Pseudodifferential Operators

Introduction to Chapter VI

This chapter starts with the theory of pseudodifferential operators (ψdo's). We choose the simplest class of symbols to make the exposition as accessible as possible.

In §40 we study the boundedness and the composition of ψdo's. In §§41, 42 we deal with the ellipticity and the Fredholm property of ψdo's. In §§43, 45 we consider the adjoints of ψdo's and the change of variables formula.

The rest of Chapter VI is devoted to the applications of ψdo's. In §46 we prove the existence and the uniqueness of the solution of the Cauchy problem for parabolic equations. In §47 we find the asymptotics of the heat kernel as $t \to +0$. The Cauchy problem for the strictly hyperbolic equation of order m is studied in §48. Separately, in §49 we consider the domains of dependence and influence for the strictly hyperbolic equations of any order. In §44 we deal with the pseudolocal property and microlocal regularity of ψdo's.

Finally, in §50 we study propagation of singularities for the equations of real principal type following Hörmander [**H1**], with applications to propagation of singularities of solutions of the Cauchy problem for hyperbolic equations. In the problem section (§51) we indicate an alternative approach to the Cauchy problem for hyperbolic equations following Taylor [**T2**], and we briefly describe a general class of pseudodifferential operators introduced by Beals and Fefferman [**B**], [**BF**].

40. Boundedness and composition of ψdo's

40.1. The boundedness theorem.

Denote by S^α the class of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ functions $A(x,\xi)$ such that

$$\left|\frac{\partial^{k+p}A(x,\xi)}{\partial x^p \partial \xi^k}\right| \le C_{pk}(1+|\xi|)^{\alpha-|k|}, \ \forall k,p, \tag{40.1}$$

and $A(x,\xi)$ is independent of x for $|x| > R$. Let $A(\infty,\xi) = A(x,\xi)$ for $|x| > R$ and $A'(x,\xi) = A(x,\xi) - A(\infty,\xi)$, i.e., $A'(x,\xi)$ has a compact support in x.

For $A(x,\xi) \in S^\alpha$ we define an operator

$$Au = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} A(x,\xi)\tilde{u}(\xi)e^{ix\cdot\xi}d\xi, \ \forall u(x) \in C_0^\infty(\mathbb{R}^n), \tag{40.2}$$

where $\tilde{u}(\xi)$ is the Fourier transform of $u(x)$. If $A(x,\xi) = \sum_{|k|=0}^m a_k(x)\xi^k$ is a polynomial in ξ, then

$$Au = \sum_{|k|=0}^m a_k(x)\left(-i\frac{\partial}{\partial x}\right)^k u \tag{40.3}$$

is a differential operator $A(x,D)u$, $D = -i\frac{\partial}{\partial x}$. We call (40.2) pseudodifferential operator and $A(x,\xi)$ its symbol. We often denote by $A(x,D)$ a pseudodifferential operator with symbol $A(x,\xi)$.

Theorem 40.1. *A pseudodifferential operator $A(x,D)$ with symbol $A(x,\xi) \in S^\alpha$ is bounded from $H_s(\mathbb{R}^n)$ to $H_{s-\alpha}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$:*

$$\|A(x,D)u\|_{s-\alpha} \le C\|u\|_s, \quad \forall u \in H_s(\mathbb{R}^n). \tag{40.4}$$

Proof: It is clear that

$$\|A(\infty,D)u\|_{s-\alpha}^2 = \int_{\mathbb{R}^n} |A(\infty,\xi)\tilde{u}(\xi)|^2(1+|\xi|)^{2(s-\alpha)}d\xi \le C_0^2\|u\|_s^2,$$

where $C_0 = \sup_\xi \frac{|A(\infty,\xi)|}{(1+|\xi|)^\alpha}$, $\forall u \in S$.

Let

$$\tilde{A}'(\eta,\xi) = \int_{\mathbb{R}^n} A'(x,\xi)e^{-ix\cdot\eta}dx. \tag{40.5}$$

Taking the Fourier transform of $v(x) = A'(x,D)u$, we get

$$\tilde{v}(\eta) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \tilde{A}'(\eta-\xi,\xi)\tilde{u}(\xi)d\xi. \tag{40.6}$$

Since $A'(x,\xi) \in C_0^\infty$ in x, we get, using integration by parts in (40.5):

$$|\tilde{A}(\eta,\xi)| \le C_N(1+|\eta|)^{-N}(1+|\xi|)^\alpha, \ \forall N.$$

Therefore

$$|\tilde{v}(\eta)| \le C_N \int_{\mathbb{R}^n} \frac{(1+|\xi|)^\alpha |\tilde{u}(\xi)|}{(1+|\xi-\eta|)^N} d\xi.$$

It follows from (13.10) that

$$(1+|\xi-\eta|)^{-|t|} \le C_t \frac{(1+|\xi|)^t}{(1+|\eta|)^t}, \ \forall t \in \mathbb{R}. \tag{40.7}$$

Taking $N \ge n+1+|s-\alpha|$ and using (40.7) with $t = s-\alpha$, we get:

$$(1+|\eta|)^{s-\alpha}|\tilde{v}(\eta)| \le C \int_{\mathbb{R}^n} \frac{(1+|\xi|)^s |\tilde{u}(\xi)|}{(1+|\xi-\eta|)^{n+1}} d\xi.$$

Now as in (13.11), (13.12) we obtain

$$\|v\|_{s-\alpha} \le C\|u\|_s, \ \forall u \in C_0^\infty(\mathbb{R}^n).$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$ (see Theorem 13.2), we can take the closure in $H_s(\mathbb{R}^n)$ and get (40.4).

Definition 40.1. We say that an operator T_α is of order $\le \alpha$ if

$$\|T_\alpha u\|_s \le C_s \|u\|_{s+\alpha} \quad \text{for every} \quad s. \tag{40.8}$$

It follows from Theorem 40.1 that $A(x, D)$ with symbol $A(x,\xi) \in S^\alpha$ has order α.

40.2. Composition of ψdo's.

Theorem 40.2. *Let $A(x,\xi) \in S^\alpha$ and $B(x,\xi) \in S^\beta$. Then for any N the composition $A(x,D)B(x,D)$ has the form*

$$A(x,D)B(x,D) = C_N(x,D) + T_{\alpha+\beta-N-1}, \tag{40.9}$$

where $C_N(x,\xi) \in S^{\alpha+\beta}$, $\operatorname{ord} T_{\alpha+\beta-N-1} \le \alpha+\beta-N-1$, and

$$C_N(x,\xi) = \sum_{|k|=0}^{N} \frac{1}{k!} \frac{\partial^k A(x,\xi)}{\partial \xi^k} D_x^k B(x,\xi), \ D_x = -i\frac{\partial}{\partial x}. \tag{40.10}$$

Proof: Note that $A(x,D)B(\infty,D)$ is a ψdo with symbol $A(x,\xi)B(\infty,\xi)$. Let

$$v(x) = B'(x,D)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} B'(x,\eta)\tilde{u}(\eta)e^{ix\cdot\eta} d\eta.$$

As in (40.6), we have:

$$\begin{aligned} A(x,D)v &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A(x,\xi)\tilde{v}(\xi)e^{ix\cdot\xi} d\xi \\ &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(x,\xi)\tilde{B}'(\xi-\eta,\eta)e^{ix\cdot\xi}\tilde{u}(\eta) d\xi d\eta, \ \forall u \in C_0^\infty(\mathbb{R}^n). \end{aligned}$$

Make the change of variables $\xi - \eta = \zeta$. Then

$$A(x,D)v = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(x,\eta+\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta.$$

Note that $\tilde{B}'(\zeta,\xi)$ is rapidly decreasing in ζ and $\tilde{u}(\eta)$ is rapidly decreasing in η. The Taylor expansion of $A(x,\eta+\zeta)$ in ζ is:

$$A(x,\eta+\zeta) = \sum_{|k|=0}^{N} \frac{1}{k!}\frac{\partial^k A(x,\eta)}{\partial \eta^k}\zeta^k + R_N(x,\eta,\zeta), \tag{40.11}$$

where $R_N(x,\eta,\zeta)$ is the remainder. Note that

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \zeta^k \tilde{B}'(\zeta,\eta)e^{ix\cdot\zeta}d\zeta = D_x^k B'(x,\eta).$$

Therefore

$$A(x,D)v = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \sum_{|k|=0}^{N} \frac{1}{k!}\frac{\partial^k A(x,\eta)}{\partial \eta^k} D_{x'}^k B'(x,\eta)\tilde{u}(\eta)e^{ix\cdot\eta}d\eta$$
$$+ T_{\alpha+\beta-N-1}u,$$

where $T_{\alpha+\beta-N-1}u = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n} R_N(x,\eta,\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta$. It remains to prove that $T_{\alpha+\beta-N-1}$ has order $\alpha+\beta-N-1$. We have

$$R_N(x,\eta,\zeta) = R_N(\infty,\eta,\zeta) + R'_N(x,\eta,\zeta), \tag{40.12}$$

where $R_N(\infty,\eta,\zeta)$ is the remainder for $A(\infty,\xi)$ and $R'_N(x,\eta,\zeta)$ is the remainder for $A'(x,\xi)$.

