Preface

This book springs from lectures on degree theory given by the authors during many years at the Departamento de Geometría y Topología at the Universidad Complutense de Madrid, and its definitive form corresponds to a three-month course given at the Dipartimento di Matematica at the Università di Pisa during the spring of 2006. Today mapping degree is a somewhat classical topic that appeals to geometers and topologists for its beauty and ample range of relevant applications. Our purpose here is to present both the history and the mathematics.

The notion of degree was discovered by the great mathematicians of the decades around 1900: Cauchy, Poincaré, Hadamard, Brouwer, Hopf, etc. It was then brought to maturity in the 1930s by Hopf and also by Leray and Schauder. The theory was fully burnished between 1950 and 1970. This process is described in Chapter I. As a complement, at the end of the book there is included an index of names of the mathematicians who played their part in the development of mapping degree theory, many of whom stand tallest in the history of mathematics. After the first historical chapter, Chapters II, III, IV, and V are devoted to a more formal proposition-proof discourse to define and study the concept of degree and its applications. Chapter II gives a quick presentation of manifolds, with special emphasis on aspects relevant to degree theory, namely regular values of differentiable mappings, tubular neighborhoods, approximation, and orientation. Although this chapter is primarily intended to provide a review for the reader, it includes some not so standard details, for instance concerning tubular neighborhoods. The main topic, degree theory, is presented in Chapters III and IV. In a simplified manner we can distinguish two approaches to the theory: the Brouwer-Kronecker degree and the Euclidean degree. The first is developed in Chapter III by differential means, with a quick diversion into the de Rham computation in cohomological terms. We cannot help this diversion: cohomology is too appealing to skip. Among other applications, we obtain in this chapter a differential version of the Jordan Separation Theorem. Then, we construct the Euclidean degree in
Chapter IV. This is mainly analytic and astonishingly simple, especially
in view of its extraordinary power. We hope this partisan claim will be
acknowledged readily, once we obtain quite freely two very deep theorems:
the Invariance of Domain Theorem and the Jordan Separation Theorem,
the latter in its utmost topological generality. Finally, Chapter V is de-
voted to some of those special results in mathematics that justify a theory
by their depth and perfection: the Hopf and the Poincaré-Hopf Theorems,
with their accompaniment of consequences and comments. We state and
prove these theorems, which gives us the perfect occasion to take a glance
at tangent vector fields.

We have included an assorted collection of some 180 problems and exer-
cises distributed among the sections of Chapters II to V, none for Chapter
I due to its nature. Those problems and exercises, of various difficulty, fall
into three different classes: (i) suitable examples that help to seize the ideas
behind the theory, (ii) complements to that theory, such as variations for
different settings, additional applications, or unexpected connections with
different topics, and (iii) guides for the reader to produce complete proofs
of the classical results presented in Chapter I, once the proper machinery
is developed.

We have tried to make internal cross-references clearer by adding the
Roman chapter number to the reference, either the current chapter number
or that of a different chapter. For example, III.6.4 refers to Proposition 6.4
in Chapter III; similarly, the reference IV.2 means Section 2 in Chapter IV.
We have also added the page number of the reference in most cases.

One essential goal of ours must be noted here: we attempt the simplest
possible presentation at the lowest technical cost. This means we restrict
ourselves to elementary methods, whatever meaning is accepted for ele-
mentary. More explicitly, we only assume the reader is acquainted with
basic ideas of differential topology, such as can be found in any text on the
calculus on manifolds.

We only hope that this book succeeds in presenting degree theory as
it deserves to be presented: we view the theory as a genuine masterpiece,
joining brilliant invention with deep understanding, all in the most accom-
plished attire of clarity. We have tried to share that view of ours with the
reader.

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