta \cdot \Delta_{d-1}\) induced by the coefficients of \(F\).

Clearly, in a tropical polynomial \(F\) of fixed degree only finitely many exponents occur, and this restricts the choice for the facet normals of the dome \(D(F)\). This way the degree of \(F\) imposes restrictions on the combinatorics of the tropical hypersurface \(T(F)\).

**Remark 1.20.** The dome of a homogeneous tropical polynomial has the all-ones vector in its lineality space. Often we will tacitly ignore these linealities. For purposes of visualization we usually identify the homogeneous point \((x_1, x_2, \ldots, x_d) + \mathbb{R}1\) with \((x_2 - x_1, x_3 - x_1, \ldots, x_d - x_1)\) in \(\mathbb{R}^{d-1}\) as in (1.6).

### 1.5. Constant Coefficients

It is worthwhile to look into the special case of a tropical polynomial \(F\) whose coefficients are all the same. Then the dual subdivision \(S(F)\) is trivial, as all the lattice points in the Newton polytope \(\mathcal{N}(F)\), which correspond to monomials, are raised to the same height. Consequently, the extended Newton polyhedron \(\mathcal{N}(F)\) is the Minkowski sum of \(\mathcal{N}(F)\) and the upward pointing ray. Clearly, in all of this, the exact common value of the coefficients does not matter. We can specialize Theorem 1.13 to constant coefficients to yield the following.

**Corollary 1.21.** Let \(F\) be a min-tropical (resp. max-tropical) polynomial with constant coefficients. Then the tropical hypersurface \(T(F)\) coincides with the codimension-1-skeleton of the inner (resp. outer) normal fan of its Newton polytope \(\mathcal{N}(F)\).

**Proof.** Let \(F(X) = \bigoplus_{u \in S} a_u \odot X^u\) be a min-tropical polynomial, with \(S = \text{supp}(F)\) and such that there exists \(c \in \mathbb{R}\) with \(a_u = c\) for all \(u \in S\). Thanks to Observation 1.5 we may assume that \(c = 0\). As in the proof of Theorem 1.13 the point \((w, F(w)) \in \mathbb{R}^{d+1}\) is contained in the facet of dome \(D(F)\) defined by \(X^u\) if and only if the minimum \(F(w)\) equals \(w_1 \cdot u_1 + \cdots + w_d \cdot u_d\). With \((w, 1) := (w_1, w_2, \ldots, w_d, 1) \in \mathbb{R}^{d+1}\) this is equivalent to the linear form \((w, 1)\) attaining its minimum on the polyhedron \(\mathcal{N}(F)\) at the
point \((u, 0)\). And because each coefficient of \(F\) is zero, the same holds true for the linear form \((w, 0)\); the latter is an inward pointing normal vector of the facet of \(\mathcal{N}(F)\) which corresponds to \(u\).

If we replace \(\min\) by \(\max\), the tropical hypersurface of a tropical polynomial with constant coefficients consists of outer normal cones. 

**Example 1.22.** A planar example is

\[
F(X,Y) = \min\left( X, 2X, 2Y, X + 2Y, 2X + Y \right),
\]

where all coefficients equal zero. This is a bivariate tropical polynomial of degree three, which is not homogeneous, and so \(F\) defines a tropical plane cubic in \(\mathbb{R}^2\); see Figure 1.7.

For instance, the edge which joins the vertices \((1,0)\) and \((0,2)\) of the Newton polygon \(\mathcal{N}(F)\) has slope \(-2\). In the plane, edges and facets are the same, and the vector \((2, 1)\) is an inner normal vector on that facet. Therefore the ray \(\mathbb{R}_{\geq 0}(2, 1)\) is contained in the tropical plane curve \(\mathcal{T}(F)\).

**1.6. Tropical Plane Curves**

After the tropical hyperplanes, which have been discussed in Section 1.4, the simplest class of tropical hypersurfaces are curves in the plane. Although this is not the main focus of this book we wish to exhibit some basic properties. In particular, we will see how tropical plane curves give rise to planar metric graphs. Let

\[
F(X, Y) = \bigoplus_{u \in S} a_u \odot X^{u_1} Y^{u_2}
\]
be a bivariate tropical polynomial with nonempty support \( S \subset \mathbb{Z}^2 \) and coefficients \( a_u \in \mathbb{R} \), and \( \mathcal{C} := \mathcal{T}(F) \) is a tropical plane curve. Here we stick to the inhomogeneous setting, while we could also study the same objects via their homogenizations in \( \mathbb{R}^3/\mathbb{R}1 \).