Let $\tilde{R}'_N(\xi,\eta,\zeta)$ be the Fourier transform of $R'_N(x,\eta,\zeta)$ in x. We prove the following estimates for $R_N(\infty,\eta,\zeta)$ and $\tilde{R}'_N(\xi,\eta,\zeta)$:

$$|R_N(\infty,\eta,\zeta)| \le C(1+|\eta|)^{\alpha-N-1}(1+|\zeta|)^{N+|\alpha|+1}, \tag{40.13}$$

$$|\tilde{R}'_N(\xi,\eta,\zeta)| \le C_M(1+|\xi|)^{-M}(1+|\eta|)^{\alpha-N-1}(1+|\zeta|)^{N+|\alpha|+1}, \ \forall M. \tag{40.14}$$

To prove (40.13), consider two cases: $|\zeta| \le \frac{1}{2}(1+|\eta|)$ and $|\zeta| > \frac{1}{2}(1+|\eta|)$. For $|\zeta| \le \frac{1}{2}(1+|\eta|)$ we use the Lagrange form of the remainder in the Taylor formula:

$$R_N(\infty,\eta,\zeta) = \sum_{|k|=N+1} \frac{1}{k!}\frac{\partial^k A(\infty,\eta+\theta\zeta)}{\partial\eta^k}\zeta^k,$$

where $|\theta| < 1$. Then

$$\left|\frac{\partial^k A(\infty,\eta+\theta\zeta)}{\partial\eta^k}\right| \le C(1+|\eta+\theta\zeta|)^{\alpha-N-1} \le C_1(1+|\eta|)^{\alpha-N-1}, \tag{40.15}$$

since $1+|\eta+\theta\zeta| \ge 1+|\eta|-|\zeta| \ge \frac{1}{2}(1+|\eta|)$ and $(1+|\eta+\theta\zeta|) \le 1+|\eta|+|\zeta| \le \frac{3}{2}(1+|\eta|)$. Therefore (40.13) holds.

If $|\zeta| > \frac{1}{2}(1+|\eta|)$, we get from (40.11):

$$|R_N(\infty,\eta,\zeta)| \le |A(\infty,\eta+\zeta)| + \sum_{|k|=0}^{N} \frac{1}{k!}\left|\frac{\partial^k A(\infty,\eta)}{\partial \eta^k}\right| |\zeta|^{|k|}$$
$$\le C(1+|\eta+\zeta|)^\alpha + C\sum_{|k|=0}^{N}(1+|\eta|)^{\alpha-|k|}|\zeta|^{|k|}.$$

Note that

$$(1+|\eta|)^{\alpha-|k|}|\zeta|^k \le (1+|\eta|)^{\alpha-N-1}(1+|\eta|)^{N-|k|+1}|\zeta|^{|k|}$$
$$\le C(1+|\eta|)^{\alpha-N-1}|\zeta|^{N+1},$$

since $(1+|\eta|)^{N-|k|+1} \le C|\zeta|^{N-|k|+1}$. Using (40.7) with $t=-\alpha,\ \ \xi=\eta+\zeta$ we get

$$(1+|\eta+\zeta|)^\alpha \le C(1+|\eta|)^\alpha(1+|\zeta|)^{|\alpha|} \le C(1+|\eta|)^{\alpha-N-1}(1+|\zeta|)^{|\alpha|+N+1}$$

since $|\zeta| \ge \frac{1}{2}(1+|\eta|)$. Therefore (40.13) holds.

The proof of (40.14) is similar. We replace the estimate $|\frac{\partial^k A(\infty,\eta)}{\partial\eta^k}| \le C(1+|\eta|)^{\alpha-|k|}$ by the estimate $|\frac{\partial^k \tilde{A}'(\xi,\eta)}{\partial\eta^k}| \le C_M(1+|\xi|)^{-M}(1+|\eta|)^{\alpha-|k|}$ for all M. □

Let

$$T_{\alpha+\beta-N-1} = T^{(1)}_{\alpha+\beta-N-1} + T^{(2)}_{\alpha+\beta-N-1},$$

where

(40.16)
$$T^{(1)}_{\alpha+\beta-N-1}u = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} R_N(\infty,\eta,\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta,$$

(40.17)
$$T^{(2)}_{\alpha+\beta-N-1}u = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} R'_N(x,\eta,\zeta)\tilde{B}'(\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta)d\eta d\zeta.$$

Since $(F_x T^{(1)}_{\alpha+\beta-N-1}u)(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} R_N(\infty,\eta,\xi-\eta)\tilde{B}'(\xi-\eta,\eta)\tilde{u}(\eta)d\eta$ and

(40.18) $$|\tilde{B}'(\xi-\eta,\eta)| \le C_M(1+|\xi-\eta|)^{-M}(1+|\eta|)^\beta,$$

we get, using (40.13):

(40.19) $$(1+|\xi|)^s\ |(F_xT^{(1)}_{\alpha+\beta-N-1}u)(\xi)|$$
$$\le C_M\int_{\mathbb{R}^n}\frac{(1+|\xi|)^s(1+|\xi-\eta|)^{N+|\alpha|+1}(1+|\eta|)^{\alpha+\beta-N-1}|\tilde{u}(\eta)|}{(1+|\xi-\eta|)^M}d\eta.$$

Taking $M > N + 1 + |\alpha| + |s| + n + 1$ and using (40.7) with $t = -s$, we get (40.20)

$$(1 + |\xi|)^s \; |(F_x T^{(1)}_{\alpha+\beta-N-1} u)(\xi)| \leq C \int_{\mathbb{R}^n} \frac{(1 + |\eta|)^{s+\alpha+\beta-N-1} |\tilde{u}(\eta)|}{(1 + |\xi - \eta|)^{n+1}} d\eta.$$

The arguments in (13.11), (13.12) applied to (40.20) give

$$\|T^{(1)}_{\alpha+\beta-N-1} u\|_s \leq C \|u\|_{s+\alpha+\beta-N-1}.$$

Analogously, we have

$$(F_x T^{(2)}_{\alpha+\beta-N-1} u)(\xi) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \tilde{R}'_N(\xi - \eta - \zeta, \eta, \zeta) \tilde{B}'(\zeta, \eta) \tilde{u}(\eta) d\eta d\zeta.$$

Using (40.14) with M replaced by M_1, we obtain

$$|(F_x T^{(2)}_{\alpha+\beta-N-1} u)(\xi)| \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(1 + |\eta|)^{\alpha-N-1} (1 + |\zeta|)^{|\alpha|+N+1} (1 + |\eta|)^{\beta} |\tilde{u}(\eta)| d\eta d\zeta}{(1 + |\xi - \eta - \zeta|)^{M_1} (1 + |\zeta|)^M}.$$

For any M_1 we have (cf. (40.7))

$$(1 + |\xi - \eta - \zeta|)^{-M_1} \leq C_{M_1} (1 + |\xi - \eta|)^{-M_1} (1 + |\zeta|)^{M_1}. \tag{40.21}$$

Taking $M \geq M_1 + N + 1 + |\alpha| + n + 1$ in (40.18) and using (40.21) leads to

$$|(1 + |\xi|)^s |(F_x T^{(2)}_{\alpha+\beta-N-1} u)(\xi)| \leq C \int_{\mathbb{R}^n} \frac{(1 + |\xi|)^s (1 + |\eta|)^{\alpha+\beta-N-1} |\tilde{u}(\eta)|}{(1 + |\xi - \eta|)^{M_1}} d\eta. \tag{40.22}$$

Treating (40.22) as in (13.11), (13.12), we get

$$\|T^{(2)}_{\alpha+\beta-N-1} u\|_s \leq C \|u\|_{\alpha+\beta-N-1+s}.$$

Therefore $T_{\alpha+\beta-N-1}$ is an operator of order $\leq \alpha + \beta - N - 1$. □

We often call an operator of a negative order a smoothing operator.

