Chapter 1

The Market

We introduce in this Chapter the setup adopted throughout this work: a financial market with a fixed number of assets, which can be thought of as stocks, and with a money market. We place ourselves in a continuous-time framework, in which asset prices are modelled by continuous semimartingales and the flow of information is assumed only to be right-continuous. Within this setup we introduce investment strategies, including portfolios, and study their properties, such as covariation, growth and return characteristics. We study these characteristics both in absolute terms (that is, measured with respect to “cash”, or a money-market) and in relative terms (that is, measured with respect to the performance of a fixed, “baseline” portfolio). In particular, how various strategies behave relative to the market portfolio is a subject of both theoretical and practical importance.

The class of functionally generated portfolios, expressed solely in terms of functions of the prevailing capitalization weights of the stocks in the market, is introduced and studied. These portfolios have remarkable properties, which lead to controlled behaviour over sufficiently long time horizons.

1.1. Probabilistic setup

1.1.1. The underlying stochastic structure. Throughout this book we fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The symbol “\(\mathbb{E}^\mathbb{P}[f]\)” is reserved for the Lebesgue integral—or expectation in probabilistic terminology—with respect to the probability measure \(\mathbb{P}\). Here, \(f : \Omega \to \mathbb{R}\) is \(\mathcal{F}\)-measurable, and it is assumed that either \(f^+ := \max(f, 0)\), the positive part of \(f\), or \(f^- := \max(-f, 0)\), the negative part of \(f\), has finite \(\mathbb{P}\)-expectation.
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Time-evolution and the flow of information are modelled via a filtration \( \mathcal{F}(\cdot) \equiv (\mathcal{F}(t) ; t \in \mathbb{R}_+) \), i.e., a family of sub-\( \sigma \)-algebras of \( \mathcal{F} \) which is nondecreasing, in the sense that \( \mathcal{F}(s) \subseteq \mathcal{F}(t) \subseteq \mathcal{F} \) holds for \( 0 \leq s \leq t < \infty \). This family is assumed to be right-continuous, that is, to satisfy

\[
\mathcal{F}(t) = \mathcal{F}(t+) := \bigcap_{s>t} \mathcal{F}(s), \quad \forall t \in \mathbb{R}_+.
\]

No completeness assumption is imposed on this filtration. We set \( \mathcal{F}(\infty) := \bigvee_{t \in \mathbb{R}_+} \mathcal{F}(t) \) and, for concreteness only, take \( \mathcal{F}(0) = \{\emptyset, \Omega\} \mod \mathbb{P} \). We shall denote by \( \mathcal{F}(\cdot) \) the collection of stopping times of \( \mathcal{F}(\cdot) \), namely, of mappings \( \tau : \Omega \to [0, \infty] \) with the property \( \{\tau \leq t\} \in \mathcal{F}(t) \) for every \( t \in \mathbb{R}_+ \).

On such a filtered probability space we shall consider random elements, that is, \((\mathcal{F}/\mathcal{B}(\mathbb{S}))\)-measurable mappings \( h : \Omega \to \mathbb{S} \) with values in some metric space \( \mathbb{S} \) equipped with the \( \sigma \)-algebra \( \mathcal{B}(\mathbb{S}) \) of its Borel sets. We shall also consider stochastic processes, i.e., \((\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))/\mathcal{B}(\mathbb{S})\)-measurable mappings \( X : \Omega \times \mathbb{R}_+ \to \mathbb{S} \). As is common practice in probability theory, the argument \( \omega \in \Omega \) in random elements and processes will be implicit, and mostly omitted. Unless otherwise explicitly stated, identities or inequalities involving random variables are interpreted as being valid almost everywhere (a.e.) under \( \mathbb{P} \), that is, for \( \mathbb{P} \)-almost every (a.e.) \( \omega \in \Omega \). Similarly—and, again, unless otherwise explicitly mentioned—comparisons between processes will be understood modulo \( \mathbb{P} \)-evanescence.\(^1\) In the same spirit, the comparison of subsets of \( \Omega \) is understood modulo \( \mathbb{P} \)-null sets.

All stochastic processes \( X \) to be encountered will be adapted to the underlying filtration \( \mathcal{F}(\cdot) \); this means that \( X(t) \) is \( \mathcal{F}(t) \)-measurable, for every \( t \in \mathbb{R}_+ \). In fact, processes which model investment decisions will be predictable, that is, measurable with respect to the predictable \( \sigma \)-algebra, discussed in §1.1.2 right below.

1.1.2. Predictability and optionality. We fix an arbitrary metric space \( \mathbb{S} \) equipped with the \( \sigma \)-algebra \( \mathcal{B}(\mathbb{S}) \) of its Borel sets.

An \( \mathbb{S} \)-valued stochastic process of the form \( \sum_{j=1}^{m} h_j 1_{(t_{j-1},t_j]} \), where \( m \in \mathbb{N} \), \( 0 = t_0 < \cdots < t_m \), and \( h_j : \Omega \to \mathbb{S} \) is \((\mathcal{F}(t_{j-1})/\mathcal{B}(\mathbb{S}))\)-measurable for every \( 1 \leq j \leq m \), will be called simple predictable. Here, the adjective “simple” connotes the piecewise-constant character of the paths of such a process; whereas the adjective “predictable” conveys the fact that the value \( h_j \) of the process over the interval \( (t_{j-1},t_j] \) is announced at the start \( t_{j-1} \) of this interval (i.e., \( h_j \) is \( \mathcal{F}(t_{j-1}) \)-measurable).

\(^1\)A subset of \( \Omega \times \mathbb{R}_+ \) is \( \mathbb{P} \)-evanescent if its projection on \( \Omega \) has zero outer \( \mathbb{P} \)-measure.
1.1. Probabilistic setup

We shall denote by $\mathcal{P}$ the smallest $\sigma$-algebra on the product space $\Omega \times \mathbb{R}_+$ with respect to which all simple predictable processes, seen as functions from $\Omega \times \mathbb{R}_+$ to $\mathbb{S}$, are measurable, and call it the predictable $\sigma$-algebra. We shall say that a given $\mathbb{S}$-valued process is predictable if it is measurable with respect to this $\sigma$-algebra $\mathcal{P}$. Given any Borel subset $E$ of $\mathbb{S}$, we use $\mathcal{P}(E)$ to denote the class of all $E$-valued, predictable processes.

Predictability is a mathematical abstraction of the requirement that decisions should be based only on past and present knowledge, without any anticipation of future events. The requirement of predictability will also be imposed on all integrands with respect to semimartingales, starting with §1.1.4 and continuing throughout this work.

Exercise 1.1. Show that all left-continuous, adapted processes are predictable.

The next exercise introduces another important notion for stochastic processes, that of optionality. We shall need it in Chapter 3.

Exercise 1.2 (Optionality). We shall call simple optional an $\mathbb{S}$-valued process of the form $\sum_{j=1}^{m} h_j 1_{[t_{j-1}, t_j)}$, where $m \in \mathbb{N}$, $0 = t_0 < \cdots < t_m < \infty$, and $h_j : \Omega \to \mathbb{S}$ is $(\mathcal{F}(t_{j-1})/\mathcal{B}(\mathbb{S}))$-measurable for every $1 \leq j \leq m$. The optional $\sigma$-algebra, denoted by $\mathcal{O}$, is the smallest $\sigma$-algebra of subsets of the product space $\Omega \times \mathbb{R}_+$, with respect to which all simple optional processes are measurable. A given $\mathbb{S}$-valued process is called optional if it is measurable with respect to $\mathcal{O}$.

With these definitions, show that

1. all right-continuous, adapted processes are optional; and that
2. all simple predictable processes are optional. In particular, conclude that all predictable processes are optional: $\mathcal{P} \subseteq \mathcal{O}$.

1.1.3. Continuous semimartingales. A real-valued stochastic process is called continuous semimartingale if it can be written as the sum of an adapted process whose paths are continuous and have finite variation on compact time intervals, and of a continuous-path local martingale.

We shall denote by $[X, X]$ the quadratic variation of a continuous semimartingale $X$; this is a continuous nondecreasing process, satisfying

$$[X, X](t) = \mathbb{P}\text{-lim}_{m \to \infty} \sum_{k=1}^{2^m} \left( X \left( \frac{k}{2^m} \land t \right) - X \left( \frac{k-1}{2^m} \land t \right) \right)^2, \quad t \in \mathbb{R}_+,$$

where "$\mathbb{P}$-lim" denotes limit in $\mathbb{P}$-measure. Written loosely, the above reads as $[X, X] = \int_0^t (dX(t))^2$, or even less rigorously as $d[X, X](t) = (dX(t))^2$ for $t \in \mathbb{R}_+$. The covariation of two continuous semimartingales $X$ and $Y$ is
defined using polarization via
\[ [X,Y] := \frac{1}{4}([X+Y,X+Y] - [X-Y,X-Y]) \]
and is a continuous process of finite variation on compact time intervals. If both \( X \) and \( Y \) are continuous local martingales, then \([X,Y]\) is the unique continuous process of finite variation with the property that \( XY - [X,Y] \) is a local martingale; see for example, [KS91, §1.5].

We now fix an integer \( n \in \mathbb{N} \) and consider a continuous, \( \mathbb{R}^n \)-valued semi-martingale \( R \equiv (R_i; 1 \leq i \leq n) \), with \( R_i(0) = 0 \) and component processes
\[
(1.1) \quad R_i = A_i + M_i, \quad \forall 1 \leq i \leq n.
\]
Here, the real-valued component processes \( A_i \) of the vector-valued process \( A \equiv (A_i; 1 \leq i \leq n) \) with \( A_i(0) = 0 \) are adapted, continuous, and have finite variation on compact time intervals, whereas the real-valued components \( M_i \) of the vector process \( M \equiv (M_i; 1 \leq i \leq n) \) are continuous local martingales with \( M_i(0) = 0 \), for all \( 1 \leq i \leq n \). We shall refer occasionally, and somewhat colloquially, to the finite-variation processes \( A_i \) in (1.1) as the \textit{drift} (or “trend”) components, and to the local martingales \( M_i \) as the \textit{noise} (or “diffusion”) components, of the semimartingale \( R \).

We note that \([R_i,R_j] = [M_i,M_j]\) holds for \( 1 \leq i, j \leq n \), and introduce the continuous, nondecreasing scalar process
\[
(1.2) \quad O := \sum_{i=1}^{n} \left( \int_{0}^{T} |dA_i(t)| + [M_i,M_i] \right),
\]
with \( \int_{0}^{T} |dA_i(t)| \) denoting the total variation of \( A_i \) on the interval \([0,T]\) for \( T \in \mathbb{R}_+ \). This scalar process \( O \) plays the rôle of an “operational clock” for the vector semimartingale \( R \): all processes \( A_i \) and \([M_i,M_j]\) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \) are absolutely continuous with respect to this clock; and time-intervals of constancy for \( O \) correspond exactly to intervals of constancy for \( R \). Any integral with respect to the nondecreasing process \( O \) in (1.2) of a strictly positive and predictable process can also be used as a “clock”. In many concrete cases, other such clocks may be more natural; for example, when \( R \) is a nondegenerate Itô process, as will be the case in almost all the examples treated in this book, it is more convenient to use the Lebesgue clock, given by \( \text{Leb}(t) = t \) for \( t \in \mathbb{R}_+ \).

The drift components and the covariations of the noise components of the processes \( R_i \) in (1.1) can be cast as integrals of appropriate, predictable \textit{rate processes} with respect to the random operational clock \( O \) of (1.2). Indeed, a straightforward application of the Radon-Nikodým theorem on the measurable space \((\Omega \times \mathbb{R}_+, \mathcal{P})\) implies the existence of predictable processes
\[
(1.3) \quad \alpha \equiv (\alpha_i; 1 \leq i \leq n), \quad c \equiv (c_{ij}; 1 \leq i, j \leq n),
\]
vector-valued and matrix-valued, respectively, such that

\[ A = \int_0^\alpha(t) dO(t), \quad \text{and} \quad C \equiv [M, M] = \int_0^c(t) dO(t). \tag{1.4} \]

Here and throughout the text, we use the notation \( C \equiv (C_{ij}; 1 \leq i, j \leq n) \) for the nonnegative-definite, matrix-valued process of covariations

\[ C_{ij} := [M_i, M_j] = [R_i, R_j], \quad 1 \leq i, j \leq n. \tag{1.5} \]

The components of the vector-valued process \( \alpha \equiv (\alpha_i; 1 \leq i \leq n) \) represent the local rates of return of the individual stocks in the market, whereas the entries of the matrix-valued process \( c \equiv (c_{ij}; 1 \leq i, j \leq n) \) stand for the local covariation rates of these stocks. All these rates are measured with respect to the operational clock \( O \) of (1.2).

We shall refer to the collection of local rates \( \alpha, c \) in (1.3), (1.4) as the local characteristics of the market. We note that these rates are unique only up to \((P \otimes O)\)-null sets, where \( P \otimes O \) is the measure

\[ (P \otimes O)[\Psi] \equiv \mathbb{E}^P \left[ \int_0^\infty 1_\Psi(t) dO(t) \right], \quad \Psi \in \mathcal{P}, \]

on the product space \((\Omega \times \mathbb{R}_+, \mathcal{P})\). By altering it on a \((P \otimes O)\)-null set if necessary, we shall also assume throughout that the process \( c \) takes values in the space of symmetric, nonnegative-definite, \( n \times n \) matrices.

**1.1.4. Vector stochastic integration.** For a given integer \( n \in \mathbb{N} \), we consider a continuous vector-valued semimartingale \( R \equiv (R_i; 1 \leq i \leq n) \) with \( R_i(0) = 0 \) for \( 1 \leq i \leq n \) as in §1.1.3, and keep the notation introduced there; e.g., the decompositions (1.1), the stochastic clock \( O \) of (1.2), the covariations of (1.5), and the rates in (1.3), (1.4), all carry over.

**Definition 1.3 (Vector integrands).** We denote by \( \mathcal{I}(R) \) the class of predictable vector processes \( \pi \equiv (\pi_i; 1 \leq i \leq n) \), which are integrable with respect to the vector semimartingale \( R \). Furthermore, for a given Borel set \( E \subseteq \mathbb{R}^n \), we denote the collection of all \( E \)-valued processes in \( \mathcal{I}(R) \) by

\[ \mathcal{I}(R; E) := \mathcal{I}(R) \cap \mathcal{P}(E). \]

The collection \( \mathcal{I}(R) \) has a convenient operational characterization. A predictable process \( \pi \equiv (\pi_i; 1 \leq i \leq n) \) belongs to \( \mathcal{I}(R) \) if and only if

\[ \int_0^T \left( |\pi'(t)\alpha(t)| + \pi'(t)c(t)\pi(t) \right) dO(t) < \infty, \quad \forall T \in \mathbb{R}_+. \tag{1.6} \]

For instance, this condition is automatically satisfied by any predictable process \( \pi \) which is locally bounded, i.e., for which there exist a localizing
sequence \((\tau_m; \, m \in \mathbb{N})\) of stopping times that increase \(\mathbb{P}\text{-}a.e.\) to infinity, and a sequence of nonnegative real constants \((k_m; \, m \in \mathbb{N})\) such that

\[
\sup_{t \in [0, \tau_m]} |\pi_i(t)| \leq k_m, \quad \forall \, m \in \mathbb{N}, \forall \, 1 \leq i \leq n.
\]

In particular, \(\mathcal{I}(R; \, E) = \mathcal{P}(E)\) holds for a bounded Borel set \(E \in \mathcal{B}(\mathbb{R}^n)\).