A lattice triangle \( \Delta \) in \( \mathbb{R}^2 \) is \textit{unimodular} if it has area \( \frac{1}{2} \). Equivalently it does not contain any lattice points other than its vertices.

\textbf{Observation 1.23.} The triangle \( \Delta \) is unimodular if and only if there is a lattice vector \( v \in \mathbb{Z}^2 \) and an integral linear transformation \( \tau \in \text{SL}_2 \mathbb{Z} \) such that
\[
v + \tau(\Delta) = \text{conv}\{0, e_1, e_2\}
\]
is the standard triangle in \( \mathbb{R}^2 \).

The elements of the special linear group \( \text{SL}_2 \mathbb{Z} \) are \( 2 \times 2 \)-matrices with integer coefficients and determinant one; they are also called \textit{unimodular transformations}. A triangulation of a finite point set in \( \mathbb{Z}^2 \) is \textit{unimodular} if each triangle is unimodular. The tropical plane curve \( \mathcal{C} \) is \textit{smooth} if its dual subdivision \( \mathcal{S}(F) \) is a unimodular triangulation of the support. Figure 1.6 shows a smooth tropical curve of degree two, whereas the tropical curve of degree four in Figure 1.4 is not smooth.

From Theorem 1.13 we know that \( \mathcal{C} \) is a polyhedral complex of dimension 1 in \( \mathbb{R}^2 \). It consists of \textit{vertices}, which are the 0-dimensional cells, and \textit{edges}, which are the 1-dimensional cells. The edges are equipped with positive edge lengths in the following way. Each bounded edge, \( \eta \), of \( \mathcal{C} \) is dual to an adjacent pair of unimodular triangles, \( \Delta \) and \( \Delta' \). Assuming that \( \Delta = \text{conv}(u, w, v) \) and \( \Delta' = \text{conv}(v, w, x) \), with their vertices in clockwise ordering, it follows from Theorem 1.13 that
\[
(1.11) \quad \det \begin{pmatrix}
1 & 1 & 1 & 1 \\
\mathbf{u}_1 & \mathbf{v}_1 & \mathbf{w}_1 & \mathbf{x}_1 \\
\mathbf{u}_2 & \mathbf{v}_2 & \mathbf{w}_2 & \mathbf{x}_2 \\
a_u & a_v & a_w & a_x
\end{pmatrix}
\]
is strictly positive, and we take that value as the \textit{lattice length} of \( \eta \). This defines positive lengths for the bounded edges of \( \mathcal{C} \). The unbounded edges have infinite lengths.

\textbf{Example 1.24.} We consider the bivariate tropical polynomial
\[
F(X, Y) = 3 \oplus 1X \oplus 1Y \oplus 2X^2 \oplus 0XY \oplus 6Y^2 \oplus 4X^3 \oplus 0X^2Y \oplus 2XY^2 \oplus 12Y^3 \oplus 8X^4 \oplus 3X^3Y \oplus 0X^2Y^2 \oplus 9XY^3 \oplus 20Y^4
\]
of degree four; we omit the tropical multiplication symbol “\( \odot \)” between a coefficient and its monomial for brevity. The resulting quartic curve \( \mathcal{C} = \mathcal{T}(F) \) is smooth; see Figure 1.8. For instance, letting \( u = (1, 1) \), \( v = (2, 2) \),
1.6. Tropical Plane Curves

Figure 1.8. Smooth tropical plane quartic and dual subdivision. The edge joining (1, −1) and (2, −3) (left) and its dual pair of triangles (right) are marked.

Let $w = (0, 1)$, and $x = (1, 2)$, the determinant (1.11) equals 1. The triangle $\Delta = \text{conv}(u, w, v)$ is dual to the point $(1, −1)$, and $\Delta' = \text{conv}(v, w, x)$ is dual to $(2, −3)$. Those two points span a bounded edge of $C$, and its lattice length is one.