Corollary 40.3. *Let $A(x, \xi) \in S^\alpha$ and $B(x, \xi) \in S^\beta$. Then the commutator $[A, B] = A(x, D)B(x, D) - B(x, D)A(x, D)$ has the form*

$$\sum_{k=1}^{N} C_k(x, D) + T_{\alpha+\beta-N-1},$$

where $\operatorname{ord} T_{\alpha+\beta-N-1} \leq \alpha + \beta - N - 1$, $C_k(x, \xi) \in S^{\alpha+\beta-k}$, *and, in particular,*

$$C_1(x, \xi) = -i \sum_{j=1}^{n} \left(\frac{\partial A(x, \xi)}{\partial \xi_j} \frac{\partial B(x, \xi)}{\partial x_k} - \frac{\partial A(x, \xi)}{\partial x_j} \frac{\partial B(x, \xi)}{\partial \xi_j} \right),$$

i.e.,

$$iC_1(x, \xi) = \{A(x, \xi), B(x, \xi)\}. \tag{40.23}$$

Here $\{A(x,\xi),B(x,\xi)\} = \sum_{j=1}^{n}\left(\frac{\partial A}{\partial \xi_1}\frac{\partial B}{\partial x_j} - \frac{\partial A}{\partial x_j}\frac{\partial B}{\partial \xi_j}\right)$ *is called the Poisson bracket of* $A(x,\xi)$ *and* $B(x,\xi)$.

The proof of the corollary follows immediately from the application of Theorem 40.2 to $A(x,D)B(x,D)$ and $B(x,D)A(x,D)$.

Remark 40.1. Let $|||A(x,D)|||_{(s)}$ be the operator norm of $A(x,D)$ acting from $H_s(\mathbb{R}^n)$ to $H_{s-\alpha}(\mathbb{R}^n)$. It follows from the proof of Theorem 40.1 that

$$|||A(x,D)|||_{(s)} \le \sup_{\xi}\frac{|A(\infty,\xi)|}{(1+|\xi|)^{\alpha}} + \sup_{\xi,\eta}\frac{(1+|\eta|)^{N}|\tilde{A}'(\eta,\xi)|}{(1+|\xi|)^{\alpha}}, \tag{40.24}$$

where $N \ge n+1+|s-\alpha|$. Note that

$$(1+|\eta|)^{N}|\tilde{A}'(\eta,\xi)| \le \sum_{|k|=0}^{N}\int_{\mathbb{R}^n}|D_x^k A'(x,\xi)|dx. \tag{40.25}$$

The estimate (10.25) follows from the equality

$$\eta^k\tilde{A}'(\eta,\xi) = \int_{\mathbb{R}^n}(D_x^k A'(x,\xi))e^{-ix\cdot\eta}d\eta.$$

Remark 40.2. We will later use the following property of the remainder $T_{\alpha+\beta-N-1}$ in (40.9): $(1+|x|^2)^M T_{\alpha+\beta-N-1}$ is an operator of order $\le \alpha+\beta-N-1$ for each M, i.e.,

$$\|(1+|x|^2)^M T_{\alpha+\beta-N-1}u\|_s \le C_M\|u\|_{s+\alpha+\beta-N-1}, \quad \forall s. \tag{40.26}$$

Proof: It follows from (40.17) that $T^{(2)}_{\alpha+\beta-N-1} = 0$ when $|x| > R$ since $R'_N(x,\eta,\zeta) = 0$ when $|x| > R$. Consider now $T^{(1)}_{\alpha+\beta-N-1}$ (see (40.16)). Integrating by parts with respect to ζ in (40.16), we get

$$(1+|x|^2)^M T^{(1)}_{\alpha+\beta-N-1}u = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1-\Delta_\zeta)^M(R'_N(\infty,\eta,\zeta)\tilde{B}'(\zeta,\eta)) \times e^{ix\cdot(\zeta+\eta)}\tilde{u}(\eta)d\eta d\zeta.$$

We have

$$|(1-\Delta_\zeta)^M(R'_N(\infty,\eta,\zeta)\tilde{B}'(\zeta,\eta)| \le C_{M_1}\frac{(1+|\eta|)^{\alpha+\beta-N-1}}{(1+|\zeta|)^{M_1}}, \ \forall M_1.$$

Therefore, as in (40.19), (40.20) we get

$$\|(1+|x|^2)^M T^{(1)}_{\alpha+\beta-N-1}u\|_s \le C\|u\|_{\alpha+\beta-N-1+s}.$$

41. Elliptic operators and parametrices

41.1. Parametrix for a strongly elliptic operator.

Let $A(x,D)$ be elliptic differential operator of degree m, i.e., $A(x,\xi) = A_0(x,\xi) + A_1(x,\xi)$, where

$$A_0(x,\xi) = \sum_{|k|=m} a_k(x)\xi^k, \quad A_1(x,\xi) = \sum_{|k|=0}^{m-1} a_k(x)\xi^k,$$

$A_0(x,\xi) \neq 0, \ \forall x \in \mathbb{R}^n, \ \xi \neq 0.$

As in §40, we assume that $A(x,\xi) = A(\infty,\xi)$ when $|x| > R$. An elliptic operator $A(x,D)$ is called strongly elliptic if $\operatorname{Re} A_0(x,\xi) > 0, \forall x \in \mathbb{R}^n, \ \xi \neq 0$. Let $C_\delta = \{\lambda \in \mathbb{C} \setminus \{0\} : -\frac{\pi}{2} - \delta \leq \arg\lambda \leq \frac{\pi}{2} + \delta\}, \ \delta > 0$.

Lemma 41.1. *Let $A(x,D)$ be a strongly elliptic operator and let $R_0(x,\xi,\lambda)$ $= (A_0(x,\xi)+\lambda)^{-1}$. Then there exists $\delta > 0$ such that $R_0(x,\xi,\lambda) \in S^{-m}$ and satisfies the estimates*

$$|R_0(x,\xi,\lambda)| \leq C(|\xi|^m + |\lambda|)^{-1} \leq C_1(|\xi| + |\lambda|^{\frac{1}{m}})^{-m}, \ \lambda \in C_\delta, \tag{41.1}$$

$$\left|\frac{\partial^{k+p} R_0(x,\xi,\lambda)}{\partial\xi^k \partial x^p}\right| \leq C_{pk}|R_0(x,\xi,\lambda)| \ (|\xi| + |\lambda|^{\frac{1}{m}})^{-|k|}, \ \forall p, \forall k, \ \lambda \in C_\delta. \tag{41.2}$$

Proof: Let $A_{01} = \operatorname{Re} A_0, \ A_{02} = \operatorname{Im} A_0$, and $\lambda = \tau + i\sigma$. Then $|\sigma| = |(\sigma + A_{02}) - A_{02}| \leq |\sigma + A_{02}| + C|\xi|^m$. Therefore, $|\sigma| + |\xi|^m \leq |\sigma + A_{02}| + (C+1)|\xi|^m \leq |\sigma + A_{02}| + C_1 A_{01}$, since $|A_{02}| \leq C|\xi|^m, A_{01} \geq C'|\xi|^m$ for some C, C'. Thus, $|\sigma| + C_1\tau + |\xi|^m \leq |\sigma + A_{02}| + C_1(A_{01} + \tau) \leq |\sigma + A_{02}| + C_1|A_{01} + \tau| \leq C_2|A_0(x,\xi) + \lambda|$. Note that $\tau \geq -|\sigma|\tan\delta$ for $\tau \leq 0, \ \tau + i\sigma \in C_\delta$. Therefore, if $\delta > 0$ is small, then $|\sigma| + C_1\tau \geq C_3(|\sigma| + |\tau|)$ for $\lambda \in C_\delta$. This proves (41.1).