Condition (1.6) ensures that every vector process \(\pi \in \mathcal{I}(R)\) is both:

- Lebesgue-integrable with respect to the vector-valued process \(A = (A_i; \, 1 \leq i \leq n)\) of finite variation, and the resulting integral process has finite variation. In fact, the first variation process associated with the integral process \(\int_0^\cdot \pi'(t)dA(t)\) of \(\pi\) with respect to \(A\) equals

\[
\int_0^\cdot |\pi'(t)|dA(t) \equiv \int_0^\cdot |\pi'(t)\alpha(t)|dO(t); \quad \text{and}
\]

- Itô-integrable with respect to the vector-valued local martingale \(M = (M_i; \, 1 \leq i \leq n)\), and the resulting integral is a continuous local martingale. In fact, the quadratic variation process associated with the stochastic integral process \(\int_0^\cdot \pi'(t)dM(t)\) of \(\pi\) with respect to \(M\) equals

\[
\int_0^\cdot \pi'(t)dC(t)\pi(t) \equiv \int_0^\cdot \pi'(t)c(t)\pi(t)dO(t).
\]

These considerations show that the condition (1.6) is necessary and sufficient for the \(R\)-integrability of a given \(\pi \in \mathcal{P}(\mathbb{R}^n)\). We then denote by

\[
(1.7) \quad \int_0^\cdot \sum_{i=1}^n \pi_i(t)dR_i(t) \equiv \int_0^\cdot \pi'(t)dR(t) = \int_0^\cdot \pi'(t)dA(t) + \int_0^\cdot \pi'(t)dM(t)
\]

the stochastic integral of the predictable, vector-valued process \(\pi\), with respect to the vector-valued semimartingale \(R\).

Stochastic integration does not require in any way the integrator \(R\) to satisfy \(R(0) = 0\). However, there is no universal agreement on how to define the initial value of the process \(\int_0^\cdot \pi'(t)dR(t)\) when \(R(0)\) and \(\pi(0)\) are non-zero. In this work we assume that all stochastic integral processes start from zero; effectively, whatever the value of \(R(0)\) may be, the integral with respect to \(R\) is the same as the integral with respect to \(R - R(0)\).

Let us note that the so-called \emph{vector stochastic integral} in (1.7) is more general than its component-wise version

\[
\sum_{i=1}^n \int_0^\cdot \pi_i(t)dR_i(t),
\]

which would require \(\pi_i \in \mathcal{I}(R_i)\) to hold for all \(1 \leq i \leq n\). Consider for example the case \(n = 2\) with \(R_2 = -R_1\); then, \emph{any} predictable process
(π₁, π₂) with π₁ = π₂ is R-integrable (with the corresponding integral being equal to zero), regardless of whether πᵢ ∈ ℐ(ℛᵢ) holds for i ∈ {1, 2}. The next remark generalizes this example.

Remark 1.4 (Null integrands). Suppose that ζ ∈ ℙ is R-null, by which we mean that ζ′α = 0 and cζ = 0 hold in the (ℙ ⊗ ℓ)-a.e. sense. Then, the integral in (1.6) for π ≡ ζ equals zero, implying that ζ ∈ ℐ(ℛ).

Now, ∫₀ |ζ′(t)dA(t)| = ∫₀ |ζ′(t)α(t)|dO(t) ≡ 0. Furthermore, the quadratic variation process of the continuous local martingale ∫₀ ζ′(t)dM(t) equals ∫₀ ζ′(t)c(t)ζ(t)dO(t) ≡ 0. As a consequence, ∫₀ ζ′(t)dM(t) ≡ 0 holds as well. It follows that

\[ \int₀ ζ′(t)dR(t) = \int₀ ζ′(t)dA(t) + \int₀ ζ′(t)dM(t) ≡ 0. \]

1.1.5. Stochastic exponentials and logarithms. Consider a continuous semimartingale Z with Z(0) = 0. We define its stochastic exponential as the unique solution ℰ(Z) of the linear stochastic integral equation

\[ ℰ(Z) = 1 + \int₀ ℰ(Z)(t) dZ(t). \]

We leave it as an exercise to show that, in fact,

\[ ℰ(Z) = \exp \left( Z - \frac{1}{2}[Z,Z] \right). \]

The operator ℰ(·) has an inverse, which we denote by ℓ(·) and call the stochastic logarithm. For any strictly positive continuous semimartingale Y with Y(0) = 1, its stochastic logarithm is given by

\[ ℓ(Y) = \int₀ \frac{dY(t)}{Y(t)}. \]

Exercise 1.5. For a continuous semimartingale Z with Z(0) = 0, show that ℓ(ℰ(Z)) = Z. Similarly, for a strictly positive, continuous semimartingale Y with Y(0) = 1, show that ℰ(ℓ(Y)) = Y.

Exercise 1.6. Let Y be a strictly positive, continuous local martingale with Y(0) = 1. Note that Y(∞) := \lim_{t→∞} Y(t) ∈ [0, ∞) exists ℙ-a.e. on account of the supermartingale convergence theorem, and let

\[ B := \frac{1}{2}[ℓ(Y), ℓ(Y)] = \frac{1}{2}[\log Y, \log Y]. \]

Show that \{Y(∞) > 0\} = \{B(∞) < ∞\} and that

\[ \lim_{T→∞} \frac{\log Y(T)}{B(T)} = -1, \quad ℙ\text{-a.e. on } \{B(∞) = ∞\}. \]
**Exercise 1.7.** For any two continuous semimartingales $X$ and $Z$ with $X(0) = 0 = Z(0)$, establish the Yor formula:

$$\mathcal{E}(X)\mathcal{E}(Z) = \mathcal{E}(X + Z + [X, Z]).$$

1.1.6. General semimartingales. Our focus throughout this work will be on continuous processes as models of asset prices. Nevertheless, the need will occasionally arise to consider processes with jump discontinuities; more precisely, processes that have rcll paths: right-continuous on $[0, \infty)$ and possess finite left-limits on $(0, \infty)$. This generality will be necessary when dealing with wealth processes in the presence of general cumulative consumption streams as well as with martingales, of a general right-continuous filtration, which can only be guaranteed to have rcll paths. A concise reference for this, more general theory, is [JS03, Chapters I–III]; for a more detailed exposition, consult [HWY92]. We do note, however, that we shall try to minimize the use of the general theory throughout this book.

A real-valued process with rcll paths is called semimartingale if it can be written as the sum of an adapted process whose paths have finite variation on compact time intervals, and of a local martingale. In particular, (local) supermartingles and submartingales are semimartingales.

The previous decomposition of a semimartingale is not unique in general: there exist local martingales of finite variation, such as the compensated Poisson process. However, for any such decomposition, the local martingale part can be written as the sum of a continuous local martingale and a so-called pure-jump local martingale; as it turns out, the continuous local martingale part is the same in any such decomposition. We can therefore associate with any semimartingale $X$ a continuous local martingale part in a unique manner.

For a general process $X$ with rcll paths, we consider its jump process $\Delta X$ defined via

$$\Delta X(t) := X(t) - X(t^-), \quad t \in \mathbb{R}_+;$$

here $X(t^-)$ is the left limit of $X$ at $t \in \mathbb{R}_+$, with the convention $X(0^-) = X(0)$, i.e., $\Delta X(0) = 0$.

The covariation process $[X, Z]$ of two semimartingales $X$ and $Z$, having continuous local martingale parts $X^c$ and $Z^c$, respectively, is defined as

$$[X, Z] := [X^c, Z^c] + \sum_{t \leq} \Delta X(t) \Delta Z(t).$$

The above sum of products of jumps converges, on the strength of the Cauchy-Schwarz inequality and of the fact that, for semimartingales, squares of jumps over compact time intervals are summable.
The stochastic exponential of a general semimartingale $Z$ with $Z(0) = 0$ and $\Delta Z > -1$ is the strictly positive semimartingale
\begin{align*}
\mathcal{E}(Z) &= \exp \left( Z - \frac{1}{2} [Z^c, Z^c] \right) \prod_{t \leq \cdot} (1 + \Delta Z(t)) \exp(-\Delta Z(t)); \\
\end{align*}
it is the unique solution of the linear stochastic integral equation
\begin{align*}
\mathcal{E}(Z) &= 1 + \int_0^\cdot \mathcal{E}(Z)(t-)dZ(t).
\end{align*}
For any two semimartingales $X$ and $Z$ with $X(0) = 0 = Z(0)$ and $\Delta X > -1$, $\Delta Z > -1$, we still have the Yor formula
\begin{align*}
\mathcal{E}(X)\mathcal{E}(Z) &= \mathcal{E}(X + Z + [X, Z]),
\end{align*}
mentioned in Exercise 1.7 in the context of continuous semimartingales.\footnote{The proof of the Yor formula is the same for the general case, using integration by parts and the definition of the stochastic exponential $Y = \mathcal{E}(Z)$ as the solution of $Y = 1 + \int_0 Y(t-)dZ(t)$.}

### 1.2. Assets and investment

We are now in a position to introduce the model for a financial market that we shall be dealing with throughout this work, and to study in its context investment strategies and their properties.

#### 1.2.1. Assets

We shall consider throughout this book a vector of continuous semimartingales $R \equiv (R_i; 1 \leq i \leq n)$ with $n \in \mathbb{N}$ and $R_i(0) = 0$ for $1 \leq i \leq n$; its component processes represent the cumulative returns of $n$ stocks. We write the semimartingale decomposition
\begin{align*}
R_i &= A_i + M_i, \quad 1 \leq i \leq n,
\end{align*}
for $R \equiv (R_i; 1 \leq i \leq n)$ as in §1.1.3. In this financial market the stock prices $S \equiv (S_i; 1 \leq i \leq n)$ are then $S_i = S_i(0) \mathcal{E}(R_i), 1 \leq i \leq n$, in the notation of §1.1.5, for some given real numbers $S_i(0) > 0, 1 \leq i \leq n$, and their dynamics are given formally by
\begin{align*}
\frac{dS_i(t)}{S_i(t)} &= dR_i(t), \quad t \in \mathbb{R}_+, \quad 1 \leq i \leq n.
\end{align*}
Together with the specified initial values $(S_i(0), 1 \leq i \leq n)$, the dynamics of (1.10) completely determine the continuous, positive semimartingales $(S_i, 1 \leq i \leq n)$. Depending on the chosen value of $S_i(0)$, the process $S_i$ may model the price per share of company $1 \leq i \leq n$, or may alternatively model this company’s total capitalization. This is understood as the product of the actual share price, times the number of shares currently outstanding. The latter point of view is developed further in §1.3.5.
In any case, given $S_i(0) > 0$, the dynamics of (1.10) are expressed equivalently in logarithmic scale as

$$\log S_i = \log S_i(0) + \Gamma_i + M_i, \quad 1 \leq i \leq n.$$  

Here, with $C_{ii} = \begin{bmatrix} R_i, R_i \end{bmatrix} = \begin{bmatrix} \log S_i, \log S_i \end{bmatrix} \sin(1.5)$, we have set

$$\Gamma_i := A_i - \frac{1}{2} C_{ii}, \quad 1 \leq i \leq n,$$

for the stocks’ cumulative growth processes.

In terms of the rate processes in (1.3), (1.4), we now have

$$\Gamma_i = \int_0^t \gamma_i(t) dO(t),$$

where $\gamma_i := \alpha_i - \frac{1}{2} c_{ii}, \quad 1 \leq i \leq n$.

The processes $(\gamma_i; 1 \leq i \leq n)$ constitute the local growth rates for the various stocks. Recall that the processes $(\alpha_i; 1 \leq i \leq n)$ stand for the local mean rates of return of these stocks, whereas the processes $(c_{ii}; 1 \leq i \leq n)$ represent the stocks’ local quadratic variation rates. All these rates are expressed with respect to the operational clock $O$ of (1.2).

**1.2.2. Investment.** In the market described in §1.2.1, we now place an investor with initial capital $x \in \mathbb{R}_+$. For a given predictable, vector-valued process $\vartheta \equiv (\vartheta_i; 1 \leq i \leq n) \in \mathcal{I}(S)$ as in Definition 1.3, we shall think of the component $\vartheta_i(t)$ of the random vector $\vartheta(t)$ as representing the number of shares held by this investor in the asset $1 \leq i \leq n$ at time $t \in \mathbb{R}_+$. With this understanding, the integrand process $\vartheta \in \mathcal{I}(S)$ acquires the significance of investment strategy, and its vector stochastic integral

$$X(\cdot; x, \vartheta) := x + \int_0^t \vartheta(t) dS(t) \equiv x + \int_0^t \sum_{i=1}^n \vartheta_i(t) dS_i(t)$$

becomes the wealth process generated by $\vartheta$.

The interpretation of the quantities in (1.12) is as follows. At each instant $t \in \mathbb{R}_+$, the investor has $\vartheta_i(t)$ shares, thus $\vartheta_i(t) S_i(t)$ units of currency, invested in stock $1 \leq i \leq n$. The total wealth $\sum_{i=1}^n \vartheta_i(t) S_i(t)$ invested in the stock market, however, may be different from the actual wealth level $X(t; x, \vartheta)$ of the investor at time $t \in \mathbb{R}_+$. The discrepancy

$$X(t; x, \vartheta) - \sum_{i=1}^n \vartheta_i(t) S_i(t), \quad t \in \mathbb{R}_+,$$

is assumed to be invested in a money market account, whenever it is positive; and whenever the quantity in (1.13) is negative, its absolute value is supposed to represent the amount borrowed from the money market. It is then implicit in (1.12) that the money market charges no interest to borrowers, and offers no interest to lenders.
1.2. Assets and investment

Even when interest rates are insignificant relative to the rates of return of stocks, it is really helpful to be able to account for a non-trivial money market in financial modelling. Fortunately, this can be achieved at no cost to the theory developed here—as long as one thinks of stock returns as being in excess of interest, and of stock prices as being discounted by the money market account. This point is discussed more thoroughly in §1.2.5.

Remark 1.8 (From shares to amounts, and back). Let us consider a predictable, vector-valued process \( \vartheta \equiv (\vartheta_i; 1 \leq i \leq n) \in \mathcal{I}(S) \), with the interpretation that its components represent numbers of shares invested in the individual stocks. We define \( \varphi_i := \vartheta_i S_i \) for \( 1 \leq i \leq n \); then \( \varphi_i(t) \) represents the amount, in units of currency, invested in stock \( i \) at time \( t \in \mathbb{R}_+ \). From (1.10), it then follows that

\[
(1.14) \quad \int_0^t \sum_{i=1}^n \vartheta_i(t) dS_i(t) = \int_0^t \sum_{i=1}^n \varphi_i(t) dR_i(t);
\]

we leave as an exercise the verification of the fact that \( \varphi \in \mathcal{I}(R) \).