**Algorithm A:** Metric skeleton of a smooth tropical plane curve.

The genus of the (smooth) tropical plane curve $C = T(F)$ is the number of interior lattice points of the Newton polygon $N(F)$. For instance, the smooth quartic from Example 1.24 and Figure 1.8 has genus three. The main

\input{bivariate tropical polynomial $F$ such that $T(F)$ is a smooth tropical plane curve of genus $g$

\output{trivalent and connected planar graph with $2g - 2$ nodes and $3g - 3$ edges, equipped with a weight function on the edges

$\Sigma \leftarrow T(F)$, as abstract graph

while $\Sigma$ has a node of degree one do
  delete that node and the unique incident edge from $\Sigma$

foreach edge $\eta$ of $\Sigma$ do
  $\ell(\eta) \leftarrow$ lattice length from (1.11)

while $\Sigma$ has a node, $v$, of degree two do
  let $\eta$ and $\eta'$ be the two edges incident with $v$
  $\ell(\eta) \leftarrow \ell(\eta) + \ell(\eta')$
  contract $\eta'$ in $\Sigma$

return $(\Sigma, \ell)$
result of this section is the following theorem, which proves the correctness of Algorithm A. Its output is a finite abstract graph which is not necessarily simple; i.e., it may have loops or multiple edges between a pair of nodes. Moreover, each edge is equipped with a weight, which is a positive real number. The degree of a node is the number of incident edges; yet here we assume that a loop contributes two to the degree of its unique node. A graph is trivalent if each node degree is three. Abstract graphs are discussed in more detail in Section 3.1.

There are minor issues concerning the terminology in Algorithm A. First, we call the vertices of \( C \) “nodes” in the abstract setting. Second, abstract graphs do not allow for edges with only one node, unless the edge is a loop; there are several ways to resolve this technically. However, this is not relevant here because the unbounded edges are disposed of in the first step of Algorithm A.

**Theorem 1.25.** Applying Algorithm A to a smooth tropical plane curve, \( C \), of genus \( g \) yields a trivalent and connected planar graph \( \Sigma \) with \( 2g - 2 \) nodes and \( 3g - 3 \) edges, and a positive weight function \( \ell \). The graph \( \Sigma \) and its weight function \( \ell \) only depend on \( C \), not on the order in which nodes and edges are processed.

**Proof.** In the first step the 1-dimensional polyhedral complex \( C \) is taken as a finite graph. Here we truncate the unbounded edges arbitrarily, introducing one new node per unbounded edge. Since \( C \) is dual to a triangulation, each of its nodes has degree at most three.

After the first while-loop, there is no node of degree one left, and we did not increase the degree of any node. So at that time \( \Sigma \) only has nodes of degree two or three. Similarly, after the second while-loop only nodes of degree three remain; i.e., \( \Sigma \) is trivalent. In this count a loop, if it exists, contributes two to the degree of its unique node. Throughout the process \( \Sigma \) remains planar and connected. All weights are positive. Within each while-loop the operations commute.

Those connected components of the complement of the planar embedding \( \Sigma \subset \mathbb{R}^2 \) which are bounded correspond to the bounded regions of \( F \), which in turn correspond to the \( g \) interior lattice points of \( \mathcal{N}(F) \). Let \( n \) be the number of nodes of the final graph \( \Sigma \), and let \( m \) be the number of edges. Each edge is in exactly two regions, and each node is contained in exactly three edges (where loops count double). Then the number of incident pairs of nodes and edges equals \( 2m - k = 3n - k \), where \( k \) is the number of loops. Hence \( 2m = 3n \). By Euler’s formula we have \( n - m + (g + 1) = 2 \); there is exactly one unbounded region. This gives the claim. \( \square \)

The weighted graph \((\Sigma, \ell)\) resulting from Algorithm A is the metric skeleton of the smooth tropical plane curve \( C \); see Figure 1.9 for an example.
Passing from a tropical plane curve to its metric skeleton is an abstraction which can be studied geometrically and axiomatically; see [IMS09, §3.3]. This leads to “abstract tropical curves” and their moduli spaces. It is interesting to analyze the degenerations of a smooth tropical curve. Here we restrict our attention to one example.