Estimates (41.2) follow from (41.1). □

Suppose that A is a strongly elliptic operator.

Theorem 41.2. *For any $N \geq 0$, there exists a ψdo operator $R^{(N)}(x,D,\lambda)$, $R^{(N)}(x,\xi,\lambda) \in S^{-m}$, such that*

$$(A(x,D) + \lambda I)R^{(N)}(x,D,\lambda) = I + T_{-N-1}, \tag{41.3}$$

where I is the identity operator, $\operatorname{ord} T_{-N-1} \leq -N-1$*, and*

$$|||T_{-N-1}|||_{(s)} \leq C_N|\lambda|^{-\frac{N+1}{m}}, \ \lambda \in C_\delta, \tag{41.4}$$

where $|||T_{-N-1}|||_{(s)}$ is the operator norm of T_{-N-1} acting from $H_s(\mathbb{R}^n)$ to $H_s(\mathbb{R}^n)$.

The operator $R^{(N)}(x, D, \lambda)$ is called the parametrix for $A(x, D) + \lambda$.

Proof: In view of Theorem 40.2 we have

$$(A(x, D) + \lambda I)R_0(x, D, \lambda) = I + C_{-1}(x, D, \lambda),$$

where $C_{-1}(x, \xi, \lambda) \in S^{-1}$ and

$$C_{-1}(x, \xi, \lambda) = \sum_{|k|=1}^{m} \frac{1}{k!}\frac{\partial A(x, \xi)}{\partial \xi^k} D_x^k R_0(x, \xi, \lambda) + A_1(x, \xi)R_0(x, \xi, \lambda). \tag{41.5}$$

Note that there is no remainder in (41.5), since $A(x, \xi)$ is a polynomial.

Let

$$R_{-1}(x, \xi, \lambda) = -R_0(x, \xi, \lambda)C_{-1}(x, \xi, \lambda).$$

Then by Theorem 40.2 we have

$$(A(x, D) + \lambda I)(R_0 + R_{-1}(x, D, \lambda)) = I + C_{-2}(x, D, \lambda), \tag{41.6}$$

where $C_{-2}(x, \xi, \lambda) \in S^{-2}$ and
(41.7)

$$C_{-2}(x, \xi, \lambda) = A_1(x, \xi)R_{-1}(x, \xi, \lambda) + \sum_{|k|=1}^{m} \frac{1}{k!}\frac{\partial^k A(x, \xi)}{\partial \xi^k} D_x^k R_{-1}(x, \xi, \lambda).$$

Analogously defining $R_{-2}(x, \xi, \lambda) = -R_0(x, \xi, \lambda)C_{-2}(x, \xi, \lambda)$, etc., we construct $R_{-p}(x, \xi, \lambda) = -R_0(x, \xi, \lambda)C_{-p}(x, \xi, \lambda)$, where $C_{-p}(x, \xi, \lambda) \in S^{-p}$, $1 \le p \le N$, and
(41.8)

$$C_{-p}(x, \xi) = A_1(x, \xi)R_{-p+1}(x, \xi, \lambda) + \sum_{|k|=1}^{m} \frac{1}{k!}\frac{\partial^k A(x, \xi)}{\partial \xi^k} D_x^k R_{-p+1}(x, \xi, \lambda).$$

Therefore, we get:

$$(A + \lambda I)(R_0 + R_{-1} + \cdots + R_{-N}) = I + T_{-N-1}, \tag{41.9}$$

where $T_{-N-1}(x, \xi, \lambda) \in S^{-N-1}$ and $|D_x^p T_{-N-1}(x, \xi, \lambda) \le \frac{C_{pN}}{|\lambda|^{\frac{N+1}{m}}}, \forall |p| \ge 0$. Hence, by Theorem 40.1 and Remark 40.1, the operator T_{-N-1} satisfies (41.4). Note that the sum

$$R^{(N)}(x, \xi, \lambda) = R_0(x, \xi, \lambda) + \cdots + R_{-N}(x, \xi, \lambda) \tag{41.10}$$

can be rewritten in the following form:

$$R^{(N)}(x, \xi, \lambda) = (A_0(x, \xi) + \lambda)^{-1} + \sum_{k=1}^{N} (A_0(x, \xi) + \lambda)^{-(k+1)} p_k(x, \xi), \tag{41.11}$$

where $p_k(x, \xi)$ are polynomials in ξ, $\deg p_k(x, \xi) \le km - \frac{k}{2}$.

41.2. The existence and uniqueness theorem.

Lemma 41.3. *Let $A(x,D)$ be the same as in Theorem 41.2. Let s be arbitrary. Suppose $|\lambda|$ is sufficiently large and $\lambda \in C_\delta$. Then for any $f \in H_{s-m}(\mathbb{R}^n)$ there exists a unique solution $u \in H_s(\mathbb{R}^n)$ of the equation*

$$(A(x,D) + \lambda I)u = f.$$

Proof: Consider (41.3) with $N = 0$. It follows from (41.4) that T_{-1} is an operator with a small norm. Therefore $I + T_{-1}$ is invertible in $H_{s-m}(\mathbb{R}^n)$ and $R = R_0(x,D,\lambda)(I+T_{-1})^{-1}$ is the right inverse to $A + \lambda I$, i.e., $(A+\lambda I)Rf = f,\ \forall f \in H_{s-m}(\mathbb{R}^n)$. Also $u = Rf \in H_s(\mathbb{R}^n)$ since $\operatorname{ord} R = -m$.

Applying Theorem 40.2, we have $R_0(x,D,\lambda)(A(x,D)+\lambda) = I + T_{-1}^{(1)}$, where $T_{-1}^{(1)}$ satisfies (41.4) with $N = 0$. Therefore $(I + T_{-1}^{(1)})^{-1}$ exists when $\lambda \in C_\delta$ is sufficiently large. Thus $R^{(1)} = (I + T_{-1}^{(1)})^{-1}R_0$ is the left inverse of $A(x,D) + \lambda I$. Since $A(x,D) + \lambda I$ has the left and the right inverses, it is invertible as an operator from $H_s(\mathbb{R}^n)$ to $H_{s-m}(\mathbb{R}^n)$, and $R = R^{(1)} = (A(x,D) + \lambda I)^{-1}$. □

41.3. Elliptic regularity.

In the case where $A(x,\xi) \in S^\alpha$ is not a polynomial in ξ we call $A(x,\xi)$ elliptic if $A(x,\xi)$ can be represented in the form

$$A(x,\xi) = A_0(x,\xi)(1 - \chi(\xi)) + A_1(x,\xi),$$

where $A_0(x,\xi)$ is homogeneous in ξ of degree α, $A_0(x,\xi) \neq 0,\ \forall \xi \neq 0$, and $A(x,\xi)$ belongs to C^∞ for $\xi \neq 0$, $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$ for $|\xi| \leq 1$, and $A_1(x,\xi) \in S^{\alpha-1}$.

Lemma 41.4 (Elliptic regularity)**.** *Let $A(x,\xi) \in S^\alpha$ be elliptic and let $u \in H_s(\mathbb{R}^n)$ be a solution of $A(x,D)u = f$ for some $s \in \mathbb{R}$. Suppose $f \in H_{s-\alpha+r}(\mathbb{R}^n)$, $r > 0$. Then $u \in H_{s+r}(\mathbb{R}^n)$. In particular, if $f \in H_N(\mathbb{R}^n)$, $\forall N$, then $u \in H_N(\mathbb{R}^n)$, $\forall N$.*

Proof: Let $R(x,\xi) = A_0^{-1}(x,\xi)(1-\chi(\xi))$. It follows from Theorem 40.2 that

$$R(x,D)A(x,D)u = u - \chi(D)u + T_{-1}u, \tag{41.12}$$

where $\operatorname{ord} T_{-1} \leq -1$. Therefore

$$u = Rf + \chi(D)u - T_{-1}u.$$

Note that $Rf \in H_{s+r}$ since $\operatorname{ord} R = -\alpha$. Also $T_{-1}u \in H_{s+1}(\mathbb{R}^n)$, $\chi(D)u \in H_N(\mathbb{R}^n), \forall N$. Therefore $u \in H_{s+r_0}$, where $r_0 = \min(1,r)$. If $r > 1$, we can repeat the same arguments with $u \in H_{s+1}(\mathbb{R}^n)$ instead of $u \in H_s(\mathbb{R}^n)$.

