# Contents

Preface ix

1 Prime numbers 1
   1.1 Primes as elementary building blocks ....................... 1
   1.2 Counting primes ........................................... 3
   1.3 Using the logarithm to count powers ....................... 7
   1.4 Approximations for $\pi(x)$ .............................. 9
   1.5 The prime number theorem ................................ 11
   1.6 Counting prime powers logarithmically .................... 11
   1.7 The Riemann hypothesis—a look ahead ..................... 14
   1.8 Additional exercises ...................................... 16

2 The zeta function 21
   2.1 Infinite sums .............................................. 21
   2.2 Series for well-known functions ........................... 26
   2.3 Computation of $\zeta(2)$ ................................ 29
   2.4 Euler’s product formula .................................... 32
   2.5 Looking back and a glimpse of what is to come ........... 34
   2.6 Additional exercises ...................................... 34

3 The Riemann hypothesis 41
   3.1 Euler’s discovery of the product formula .................. 41
   3.2 Extending the domain of the zeta function ................ 43
   3.3 A crash course on complex numbers .......................... 45
   3.4 Complex functions and powers ............................. 47
   3.5 The complex zeta function ................................ 50
   3.6 The zeroes of the zeta function ........................... 51
   3.7 The hunt for zeta zeroes .................................. 54
   3.8 Additional exercises ...................................... 55

4 Primes and the Riemann hypothesis 59
   4.1 Riemann’s functional equation ............................. 60
   4.2 The zeroes of the zeta function ............................ 63
   4.3 The explicit formula for $\psi(x)$ ......................... 66
   4.4 Pairing up the non-trivial zeroes ......................... 69
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5 The prime number theorem</td>
<td>72</td>
</tr>
<tr>
<td>4.6 A proof of the prime number theorem</td>
<td>73</td>
</tr>
<tr>
<td>4.7 The music of the primes</td>
<td>76</td>
</tr>
<tr>
<td>4.8 Looking back</td>
<td>78</td>
</tr>
<tr>
<td>4.9 Additional exercises</td>
<td>81</td>
</tr>
<tr>
<td>Appendix A. Why big primes are useful</td>
<td>87</td>
</tr>
<tr>
<td>Appendix B. Computer support</td>
<td>91</td>
</tr>
<tr>
<td>Appendix C. Further reading and internet surfing</td>
<td>99</td>
</tr>
<tr>
<td>Appendix D. Solutions to the exercises</td>
<td>101</td>
</tr>
<tr>
<td>Index</td>
<td>143</td>
</tr>
</tbody>
</table>