Conversely, let us start with a predictable, vector-valued process \( \varphi \equiv (\varphi_i; 1 \leq i \leq n) \in \mathcal{I}(R) \), whose components represent currency amounts invested in individual stocks, and define the number-of-shares processes \( \vartheta_i := \varphi_i / S_i \) for \( 1 \leq i \leq n \). It is straightforward to check, and we leave again as exercises, the fact that the resulting process \( \vartheta \equiv (\vartheta_i; 1 \leq i \leq n) \) belongs to \( \mathcal{I}(S) \) and that (1.14) holds once again.

The upshot of this discussion is that wealth processes can be defined alternatively in terms of stochastic integrals with respect to stock returns, rather than stock prices. This topic is taken up again in Section 1.3.

1.2.3. Admissibility. In practice, credit constraints preclude the implementation of any investment strategy imaginable. The notion introduced below reflects this reality.

Definition 1.9. For given initial capital \( x \in \mathbb{R}_+ \) and investment strategy \( \vartheta \equiv (\vartheta_i; 1 \leq i \leq n) \in \mathcal{I}(S) \), the wealth process \( X(\cdot; x, \vartheta) \) will be called admissible if solvency \( X(\cdot; x, \vartheta) \geq 0 \) holds.

Restricting attention to nonnegative wealth processes automatically enforces a mechanism for avoiding insolvency, that is, negative wealth, a possibility that has to be taken seriously into consideration in economic theory and in financial modelling. From a mathematical viewpoint, the credit constraints implicit in this admissibility requirement ensure that so-called doubling strategies, as the ones in the following example, are not allowed.

Example 1.10 (Doubling strategies). Let \( n = 1 \) and \( R \equiv R_1 \) be a standard Brownian motion. Restricting attention to the time-interval \([0, 1)\),
we define a predictable process $\eta$ and a continuous martingale $N$ via

$$\eta(t) := \frac{1}{S(t)\sqrt{1-t}}, \quad t \in [0, 1), \quad N := \int_0^t \eta(s) dS(s) = \int_0^t \frac{dR(t)}{\sqrt{1-t}}.$$  

Note that $[N, N](t) = -\log(1-t)$ holds for $t \in [0, 1)$, and thus the process $W$ defined via $W(u) := N(1 - \exp(-u))$, $u \in \mathbb{R}_+$, is standard scalar Brownian motion in its own filtration, by Lévy’s characterization ([KS91, Theorem 3.3.16]). It then follows that the stopping times

$$\tau^m := \inf \{t \in [0, 1) \mid N(t) \geq m\}, \quad m \in \mathbb{N},$$

are well defined and $[0, 1)$-valued. For every $m \in \mathbb{N}$, consider the strategy $\vartheta^m := \eta 1_{(0, \tau^m]} \in \mathcal{I}(S)$, and note that $X(1, \vartheta^m) = 1 + m$ holds for all $m \in \mathbb{N}$. In other words, starting with one currency unit at time $t = 0$, arbitrarily large levels of wealth can be reached at time $t = 1$.

All wealth processes $X(\cdot; 1, \vartheta^m) \equiv 1 + N(\cdot \wedge \tau^m)$, $m \in \mathbb{N}$, are local martingales and unbounded from below, signifying the existence of an “infinitely deep credit line”. For suppose the local martingale $X(\cdot; 1, \vartheta^m)$ were bounded from below by a real constant; then this local martingale would be a supermartingale, leading to the absurd conclusion

$$1 + m = \mathbb{E}^P[X(1; 1, \vartheta^m)] \leq X(0; 1, \vartheta^m) = 1.$$  

In other words, in the absence of credit constraints, the use of appropriate strategies allows one to attain any target level of wealth imaginable, at any given, fixed time $T \in (0, \infty)$.

The situation is very different when credit constraints are imposed, in recognition of the hard reality that “eventually one runs out of other people’s money”. This is the case, for instance, with the admissibility constraint of Definition 1.9, which requires solvency at all times. Then every admissible wealth process $X(\cdot; x, \vartheta)$ in this example is a nonnegative local martingale, thus a supermartingale; and the Optional Sampling Theorem implies that $\mathbb{E}^P[X(\tau; x, \vartheta)] \leq x$ holds for every $\mathbb{P}$-a.e. finite stopping time $\tau \in \mathcal{T}$, drastically limiting the possibilities for gain.

Let us mention that the admissibility conditions of Definition 1.9 do not completely proscribe the possibility of making sure gains with limited downside. We take up this important subject in Chapter 2.

**Remark 1.11 (Short-selling restrictions).** In addition to imposing solvency in the form of nonnegative wealth, other types of constraints could be incorporated in the model. For example, potential prohibition of selling stocks short leads to consideration of *long-only* investment strategies $\vartheta \equiv (\vartheta_i; 1 \leq i \leq n)$, which satisfy $\vartheta_i \geq 0$ for $1 \leq i \leq n$; in other words, $\vartheta \in \mathcal{I}(S; \mathbb{R}_+^n)$. Proscribing short (that is, negative) positions in the stock
market is not always such an onerous restriction; on the contrary, it reflects quite adequately the mandates and realities under which many institutional money managers, such as pension funds, are required to operate.

Suppose now that $x \in \mathbb{R}_+$ is the investor’s initial capital. If, in addition to proscribing short-selling of stocks, we also prohibit borrowing from the money market, we face the extra constraint

$$X(t; x, \vartheta) - \sum_{i=1}^{n} \vartheta_i(t)S_i(t) \geq 0, \quad \forall t \in \mathbb{R}_+.$$  

The quantity on the left-hand side of the above inequality expresses the amount of cash held in the money market; see (1.13), as well as §1.2.5 below. The constraint (1.15) appears operationally complicated, as it involves not only the current positions in stocks, but also the current overall level of wealth. However, we shall see in §1.3.2 that this constraint becomes extremely simple when cast in terms of proportional investment rules.

1.2.4. Capital withdrawals. We now extend the description of wealth processes to accommodate withdrawals of capital. These may happen for reasons either external (for instance, a payment stream representing a liability that the investor faces) or internal (for instance, consumption). Both of these cases will be important in the sequel.

Withdrawal of capital is modelled through elements in $\mathcal{K}$, defined as the set of all nondecreasing, adapted and right-continuous processes $K$ with $K(0) = 0$. Here, $K(t)$ stands for the cumulative (or aggregate) capital withdrawn up to time $t \in \mathbb{R}_+$; actual withdrawals in each infinitesimal interval $(t, t + dt]$ are formally represented as $dK(t)$.

Elementary examples of processes in $\mathcal{K}$ are those of the form $h1_{(\tau, \infty)}$, where $\tau$ is a stopping time with $\tau > 0$ and $h$ is a non-negative and $\mathcal{F}(\tau)$-measurable random variable. They usually go under the rubric of European contingent claims; we shall study these in more detail in Section 3.3. Convex combinations of such processes may represent payment streams at discrete points in time, or consumption in “gulps”.

Another example involves processes of the form $\int_{0}^{\cdot} h(t)dt$, where $h$ is predictable (or optional; see Exercise 1.2), non-negative and locally integrable. Such processes correspond to continuous payment streams, or to consumption patterns, that occur continuously in time in a “smooth” manner.

Let us now consider an investor with initial capital $x \in \mathbb{R}_+$. For a given predictable investment strategy $\vartheta = (\vartheta_i; 1 \leq i \leq n) \in \mathcal{I}(S)$ as in §1.2.2, and a given process $K \in \mathcal{K}$, we define

$$X(\cdot; x, \vartheta, K) := x + \int_{0}^{\cdot} \sum_{i=1}^{n} \vartheta_i(t)dS_i(t) - K$$
to be the \textbf{wealth process} generated by the investment strategy \( \vartheta \) and the cumulative withdrawal stream \( K \). Comparing (1.16) with (1.12), we have for \( x \in \mathbb{R}_+ \), \( \vartheta \in \mathcal{I}(S) \) and \( K \in \mathcal{K} \) the identity
\[
X(\cdot; x, \vartheta, K) = X(\cdot; x, \vartheta, 0) - K = X(\cdot; x, \vartheta) - K.
\]

\textbf{Notational Convention 1.12.} Whenever the argument “\( K \)” is missing from the expression \( X(\cdot; x, \vartheta, K) \), that is, whenever we write \( X(\cdot; x, \vartheta) \), it will be understood that there are no capital withdrawals and that the notation of (1.12) is in force.

Capital withdrawals are allowed as long as the investor remains solvent. We make this mathematically precise, as is shown below.

\textbf{Definition 1.13.} For \( x \in \mathbb{R}_+ \), \( \vartheta \in \mathcal{I}(S) \) and \( F \in \mathcal{K} \), the wealth process \( X \equiv X(\cdot; x, \vartheta, F) \) is said to \textbf{finance} a given cumulative withdrawal stream \( K \in \mathcal{K} \), if \( X \geq K \) holds. In this case, the process \( K \in \mathcal{K} \) will be called \textbf{financeable from initial capital} \( x \in \mathbb{R}_+ \).

We denote by \( \mathcal{K}(x) \) the subset of \( \mathcal{K} \) consisting of cumulative capital withdrawal streams financeable from initial capital \( x \in \mathbb{R}_+ \); formally,
\[
\mathcal{K}(x) := \{ K \in \mathcal{K} | \exists \vartheta \in \mathcal{I}(S), F \in \mathcal{K}, \text{ such that } X(\cdot; x, \vartheta, F) \geq K \}.
\]

Some remarks are in order. Suppose that for some given \( x \in \mathbb{R}_+ \), \( \vartheta \in \mathcal{I}(S) \) and \( F \in \mathcal{K} \), the wealth process \( X \equiv X(\cdot; x, \vartheta, F) \) finances some \( K \in \mathcal{K} \), if \( X \geq K \) holds. In this case, the process \( K \in \mathcal{K} \) will be called \textbf{financeable from initial capital} \( x \in \mathbb{R}_+ \).

The following observations are immediate from the definition (1.17) of \( \mathcal{K}(x) \), \( x \in \mathbb{R}_+ \), and from its characterization (1.18):
\begin{itemize}
  \item \( K \equiv 0 \in \mathcal{K}(0) \).
  \item For every \( x \in \mathbb{R}_+ \), the set \( \mathcal{K}(x) \) is convex.
  \item For every \( x \in \mathbb{R}_+ \), \( K \in \mathcal{K}(x) \), and \( \overline{K} \in \mathcal{K} \) with \( \overline{K} \leq K \), we have \( \overline{K} \in \mathcal{K}(x) \).
  \item For every \( 0 \leq x_1 \leq x_2 < \infty \), we have \( \mathcal{K}(x_1) \subseteq \mathcal{K}(x_2) \).
  \item For every \( x \in (0, \infty) \), we have \( \mathcal{K}(x) = x\mathcal{K}(1) \).
\end{itemize}
1.2.5. **Money market and discounting.** We elaborate here on the discussion following (1.12) and (1.13) regarding money market accounts. Assume that $\tilde{R}_0$ is an adapted and continuous process of finite variation on compact time-intervals with $\tilde{R}_0(0) = 0$; for $t \in \mathbb{R}_+$, the differential $d\tilde{R}_0(t)$ denotes the interest paid in the interval $[t, t+dt]$ per unit of account. No assumption is made about the sign of this interest. Similarly, let $\tilde{R}_i$, $1 \leq i \leq n$, be continuous semimartingales with $\tilde{R}_i(0) = 0$ representing the cumulative returns of $n$ stocks.

The dynamics of the processes $(\tilde{S}_i; 0 \leq i \leq n)$ satisfy

$$\frac{d\tilde{S}_i(t)}{\tilde{S}_i(t)} = d\tilde{R}_i(t), \quad t \in \mathbb{R}_+, \quad 0 \leq i \leq n.$$ 

Here $\tilde{S}_i(0) > 0$ for $1 \leq i \leq n$, and we set $\tilde{S}_0(0) = 1$ in order for the process $\tilde{S}_0$ to represent the time evolution of a unit of currency invested in the money market at time $t = 0$. Define

$$(1.19) \quad R_i := \tilde{R}_i - \tilde{R}_0, \quad S_i := \frac{\tilde{S}_i}{\tilde{S}_0}, \quad 0 \leq i \leq n,$$

and note that, since $\tilde{R}_0$ is of finite variation, we have

$$\frac{dS_i(t)}{S_i(t)} = dR_i(t), \quad t \in \mathbb{R}_+, \quad 0 \leq i \leq n,$$

after some simple calculus; this is exactly (1.10) with the added requirements $R_0 \equiv 0$ and $S_0 \equiv 1$. The processes $(R_i; 1 \leq i \leq n)$ are the cumulative returns of the $n$ stocks in excess of the interest paid by the money market, and the processes $(S_i; 1 \leq i \leq n)$ are the stock prices discounted by the money market.

Now consider an investor starting out with initial capital $x \in \mathbb{R}_+$, using an investment strategy $\vartheta \equiv (\vartheta_i; 1 \leq i \leq n)$ with the same interpretation as in §1.2.2, and implementing a capital withdrawal plan $\tilde{K}$ as in §1.2.4. If $\tilde{X} \equiv \tilde{X}(\cdot; x, \vartheta, \tilde{K})$ is the corresponding generated wealth process, then

$$(1.20) \quad \tilde{X} = x + \int_0^t \sum_{i=1}^n \vartheta_i(t) d\tilde{S}_i(t) + \int_0^t \left( \tilde{X}(t) - \sum_{i=1}^n \vartheta_i(t)\tilde{S}_i(t) \right) d\tilde{R}_0(t) - \tilde{K}.$$ 

Here, the amount of cash held in the money market throughout time is given by the process

$$\tilde{X}(\cdot) - \sum_{i=1}^n \vartheta_i(\cdot)\tilde{S}_i(\cdot) \equiv \tilde{X}(\cdot; x, \vartheta, \tilde{K}) - \sum_{i=1}^n \vartheta_i(\cdot)\tilde{S}_i(\cdot).$$
Exercise 1.14. With the above notation, set
\[ K := \int_0^\cdot \frac{d\tilde{K}(t)}{\tilde{S}_0(t)}, \quad X(\cdot ; x, \vartheta, K) := \frac{\tilde{X}(\cdot ; x, \vartheta, \tilde{K})}{\tilde{S}_0(\cdot)}. \]

Integrating by parts (1.20), and recalling that $\tilde{R}_0$ is of finite variation, show that
\[ X(\cdot ; x, \vartheta, K) = x + \int_0^\cdot \sum_{i=1}^n \vartheta_i(t) dS_i(t) - K. \]
This is exactly (1.16) with $S \equiv (S_i; 1 \leq i \leq n)$ given in (1.19).