After at most $[r] + 1$ steps we prove Lemma 41.4. □

42. Compactness and the Fredholm property

42.1. Compact operators.

Theorem 42.1. *Let* $\operatorname{ord} T' \leq -\delta$ *and*

$$\|(1+|x|^2)^M T'u\|_{s+\delta} \leq C\|u\|_s, \ \forall s, \tag{42.1}$$

where $M > 0$. *Then* T' *is compact in* $H_s(\mathbb{R}^n)$, $\forall s$.

Proof: We have the identity:

$$T' = (1+|x|^2)^{-M}\Lambda^{-\delta}\Lambda^{\delta}(1+|x|^2)^M T'.$$

Denote $T_1 = \Lambda^\delta(1+|x|^2)^M T'$, where Λ^δ is a ψdo with symbol $\Lambda^\delta(\xi) = (1+|\xi|^2)^{\frac{\delta}{2}}$. It follows from (42.1) that T_1 is bounded in $H_s(\mathbb{R}^n)$, $\forall s$. Therefore it is enough to prove that $T_0 = (1+|x|^2)^{-M}\Lambda^{-\delta}$ is a compact operator in $H_s(\mathbb{R}^n)$ since the product of a compact and of a bounded operator is compact.

The compactness of T_0 in $L_2(\mathbb{R}^n)$ was proven in §30 (see the proof of Theorem 30.3). The proof of the compactness in $H_s(\mathbb{R}^n)$ is similar.

Let $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$, $|\chi(\xi)| \leq 1$, and $\chi(\xi) = 1$ for $|\xi| \leq 1$. Denote by T_N the operator $(1+|x|^2)^{-M}\chi(\frac{x}{N})\Lambda^{-\delta}(D)\chi(\frac{D}{N})$. We have

$$\tilde{v}(\xi) = \int_{\mathbb{R}^n} \psi_N(\xi-\eta)\Lambda^{-\delta}(\eta)\chi\left(\frac{\eta}{N}\right)\tilde{u}(\eta)d\eta, \tag{42.2}$$

where

$$v = T_N u, \qquad \psi_N(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|x|^2)^{-M}\chi\left(\frac{x}{N}\right)e^{-ix\cdot\xi}d\xi.$$

Therefore

$$\Lambda^s(\xi)\tilde{v}(\xi) = \int_{\mathbb{R}^n} K_N(\xi,\eta)\Lambda^s(\eta)\tilde{u}(\eta)d\eta,$$

where

$$K_N(\xi,\eta) = \Lambda^s(\xi)\psi_N(\xi-\eta)\Lambda^{-s-\delta}(\eta)\chi\left(\frac{\eta}{N}\right). \tag{42.3}$$

Note that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|K_N(\xi,\eta)|^2 d\xi d\eta < +\infty,$$

i.e., the integral operator $K_N w = \int_{\mathbb{R}^n} K_N(\xi,\eta)w(\eta)d\eta$ is a Hilbert-Schmidt operator. Since Hilbert-Schmidt operators are compact in $L_2(\mathbb{R}^n)$, we conclude that the operator (42.2) is compact in $\tilde{H}_s(\mathbb{R}^n) = FH_s(\mathbb{R}^n)$, and therefore T_N is compact in $H_s(\mathbb{R}^n)$. We have

$$|||T_0 - T_N|||_{(s)} \to 0 \quad \text{as} \quad N \to \infty, \tag{42.4}$$

where $|||T_0 - T_N|||_{(s)}$ is the operator norm of $T_0 - T_N$ acting in $H_s(\mathbb{R}^n)$. Therefore T_0 is a compact operator in $H_s(\mathbb{R}^n)$ since the limit of compact operators in the operator norm is a compact operator. To prove (42.4) we have used that the operator of multiplication by $(1+|x|^2)^{-M}$ is bounded in $H_s(\mathbb{R}^n)$ (see Problem 15.17) and that the norm of the operator $(1+|x|^2)^{-M}(1-\chi(\frac{x}{N}))$ tends to zero as $N\to\infty$.

Theorem 42.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and let T be a bounded operator from $H_s(\Omega)$ to $H_{s+\varepsilon}(\Omega)$, $\varepsilon>0$. Then T is a compact operator in $H_s(\Omega)$.*

Proof: Let l be an extension operator from Ω to $\mathbb{R}^n$ such that

$$\|lu\|_{s,\mathbb{R}^n}\le C\|u\|_{s,\Omega}\quad\text{for all}\quad u\in H_s(\Omega)$$

and $lu=0$ for $|x|>R$. We assume that $\overline{\Omega}$ is contained in the ball $\{x : |x|<R\}$.

Denote by $\psi_0(x)$ a $C_0^\infty(\mathbb{R}^n)$ function such that $\psi_0(x)=1$ for $|x|\le R$. Let p be the restriction operator to the domain Ω. We have, for any $u\in H_s(\Omega)$:

$$Tu=pl(Tu)=p\psi_0 l(Tu)=p\psi_0\Lambda^{-\varepsilon}\Lambda^{\varepsilon}l(Tu).$$

The operator $\Lambda^{\varepsilon}l(Tu)$ is bounded from $H_s(\Omega)$ to $H_s(\mathbb{R}^n)$ and the operator p is bounded from $H_s(\mathbb{R}^n)$ to $H_s(\Omega)$. It follows from the proof of Theorem 42.1 that $\psi_0(x)\Lambda^{-\varepsilon}$ is compact in $H_s(\mathbb{R}^n)$. Therefore T is compact in $H_s(\Omega)$ as a product of bounded operators and a compact operator. □

Remark 42.1. Analogously to the proof of Theorem 42.2 one can show that the embedding of $H_{s+\varepsilon}(\Omega)$ into $H_s(\Omega)$ is compact: we have, as above, $u=p\psi_0 lu=p\psi_0\Lambda^{-\varepsilon}\Lambda^{\varepsilon}lu$ for any $u\in H_{s+\varepsilon}(\Omega)$. The operator $\Lambda^{\varepsilon}lu$ is bounded from $H_{s+\varepsilon}(\Omega)$ to $H_s(\mathbb{R}^n)$. Therefore $Iu=p(\psi_0\Lambda^{-\varepsilon})(\Lambda^{\varepsilon}lu)$ is a compact operator from $H_{s+\varepsilon}(\Omega)$ to $H_s(\Omega)$ as the product of the bounded operators p and $\Lambda^{\varepsilon}lu$ and the compact operator $\psi_0\Lambda^{-\varepsilon}$. □

42.2. Fredholm operators.

Let A be a bounded operator from the Banach space B_1 to the Banach space B_2 and let B_k^* be the dual spaces to B_k, $k=1,2$. Denote by $(u,v^*)_k$ the pairing between B_k and B_k^*, $k=1,2$. For example, when $B=H_s(\mathbb{R}^n), B^*=H_{-s}(\mathbb{R}^n)$ and (u,v) is the extension of the scalar product in $L_2(\mathbb{R}^n)$. The operator A^* is called the adjoint of A if

$$(Au,v^*)_2=(u,A^*v^*)_1\quad\text{for all}\quad u\in B_1,\ v^*\in B_2^*. \tag{42.5}$$

A bounded operator A from B_1 to B_2 is called a Fredholm operator if

a) $\ker A=\{u\in B_1 : Au=0\}$ is finite-dimensional.

b) $\operatorname{Im} A=\{v\in B_2 : v=Au \text{ for some } u\in B_1\}$ is closed in B_2, i.e., the image of A is closed.

c) $\ker A^*=\{v^*\in B_2^* : A^*v^*=0\}$ is finite-dimensional.

It follows from (42.5) that if $v_2^* \in \ker A^*$, i.e., $A^* v_2^* = 0$, then $(Au, v^*)_2 = 0$ for all $u \in B_1$. Vice versa, if $(Au, v^*)_2 = 0$ for all $u \in B_1$, then $(u, A^* v^*) = 0$, $\forall u \in B_1$, and therefore, $A^* v^* = 0$. Thus $\ker A^*$ is "orthogonal" to $\operatorname{Im} A$. We say that $\ker A^*$ is the cokernel of A and that $\operatorname{Im} A$ has a finite codimension if $\dim \ker A^* < +\infty$.