**Example 1.26.** We consider the dual subdivision $S(F)$ of the bivariate quartic tropical polynomial $F$ from Example 1.24. Its secondary cone has dimension 15, with a 3-dimensional lineality space; cf. Section A.4. It turns out that, modulo linealities, that secondary cone is simplicial; i.e., it has 13 rays. These lead to the 13 most degenerate tropical plane quartics which are shown in Figure 1.10. For instance, the degenerate curve in the top left is given by

$$R(X,Y) = 3 \oplus 3X \oplus 2Y \oplus 3X^2 \oplus 2XY \oplus 1Y^2 \oplus 3X^3 \oplus 2X^2Y \oplus 1XY^2 \oplus 0Y^3 \oplus 3X^4 \oplus 2X^3Y \oplus 1X^2Y^2 \oplus 0XY^3 \oplus 3Y^4.$$ 

From a geometric point of view it may not suffice to look at tropical plane curves as mere subsets of $\mathbb{R}^2$ or $\mathbb{R}^3/\mathbb{R}$. There is an underlying arithmetic structure which originates from the integer lattice spanned by the support. A glimpse of this became visible in the definition of lattice length via (1.11). Going further in this direction would lead to concepts such as the “multiplicity” of a vertex and the “weight” of an edge, all of which are positive integers. In the smooth case all multiplicities and weights are equal to one. However, multiplicities and weights help to distinguish degenerate curves, like those in Figure 1.10. The related “balancing condition” is discussed in Problem 1.35. For further reading on this subject we recommend [GM07, §2.2] and [MS15, §3.4].
Figure 1.10. The 13 most degenerate tropical plane quartics from Example 1.26.
Problems

The sequence of Fermat numbers $2^{2^n} + 1$ has the property that any two of these numbers are coprime. This gives one way of showing that there are infinitely many primes; cf. [AZ18, Chapter 1].

**Problem 1.27.** Compute the tropical Fermat numbers $2^\circ 2^n \oplus 1$, both for $\oplus = \text{min}$ and $\oplus = \text{max}$.

**Problem 1.28.** Compute $(x \oplus y)^\circ n$.

**Problem 1.29.** Let $F(X) = \bigoplus_{u \in S} a_u \odot X_1^{u_1} X_2^{u_2} \ldots X_d^{u_d}$ be a $d$-variate tropical polynomial. For $v \in \mathbb{Z}^d$ we consider the tropical polynomial

$$F_v(X) = \bigoplus_{u \in S} a_u \odot X_1^{u_1+v_1} X_2^{u_2+v_2} \ldots X_d^{u_d+v_d}.$$ 

In which way is the tropical hypersurface $\mathcal{T}(F_v)$ related to $\mathcal{T}(F)$? What is the connection between the dual subdivisions $S(F_v)$ and $S(F)$?

**Problem° 1.30.** Draw the tropical hypersurface defined by the homogeneous tropical polynomial

$$(4 \odot X^3) \oplus (1 \odot XYZ) \oplus (4 \odot Y^3) \oplus (1 \odot Y^2Z) \oplus (1 \odot YZ^2) \oplus (6 \odot Z^3).$$

What does the dual subdivision of the Newton polytope look like?

**Problem° 1.31.** Let $A$ be the set of vertices of some convex $n$-gon for $n \geq 3$. How many labeled triangulations does $A$ have?

It can be shown that all triangulations of the vertex set of a convex polygon are regular, but there are also point sets which are different.

**Problem° 1.32.** Give an example of a point configuration and a subdivision which is not regular.

A **tropical conic** is a tropical hypersurface in $\mathbb{R}^3/\mathbb{R}1$ of a homogeneous tropical polynomial of degree two.

**Problem° 1.33.** What are the combinatorially distinct types of tropical conics in $\mathbb{R}^3/\mathbb{R}1$? By the way, what is a good definition for “combinatorially distinct” in this context?

The natural way of defining the multiplication of matrices makes sense tropically as well. We use “$\odot$” as a symbol for tropical matrix multiplication.

**Problem° 1.34.** Let $A \in \mathbb{R}^{2 \times 2}$ be a square matrix with two rows and columns. For which vectors $b \in \mathbb{R}^2$ does the tropical linear system of equations $A \odot x = b$ have a solution?
A lattice vector in $\mathbb{Z}^2$ is \textit{primitive} if its coefficients are relatively prime. Each rational vector can be rescaled to integer coordinates, and dividing by the greatest common divisor of the coefficients gives a primitive lattice vector. Consequently, each facet of a lattice polygon has a unique inward pointing normal vector which is primitive.

\textbf{Problem} ° 1.35. Show that for each unimodular triangle the (inward) pointing primitive facet normal vectors add to zero.

