## Contents

Preface vii
Notation ix

Chapter 1. Background material 1
1.1. Rings and modules 1
1.2. Simple and semisimple modules and algebras 6
   1.2.1. Semisimple modules 6
   1.2.2. Simple and semisimple rings 6
   1.2.3. Simple and semisimple algebras 8
   1.2.4. Quaternion algebras 16
   1.2.5. The Brauer group 17
   1.2.6. Compositums and tensor products of fields 20
1.3. Involutions on algebras 23
   1.3.1. Involutions on simple algebras 27
   1.3.2. Involutions on semisimple algebras 36
1.4. Formally real fields 36
1.5. Bilinear and sesquilinear forms 43
   1.5.1. Definitions and basic properties 43
   1.5.2. Isometries 48
   1.5.3. Witt rings 50
   1.5.4. $K$ real closed or algebraically closed 52
   1.5.5. Local and global fields 53
   1.5.6. Trace forms over local fields 57
   1.5.7. Equivalence of reflexive forms over division algebras 61
1.6. Grothendieck groups 64
1.7. Linear representations of finite groups 68
   1.7.1. Basic facts about linear representations 68
   1.7.2. Characters 71
   1.7.3. The twisted contragredient representation 74
   1.7.4. Induced representations 76
   1.7.5. Simple components of $KG$ and irreducible representations 77
   1.7.6. Representations over the real and complex numbers 80

Chapter 2. Isometry representations of finite groups 83
2.1. Introduction 83
2.2. Notes on Chapter 2 90
Chapter 3. Hermitian forms over semisimple algebras 91
  3.1. Introduction 91
    3.1.1. Basic properties of $\varepsilon$-Hermitian forms over semisimple algebras 91
    3.1.2. Hyperbolic forms 96
    3.1.3. Witt's Theorem 100
  3.2. The algebra of forms 104
    3.2.1. Sums and products of forms 104
    3.2.2. Going up 109
    3.2.3. Going down 111
    3.2.4. Indecomposable forms 114
    3.2.5. The twisted contragredient representation revisited 115
    3.2.6. Adjoint involutions 116
    3.2.7. Forms over simple algebras in terms of matrices 118
    3.2.8. Transfers of forms over simple algebras 127
    3.2.9. Morita theory for forms over simple algebras 131
  3.3. The Hasse Principle 134
  3.4. Lifting a local representation to a global representation 139
  3.5. Existence of isometry representations 146
  3.6. Witt's Theorem for isometry representations 147
  3.7. Notes on Chapter 3 148

Chapter 4. Equivariant Witt-Grothendieck and Witt groups 151
  4.1. The Witt-Grothendieck and Witt groups of Hermitian forms over semisimple algebras 151
  4.2. Functorial properties of the Witt-Grothendieck and Witt groups of isometry representations 155
  4.3. The graded Witt-Grothendieck and Witt rings 168
  4.4. $K_0$ algebraically closed 169
  4.5. The field $K_0$ real closed 171
  4.6. $K/K_0$ definite 182
  4.7. The involutory Schur subgroup 184
  4.8. Similarity of isometry representations 188
  4.9. Structure of $\hat{W}_\varepsilon(K,G)$ when $K_0$ is not formally real 196
  4.10. Structure of $\hat{W}_\varepsilon(K,G)$ when $K/K_0$ is a definite extension of algebraic number fields 205
  4.11. Structure of the Witt group $W_\varepsilon(K,G)$ 211
  4.12. Notes on Chapter 4 216

Chapter 5. Representations over finite, local and global fields 219
  5.1. Outline 219
  5.2. Symplectic representations 222
  5.3. Unitary and orthogonal representations over finite fields 225
  5.4. Unitary and orthogonal representations over local fields 227
## Contents

5.4.1. Unitary representations 228  
5.4.2. Orthogonal representations 229  
5.5. Global fields 240  
5.6. Isometry representations of the symmetric group 249

Chapter 6. Fröhlich’s invariant, Clifford algebras and the equivariant Brauer-Wall group 253  
6.1. Graded algebras and the Brauer-Wall group 253  
6.2. Clifford algebras 258  
6.2.1. The Clifford group 259  
6.3. Equivariant graded algebras 261  
6.4. The equivariant Clifford algebra and Witt group 273  
6.5. A formula for Fröhlich’s cohomology class 276  
6.6. Notes on Chapter 6 277

Bibliography 279

Glossary 283

Index 287