This discussion validates the setup of §1.2.1–§1.2.4 in the context of a non-trivial money market, as long as everything is expressed in relative terms, in the manner of (1.19). Since all notation and developments will be significantly simpler and cleaner this way, from now on we adopt the convention that the money market pays—and charges—zero interest. We keep throughout the notation
\[ (1.21) \quad R_0 \equiv 0, \quad S_0 \equiv 1, \]
reinforcing the fact that wealth is discounted in units of the money market.

1.3. Proportional investment

We now discuss investment decisions from the point of view of investors who are not so much interested in how many shares of any given asset they buy or sell at any given time, or even in the currency amounts involved in such transactions, as they are in requiring that total wealth stay strictly positive at all times, and in keeping track of the proportions of this wealth which get invested in the individual assets. Such a point of view leads naturally to the notions of numéraire in Definition 1.15, and of portfolio in Definition 1.16.

1.3.1. Numéraires and portfolios. Wealth processes that remain strictly positive at all times may be used as “baselines” in terms of which all other wealth is denominated, thus allowing for relative comparisons. Such wealth processes, which in addition do not withdraw any capital, will be of central importance throughout this book; they merit a separate appellation.

Definition 1.15. A wealth process $X \equiv X(\cdot ; 1, \vartheta)$ as in (1.12) will be called numéraire if it remains strictly positive at all times: $X > 0$. The collection of all numéraires will be denoted by $X'$.

\[^3\]Whenever $\mathbb{R}_+ \ni s \leq t \in \mathbb{R}_+$, integrals of the form “$\int_s^t$” with respect to cumulative withdrawal processes are always assumed to be equal to “$\int_{[s,t]}$.”
1.3. Proportional investment

Numéraires are admissible wealth processes but satisfy a more stringent, strict positivity (as opposed to mere non-negativity) constraint. The requirement $X(0) = 1$ is a simple normalization, ensuring that relative and actual wealth coincide at time $t = 0$. By definition, there exists for any given numéraire $X \in \mathcal{X}$ a predictable, vector-valued process $\vartheta \in \mathcal{I}(S)$ with

$$X = 1 + \int_0^t \sum_{i=1}^n \vartheta_i(t) dS_i(t) \equiv X(\cdot;1,\vartheta) > 0.$$  

(1.22)

Since $X > 0$, one may then define a new predictable, vector-valued process $\pi \equiv (\pi_i, 1 \leq i \leq n) \in \mathcal{P}(\mathbb{R}^n)$ via

$$\pi_i := S_i \vartheta_i / X, \quad \text{for } 1 \leq i \leq n.$$  

(1.23)

Because $\vartheta_i(t)$ represents the units of investment (or numbers of shares) held in stock $i$ at time $t$, the quantity $\pi_i(t)$ gives the fraction (or proportion) of current wealth invested in stock $i$, and $\sum_{i=1}^n \pi_i(t)$ the proportion invested in the entire stock market, at time $t \in \mathbb{R}_+$. The remaining proportion

$$\pi_0 := 1 - \sum_{i=1}^n \pi_i$$  

(1.24)

of wealth is placed in the money market, which, we recall, pays and charges zero interest. As all wealth is discounted with respect to the money market account in accordance with §1.2.5, this proportion of wealth may be thought of as placed under the proverbial mattress. Negative weights in stocks signify short-selling; negative weights in the money market signify borrowing from it. Weights in stocks in excess of 1 indicate leverage.

It is straightforward to check that the process $\pi$ defined from $\vartheta$ as in (1.23) is $\mathcal{P}$-integrable. Thus, we may then cast (1.22) as

$$X = 1 + \int_0^t X(t) \sum_{i=1}^n \pi_i(t) dR_i(t) = 1 + \int_0^t X(t) \pi'(t) dR(t)$$  

in terms of the resulting process $\pi \in \mathcal{I}(R)$ on account of (1.10), or more schematically, and in differential form, as

$$\frac{dX(t)}{X(t)} = \sum_{i=1}^n \pi_i(t) dR_i(t) = \sum_{i=1}^n \pi_i(t) \frac{dS_i(t)}{S_i(t)}, \quad X(0) = 1.$$  

(1.25)

In particular, we have $X = \mathcal{E}(R_\pi)$ in the stochastic exponential notation of §1.1.5, where the cumulative return from investing according to the process $\pi \in \mathcal{I}(R)$ is given by

$$R_\pi := \int_0^t \pi'(t) dR(t) = \int_0^t \sum_{i=1}^n \pi_i(t) dR_i(t).$$  

(1.26)
Conversely, starting with any given predictable, vector-valued process \( \pi \in \mathcal{I}(R) \), one may define the vector stochastic integral \( \int \pi(t) \, dR_t(t) \), and in terms of it the strictly positive, continuous semimartingale
\[
X_\pi := \mathcal{E}(R_\pi) = \mathcal{E}\left( \int_0^t \sum_{i=1}^n \pi_i(t) \, dR_i(t) \right), \quad \pi \in \mathcal{I}(R).
\]

From §1.1.5 and (1.10), this stochastic exponential process satisfies the dynamics \( dX_\pi(t) = X_\pi(t) \, dR_\pi(t) \) of (1.25). Then, with the predictable, vector-valued process \( \vartheta \equiv (\vartheta_i 1 \leq i \leq n) \in \mathcal{I}(S) \) defined via
\[
\vartheta_i := \frac{X_\pi \pi_i}{S_i}, \quad 1 \leq i \leq n,
\]
it follows that the equation (1.22) holds with \( X_\pi \) replacing \( X \).

To recapitulate: the previous discussion gives us an alternative, multiplicative representation of numéraires. Indeed, the collection \( \mathcal{X} \) of numéraires consists precisely of all processes of the form \( X_\pi \) as in (1.27), with \( \pi \in \mathcal{I}(R) \).

The notion is important and merits its own appellation.

**Definition 1.16.** We shall use the term **portfolio** to refer to any predictable, vector-valued processes \( \pi \in \mathcal{I}(R) \) as in Definition 1.3, generating the corresponding numéraire \( X_\pi \) as in (1.27).

For a given portfolio \( \pi \in \mathcal{I}(R) \), suppose one tries to solve the stochastic equation (1.25) for \( X \) but now with an arbitrary initial capital \( x \in (0, \infty) \) rather than \( x = 1 \). It is clear that the solution is then \( xX_\pi = x\mathcal{E}(R_\pi) \): for portfolios, wealth scales multiplicatively.

**Example 1.17 (Constant-proportion portfolios).** Fix a vector \( p \equiv (p_i; 1 \leq i \leq n) \in \mathbb{R}^n \), and let \( \pi \equiv p \in \mathcal{I}(R) \) be the portfolio that keeps a fixed proportion \( p_i \) of wealth in each stock \( 1 \leq i \leq n \) at all times. In this case, it is clear that \( R_p = p' R = \sum_{i=1}^n p_i R_i \).

An important special case is when \( p = 0 \), meaning \( \pi_i \equiv 0 \) for all \( 1 \leq i \leq n \). This portfolio never touches the stock market, keeping all wealth in the money market. Since wealth is denominated in terms of the money market, it is clear that \( X_0 \equiv 1 \); this is confirmed by (1.27), since \( R_0 \equiv 0 \).

When \( p \) coincides with one of the *unit vectors* \( e^i \), \( 1 \leq i \leq n \), of \( \mathbb{R}^n \), whose coordinates are all zero except for coordinate \( i \) which is 1, the entire wealth is invested in stock \( i \) at all times. In this case, \( X_{e^i} = S_i/S_i(0) \) holds.

Another special case is the *equal-weighted portfolio*, that assigns constant and equal weights to all stocks at all times: \( p_i = 1/n \), for all \( 1 \leq i \leq n \). Just like the portfolios of the previous paragraph, this portfolio never touches the money market. We shall study it in more detail in Section 1.5.
1.3. Portfolio constraints. The introduction of numéraires and proportional investment (i.e., portfolios) facilitates the description of certain common constraints on investment choice. To begin with, the representation (1.27) already enforces admissibility in the sense of Definition 1.9, actually in the even stronger form of strict positivity.

Recalling Definition 1.3 on integrands, we see that prohibiting short sales of stocks for a portfolio \( \pi \) requires \( \pi \in \mathcal{I}(R; \mathbb{R}_n^+) \). If borrowing from the money market is further prohibited for \( \pi \), the constraint \( \pi_0 \geq 0 \) in the notation of (1.24) should also hold; the requirement then becomes \( \pi \in \mathcal{I}(R; \triangle_n) \), where the \( n \)-dimensional simplex \( \triangle_n \) is defined as

\[
\triangle_n := \left\{ (x_i; 1 \leq i \leq n) \in \mathbb{R}_n^+ \mid \sum_{i=1}^n x_i \leq 1 \right\}.
\]

Such portfolios, which never sell any stock short and never borrow from the money market, will be called long-only portfolios. We note that \( \triangle_n \)-valued predictable processes are a fortiori bounded and, in particular, \( R \)-integrable: \( \mathcal{I}(R; \triangle_n) = \mathcal{P}(\triangle_n) \).

A portfolio that never touches (i.e., neither borrows from, nor lends to) the money market will be called a stock portfolio. Clearly, \( \pi \) being a stock portfolio is equivalent to the requirement \( \pi \in \mathcal{I}(R; H_{n-1}) \), where

\[
H_{n-1} := \left\{ (x_i; 1 \leq i \leq n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1 \right\}
\]

is an \((n - 1)\)-dimensional affine subspace of \( \mathbb{R}^n \).

If, additionally, no short-selling of stocks is allowed, the portfolio \( \pi \) will be called a long-only stock portfolio. This requirement is cast as \( \pi \in \mathcal{I}(R; \triangle_{(n-1)}) = \mathcal{P}(\triangle_{(n-1)}) \), where

\[
\triangle_{(n-1)} := H_{n-1} \cap \mathbb{R}_n^+ = \left\{ (x_i; 1 \leq i \leq n) \in \mathbb{R}_n^+ \mid \sum_{i=1}^n x_i = 1 \right\}
\]

is the lateral face of the unit simplex \( \triangle_n \) in (1.28).

**Example 1.18 (Universal portfolio).** For every element \( p \in \triangle_{(n-1)} \) of the set in (1.30), denote by \( X_p \) the wealth process generated as in (1.25), (1.27) by the long-only, constant proportion stock portfolio \( \pi \equiv p \). In terms of these processes, now define the long-only so-called universal portfolio \( \pi^U = (\pi^U_i; 1 \leq i \leq n) \) of [Cov91] and [Jam92], whose components

\[
\pi^U_i(t) := \frac{\int_{\triangle_{(n-1)}} p_i X_p(t) dp}{\int_{\triangle_{(n-1)}} X_p(t) dp}, \quad t \in \mathbb{R}_+, \quad 1 \leq i \leq n,
\]
gives the proportions of the thereby-generated wealth
\[ X_{\pi^U}(t) = \frac{1}{\text{Leb}(\triangle_{n-1})} \int_{\triangle_{n-1}} X_p(t) dp, \quad t \in \mathbb{R_+} \]
that get invested in the various assets \( 1 \leq i \leq n \).

1.3.3. Return versus growth. Given any two portfolios \( \pi \in \mathcal{I}(R) \) and \( \rho \in \mathcal{I}(R) \), let us recall the notation of (1.4), (1.26) and define
\[ c_{\pi\rho} := \pi' c\rho = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i c_{ij} \rho_j, \quad C_{\pi\rho} := [R_\pi, R_\rho] = \int_0^\cdot c_{\pi\rho}(t) dO(t). \]
As a matter of convention, and in order to ease notation, we shall simply use the subscript “\( i \)” instead of \( e_i \) when one of the portfolios in (1.31) is the unit vector \( e_i \) of \( \mathbb{R}^n \) for some \( 1 \leq i \leq n \); for example, we shall write \( c_{e_i \rho} \equiv c_{e_i e_j} \) and \( C_{e_i \rho} \equiv C_{e_i e_j} \). This convention is consistent with the actual equalities \( c_{ij} = c_{e_i e_j} \) and \( C_{ij} = C_{e_i e_j} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \).

The definition (1.27) for \( X_\pi = \mathcal{E}(R_\pi) \), the properties of stochastic exponentials in §1.1.5, and the notation of (1.26) and (1.31), express the log-wealth process generated by a portfolio \( \pi \in \mathcal{I}(R) \) as
\[ \log X_\pi = R_\pi - \frac{1}{2} C_{\pi\pi} = \int_0^\cdot \pi'(t) dA(t) - \frac{1}{2} C_{\pi\pi} + \int_0^\cdot \pi'(t) dM(t). \]
The finite-variation part of the above decomposition is called the cumulative growth of the portfolio \( \pi \in \mathcal{I}(R) \) and is denoted
\[ \Gamma_\pi := A_\pi - \frac{1}{2} C_{\pi\pi}, \quad \text{where} \quad A_\pi := \int_0^\cdot \pi'(t) dA(t) = \int_0^\cdot \sum_{i=1}^{n} \pi_i(t) dA_i(t), \]
by analogy with (1.11). In a similar vein, the local martingale part of the decomposition in (1.32) is denoted by
\[ M_\pi := \int_0^\cdot \pi'(t) dM(t) = \int_0^\cdot \sum_{i=1}^{n} \pi_i(t) dM_i(t). \]
It is useful to recall at this point the definition of the stochastic logarithm from §1.1.5, and obtain
\[ \log X_\pi = \Gamma_\pi + M_\pi, \quad \mathcal{L}(X_\pi) = A_\pi + M_\pi = R_\pi = \int_0^\cdot \frac{dX_\pi(t)}{X_\pi(t)}. \]
From these representations, the cumulative growth process \( \Gamma_\pi \) emerges as the “drift” (i.e., finite-variation) component of the natural logarithm \( \log X_\pi \) of \( X_\pi \), and the cumulative mean-returns process \( A_\pi \) as the drift component of the stochastic logarithm \( \mathcal{L}(X_\pi) \) of \( X_\pi \). The quantities in (1.33) have


1.3. Proportional investment

equivalent representations in terms of rates with respect to the operational clock $O$ in (1.2), as we now explain. We define the predictable processes

\begin{equation}
(1.36) \quad \alpha_\pi := \pi' \alpha, \quad \gamma_\pi := \pi' \alpha - \frac{1}{2} \pi' \gamma \pi = \alpha_\pi - \frac{1}{2} c_\pi \pi.
\end{equation}

Here $\alpha_\pi$ is the **mean rate of return** of the portfolio $\pi \in \mathcal{I}(R)$, and $\gamma_\pi$ is this portfolio’s **growth rate**. These rates are connected to their cumulative versions in (1.33) via

\begin{equation}
A_\pi = \int_0^\cdot \alpha_\pi(t) dO(t), \quad \Gamma_\pi = \int_0^\cdot \gamma_\pi(t) dO(t).
\end{equation}

### 1.3.4. Excess growth.

It follows from (1.36) that the mean rate of return of the wealth generated by a given portfolio $\pi \in \mathcal{I}(R)$ is the weighted average of the mean rates of return of the individual stocks, where the weights are given by the fractions in the portfolio $\pi$, namely, $\alpha_\pi = \sum_{i=1}^n \pi_i \alpha_i$.

