Preface

This book develops a mathematical theory for finance based on the following “viability” principle: that it should not be possible to fund cumulative capital withdrawal streams which are nontrivial, starting with arbitrarily small amounts of initial wealth.

The underlying framework consists of a given, finite number of risky assets, already discounted by a money-market or “cash”. The asset prices are modelled via continuous semimartingales with respect to an arbitrary right-continuous filtration, or flow of information. In this context, proscribing such egregious forms of what is commonly called arbitrage, as the one described above, turns out to be equivalent to the existence of a portfolio with wealth process possessing the local martingale numéraire and growth optimality properties; whereas, the reciprocal of this wealth process is a particular case of a so-called local martingale deflator.

A precise meaning is assigned to these terms, and it is shown that the above equivalent conditions can be formulated entirely, actually very simply, in terms of the local characteristics of the underlying asset prices: their drift and covariation components.

In this framework, full-fledged theories are developed for the most central problems of the field: the hedging of liabilities, and portfolio/consumption optimization. The important notion of market completeness is also introduced, and is characterized equivalently as the martingale representation property for the underlying flow of information, in terms of the fundamental local martingales that represent the assets’ noise components.

As it turns out, the semimartingale property of these processes is necessary and sufficient for viability, when investment occurs only along a
discrete-time schedule and is constrained to be “long-only”—that is, to avoid
negative (“short”) positions in the risky assets, and never to borrow cash. In
this manner, the semimartingale property for the different asset prices in the
market, and with it the finiteness of their quadratic- and cross-variations,
emerge as consequences of purely economic considerations, rather than being
postulated a priori.

The “viability” approach to the financial market adopted here has a
long lineage; this, along with the historical development of the field, can be
traced in the Notes to the various chapters, particularly Chapters 2 and 3.

Preview

Chapter 1 sets up the model for a financial market with continuous asset
prices, and develops in its context the notions of investment strategies and
of portfolios. The wealth, covariation and growth characteristics of port-
folios are studied, in terms both absolute (with respect to the underlying
money-market) and relative (with respect to some given baseline portfolio).
The market portfolio is introduced and studied, along with a class of portfo-
lios generated functionally in terms of its components, the so-called market
weights of the different assets.

Chapter 2 is foundational. The notion of market viability is introduced
here, and is characterized in terms of several equivalent conditions. Among
others, these involve local martingale deflators, strictly positive local mar-
tingales whose products with all asset prices are also local martingales; the
notion of local martingale numéraire, a strictly positive wealth process whose
reciprocal is a local martingale deflator; and the notions of growth optimal
(or “Kelly”) and log-optimal portfolios. Market viability implies that the
random vector process $\alpha$ of the assets’ local rates of return should be in the
range of the random matrix process $c$ of the assets’ local covariation rates.
Any portfolio $\nu$ with the local martingale numéraire property should satisfy
$\alpha = cv$, and the resulting maximal growth rate $(1/2)\nu'cv$ should be locally
integrable with respect to the market’s operational clock, under which these
rates are determined. The above structural conditions are in fact necessary
and sufficient for market viability, thus showing that this notion can be
formulated purely in terms of market characteristics.

Examples are presented which illustrate what can happen when viabil-
ity fails, including the possible existence of wealth processes which start
with zero initial capital and rise, monotonically and inexorably, to spectac-
ular levels of wealth. We also study the implications of imposing the local
martingale numéraire property, either on the market portfolio itself, or on
“mutual funds” that involve only the market portfolio and cash. This leads,
respectively, to markets we call balanced and which exhibit a striking failure
of long-term diversity; and to markets satisfying the conditions of the capital asset-pricing model.

Two of the most fundamental questions in the field of finance are those of financing or hedging, and of portfolio optimization. These can be described, respectively and in very broad brushes, as follows:

(i) Suppose we have to fund a given capital withdrawal stream, thought of as consumption or liability, that extends into the future. How much capital do we need to initially set aside for this funding to become possible via judicious trading in the market, and which is the smallest such amount of initial capital?

(ii) Suppose, conversely, that we start with a given amount of initial capital. Which are the capital withdrawal streams that we can fund into the future, starting with this given amount, and among those, how do we select the best one, according to a given criterion for optimality?

Both of these questions are tackled in Chapter 3. Complete answers are provided for them, always in the context of continuous asset prices and under conditions of viability, as follows:

(i) Given a cumulative withdrawal stream, we consider its expected value into the future, discounted by each possible local martingale deflator in this viable market. Then the smallest initial capital that can fund this stream exists, and is given by a “worst-case scenario”; i.e., as the supremum of these expected discounted values over all possible local martingale deflators, when this supremum is finite.

(ii) Conversely, the capital withdrawal streams that can be financed starting from a given initial capital are precisely those for which all such expected discounted values are dominated by the amount of initial capital in question. Then, given a suitable criterion for measuring the utility of future consumption, in Section 3.4 we use methods of functional analysis to characterize the optimal cumulative withdrawal stream.

The functional-analytic tools needed for this study are developed in the context of the nonnegative orthant in the space of measurable functions, equipped with the topology of convergence in measure. They take the form of concave duality, whereby the maximizing consumption stream is characterized in terms of an auxiliary, dual “minimizing deflator”, and vice-versa. The underlying abstract concave duality is developed in Section A.3 of the Appendix; it is completely self-contained and general.