The following proposition is a well-known fact of functional analysis (see Rudin [**R**]).

Proposition 42.3. *Let T be a compact operator in B. Then the operator $I + T$ is Fredholm in B, where I is the identity operator. Moreover, $\dim \ker(I + T) = \dim \ker(I + T^*)$.*

It follows from Proposition 42.3 that the equation $u + Tu = f$ has a solution for any $f \in B$ iff the homogeneous equation $v + Tv = 0$ has only a trivial solution. Indeed, $\dim \ker(I + T) = 0$ implies $\dim \ker(I + T^*) = 0$, and therefore the range of $I + T$ is the whole space B.

Proposition 42.3 is called the Fredholm alternative.

We have already used this proposition in §30.

Proposition 42.4. *A bounded operator A from B_1 to B_2 is Fredholm if and only if there exists a bounded operator R from B_2 to B_1 such that*

$$RA = I + T_1, \quad AR = I + T_2, \tag{42.6}$$

where T_k are compact operators in B_k, $k = 1, 2$.

The operator R is called a regularizer of A.

Proof:

a) Let A be a Fredholm operator. Denote by $B_{02} \subset B_2$ the image of A. Let $f_1^*, \dots, f_{m_-}^*$ be a basis in $\ker A^* \subset B_2^*$. Choose $f_1, \dots, f_{m-}$ in B_2 such that $(f_j, f_k^*) = \delta_{jk}$, $1 \le j,\ k \le m_-$. Let $B_{02}^{\perp}$ be the linear span of $f_1, \dots, f_{m_-}$. Then $B_2 = B_{02} \dotplus B_{02}^{\perp}$ is the direct sum of B_{02} and $B_{02}^{\perp}$, i.e., any $v \in B_2$ has a unique decomposition

$$v = v_1 + v_2, \tag{42.7}$$

where $v_2 = \sum_{k=1}^{m_-} (v, f_k^*)_2 f_k \in B_{02}^{\perp}$, and $v_1 = (v - v_2) \in B_{02}$. Note that in the case where B_2 is a Hilbert space, (42.7) is the orthogonal decomposition of $\operatorname{Im} A$ and $\ker A^*$. Let

$$P_2 v = \sum_{k=1}^{m_-} (v, f_k^*)_2 f_k. \tag{42.8}$$

Note that $P_2^2 v = P_2 \sum_{k=1}^{m_-} (P_2 v, f_k^*)_2 f_k = P_2 v$, i.e., P_2 is the projector on $B_{02}^{\perp}$. Therefore, (42.7) has the form:

$$v = (I - P_2)v + P_2 v, \tag{42.9}$$

where P_2 is the projection on $B_{02}^{\perp}$ and $(I - P_2)$ is the projection on B_{02}.

Analogously, let $e_1, \dots, e_{m_+}$ be a basis in $\ker A \subset B_1$. Let $e_j^* \in B_1^*$ be such that $(e_j, e_k^*)_1 = \delta_{jk},\ 1 \le j,k \le m_+$.

Denote by P_1 the following projection on $\ker A$:

$$P_1 u = \sum_{j=1}^{m_+} (u, e_j^*)_1 e_j, \tag{42.10}$$

and let $B_{01} = \{u \in B_1 : (u, e_j^*)_1 = 0,\ 1 \le j \le m_+\}$. Then we have a direct sum $B_1 = B_{10} \dot{+} \ker A$,

$$u = (I - P_1)u + P_1 u,$$

where $P_1 u \in \ker A$ and $(I - P_1)u \in B_{10}$. The operator A is a bounded one-to-one map of B_{01} onto B_{02}. By the open map theorem (see Rudin [**R**]), there exists a bounded inverse map R_0 of B_{02} onto B_{01}. Denote by R the following bounded operator from B_2 to B_1:

$$Rf = R_0 f \text{ for } f \in B_{02}, \quad Rf = 0 \text{ for } f \in B_{02}^{\perp}. \tag{42.11}$$

Then we have

$$\begin{aligned} ARf &= f - P_2 f, \quad \forall f \in B_2, \\ RAu &= u - P_1 u, \quad \forall u \in B_1. \end{aligned} \tag{42.12}$$

Note that P_1, P_2 are finite rank and hence compact operators. Therefore R is a regularizer of A.

b) Suppose there exists R such that (42.6) holds. Since $\ker A \subset \ker(I+T_1)$ and $\ker(I+T_1)$ is finite-dimensional, we see that $\dim\ker A < +\infty$. Analogously, from $R^*A^* = I + T_2^*$ we conclude that $\dim\ker A^* < +\infty$ since $\ker A^* \subset \ker(I + T_2^*)$.

As in (42.9) we have the following decomposition of B_2:

$$B_2 = \operatorname{Im}(I + T_2) + B_3,$$

where B_3 is finite-dimensional. Since $\operatorname{Im} A \supset \operatorname{Im}(I + T_2)$, we have $\operatorname{Im} A = \operatorname{Im}(I + T_2) + B_{30}$, where B_{30} is a subspace of B_3. Hence $\operatorname{Im} A$ is closed since $\operatorname{Im}(I + T_2)$ is closed and B_{30} is finite-dimensional. Therefore A is Fredholm. □

Remark 42.2. Sometimes it is easier to construct separately the left and the right regularizers, i.e., bounded operators R_1 and R_2 such that $R_1 A = I + T_1, AR_2 = I + T_2$, where T_k are compact operators. Then the proof of Proposition 42.4 implies that A is Fredholm. Moreover, we have $R_1 A R_2 = R_1 + R_1 T_2 = R_2 + T_1 R_2$, and therefore, $R_2 - R_1 = R_1 T_2 - T_1 R_2 = T$, where T is a compact operator from B_2 to B_1. Thus the left and the right regularizers differ by a compact operator.

42.3. Fredholm elliptic operators in $\mathbb{R}^n$.

Not every elliptic operator in $\mathbb{R}^n$ is Fredholm. For example, $-\Delta - k^2$ is not Fredholm from $H_s(\mathbb{R}^n)$ to $H_{s-2}(\mathbb{R}^n)$ for any s.

We consider a class of symbols $A(x,\xi) \in S^\alpha$ such that

$$|A(x,\xi)| \geq C(1+|\xi|)^\alpha \quad \text{if} \quad |x|^2+|\xi|^2 \geq R^2 \quad \text{for some} \quad R>0. \tag{42.13}$$

Theorem 42.5. *Let $A(x,\xi) \in S^\alpha$ and let (42.13) holds. Then $A(x,D)$ is a Fredholm operator from $H_s(\mathbb{R}^n)$ to $H_{s-\alpha}(\mathbb{R}^n)$, $\forall s$.*

Proof: Suppose that $\chi(\xi)$ is as in Theorem 42.1. Define $R_0(x,\xi) = A^{-1}(x,\xi)(1-\chi(\frac{x}{R})\chi(\frac{\xi}{R}))$. Note that $R_0(x,\xi) \in S^{-\alpha}$. Applying Theorem 40.2 we have:

$$A(x,D)R_0(x,D) = \Big(I - \chi\Big(\frac{x}{R}\Big)\chi\Big(\frac{D}{R}\Big)\Big) + T_{-1}^{(1)}, \tag{42.14}$$

where $\operatorname{ord} T_{-1}^{(1)} \leq -1$. Moreover, by Remark 40.2, $\operatorname{ord}\,(1+|x|^2)^M T_{-1}^{(1)} \leq -1$. By Theorem 42.1, the operator $T_1^{(1)} - \chi(\frac{x}{R})\chi(\frac{D}{R})$ is compact in $H_{s-\alpha}(\mathbb{R}^n)$. Therefore $R_0(x,D)$ is the right regularizer of $A(x,D)$. Analogously it follows from Theorem 42.1 that $R_0(x,D)$ is a left regularizer of $A(x,D)$ since $R_0(x,D)A(x,D) = I - \chi(\frac{x}{R})\chi(\frac{D}{R}) + T_{-1}$ and $-\chi\big(\frac{x}{R}\big)\chi\big(\frac{D}{R}\big) + T_{-1}$ is a compact operator in $H_s(\mathbb{R}^n)$. Therefore $A(x,D)$ is Fredholm.