The same is **not** true for the growth rate. In fact, from (1.36) one obtains

\begin{equation}
(1.37) \quad \gamma_\pi = \sum_{i=1}^n \pi_i \gamma_i + \gamma_\pi^*, \quad \text{where} \quad \gamma_\pi^* := \frac{1}{2} \left( \sum_{i=1}^n \pi_i c_{ii} - \sum_{i=1}^n \sum_{j=1}^n \pi_i c_{ij} \pi_j \right)
\end{equation}

or equivalently, in cumulative form:

\begin{equation}
(1.38) \quad \Gamma_\pi = \sum_{i=1}^n \int_0^\cdot \pi_i(t) d\Gamma_i(t) + \Gamma_\pi^*, \quad \Gamma_\pi^* := \int_0^\cdot \gamma_\pi^*(t) dO(t).
\end{equation}

In other words, to obtain the growth rate of a given portfolio $\pi$ one needs to add to the weighted average $\sum_{i=1}^n \pi_i \gamma_i$ of the individual stock growth rates, the quantity $\gamma_\pi^*$ in (1.37). This is determined entirely from the portfolio’s weights and from the covariation structure of the stocks in the market.

Following [Fer02], we shall call the process $\gamma_\pi^*$ in (1.37) the **excess growth rate** (with respect to the clock $O$ in (1.2)) associated with the portfolio $\pi \in \mathcal{I}(R)$. We shall call the process $\Gamma_\pi^*$ in (1.38) this portfolio’s **cumulative excess growth**. As we shall see in Exercise 3.78, the excess growth plays an important rôle in the study of relative portfolio (out)performance.

**Exercise 1.19.** Suppose that the portfolio $\pi$ is long-only, i.e., $\pi \in \mathcal{I}(R) \cap \mathcal{P}(\Delta_n)$ in the notation of (1.28). Then show that the excess growth rate of (1.37) satisfies $\gamma_\pi^* \geq 0$.

**Exercise 1.20 (Fund-of-funds investing).** Let $(\pi^k; 1 \leq k \leq l)$, $l \in \mathbb{N}$, be portfolios in the market of Section 1.2, and denote by $V^k \equiv X_{\pi^k}$ the corresponding wealth processes generated with $V^k(0) = 1$, for $1 \leq k \leq l$.

Now regard the continuous, strictly positive semimartingales $(V^k; 1 \leq k \leq l)$ as the assets of a new, **aggregated market**, in which one can invest in a “fund-of-funds” fashion using a portfolio $\rho = (\rho^k; 1 \leq k \leq l)$. Such
investment induces a portfolio \( \Pi = (\Pi_i; 1 \leq i \leq n) \) in the original market, with weights in the individual stocks given by \( \Pi_i = \sum_{k=1}^{l} \rho^k \pi_{ik} \).

Show that the excess growth rate \( \gamma^*_\Pi \) of this induced portfolio in the original market is

\[
\gamma^*_\Pi = \sum_{k=1}^{l} \rho^k \gamma^*_\pi_k + g^*_\rho,
\]

where \( g^*_\rho := \frac{1}{2} \left( \sum_{k=1}^{l} \rho^k c_{\pi_k} \pi_k - \sum_{k=1}^{l} \sum_{j=1}^{l} \rho^k c_{\pi_k \pi_j} \rho^j \right) \)

is the excess growth rate of the fund-of-funds \( \rho \) in the aggregated market.

### 1.3.5. Market portfolio

As mentioned right after (1.10), the processes \((S_i; 1 \leq i \leq n)\) may be thought of as representing the capitalizations of \( n \) different companies, whose stocks are traded in an equity market. Capitalization is understood as the product of the actual stock price times the number of shares currently outstanding.

With this interpretation, the sum \( S_1 + \cdots + S_n \) represents the total capitalization of the market, and

\[
(1.39) \quad \mu_i := \frac{S_i}{\Sigma}, \quad 1 \leq i \leq n, \quad \text{where} \quad \Sigma := S_1 + \cdots + S_n,
\]

the relative market capitalizations, or market weights, of individual companies. The resulting vector process \( \mu \equiv (\mu_i; 1 \leq i \leq n) \) is adapted and continuous, thus predictable, and takes values in the set \( \Delta_{(n-1)} \) of (1.30); therefore it is also bounded and \( R \)-integrable. Consequently, this process \( \mu \in \mathcal{P}(\Delta_{(n-1)}) \) can be used as a stock portfolio. It is checked with the help of (1.25) that investing according to this stock portfolio \( \mu \) amounts to owning a slice of the entire market capitalization, proportional to the initial investment. Namely, in the notation of (1.39), we have

\[
(1.40) \quad X_\mu = \frac{S_1 + \cdots + S_n}{S_1(0) + \cdots + S_n(0)} = \frac{\Sigma}{\Sigma(0)}.
\]

In view of this representation (1.40), we shall call \( \mu \) the **market portfolio**. For it, the actual units of investment, that is, the numbers of shares

\[
\vartheta_i = \frac{\mu_i X_\mu}{S_i} = \frac{1}{\Sigma(0)}, \quad 1 \leq i \leq n,
\]

held in the various assets, are constant across all times and stocks. In other words, the market portfolio buys a fixed number of shares in each stock at time \( t = 0 \), the same for all of them, and then holds on to these shares indefinitely into the future. It is a quintessential **buy-and-hold** investment.

### 1.4. Relative performance

Consider a portfolio \( \rho \in \mathcal{I}(R) \), which we regard as some sort of “baseline”. We want to compare the performance of another portfolio \( \pi \in \mathcal{I}(R) \) to that
of $\rho$, so we need to understand the behaviour of the relative wealth process
\begin{align}
X^\rho_\pi := \frac{X_\pi}{X_\rho}, \quad \pi \in \mathcal{I}(R), \quad \rho \in \mathcal{I}(R).
\end{align}

In this relative wealth notation $X^\rho_\pi$, the superscript $\rho$ denotes the baseline portfolio, and the subscript $\pi$ denotes the portfolio whose wealth is compared to the wealth generated by the baseline.

1.4.1. Wealth ratios. It turns out that the relative wealth $X^\rho_\pi$ of (1.41) can be seen as the result of investing according to the portfolio $\pi$ in an auxiliary market, in which all primary assets (that is, money market $i = 0$, and stocks $1 \leq i \leq n$) are denominated in units of $X_\rho$; namely,
\begin{align}
S^\rho_i := \frac{S_i}{X_\rho}, \quad 0 \leq i \leq n,
\end{align}
with $S_0(\cdot) \equiv 1$ as in (1.21) and consequently $S^\rho_0 = 1/X_\rho$. Proposition 1.22 below shows that these processes can be expressed as stochastic exponentials
\begin{align}
S^\rho_i = S_i(0) \mathcal{E}(R^\rho_i), \quad \text{thus} \quad \frac{dS^\rho_i(t)}{S^\rho_i(t)} = dR^\rho_i(t), \quad 0 \leq i \leq n,
\end{align}
in terms of the vector semimartingale $R^\rho \equiv (R^\rho_i; 0 \leq i \leq n)$ with components
\begin{align}
R^\rho_0 := C_{\rho\rho} - R_\rho \quad \text{and} \quad R^\rho_i := R^\rho_0 + (R_i - C_{i\rho}), \quad 1 \leq i \leq n.
\end{align}
These processes are then seen to play the rôle of cumulative returns in the new, auxiliary market. We recall here the notation of (1.26), as well as of (1.31) and the paragraph following it.

Remark 1.21. It is very important to stress that, in this auxiliary market, the index 0 plays a rôle whose importance is exactly the same as that of all other indices $1 \leq i \leq n$, because the processes
\[ R^\rho_0 = C_{\rho\rho} - R_\rho \quad \text{and} \quad S^\rho_0 = 1/X_\rho = \mathcal{E}(R^\rho_0) \]
are in general no longer trivial. This is in contrast to the situation in (1.21), with $R_0 \equiv 0$ and $S_0 \equiv 1$, where the money market is the baseline.

The “change of numéraire” representation (1.45) that follows generalizes the stochastic exponential representation $X_\pi = \mathcal{E}(R_\pi)$ from (1.27) and will be crucial for further developments. We recall for it the notation of (1.44), (1.31), (1.26), as well as $\pi_0 = 1 - \sum_{i=1}^n \pi_i$ from (1.24).

Proposition 1.22. For any two portfolios $\pi \in \mathcal{I}(R)$ and $\rho \in \mathcal{I}(R)$, the relative wealth process of (1.41) admits the representation
\begin{align}
X^\rho_\pi = \mathcal{E}(R^\rho_\pi), \quad R^\rho_\pi := R_{\pi - \rho} - C_{\pi - \rho} = \int_0^t \sum_{i=0}^n \pi_i(t)dR^\rho_i(t).
\end{align}
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Proof. We begin by noting

\[ \int_0^t \sum_{i=1}^n \pi_i(t) dR_i^\rho(t) = \int_0^t \left( \sum_{i=1}^n \pi_i(t) \right) dR_0^\rho(t) + \int_0^t \sum_{i=1}^n \pi_i(t) d(R_i(t) - C_i^\rho(t)) \]

\[ = \int_0^t (1 - \pi_0(t)) dR_0^\rho(t) + (R_\pi - C_\pi^\rho), \]

which gives

\[ \int_0^t \sum_{i=1}^n \pi_i(t) dR_i^\rho(t) = \int_0^t \pi_0(t) dR_0^\rho(t) + \int_0^t \sum_{i=1}^n \pi_i(t) dR_i^\rho(t) \]

\[ = R_0^\rho + R_\pi - C_\pi^\rho = R_{\pi-\rho} - C_{\pi-\rho}. \]

Thus, in order to establish (1.45), it suffices to show the validity of

\[ \int_0^t \sum_{i=0}^n \pi_i(t) dR_i^\rho(t) = \int_0^t \pi_0(t) dR_0^\rho(t) + \int_0^t \sum_{i=1}^n \pi_i(t) dR_i^\rho(t) \]

\[ = R_0^\rho + R_\pi - C_\pi^\rho = R_{\pi-\rho} - C_{\pi-\rho}. \]

which completes the argument. □

Remark 1.23. We note from (1.45), (1.43) the relative wealth dynamics

\[ \frac{dX_\pi}{X_\pi^\rho(t)} = dR_i^\rho(t) = \sum_{i=0}^n \pi_i(t) dR_i^\rho(t) = \sum_{i=0}^n \pi_i(t) \frac{dS_i^\rho(t)}{S_i^\rho(t)}, \]

by analogy with (1.25). It is readily checked that the quantity \( R_0^\rho \) of (1.44) coincides with the quantity \( R_\pi^\rho \) of (1.45), when \( \pi \equiv 0 \), and that the quantity \( R_i^\rho \) in (1.44) coincides with \( R_i^\rho \) in (1.45), for \( 1 \leq i \leq n \). Thus, the notation introduced in this Subsection is consistent with earlier developments.

It is also useful and instructive to write down the version

\[ X_\pi \equiv X_\pi^0 = \mathcal{E} \left( \int_0^t \sum_{i=0}^n \pi_i(t) dR_i(t) \right), \quad \pi \in \mathcal{I}(R), \]

of (1.45) that corresponds to taking the money-market \( \rho \equiv 0 \) as a baseline. This expression is exactly (1.27), but with the “extra” additive term for the index \( i = 0 \) in the integral. We can exclude from (1.47), (1.46) or (1.45) the extra term for \( i = 0 \) with impunity—and thus recover the expression of (1.27)—when calculating \( X_\pi \equiv X_\pi^0 \), that is, when the money-market \( \rho = 0 \) is the baseline, because \( R_0 \equiv R_0^0 \equiv 0 \). We can also exclude the term corresponding to \( i = 0 \) from (1.45) or (1.46) for a general baseline \( \rho \) and a stock portfolio \( \pi \), because then \( \pi_0 \equiv 0 \). However, for arbitrary portfolio \( \pi \in \mathcal{I}(R) \) and baseline \( \rho \in \mathcal{I}(R) \), excluding the term corresponding to the
1.4. Relative performance

index $i = 0$ from the last expression in (1.45) will violate, in general, the validity of this equation.

1.4.2. Relative covariations. For a given portfolio $\rho \in \mathcal{I}(R)$ fixed as a “baseline”, let us introduce the predictable matrix-valued process $c^\rho \equiv (c^\rho_{ij}; 0 \leq i \leq n, 0 \leq j \leq n)$ via

\[
\begin{align*}
\text{c}^\rho_{ij} & := \text{c}_{ij} - \text{c}_{i\rho} - \text{c}_{\rho j} + \text{c}_{\rho \rho}, \quad 0 \leq i, j \leq n.
\end{align*}
\]

We recall here the notation of (1.31) and of the paragraph following it, and adopt the convention $c_{ij} = c_{i\rho} = 0$ for $i = 0$, and $c_{ij} = c_{\rho j} = 0$ for $j = 0$. This is always in accordance with the fact that $S_0 \equiv 1$ holds in the original market.

The quantities of (1.48) are the covariation rates of the individual assets, expressed not in absolute terms, but relative to the performance of the baseline portfolio $\rho$. Indeed, we have from the equations of (1.42), (1.43) the computations

\[
\begin{align*}
[R^\rho_\pi, R^\rho_\kappa] & = [\log S^\rho_\pi, \log S^\rho_\kappa] = \int_0^t c^\rho_{\pi \kappa}(t) dO(t) =: C^\rho_{\pi \kappa}, \quad 0 \leq i, j \leq n,
\end{align*}
\]

for the covariation processes of the various assets in the new, auxiliary market of the present Section 1.4, as in (1.42)–(1.44). In particular,

\[
\begin{align*}
[R^\rho_0, R^\rho_0] & = C_{\rho \rho}, \quad [R^\rho_i, R^\rho_0] = C_{\rho \rho} - C_{i\rho} \quad \text{for} \quad 1 \leq i \leq n,
\end{align*}
\]

\[
\begin{align*}
[R^\rho_i, R^\rho_j] & = C_{ij} - C_{i\rho} - C_{j\rho} + C_{\rho \rho} \quad \text{for} \quad 1 \leq i, j \leq n.
\end{align*}
\]

We note that the processes $c_\rho$ and $C_{\rho \rho}$ take values in the space of symmetric, nonnegative-definite matrices of size $(1+n) \times (1+n)$.

Once again, the extra terms in $c_\rho$ and $C_{\rho \rho}$, which involve indices including 0, are important. Only when the money market acts as a baseline will these terms be superfluous, because then the covariation processes of $R_0$ with the return of all the other stocks are trivially zero.