This property, known as the \textit{balancing condition}, is satisfied locally at each vertex of a smooth tropical plane curve. It is crucial for the metric properties of (abstract or plane) tropical curves (with respect to min). Note that, since the outward pointing normal vectors are the negatives of the inward pointing ones, the claim in Problem 1.35 also holds for outward pointing normals. Hence the same argument shows that max-tropical plane curves satisfy the balancing condition, too.

\textbf{Remarks}

A first occurrence of the actual term “tropical semiring” might be an article \cite{Sim88}. Yet the underlying ideas are much older, and it seems to be difficult to trace the history back to a single source. Already in the discussion of the monograph \cite{CG79} by Cuninghame-Green in \textit{Mathematical Reviews}, MR0580321, the reviewer Rudeanu points out that the author forgets to mention relevant work of Lunc \cite{Lun52} and Moisil \cite{Moi60}. We give more details on the history of (max, +)-linear algebra at the end of Chapter 3.

Since the exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is a monotonic bijection and since we further have $\exp(x+y) = \exp(x) \cdot \exp(y)$ the tropical semiring is also isomorphic to $(\mathbb{R}_{>0}, \min, \cdot)$ and $(\mathbb{R}_{>0}, \max, \cdot)$. For instance, in \cite[§3.1]{Mik05} the triplet $(\mathbb{T}, \oplus, \odot)$ is called the “tropical semifield” in order to stress that the tropical multiplication $\odot$ does have inverses; i.e., $(\mathbb{T}, \oplus, \odot)$ satisfies all axioms of a field save the existence of inverses with respect to the tropical addition $\oplus$.

For the foundations of the theory of polyhedral subdivisions as well as its applications we refer to the monograph of De Loera, Rambau, and Santos \cite{DLRS10}. Delaunay subdivisions and Voronoi diagrams are discussed, e.g., in \cite[§2.2.2]{DLRS10} and \cite[Chapters 6 and 7]{JT13}. The influential book by Gel’fand, Kapranov, and Zelevinsky \cite{GKZ08}, where they use the term “coherent” subdivision, presents many ideas which are now fundamental parts of tropical geometry. Regular subdivisions and secondary cones
were introduced shortly before that book in the article [GZK90]. Theorem 1.13 will be extended to tropical hypersurfaces of Hahn polynomials in Corollary 10.58 at the end of this book.

In some early papers on tropical geometry, such as [RGST05], the tropical projective torus is sometimes called “tropical projective space”. We reserve the latter name for a suitable compactification studied in Section 5.1. A version of Theorem 1.13 and its immediate consequences already occurs in [RGST05].

Research on tropical curves was initiated by Mikhalkin [Mik05, Mik06]; see also Itenberg, Mikhalkin, and Shustin [IMS09] and Gathmann and Markwig [GM07]. Mikhalkin was motivated by work of Kontsevich on Gromov–Witten invariants of symplectic manifolds [Kon92]. Kontsevich related these invariants, which play a role in mathematical physics, to counting algebraic curves of given degree passing through a certain number of points. Mikhalkin’s Correspondence Theorem [IMS09, §3.5] says that it amounts to the same as counting the corresponding tropical curves; see also [MS15, §1.7]. Later Gathmann, Kerber, and Markwig [GKM09] and Abramovich, Caporaso, and Payne [ACP15] investigated moduli spaces of tropical curves. The article [BJMS15] is concerned with the moduli of tropical plane curves; this comes with a substantial amount of data available at https://github.com/micjoswig/TropicalModuliData. In fact, the smooth tropical plane quartic in Example 1.24 is g3b0−254 in that data set. It is an interesting question to ask which trivalent metric graphs occur from tropical plane curves via Algorithm A. Birkmeyer, Gathmann, and Schmitz proposed a decision algorithm which takes an arbitrary metric graph as input [BGS17]. Combinatorial obstructions for an abstract planar graph to be realizable as a tropical plane curve are the topic of [CDJ+20] and [JT21].

A triangulation of the point set $S \subset \mathbb{Z}^d$ is called full if each point in the set $\text{conv}(S) \cap \mathbb{Z}^d$ occurs as a vertex of a cell. For $d = 2$ this is exactly our definition of a unimodular triangulation in Section 1.6. The proper generalization of unimodularity to arbitrary dimension $d$ requires that the Euclidean volume of each maximal simplex equals $1/(d!)$. In general, unimodular implies full, but these two notions are equivalent only if $d \leq 2$; see [DLRS10, §3.1]. Unimodularity will also play a role in Theorem 4.36.