Remark 42.3. If $A(x,\xi) \in S^\alpha$ and (42.13) holds, then, starting from $R_0(x,\xi) = A^{-1}(x,\xi)(1-\chi(\frac{x}{R})\chi(\frac{\xi}{R}))$ instead of $R_0(x,\xi,\lambda) = (A_0(x,\xi)+\lambda)^{-1}$, we can construct, similarly to the proof of Theorem 41.2, an operator $R^{(N)}(x,D)$ such that

$$A(x,D)R^{(N)}(x,D) = I + T_{-N-1}, \quad \forall N,$$

where $R^{(N)}(x,\xi) \in S^{-\alpha}$ and $\operatorname{ord} T_{-N-1} \leq -N-1$. Analogously, we can construct $R_1^{(N)}(x,D)$ such that

$$R_1^{(N)}(x,D)A(x,D) = I + T_{-N-1}^{(1)},$$

where $R_1^{(N)}(x,\xi) \in S^{-\alpha}$ and $\operatorname{ord} T_{-N-1}^{(1)} \leq -N-1$. As in Remark 42.2, we have $R^{(N)}(x,D) - R_1^{(N)}(x,D) = T_{-\alpha-N-1}^{(2)}$, where $\operatorname{ord} T_{-\alpha-N-1}^{(2)} \leq -\alpha-N-1$.

43. The adjoint of a pseudodifferential operator

43.1. A general form of ψdo's.

We rewrite expression (40.2) for a ψdo as follows:

$$Au = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A(x,\xi)e^{i(x-y)\cdot\xi}u(y)dyd\xi, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{43.1}$$

where the integral is understood as a repeated integral: first integration is performed with respect to y and then, with respect to ξ. It is convenient to consider, instead of (43.1), a more general form of a pseudodifferential operator:

$$Au = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, y, \xi) e^{i(x-y)\cdot\xi} u(y) dy d\xi, \tag{43.2}$$

where the integral is understood as a repeated integral and $a(x, y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with the following properties:

$$\left| \frac{\partial^{p+k+r} a(x, y, \xi)}{\partial x^p \partial y^k \partial \xi^r} \right| \leq C_{pkr}(1 + |\xi|)^{\alpha - |r|}, \ \forall p, k, r, \tag{43.3}$$

$$\begin{aligned} a(x, y, \xi) &= a(\infty, y, \xi) \quad \text{for} \quad |x| \geq R, \\ a(x, y, \xi) &= a(x, \infty, \xi) \quad \text{for} \quad |y| \geq R. \end{aligned} \tag{43.4}$$

We represent $a(x, y, \xi)$ in the following form:

$$a(x, y, \xi) = a(x, \infty, \xi) + a'(x, y, \xi), \tag{43.5}$$

where $a'(x, y, \xi) = 0$ for $|y| \geq R$.

The following theorem shows that an operator of the form (43.2) can be represented in the form (40.2) up to an operator of an arbitrary low order.

Theorem 43.1. *Let A be an operator of the form* (43.2)*. Then for any N,*

$$Au = \sum_{|k|=0}^{N} A_k(x, D)u + T_{\alpha-N-1}u, \tag{43.6}$$

where the $A_k(x, D)$ are operators of the form (40.2)*,*

$$A_0(x, \xi) = a(x, \infty, \xi) + a'(x, x, \xi) = a(x, x, \xi), \tag{43.7}$$

(43.8)

$$A_k(x, \xi) = \frac{1}{k!} D_y^k \frac{\partial^k}{\partial \xi^k} a'(x, y, \xi)|_{y=x}, \ |k| \geq 1, \quad \operatorname{ord} T_{\alpha-N-1} \leq \alpha - N - 1.$$

Proof: We have, for any $u \in C_0^\infty(\mathbb{R}^n)$:

$$Au = a(x, \infty, D)u + A'u, \tag{43.9}$$

where

$$A'u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a'(x, y, \xi) u(y) e^{i(x-y)\cdot\xi} dy d\xi. \tag{43.10}$$

Compute the integral in y using the convolution formula:

$$\int_{\mathbb{R}^n} a'(x, y, \xi) u(y) e^{-iy\cdot\xi} dy = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{a}'(x, \xi - \eta, \xi) \tilde{u}(\eta) d\eta, \tag{43.11}$$

where $\tilde{a}'(x,\eta,\xi) = F_y a'(x,y,\xi)$. We get

$$A'u = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{a}'(x,\xi-\eta,\xi)\tilde{u}(\eta)e^{ix\cdot\xi} d\eta d\xi. \tag{43.12}$$

Making the change of variables $\xi - \eta = \zeta$, we obtain

$$A'u = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{a}'(x,\zeta,\eta+\zeta)\tilde{u}(\eta)e^{ix\cdot(\eta+\zeta)} d\eta d\zeta. \tag{43.13}$$

The Taylor expansion of $\tilde{a}'(x,\zeta,\eta+\zeta)$ gives

$$\tilde{a}'(x,\zeta,\eta+\zeta) = \sum_{|k|=0}^{N} \frac{1}{k!} \frac{\partial^k \tilde{a}'(x,\zeta,\eta)}{\partial \eta^k} \zeta^k + R_N(x,\zeta,\eta). \tag{43.14}$$

Taking into account that

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\partial^k \tilde{a}'(x,\zeta,\eta)}{\partial \eta^k} \zeta^k e^{ix\cdot\zeta} d\zeta = \frac{\partial^k}{\partial \eta^k} D_y^k a'(x,y,\eta)|_{y=x}, \tag{43.15}$$

we get (43.6) with $T_{\alpha-N-1}u$ of the form

$$T_{\alpha-N-1}u = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R_N(x,\zeta,\eta)e^{ix\cdot(\eta+\zeta)}\tilde{u}(\eta) d\eta d\zeta. \tag{43.16}$$

It remains to show that $\operatorname{ord} T_{\alpha-N-1} \le \alpha - N - 1$. Note that

$$\tilde{a}'(x,\zeta,\eta) = \tilde{a}'(\infty,\zeta,\eta) + \tilde{a}''(x,\zeta,\eta), \tag{43.17}$$

where $\tilde{a}''(x,\zeta,\eta) = 0$ for $|x| \ge R$. Then

$$R_N(x,\zeta,\eta) = R_{N1}(\infty,\zeta,\eta) + R_{N2}(x,\zeta,\eta), \tag{43.18}$$

where R_{N1} corresponds to $\tilde{a}'(\infty,\zeta,\eta)$ and R_{N2} corresponds to $\tilde{a}''(x,\zeta,\eta)$. The Fourier transform in x of $T_{\alpha-N-1}u$ is

$$\begin{aligned}(F_x(T_{\alpha-N-1}u))(\xi) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} R_{N1}(\infty,\xi-\eta,\eta)\tilde{u}(\eta) d\eta \\ &+ \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{R}_{N2}(\xi-\eta-\zeta,\zeta,\eta)\tilde{u}(\eta) d\eta d\zeta.\end{aligned} \tag{43.19}$$

It follows from (40.13), (40.14), and (43.14) that

$$\begin{aligned}&|R_{N1}(\infty,\xi-\eta,\eta)| \\ &\quad \le C_M(1+|\xi-\eta|)^{-M}(1+|\eta|)^{\alpha-N-1}(1+|\xi-\eta|)^{N+1+|\alpha|}, \ \forall M,\end{aligned} \tag{43.20}$$

$$\begin{aligned}&|\tilde{R}_{N2}(\xi-\eta-\zeta,\zeta,\eta)| \\ &\quad \le C_{M_1,M}(1+|\xi-\eta-\zeta|)^{-M_1}(1+|\zeta|)^{-M} \\ &\qquad \times (1+|\eta|)^{\alpha-N-1}(1+|\zeta|)^{N+1+|\alpha|}, \quad \forall M_1, \ \forall M.\end{aligned} \tag{43.21}$$

Estimating the term in (43.19) containing R_{N1} as in (40.19), (40.20) and the term in (43.19) containing R_{N2} as in (40.21), (40.22), we get

$$\|T_{\alpha-N-1}u\|_s \le C_s\|u\|_{s+\alpha-N-1}, \ \forall s. \tag{43.22}$$

□

43.2. The adjoint operator.

Let A^* be the formally adjoint operator to the ψdo $A(x,D)$, i.e.,

$$(Au,v) = (u, A^*v), \ \forall u \in C_0^\infty(\mathbb{R}^n), \ v \in C_0^\infty(\mathbb{R}^n),$$

where (u,v) is the L_2 scalar product.