It is straightforward to extend the computations of (1.49), from relative covariations of individual assets to those of portfolios of assets, always with respect to a given baseline portfolio $\rho \in \mathcal{I}(R)$. Indeed, from (1.45) and with

\[
\begin{align*}
\text{c}^\rho_{\pi \kappa} & := \text{c}_{\pi \kappa} - \text{c}_{\pi \rho} - \text{c}_{\rho \kappa} + \text{c}_{\rho \rho}, \quad \pi, \kappa \in \mathcal{I}(R),
\end{align*}
\]

in the notation in (1.31), we obtain by analogy with (1.49) the covariations

\[
\begin{align*}
[R^\rho_\pi, R^\rho_\kappa] = [\log X^\rho_\pi, \log X^\rho_\kappa] = \int_0^t c^\rho_{\pi \kappa}(t) dO(t) =: C^\rho_{\pi \kappa}.
\end{align*}
\]

As was the case with the notation in (1.31), here too, if any of the portfolios in the expressions $c^\rho_{\pi \kappa}$ and $C^\rho_{\pi \kappa}$ above is a unit vector $e^i \in \mathbb{R}^n$, we use the subscript “$i$” instead of “$e^i$”.
In the notation of (1.48) and (1.50), another use of (1.45) gives

\[ c^\rho = (\pi - \rho)'c(\kappa - \rho) = \sum_{i=0}^{n} \sum_{j=0}^{n} \pi_i c^\rho_{ij} \kappa_j \]

for portfolios \( \rho \in \mathcal{I}(R) \), \( \pi \in \mathcal{I}(R) \), \( \kappa \in \mathcal{I}(R) \). One also obtains the following invariance property for the excess growth rates of portfolios.

**Exercise 1.24 (Numéraire invariance of excess growth).** For arbitrary portfolios \( \rho \in \mathcal{I}(R) \), \( \pi \in \mathcal{I}(R) \) and with the notation of (1.48), show that the excess growth rate \( \gamma^*_\pi \) of (1.37) is given as

\[ (1.52) \quad \gamma^*_\pi = \frac{1}{2} \left( \sum_{i=0}^{n} \pi_i c^\rho_{ii} - \sum_{i=0}^{n} \sum_{j=0}^{n} \pi_i c^\rho_{ij} \pi_j \right). \]

The choice \( \rho \equiv \pi \) in (1.52) leads to the beautiful representation

\[ (1.53) \quad \gamma^*_\pi = \frac{1}{2} \sum_{i=0}^{n} \pi_i c^\pi_{ii}, \quad \pi \in \mathcal{I}(R), \]

for the excess growth rate, due to \([Fer02]\). This representation follows from Exercise 1.24, as \( \sum_{i=0}^{n} \sum_{j=0}^{n} \pi_i c^\pi_{ij} \pi_j = 0 \) holds for every portfolio \( \pi \in \mathcal{I}(R) \); it implies, in particular, the result of Exercise 1.19. In cumulative terms, and with the notation of (1.49), the representation (1.53) becomes

\[ (1.54) \quad \Gamma^*_\pi = \frac{1}{2} \int_0^t \sum_{i=0}^{n} \pi_i(t) dC^\pi_{ii}(t), \quad \pi \in \mathcal{I}(R). \]

**1.4.3. Change of numéraire.** Proposition 1.22 deals with wealth ratios of numéraires; these are generated using portfolios in a “multiplicative” manner. There is also an “additive” version for the change-of-numéraire formula (1.45) for wealth processes as in Section 1.2.

To present this alternative version let us place, in a market with stock price-processes \( S^\rho_i = S_i/X_\rho \), \( 1 \leq i \leq n \), as in (1.42) for some “baseline” portfolio \( \rho \in \mathcal{I}(R) \), an investor endowed with initial capital \( x \in \mathbb{R}_+ \). We assume that this investor uses a strategy \( \vartheta = (\vartheta_i, 1 \leq i \leq n) \) with the same interpretation as in §1.2.2, and implements a cumulative capital withdrawal stream \( K \in \mathcal{K} \) as in §1.2.4. The wealth process \( X \equiv X^\rho(\cdot; x, \vartheta, K) \) thus generated will then satisfy

\[ \overline{X} = x + \int_0^t \sum_{i=1}^{n} \vartheta_i(t) dS^\rho_i(t) + \int_0^t \left( \overline{X}(t) - \sum_{i=1}^{n} \vartheta_i(t) S^\rho_i(t) \right) \frac{dS^\rho_0(t)}{S^\rho_0(t)} - K. \]

Here, the quantity

\[ \overline{X}(t) - \sum_{i=1}^{n} \vartheta_i(t) S^\rho_i(t) \equiv \frac{X^\rho(t; x, \vartheta, K) - \sum_{i=1}^{n} \vartheta_i(\cdot) S^\rho_i(t)}{S^\rho_0(t)} \]
represents the units of the 0th asset held at time \( t \in \mathbb{R}_+ \). Exactly as in Exercise 1.14, we set
\[
K := \int_0^t \frac{dK(t)}{S^\rho_0(t)} = \int_0^t X_\rho(t)dK(t)
\]
and use integration-by-parts to obtain
\[
\frac{X^\rho(\cdot; x, \vartheta, K)}{S^\rho_0(\cdot)} = X_\rho(\cdot)X^\rho(\cdot; x, \vartheta, K) = X_\rho(\cdot)X(\cdot)
\]
\[
= x + \int_0^t \sum_{i=1}^n \vartheta_i(t)dS_i(t) - K \equiv X(\cdot; x, \vartheta, K).
\]

Conversely, if we start with \( \vartheta \in \mathcal{I}(\mathbb{R}) \) and \( K \in \mathcal{K} \), and define the process
\[
(1.55) \quad K^\rho := \int_0^t \frac{dK(t)}{X_\rho(t)} = \int_0^t S^\rho_0(t)dK(t),
\]
straightforward calculation shows that
\[
(1.56) \quad X(\cdot; x, \vartheta, K) = X_\rho(\cdot)X^\rho(\cdot; x, \vartheta, K^\rho).
\]

Let us denote by \( \mathcal{K}^\rho(x) \) the collection consisting of all cumulative capital withdrawal streams, which are financeable starting from initial capital \( x \in \mathbb{R}_+ \) in the sense of Definition 1.13, but in the market with assets as in (1.42) denominated in units of \( X_\rho \). The above considerations then imply
\[
(1.57) \quad \mathcal{K}^\rho(x) = \left\{ \int_0^t \frac{dK(t)}{X_\rho(t)} \bigg| K \in \mathcal{K}(x) \right\}, \quad x \in \mathbb{R}_+.
\]

**1.4.4. Bounds on the excess growth.** Let us denote by \( \lambda \) the \( \mathbb{R}_+ \)-valued predictable process giving the maximum eigenvalue of the random covariation matrix \( c \). Furthermore, we let \( \ell \) denote the \([0,1]\)-valued predictable process such that \( \ell \lambda \) is the minimum eigenvalue of \( c \), with the convention \( \ell = 1 \) whenever \( \lambda = 0 \).

**Lemma 1.25.** With the above notation, the excess growth rate of any long-only portfolio \( \pi \) satisfies the bounds
\[
\frac{\ell}{2} \lambda \left( 1 - \max_{0 \leq i \leq n} \pi_i \right) \leq \gamma^*_\pi \leq \lambda \left( 1 - \max_{0 \leq i \leq n} \pi_i \right), \quad \pi \in \mathcal{I}(\mathbb{R}; \Delta_n).
\]

**Proof.** All vectors below are considered in \( \mathbb{R}^{1+n} \); we set \( e^0 \) to be the unit vector corresponding to the 0th coordinate. Letting \( \| \cdot \| \) denote Euclidean distance on \( \mathbb{R}^{1+n} \), we start by noting the inequality
\[
c^\pi_{ii} \geq \ell \lambda \| e^i - \pi \|^2 = \ell \lambda \left( (1 - \pi_i)^2 + \sum_{j \neq i} \pi_j^2 \right), \quad 0 \leq i \leq n,
\]
for the individual stocks’ relative variation rates as in (1.48), measured with respect to the performance of the portfolio \( \pi \). Thus, from the representation (1.53) and the fact that \( \sum_{i=0}^{n} \pi_i = 1 \), it follows that

\[
\gamma^*_\pi \geq \frac{\ell}{2} \left( \sum_{i=0}^{n} \pi_i (1 - \pi_i)^2 + \sum_{j \neq i}^{n} (1 - \pi_j) \pi_j^2 \right)
\]

\[
= \frac{\ell}{2} \lambda \left( \sum_{i=0}^{n} \pi_i (1 - \pi_i)^2 + \sum_{j=0}^{n} (1 - \pi_j) \pi_j^2 \right)
\]

\[
= \frac{\ell}{2} \lambda \sum_{i=0}^{n} \pi_i (1 - \pi_i) \geq \frac{\ell}{2} \lambda \left( 1 - \max_{0 \leq i \leq n} \pi_i \right).
\]

In an entirely similar manner we obtain the upper bounds

\[
c^*_\pi \leq \lambda \left\| e^i - \pi \right\|^2 = \lambda \left( (1 - \pi_i)^2 + \sum_{j \neq i}^{n} \pi_j^2 \right), \quad 0 \leq i \leq n,
\]

which lead to \( \gamma^*_\pi \leq (\lambda/2) \sum_{j=0}^{n} \pi_j (1 - \pi_j) \). For a fixed index \( 0 \leq i \leq n \), this last inequality gives

\[
\gamma^*_\pi \leq \frac{\lambda}{2} \left( \pi_i (1 - \pi_i) + \sum_{j \neq i} \pi_j \right) = \frac{\lambda}{2} \left( 1 + \pi_i \right) (1 - \pi_i) \leq \pi_i (1 - \pi_i),
\]

and the result follows by minimizing over \( 0 \leq i \leq n \) in the extreme sides of the inequality right above. \( \square \)

With \( \lambda \in \mathcal{P}(\mathbb{R}_+) \) as in Lemma 1.25, we introduce the “cumulative largest eigenvalue” and the “cumulative smallest eigenvalue” processes

(1.58) \[ \Lambda := \int_0^\cdot \lambda(t) dO(t), \quad \int_0^\cdot \ell(t) d\Lambda(t), \]

respectively. Lemma 1.25 then leads to the upper and lower bounds

(1.59) \[ \frac{1}{2} \int_0^\cdot \left( 1 - \max_{0 \leq i \leq n} \pi_i(t) \right) \ell(t) d\Lambda(t) \leq \Gamma^*_\pi \leq \int_0^\cdot \left( 1 - \max_{0 \leq i \leq n} \pi_i(t) \right) d\Lambda(t) \]

on the cumulative excess growth of a long-only portfolio \( \pi \in \mathcal{I} (R; \triangle_n) \).

1.4.5. The market as a baseline. The discussion in §1.3.5, and in particular the representation (1.40), suggests that measuring performance relative to the market portfolio \( \mu \) of (1.39) is tantamount to comparing wealth with respect to the entire market, normalized to have unit total capital. This special case is of great practical importance: many index funds exist that track large swaths of various developed capital markets.
In the notation of (1.42)–(1.44), it follows that the market weight processes satisfy

\[(1.60)\quad \mu_i = \frac{S_i}{\Sigma(0)} = \mu_i(0) \mathcal{E}\left( R_i^\mu \right), \quad \text{thus} \quad \frac{d\mu_i(t)}{\mu_i(t)} = dR_i^\mu(t), \quad 1 \leq i \leq n.\]

Therefore, for a stock portfolio \( \pi \in \mathcal{I}(R; H_{n-1}) \) in the notation of (1.29), the representation (1.45) of wealth relative to the market gives

\[(1.61)\quad X^\mu_\pi = \mathcal{E} \left( \int_0^t \sum_{i=1}^n \pi_i(t) \frac{d\mu_i(t)}{\mu_i(t)} \right), \quad \pi \in \mathcal{I}(R; H_{n-1}).\]

Here we are using the fact that \( \pi_0 \equiv 1 - \sum_{i=1}^n \pi_i = 0 \) holds for stock portfolios, to pass from (1.45) to (1.61). It is instructive also to write (1.61) in differential form and in the manner of (1.46), namely, as

\[(1.62)\quad \frac{dX^\mu_\pi(t)}{X^\mu_\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{d\mu_i(t)}{\mu_i(t)}.\]

Note that the representations (1.61) and (1.62) are only valid for stock portfolios \( \pi \). Indeed, erroneously plugging \( \pi = 0 \) into both sides of (1.61) leads to the absurdity \( X^\mu_0 \equiv 1 \), i.e., \( X_\mu \equiv 1 \).

Now let us specialize the cumulative excess growth expression of (1.54) to \( \pi \equiv \mu \), the market portfolio of §1.3.5. Since \( \mu_0 \equiv 1 - \sum_{i=1}^n \mu_i = 0 \), we obtain with the help of (1.49), (1.60) the representation

\[(1.63)\quad \Gamma^*_\mu = \frac{1}{2} \sum_{i=1}^n \int_0^t \mu_i(t) \, d[\log \mu_i, \log \mu_i](t),\]

or, equivalently,

\[(1.64)\quad \Gamma^*_\mu = \frac{1}{2} \sum_{i=1}^n \int_0^t \frac{d[\mu_i, \mu_i](t)}{\mu_i(t)} = \frac{1}{2} \sum_{i=1}^n [\mu_i, \log \mu_i].\]

The representation (1.63) casts \( 2\Gamma^*_\mu \) as the capitalization-weighted cumulative average stock variation, relative to the market. In light of (1.63), we think of \( 2\Gamma^*_\mu \) as the entire market’s cumulative intrinsic variation. As shown in Exercise 3.78, this quantity plays an important rôle in the context of outperforming the market portfolio, under appropriate structural conditions. The expressions (1.63), (1.64) allow for \( \Gamma^*_\mu \) to be estimated historically in a straightforward manner; for an alternative way to do this, see (1.85) below.

**Remark 1.26 (Market diversity).** Recalling from (1.58) the cumulative largest and smallest eigenvalue processes \( \Lambda \) and \( \int_0^t \ell(t) d\Lambda(t) \), respectively,
we obtain from the double inequality of (1.59) the upper and lower bounds (1.65)
\[
\frac{1}{2} \int_0^t \left( 1 - \max_{1 \leq i \leq n} \mu_i(t) \right) \ell(t) d\Lambda(t) \leq \Gamma_\mu^* \leq \int_0^t \left( 1 - \max_{1 \leq i \leq n} \mu_i(t) \right) d\Lambda(t)
\]
for the cumulative intrinsic variation of the market in (1.63). We regard the [0, 1]-valued process \(1 - \max_{1 \leq i \leq n} \mu_i\) that appears in (1.65) as a measure of market diversity, with higher values indicating a more diverse market. Requiring this measure of market diversity to be bounded away from zero mandates that no single company can ever come close to dominating the entire market in terms of capitalization.

With the above discussion in mind, the double inequality of (1.65) gives precise quantitative meaning to the qualitative statement that “market diversity and intrinsic market variation are different sides of the same coin”. Indeed, under the assumption that the process \(\ell\) does not take values too close to zero, which is certainly valid under the assumption of non-singularity and boundedness of the local covariation matrix of the stocks, (1.65) states that the growth of the market’s intrinsic variation \(\Gamma_\mu^*\) is controlled by the index of market diversity \(1 - \max_{1 \leq i \leq n} \mu_i\), and vice versa. It is shown in [FKK05, Theorem 6.1] that the diversity condition
\[
(1.66) \quad 1 - \max_{1 \leq i \leq n} \mu_i \geq \delta, \quad \text{for some } \delta \in (0, 1),
\]
is compatible with strong non-degeneracy of the local covariation matrix in an Itô process model for the asset prices.

