Introduction

This book has been conceived as a self-contained introduction to differential Galois theory. The self-teaching reader or the teacher wanting to give a course on this subject will find complete proofs of all results included. We have chosen to make a classical presentation of the theory. We refer to the Picard-Vessiot extension as a field rather than introducing the notion of Picard-Vessiot ring so as to keep the analogy with the splitting field in the polynomial Galois theory. We also refer to differential equations rather than differential systems, although the differential systems setting is given in the exercises.

The chapters on algebraic geometry and algebraic groups include all questions which are necessary to develop differential Galois theory. The differential Galois group of a linear differential equation is a linear algebraic group, hence affine. However, the construction of the quotient of an algebraic group by a subgroup needs the notion of abstract affine variety. Once we introduce the notion of geometric space, the concept of algebraic variety comes naturally. We also consider it interesting to include the notion of projective variety, which is a model for algebraic varieties, and present a classical example of an algebraic group which is not affine, namely the elliptic curve.

The chapter on Lie algebras aims to prove the equivalence between the solvability of a connected linear algebraic group and the solvability of its Lie algebra. This fact is used in particular to determine the algebraic subgroups of SL(2, C). We present the characterization of differential equations solvable by quadratures. In the last chapter we consider differential equations defined over the field of rational functions over the complex field and present
classical notions such as the monodromy group, Fuchsian equations and hypergeometric equations. The last section is devoted to Kovacic’s powerful algorithm to compute Liouvillian solutions to linear differential equations of order 2. Each chapter ends with a selection of exercise statements ranging in difficulty from the direct application of the theory to dealing with some topics that go beyond it. The reader will also find several illuminating examples. We have included a chapter with a list of further reading outlining the different directions in which differential Galois theory and related topics are being developed.

As guidance for teachers interested in using this book for a postgraduate course, we propose three possible courses, depending on the background and interests of their students.

(1) For students with limited or no knowledge of algebraic geometry who wish to understand Galois theory of linear differential equations in all its depth, a two–semester course can be given using the whole book.

(2) For students with good knowledge of algebraic geometry and algebraic groups, a one–semester course can be given based on Part 3 of the book using the first two parts as reference as needed.

(3) For students without a good knowledge of algebraic geometry and eager to learn differential Galois theory more quickly, a one–semester course can be given by developing the topics included in the following sections: 1.1, 3.1, 3.2, 3.3, 4.4 (skipping the references to Lie algebra), 4.6, and Part 3 (except the proof of Proposition 6.3.5, i.e. that the intermediate field of a Picard-Vessiot extension fixed by a normal closed subgroup of the differential Galois group is a Picard-Vessiot extension of the base field). This means introducing the concept of affine variety, defining the algebraic group and its properties considering only affine ones, determining the subgroups of SL(2, C) assuming as a fact that a connected linear group of dimension less than or equal to 2 is solvable, and developing differential Galois theory (skipping the proof of Proposition 6.3.5).