The solutions to expected utility maximization problems lead to optimal capital withdrawal streams which are maximal, in that they cannot
be dominated by another stream which can be financed starting with the
same initial capital (a concept also known as Pareto optimality in economic
theory), as well as complementable, meaning that the supremum of their
expected total deflated values is attained by some local martingale deflator.
It is shown in Section 3.5 that these two properties are equivalent; their
importance is highlighted in the study of outperformance of one portfolio
by another and, more crucially, in the existence of so-called equivalent lo-
cal martingale measures over finite time horizons. The existence of such
equivalent local martingale measures is not necessary for viability, or for the
theories of financing (hedging) and portfolio optimization. It becomes im-
portant, though, in the study of maximal capital withdrawal streams, and
of portfolio outperformance.

A portfolio with the local martingale numéraire property maximizes
growth, both in a local “myopic” way, as well as in an asymptotic long-term
manner. Suppose now that we want to limit ourselves to portfolios which
never allow current wealth to fall below a fixed percentage of the maximum
wealth-to-date. This is the essence of what is commonly called “portfolio
insurance”. It is demonstrated in Section 4.1 of Chapter 4 that a simple
transformation of the unconstrained growth-optimal wealth, due to Azéma
& Yor, allows a very explicit description of a portfolio that maximizes long-
term growth from investment under this so-called drawdown constraint; and
in a manner which amounts to a mutual fund between the unconstrained
growth-optimal strategy, and cash.

What about financial markets that contain infinitely many assets with
continuous prices, possibly an uncountable infinity of them? This is not
idle speculation: debt markets consisting of zero-coupon bonds, risky in-
struments with known payoff at an arbitrary, fixed date into the future, are
prime examples. In order to extend the viability-based approach to such
a setting, one first needs a theory for stochastic integration with respect
to an arbitrary collection of continuous semimartingales. Such an integra-
tion theory is developed in Section 4.3 of Chapter 4 using the covariations
of the underlying integrator semimartingales as basic building blocks, and
requiring that the collection of finite-variation components belong to the
stochastic reproducing kernel Hilbert space generated by these covariations.
This requirement is the exact analogue of the equivalent structural con-
dition for viability in the case of a finite number of assets—and permits
the development of a theory for viable markets with an arbitrary number
of risky assets. The theory developed in Section 4.3 is new; the necessary
functional-analytic tools, based on reproducing kernel Hilbert space consid-
erations, are presented in an entirely self-contained manner in Section A.5
of the Appendix.
Prerequisites

No prior familiarity with finance is required, but we assume that our reader has a good working knowledge of real analysis, measure theory, as well as of basic probability theory. Familiarity with stochastic analysis is also assumed, and knowledge of integration with respect to continuous semimartingales is taken for granted. We develop in the Appendix, in a self-contained manner and sometimes using a novel approach, the necessary results from convex and functional analysis, including Fenchel duality and reproducing kernel Hilbert space. The Appendix can be read independently of the remainder of the text.

Because the underlying filtration is general, we have to own up on several occasions to the possibility that its martingales may have discontinuities. Then, recourse to the general theory of stochastic integration, relative to semimartingales with possible jumps, becomes unavoidable, and we send our reader to specific sources for the necessary background. A prime example of this situation occurs in Section 3.1, where we develop an optional decomposition result used extensively throughout the text.

This book contains a significant amount of new research and results. For instance, the notion of model-consistent probability measure and all results pertaining to it; the treatment of the Capital Asset Pricing Model in Section 2.5, and in particular the interpretation in Exercise 2.72; the development of stochastic integration (resp., of investment theory) with respect to an arbitrary, possibly uncountably infinite, collection of continuous semimartingales (resp., assets), via a stochastic version of reproducing kernel Hilbert space.

At the same time, the book is sprinkled with a large number of exercises, some of them integral to the development of the theory and the understanding of its ramifications. In this manner, the book is suitable for second-year graduate courses on stochastic analysis and/or the mathematics of finance.

Further topics

In order to keep the presentation short and accessible, we have not touched several important topics. We work in a frictionless setting, eschewing transaction costs on investment; for in-depth treatments of the topic, we send the interested reader to the monographs [KS09] and [Sch17]. A natural continuation to portfolio choice and utility maximization problems is financial equilibrium, the crucial procedure towards the formation of prices via supply-demand balance; see, for instance [KS98, Chapter 4], [DJ03, Chapters 6-7], and the references cited there. Finally, with the exception of §2.3.7, we do not discuss the notion of robustness in investment and risk management,
an actively studied subject with connections to non-linear expectations, the
theory of analytic sets, as well as optimal transportation. We hope to be
able to cover some of these topics in the future.

As mentioned already, we have made a conscious choice to consider only
asset prices with continuous paths. We refer the interested reader to [DS06]
for an excellent overview of arbitrage theory in general semimartingale finan-
cial models. For a treatise covering broader topics in mathematical finance,
as well as jumps in asset prices, one may consult [CT04], as well as the
recent publication [EK19].

Suggested reading pathways

The topics in the book are arranged according to a natural progression.
When an abbreviated overview of the essentials is desired, or under time-
constraints in teaching, a somewhat shorter pathway may be more appro-
priate.

For instance, one can proceed via Sections 1.1-1.4, 2.1, 2.2, 3.1-3.3 in
succession, followed by Section 3.4 with motivation from selected topics in
Section 2.3, then by the related Sections 1.5 and 3.5. Time permitting,
one can then continue selectively through Section 2.4, followed by 2.5 and
by parts of Chapter 4 (where Section 4.2 should preferably precede Section
4.3).

The Appendix can be read completely independently of the rest of the
text. Its Sections A.1-A.4 are needed for developments in Sections 3.4, 3.5
and 4.2. Section 4.3 relies on the reproducing kernel Hilbert space setting
of Section A.5.