1.5. Functional generation of stock portfolios

Here we introduce the notion of a “functional generation” of stock portfolios, in the manner of [Fer02]. This methodology leads to portfolios which can be constructed easily from observables, without any need for estimation or optimization. The so-constructed portfolios have controlled behaviour, and their performance relative to the market can be analysed fairly thoroughly.

1.5.1. Motivation. A main objective in portfolio selection is the optimization of relative performance with respect to the market. For a stock portfolio \(\pi\), the dynamics of the relative wealth of \(\pi\) with respect to the market portfolio, namely, of \(X_\pi^\mu := X_\pi/X_\mu\) in (1.41), are given by
\[
\frac{dX_\pi^\mu(t)}{X_\pi^\mu(t)} = \sum_{i=1}^n \pi_i(t) \frac{d\mu_i(t)}{\mu_i(t)}
\]
as in (1.62). But these dynamics cannot shed much light, or quantitative insight, on the actual performance of \(\pi\) with respect to the market.
1.5. Functional generation of stock portfolios

Ideally, we would like to find a class of stock portfolios $\pi$ which are easy to implement, are based solely on observable quantities, and for which a decomposition of the form

\begin{equation}
\log X^\mu_\pi = J^\mu_\pi + H^\mu_\pi
\end{equation}

holds for some nondecreasing continuous process $J^\mu_\pi$ and for a semimartingale $H^\mu_\pi$ “dominated” by $J^\mu_\pi$, in the sense that the ratio $H^\mu_\pi / J^\mu_\pi$ is “small” whenever $J^\mu_\pi$ is “large”. If such a decomposition as in (1.67) can be established, then the long-run growth of $X^\mu_\pi$ will be given in the dominant order by $J^\mu_\pi$, at least on those paths where $J^\mu_\pi$ increases to infinity.

Somewhat surprisingly, there is a way to carry all this out. The resulting approach leads to stock portfolios which depend only on the prevalent market weights, i.e., on quantities readily observable in markets; and for which stochastic integrals disappear entirely from the right-hand side of (1.67).

The simplest way to achieve a decomposition as in (1.67) is to consider processes $H^\mu_\pi$ of the form

\begin{equation}
\log \left( \frac{F(\mu)}{F(\mu(0))} \right), \text{ where } \text{ri} \left( \Delta_{(n-1)} \right) \ni x \mapsto F(x) \in (0, \infty)
\end{equation}

is a twice continuously differentiable function. Here and below, $\text{ri} \left( \Delta_{(n-1)} \right)$ is the relative interior of the $(n-1)$-dimensional simplex $\Delta_{(n-1)}$ of (1.30). In this manner, $H^\mu_\pi(t)$ is a simple function of the vector $\mu(t)$ of market weights prevalent at time $t$, and depends on nothing else.

**Remark 1.27.** The function $F$ in (1.68) is defined on $\text{ri} \left( \Delta_{(n-1)} \right)$, which is an open subset in the relative topology of $H_{n-1}$ in (1.29); this, in turn, is a lower-dimensional affine subspace of $\mathbb{R}^n$. Requiring $F$ to be twice continuously differentiable on $\text{ri} \left( \Delta_{(n-1)} \right)$ suffices for subsequent applications of Itô’s rule to the process $F(\mu)$, since $\mu$ is actually $\text{ri} \left( \Delta_{(n-1)} \right)$-valued.

However, in all examples discussed below, it is possible to extend $F$ to a twice continuously differentiable function in an open neighbourhood of $\text{ri} \left( \Delta_{(n-1)} \right)$ in $\mathbb{R}^n$. This allows us to consider derivatives of $F$ in all directions of $\mathbb{R}^n$, and Itô’s formula takes more “familiar” forms in (1.70) below and similar equations that follow. Therefore, and in order to keep standard notation, we shall adopt the convention that such an extension of $F$ is always possible in our context.

Plugging $\log \left( \frac{F(\mu)}{F(\mu(0))} \right)$ for $H^\mu_\pi$ into the “desideratum” (1.67) results in

\begin{equation}
X^\mu_\pi = \frac{F(\mu)}{F(\mu(0))} \exp(J^\mu_\pi),
\end{equation}

where $J^\mu_\pi$ is some nondecreasing continuous process and $F(\mu(0))$ is a constant.
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and an application of Itô’s rule leads to the dynamics

\[
\frac{dX^\mu_\pi(t)}{X^\mu_\pi(t)} = dJ^\mu_\pi(t) + \sum_{i=1}^n \frac{\partial_i F(\mu(t))}{F(\mu(t))} d\mu_i(t) \\
+ \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2_{ij} F(\mu(t))}{2F(\mu(t))} d[\mu_i, \mu_j](t), \quad t \in \mathbb{R}_+.
\]

Here, and in computations that follow, “\(\partial\)” denotes differentiation with respect to the coordinates specified as subscripts.

A comparison of the above dynamics with those of (1.62) suggests that the putative portfolio \(\pi\) should satisfy

\[
\frac{\pi_i}{\mu_i} = \partial_i F(\mu(t)) / F(\mu(t)) \quad \text{for} \quad 1 \leq i \leq n.
\]

Such a recipe, however, does not ensure that \(\pi\) is a stock portfolio. Using instead \(\pi^F = (\pi^F_i; 1 \leq i \leq n)\) defined via

\[
\pi^F_i = \mu_i \left( \frac{\partial_i F(\mu(t))}{F(\mu(t))} + 1 - \sum_{j=1}^n \mu_j \frac{\partial_j F(\mu(t))}{F(\mu(t))} \right), \quad 1 \leq i \leq n,
\]

we note that \(\pi^F \in \mathcal{I}(R; H_{n-1})\) and, since the equality \(\int_0^t \sum_{i=1}^n k(t) d\mu_i(t) = \int_0^t k(t) d(\sum_{i=1}^n \mu_i(t)) = 0\) holds for any predictable process \(k\), we have

\[
\int_0^t \sum_{i=1}^n \frac{\pi^F_i(t)}{\mu_i(t)} d\mu_i(t) = \int_0^t \sum_{i=1}^n \frac{\partial_i F(\mu(t))}{F(\mu(t))} d\mu_i(t).
\]

According to (1.70) and the discussion following it, the equation (1.69) is valid with \(\pi \equiv \pi^F\), and with \(J^\mu_\pi \equiv J^F\) given by

\[
J^F := -\frac{1}{2} \int_0^t \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2_{ij} F(\mu(t))}{F(\mu(t))} d[\mu_i, \mu_j](t).
\]

1.5.2. Functional generation. The construction described in §1.5.1 is important enough to merit note and a separate appellation.

Definition 1.28 (Functionally generated stock portfolios). Let \(F\) be a twice continuously differentiable function as in (1.68). Then the predictable, vector-valued process \(\pi^F \equiv (\pi^F_i; 1 \leq i \leq n) \in \mathcal{I}(R; H_{n-1})\), defined via (1.71), is called the stock portfolio generated by the function \(F\).

The market capitalization of any given company is the product of its current stock price times the number of shares outstanding; such information is publicly available on a daily basis. Therefore, the computation of the market weights in (1.39) and the implementation of functionally generated stock portfolios, such as those of Definition 1.28, is straightforward.

We present an illustrative list of very simple functionally generated stock portfolios. More elaborate constructions will appear later.
1.5. Functional generation of stock portfolios

- For any given \( w \equiv (w_i; 1 \leq i \leq n) \in \triangle_{n-1} \), consider the function \( r_i (x_i; 1 \leq i \leq n) \mapsto w_1 x_1 + \cdots + w_n x_n \); this generates the stock portfolio \( \pi^w \equiv (\pi^w_i; 1 \leq i \leq n) \) with

\[
\pi^w_i = \frac{w_i \mu_i}{\sum_{j=1}^{n} w_j \mu_j} = \frac{w_i S_i}{\sum_{j=1}^{n} w_j S_j}, \quad 1 \leq i \leq n.
\]

This portfolio buys at time \( t = 0 \) the number \( w_i \geq 0 \) of shares in stock \( 1 \leq i \leq n \), then holds on to all those shares for all times. It is the ultimate “buy-and-hold” strategy.

- With a fixed vector \( p \equiv (p_i; 1 \leq i \leq n) \in \triangle_{n-1} \), the function

\[
ri (x_i; 1 \leq i \leq n) \mapsto F(x) = \prod_{i=1}^{n} (x_i)^{p_i}
\]

generates via (1.71) the constant-proportion portfolio of Example 1.17, with \( \pi^F \equiv p_i \geq 0 \) for \( 1 \leq i \leq n \), which keeps a fixed proportion \( p_i \) of wealth in each stock \( 1 \leq i \leq n \) at all times.

The equal-weighted portfolio corresponds to the special case of weights \( p_i = 1/n \) for each \( 1 \leq i \leq n \). This portfolio sells stocks that have gone up in value to buy stocks that have gone down, in order to maintain equal weights across all stocks. This built-in “buy-low, sell-high” feature makes the equal-weighted portfolio useful in the study of so-called statistical arbitrage strategies; see [DGU0712] for a comparative study along these lines.

- If \( F_1, \ldots, F_m \) are generating functions, and \( \pi_1, \ldots, \pi_m \) are the stock portfolios generated by these functions, then the function

\[
ri (x_i; 1 \leq i \leq n) \mapsto F(x) = \prod_{i=1}^{m} (F_i(x))^{p_i}
\]

generates the “mixture” \( \pi := p_1 \pi_1 + \ldots + p_m \pi_m \) of these portfolios.

The discussion of §1.5.1 already provides a proof for the validity of the next remarkable result.

**Theorem 1.29** ([Fer02, Theorem 3.1.5]). Let \( ri (x_i; 1 \leq i \leq n) \mapsto F(x) \in (0, \infty) \) be twice continuously differentiable, and \( \pi^F \in I (R; H_{n-1}) \) the stock portfolio generated by \( F \) via (1.71). We then have the decomposition (1.74)

**Equation (1.74) provides the performance of the stock portfolio \( \pi^F \) in (1.71) with respect to the market, in terms of the trajectories of market**
weights and of their covariations. It does this in a totally pathwise fashion, without any need for stochastic integration.

This construction is important in portfolio theory, and not least in the study of market outperformance. First, it provides a powerful tool for constructing portfolios with controlled behaviour; see Exercise 1.31 below. Secondly, it makes possible pathwise comparisons of stock portfolios’ relative performance, and exposes structural conditions that allow well-chosen stock portfolios to outperform the market in the long run; we make this point in Remark 1.34. Finally, this construction implements such outperformance using stock portfolios which are, at any given time $t \in \mathbb{R}^+$, simple functions of the market weights $\mu_i(t), 1 \leq i \leq n$, prevailing at that time.

1.5.3. The rôle of concavity. Portfolio generating functions that are concave have several appealing properties. Foremost among them are the facts mentioned in the next exercise.

**Exercise 1.30** ([Fer02, Proposition 3.1.15]). Let $F : \text{ri} \left( \Delta_{(n-1)} \right) \to (0, \infty)$ be twice continuously differentiable and concave. Then:

- It holds that $\pi^F \equiv (\pi_i^F; 1 \leq i \leq n) \in \mathcal{P} \left( \Delta_{(n-1)} \right)$, in the notation of (1.71), (1.30); that is, $\pi^F$ is a long-only stock portfolio.
- The process $J^F$ of (1.72) is nondecreasing.

Combining Exercise 1.30 with equation (1.74), precise knowledge about the asymptotics of relative performance of portfolios may be obtained, as was alluded to in the motivational §1.5.1.

**Exercise 1.31.** Let the function $F : \text{ri} \left( \Delta_{(n-1)} \right) \to (0, \infty)$ be twice continuously differentiable and concave, and assume it satisfies the property

$$\lim_{m \to \infty} \sup_{t \in \mathbb{R}^+} \mathbb{P} \left[ F(\mu(t)) \leq 1/m \right] = 0.$$  

(1.75)

Under these provisos, show that

$$\lim_{T \to \infty} \frac{1}{J^F(T)} \log \left( \frac{X_{\pi^F}(T)}{X_\mu(T)} \right) = 1, \quad \text{on} \quad \{ J^F(\infty) = \infty \},$$

where the limit is in $\mathbb{P}$-measure.

Condition (1.75) can be restated in various guises, depending on the part of the relative boundary of $\Delta_{(n-1)}$ where $F$ vanishes; this is demonstrated in Exercise 1.32, as well as §1.5.6 and §1.5.7. Note that condition (1.75) holds trivially if $F$ is bounded away from zero on $\text{ri} \left( \Delta_{(n-1)} \right)$; in particular for diversity weighting, discussed in §1.5.4.

**Exercise 1.32** (Constant-proportion portfolio generation). Consider the constant-proportion portfolio generating function of (1.73).
1.5. Functional generation of stock portfolios

- Show that condition (1.75) may be cast in this case as

\[
\lim_{m \to \infty} \sup_{t \in \mathbb{R}^+} \mathbb{P} \left[ \min_{1 \leq i \leq n} \mu_i(t) \leq 1/m \right] = 0.
\]

Loosely speaking, no company becomes insignificant in the long run.

- Show that the process \( J^F \) of (1.72) is given by

\[
J^F = \frac{1}{2} \sum_{i=1}^{n} p_i [\log \mu_i, \log \mu_i] - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_ip_j [\log \mu_i, \log \mu_j] = \Gamma^*_{\pi_F}.
\]

1.5.4. Diversity weighting. Define the diversity function

\[
(1.78) \quad ri (\triangle(n-1)) \ni x = (x_i; 1 \leq i \leq n) \mapsto D(x) = \left( \sum_{i=1}^{n} (x_i)^p \right)^{1/p}
\]

for fixed \( p \in (0, 1) \). This concave function takes values in the interval \((1, n^{(1-p)/p}]\). It does not attain the value 1, but approximates it when \( x \in ri (\triangle(n-1)) \) is close to one of the unit vectors \((e_i; 1 \leq i \leq n)\). The value \( n^{(1-p)/p} \), the highest possible, is attained when \( x_i = 1/n \) for all \( 1 \leq i \leq n \).

The function \( D \) generates the so-called diversity-weighted portfolio

\[
(1.79) \quad \pi_i^D := \frac{(\mu_i)^p}{\sum_{j=1}^{n} (\mu_j)^p}, \quad 1 \leq i \leq n.
\]

This stock portfolio \( \pi^D \) can be thought of as an “intermediary” between capitalization weighting (that is, the market portfolio, corresponding to \( p = 1 \)) and equal weighting (namely, assigning weight \( 1/n \) to every stock \( 1 \leq i \leq n \), corresponding to \( p = 0 \); see Exercise 1.32).

Straightforward computation establishes the properties

\[
(1.80) \quad \min_{1 \leq i \leq n} \mu_i \leq \min_{1 \leq i \leq n} \pi_i^D \leq \max_{1 \leq i \leq n} \pi_i^D \leq \max_{1 \leq i \leq n} \mu_i
\]

of the diversity-weighted portfolio in (1.79). To wit, diversity-weighting tends to de-emphasize the upper end of the capitalization scale, and to emphasize the lower end, relative to the market weights.

**Exercise 1.33.** Show that the relative performance of the diversity-weighted stock portfolio \( \pi \equiv \pi^D \) of (1.79) with respect to the market is given as

\[
(1.81) \quad \log \left( \frac{X_\pi}{X_\mu} \right) = \log \left( \frac{D(\mu)}{D(\mu(0))} \right) + (1-p)\Gamma^*_\pi,
\]

where \( \Gamma^*_\pi \) is the cumulative excess growth of \( \pi \equiv \pi^D \) as in (1.38), (1.37).
Setting \( \pi \equiv \pi^D \) to ease notation, and since the function \( D \) of (1.78) is bounded away from zero, (1.76) reads as

\[
\lim_{T \to \infty} \frac{1}{\Gamma^*_\pi(T)} \log \left( \frac{X_\pi(T)}{X_\mu(T)} \right) = 1 - p, \quad \text{on} \quad \{ \Gamma^*_\pi(\infty) = \infty \},
\]

with the limit understood in the \( \mathbb{P} \)-a.e. sense. Furthermore, in view of (1.59) and (1.80), it follows with \( \pi \equiv \pi^D \) that

\[
\Gamma^*_\pi \geq \frac{1}{2} \int_0^\infty \left( 1 - \max_{1 \leq i \leq n} \pi_i(t) \right) \ell(t) d\Lambda(t) \geq \frac{1}{2} \int_0^\infty \left( 1 - \max_{1 \leq i \leq n} \mu_i(t) \right) \ell(t) d\Lambda(t).
\]

**Remark 1.34.** Let us now suppose that the strong diversity condition (1.66) holds, and that the cumulative “smallest eigenvalue” process \( \int_0^\infty \ell(t) d\Lambda(t) \) of (1.58) satisfies \( \int_0^\infty \ell(t) d\Lambda(t) = \infty \). With \( \pi \equiv \pi^D \), we then deduce \( \Gamma^*_\pi \geq (\delta/2) \int_0^\infty \ell(t) d\Lambda(t) \), from the inequality right above, whereas the decomposition (1.81), and the fact that the function \( D \) of (1.78) is bounded away from both zero and infinity, show that the diversity-weighted portfolio \( \pi^D \) will then eventually overtake, and outperform, the market portfolio.

### 1.5.5. Shifts

The concave function in §1.5.4 is bounded away from zero, allowing use of (1.76) to calculate the long-term growth of the relative wealth process with respect to the market portfolio. Many other useful generating functions, for instance those in §1.5.6 and §1.5.7 below, are not bounded away from zero; thus, the asymptotics of (1.76) do not necessarily apply to them in the absence of the “tightness” property (1.75) for the market weight process \( \mu \). A possible remedy we discuss below is to use an additive shift of the generating function. This operation is not innocuous and comes at a cost: it reduces the growth that can be obtained by (1.76).

Consider a twice continuously differentiable function \( F_0 : \triangle_{(n-1)} \to (0, \infty) \), and for \( v \in (0, \infty) \) define \( F_v : \triangle_{(n-1)} \to (0, \infty) \) via the additive shift \( F_v := v + F_0 \). With \( \pi^{F_v} \in I(R; \triangle_{(n-1)}) \) representing the associated generated stock portfolios in the manner of Definition 1.28, we obtain

\[
\pi^{F_v} = \frac{v}{v + F_0(\mu)} \mu + \frac{F_0(\mu)}{v + F_0(\mu)} \pi^{F_0}, \quad v \in (0, \infty),
\]

from (1.71). This portfolio \( \pi^{F_v} \) is a “compromise” between the portfolio \( \pi^{F_0} \) and the market, interpolating linearly between \( \pi^{F_0} \) and the market portfolio \( \mu \) in a market-dependent, but precise and explicit, fashion. Note that at places where \( F_0(\mu) \) is close to zero, the portfolio \( \pi^{F_v} \) places almost all its weight on the market portfolio. Furthermore, it follows from (1.72) that

\[
J^{F_v} = \int_0^\infty \frac{F_0(\mu(t))}{v + F_0(\mu(t))} dJ^{F_0}(t), \quad v \in (0, \infty).
\]
Thus, the fraction $F_0(\mu) / (v + F_0(\mu))$ is precisely the local “dampening” needed in order to pass from the drift process $J^F_0$ to $J^F_v$ for $v \in (0, \infty)$. When $F_0$ is concave, as in all our examples, the growth of the (nondecreasing) functions $J^F_v$ decreases with increasing $v$. While increasing $v$ helps control the term $\log (F(\mu(\cdot))/F(\mu(0)))$ in (1.74), it also reduces the relative asymptotic growth. This highlights the importance of Exercise 1.31.

1.5.6. Quadratic portfolio generation. Consider the concave function

$$ri(\triangle_{(n-1)}) \ni x = (x_i; 1 \leq i \leq n) \mapsto Q(x) = \frac{1}{2} \left( 1 - \sum_{i=1}^{n} (x_i)^2 \right),$$

which attains its maximum value of $(n-1)/(2n)$ when $x_i = 1/n$ for all $1 \leq i \leq n$, while being close to its infimum value of 0 when $x$ approaches one of the unit vectors $e^i$, $1 \leq i \leq n$. The stock portfolio $\pi^Q$ generated by the function $Q$ is given as

$$\pi^Q_i = \frac{\mu_i (1 - \mu_i)}{2Q(\mu)}, \quad 1 \leq i \leq n,$$

while

$$J^Q = \int_0^\infty \frac{1}{2Q(\mu(t))} \, d\left( \sum_{i=1}^{n} [\mu_i, \mu_i] \right)(t) \geq \frac{n}{n-1} \sum_{i=1}^{n} [\mu_i, \mu_i].$$

As $Q$ is not bounded away from zero, we can use any of its shifts and obtain the growth of the generated wealth. Alternatively, the requirement (1.75) of Exercise 1.31 may be written equivalently as

$$\lim_{m \to \infty} \sup_{t \in \mathbb{R}_+} \mathbb{P} \left[ \max_{1 \leq i \leq n} \mu_i(t) \geq 1 - (1/m) \right] = 0;$$

this is because $Q(x)$ approaches zero if, and only if, $x$ approaches one of the unit vectors. Condition (1.82) is an extremely mild diversity requirement on the market, considerably weaker than (1.77). Under it, we obtain

$$\lim_{T \to \infty} \frac{1}{J^Q(T)} \log \left( \frac{X_{\pi^Q}(T)}{X_{\mu}(T)} \right) = 1, \quad \text{on} \quad \left\{ \sum_{i=1}^{n} [\mu_i, \mu_i] (\infty) = \infty \right\}$$

in the manner of (1.76), where the limit is in $\mathbb{P}$-measure.

1.5.7. Entropy weighting. Consider the Gibbs-Shannon entropy

$$\text{ri}(\triangle_{(n-1)}) \ni x = (x_i; 1 \leq i \leq n) \mapsto H_0(x) := \sum_{i=1}^{n} x_i \log (1/x_i).$$

This function $H_0$ generates, via (1.71), the entropy-weighted portfolio

$$\pi^H_i := \frac{\mu_i \log (1/\mu_i)}{\sum_{j=1}^{n} \mu_j \log (1/\mu_j)}, \quad 1 \leq i \leq n.$$
In this case it is straightforward to compute the process \( J^{H_0} \) of (1.72) as
\[
J^{H_0} = \frac{1}{2} \int_0^1 H_0(\mu(t)) \sum_{i=1}^n \frac{1}{\mu_i(t)} d[\mu_i, \mu_i](t) = \int_0^1 \frac{d\Gamma^*_\mu(t)}{H_0(\mu(t))},
\]
for the market excess growth \( \Gamma^*_\mu \) of (1.63). As was the case for the quadratic-generating function in §1.5.6, \( H_0(x) \) approaches zero if and only if \( x \) approaches one of the unit vectors. Therefore, under the very mild diversity condition (1.82), the limiting property (1.76) will hold with \( F = H_0 \).

Recalling from §1.5.5 the shifts \( H_v = v + H_0 \) for \( v \in (0, \infty) \), note that
\[
J^{H_v} = \int_0^1 \frac{d\Gamma^*_\mu(t)}{H_v(\mu(t))}, \quad v \in (0, \infty).
\]
The bounds \( v \leq H_v(x) \leq v + \log n \), valid for \( x \in \triangle_{(n-1)} \), lead to the identity \( \{ J^{H_v}(\infty) = \infty \} = \{ \Gamma^*_\mu(\infty) = \infty \} \) for all \( v > 0 \). By Exercise 1.31, we obtain
\[
\lim_{T \to \infty} \frac{1}{J^{H_v}(T)} \log \left( \frac{X_{\pi H_v}(T)}{X_{\mu}(T)} \right) = 1 \quad \text{on} \quad \{ \Gamma^*_\mu(\infty) = \infty \},
\]
where the limit is understood in \( \mathbb{P} \)-measure. In fact, the growth of the ratio \( X_{\pi H_v}(T)/X_{\mu}(T) \) is comparable to the growth of \( \Gamma^*_\mu(T) \) as \( T \to \infty \).

Finally, let us note also that here (1.74) takes the form
\[
\log \left( \frac{X_{\pi H_v}}{X_{\mu}} \right) = \log \left( \frac{H_v(\mu)}{H_v(\mu(0))} \right) + \int_0^1 \frac{d\Gamma^*_\mu(t)}{H_v(\mu(t))}.
\]
The above expression provides a "tradeable" way to compute
\[
(1.85) \quad \Gamma^*_\mu = \int_0^1 H_v(\mu(t)) d \log \left( \frac{X_{\pi H_v}(t)}{X_{\mu}(t)} \frac{H_v(\mu(0))}{H_v(\mu(t))} \right),
\]
the cumulative market excess growth, in terms of the historical performance of the entropy-weighted portfolio in (1.84) relative to the market portfolio.

Notes and Complements

Section 1.1. The right-continuity of the underlying filtration is a very mild technical condition, required for the applicability of basic properties of martingales in continuous time—see, for instance, [KS91, Chapter 1]. Intuitively, it means that “no informational gain can be gleaned by peeking infinitesimally into the future”. We stress that no further assumption is imposed here on the filtration, not even the usual hypothesis of augmentation by null sets. The reason is that the theory of integration with respect to semimartingales can be developed without the augmentation assumption on the underlying filtration; see [SV79, Section 4.3] or [JS03, I,§4d].

The theory of semimartingales with continuous paths is covered in several sources: [IW89], [KS91] and [RY99], to name only a few. In [JS03],
the theory of semimartingales is developed from the point of view of “characteristics” or “rates”, and this approach leads naturally to vector stochastic integration. We follow a similar approach in a much simpler context here. Whenever stochastic integration with respect to continuous semimartingales is involved, we confine ourselves to integrands that are predictable.

**Section 1.2.** The continuity of asset prices, or capitalizations, is not a very appropriate assumption when the time horizon for investment decisions is days, weeks or months. It becomes a very reasonable assumption, though, when this horizon is measured in years or decades. Several of the issues raised in this work are indeed posed over long time horizons, and this makes the continuity condition more than just defensible from that point of view.

Modelling price processes by continuous semimartingales seems to us also a very reasonable compromise between generality on the one hand, and accessibility and readability on the other. A more general treatment of some of the basic issues in this work, in the context of semimartingales with possible jumps and of convex constraints on investment rules, appears in [KK07] and [KP11].

**Section 1.3.** Our definition of investment strategy assumes that one can trade continuously in time. This is a useful idealization, as it allows us to deploy the powerful arsenal of stochastic calculus for semimartingales. But it is an idealization nonetheless; in practice, trading takes place at discrete time-points. The setting we adopt in Section 4.2, with its emphasis on so-called “simple strategies” that trade only at discrete time-points in the manner of the simple predictable processes in §1.1.2, reflects this reality.

The reader may reasonably ask at this point: Is there any reason why we use semimartingale models in the first place, beyond their mathematical tractability? Suppose we are allowed to use only the simple strategies of the previous paragraph, and insist further that these never take any negative positions in either the money market or the individual stocks. Both these assumptions are fairly reasonable from the point of view of many asset managers. We shall see in Section 4.2 that proscribing certain forms of arbitrage, when using only such simple strategies with no short (i.e., negative) positions, forces the asset prices to be semimartingales.

Investment strategies of the form considered in this work are very commonly used in mathematical finance. We refer to the early papers [HP81, HP83] and to the monographs [Duf01], [KS98] and [Fer02] for accounts and examples of the various formulations that have been employed to good effect over the years, depending on the context.

The notion of excess growth rate, and the formula (1.37) for it, appear for the first time in [FS82]. The properties of this quantity, including
its numéraire invariance as in Exercise 1.24, are studied in detail in the monograph \cite{Fer02}. Lemma 1.25 is established in \cite{FKK05}. The lineage of Proposition 1.22 goes back to \cite{GEKR95} and \cite{DS95c}. The universal portfolio of Example 1.18 was introduced in \cite{Cov91}, and its properties were studied further in \cite{Jam92} and more recently in \cite{CSW19}, among others.

**Section 1.4.** Portfolio Theory begins with the pioneering work \cite{Mar52}, as does most of what is now called “Mathematical Finance”. Important subsequent milestones include the capital asset pricing models of \cite{Sha64, Lin65, Mos66}; the portfolio optimization theories of \cite{Sam69, Mer69, Mer71}; the option valuation theories of \cite{Sam65, BS73, Mer73}; the arbitrage pricing theory of \cite{Ros76}; the martingale approach of \cite{HK79, HP81, Kre81} to option valuation; the convex duality approach to portfolio optimization of \cite{KLSX91, CK92, KS99}; the study of arbitrage in \cite{DS94, DS98a}; and the stochastic portfolio theory of \cite{Fer02}.

A graphical depiction of the behaviour of the cumulative excess growth $\Gamma^*_\mu$ over an extensive period for the U.S. stock market can be found in \cite{FK09, page 140}. Exercise 1.20 was prompted by a question of A. Banner.

**Section 1.5.** Functionally generated stock portfolios appeared for the first time in the work of \cite{Fer99}, and were studied systematically in the seminal monograph \cite{Fer02} by the same author. For a generalization of this theory to the functional generation of investment strategies, see \cite{KR17}. The approach leading to the “master equation” of Theorem 1.29, as well as the asymptotic properties that flow from it and are studied in Exercise 1.31, are new. Remark 1.34 is taken from \cite{FKK05}.

One may consult \cite{FK09, page 138} for plots of the various terms of the decomposition (1.81) for the U.S. market over a 50-year period.