Writing $A(x,D)u$ in the form (43.1), we get

$$(Au,v) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A(x,\xi)u(y)e^{i(x-y)\cdot\xi}dyd\xi\right)\overline{v(x)}dx. \tag{43.23}$$

We can rewrite (43.23) in the form

$$(Au,v) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} u(y)\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A(x,\xi)e^{i(x-y)\cdot\xi}\overline{v(x)}dxd\xi\right)dy. \tag{43.24}$$

Therefore

$$(A^*v)(y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\overline{A(x,\xi)}v(x)e^{i(y-x)\cdot\xi}dxd\xi. \tag{43.25}$$

Changing the notation from y to x and vice versa, we get

$$(A^*v)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\overline{A(y,\xi)}v(y)e^{i(x-y)\cdot\xi}dyd\xi, \tag{43.26}$$

i.e., A^*v has the form (43.2). Theorem 43.1 applied to (43.26) gives

Theorem 43.2. *Let $A(x,D)$ be a ψdo with symbol $A(x,\xi)\in S^\alpha$. Then the adjoint operator A^* is also a ψdo operator and*

$$(A^*v)(x) = \sum_{|k|=0}^{N} A_k(x,D)v + T_{\alpha-N-1}v, \tag{43.27}$$

where

$$A_0(x,\xi) = \overline{A(x,\xi)}, \tag{43.28}$$

$$A_k(x,\xi) = \frac{1}{k!}D_x^k\frac{\partial^k}{\partial\xi^k}\overline{A(x,\xi)}, \quad 1\le |k|\le N, \tag{43.29}$$

$$\operatorname{ord} T_{\alpha-N-1} \le \alpha - N - 1.$$

□

43.3. Weyl's ψdo's.

Let $A(x,\xi) \in S^\alpha$. Denote by $A_w(x,D)$ an operator of the form (43.2):

$$A_w(x,D)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\cdot\xi} u(y)dyd\xi. \tag{43.30}$$

The operators of the form (43.30) are called Weyl's pseudodifferential operators.

Applying Theorem 43.1 to (43.30), we obtain

$$A_w(x,D) = A(x,D) + A_1(x,D) + T_{\alpha-N-1}, \tag{43.31}$$

where $\operatorname{ord} T_{\alpha-N-1} \le \alpha - N - 1$ and

$$A_1(x,\xi) = \sum_{|k|=1}^{N} \frac{1}{2^k k!} D_x^k \frac{\partial^k}{\partial \xi^k} A(x,\xi).$$

Therefore $A_w(x,D)$ and $A(x,D)$ differ by the terms of order $\alpha - 1$.

Consider the case where $A(x,\xi)$ is real-valued. Then the ψdo $A(x,D)$ is not selfadjoint if the terms (43.29) are not zero. However, $A_w(x,D)$ is selfadjoint if $A(x,\xi)$ is real since

$$\begin{aligned}(A_w(x,D)u,v) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{x+y}{2},\xi\right) u(y) e^{i(x-y)\cdot\xi} dyd\xi \overline{v(x)}dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{x+y}{2},\xi\right) u(y) \overline{e^{i(y-x)\cdot\xi} v(x)} dydxd\xi \\ &= (u, A_w(x,D)v).\end{aligned}$$

44. Pseudolocal property and microlocal regularity

44.1. The Schwartz kernel.

Let A be a linear continuous operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$. The Schwartz kernel of A is the distribution $E \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that $(Au,v) = E(u(y)\overline{v(x)})$ for all $u \in C_0^\infty(\mathbb{R}^n)$, $v \in C_0^\infty(\mathbb{R}^n)$. Let Au be a ψdo of the form (43.2). Consider

$$E_\varepsilon(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,y,\xi) e^{i(x-y)\cdot\xi} \chi(\varepsilon\xi) d\xi, \tag{44.1}$$

where $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$ when $|\xi| < 1$, $\chi(\xi) = 0$ when $|\xi| > 2$. Note that $E_\varepsilon(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Let $w(x,y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then

$$E_\varepsilon(w(x,y)) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} E_\varepsilon(x,y) w(x,y) dxdy$$

converges in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ to $E \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, where

$$(44.2) \qquad E(w(x,y)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,y,\xi) e^{i(x-y)\cdot\xi} w(x,y) dx dy d\xi.$$

The integral in (44.2) is understood as a repeated integral: first integration is with respect to x and y and then with respect to ξ. Since $(Au, v) = E(u(y)\overline{v(x)})$, the distribution E is the Schwartz kernel of A. When $\alpha < -n$, the integral

$$(44.3) \qquad E(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,y,\xi) e^{i(x-y)\cdot\xi} d\xi$$

converges absolutely and defines the Schwartz kernel of A. When $\alpha \geq -n$, we understand (44.3) as a limit in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ of (44.1) as $\varepsilon \to 0$.

Theorem 44.1. *For an arbitrary $\delta > 0$, the distribution $(1 - \chi(\frac{x-y}{\delta}))E$ belongs to $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof: Since $\frac{\partial}{\partial \xi_k} e^{i(x-y)\cdot\xi} = i(x_k - y_k) e^{i(x-y)\cdot\xi}$, we have the following identity:

$$(44.4) \qquad e^{i(x-y)\cdot\xi} = (-\Delta_\xi)^N \frac{e^{i(x-y)\cdot\xi}}{|x-y|^{2N}},$$

where $\Delta_\xi = \sum_{j=1}^n \frac{\partial^2}{\partial \xi_j^2}$. Substituting (44.4) into (44.1) and integrating by parts with respect to ξ, we get

$$(44.5) \qquad \Big(1 - \frac{\chi(x-y)}{\delta}\Big) E_\varepsilon(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-\Delta_\xi)^N (a(x,y,\xi)\chi(\varepsilon\xi)) \frac{(1 - \chi(\frac{x-y}{\delta}))}{|x-y|^{2N}} e^{i(x-y)\cdot\xi} d\xi.$$

Let

$$(44.6) \qquad E^{(\delta)}(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} ((-\Delta_\xi)^N a(x,y,\xi)) \frac{(1 - \chi(\frac{x-y}{\delta}))}{|x-y|^{2N}} e^{i(x-y)\cdot\xi} d\xi.$$

We will study the limit of (44.5) as $\varepsilon \to 0$. Fix $M > 0$ and assume that $2N > 2M + n + 1 + \alpha$. Note that $|\frac{\partial^k}{\partial \xi^k} \chi(\varepsilon\xi)| \leq C\varepsilon^{|k|}$ and $\frac{\partial^k}{\partial \xi^k}\chi(\varepsilon\xi)$ differ from zero only when $\frac{1}{\varepsilon} < |\xi| < \frac{2}{\varepsilon}$ for $|k| \geq 1$. Also $1 - \chi(\varepsilon\xi) = 0$ as $|\xi| < \frac{1}{\varepsilon}$. Therefore

$$\left| \left(1 - \frac{\chi(x-y)}{\delta}\right) E_\varepsilon(x,y) - E^{(\delta)}(x,y) \right| \leq C \sum_{|k|=0}^{2N} \int_{\frac{1}{\varepsilon} < |\xi|} \varepsilon^{|k|} (1 + |\xi|)^{\alpha - 2N + |k|} d\xi \to 0$$

as $\varepsilon \to 0$, because $2N > n+1+\alpha$. Since $(1 - \frac{\chi(x-y)}{\delta})E_\varepsilon(x,y)$ also converges to $(1 - \frac{\chi(x-y)}{\delta})E$ in the distribution sense, we see that

$$\Big(1 - \frac{\chi(x-y)}{\delta}\Big)E = E^{(\delta)}, \tag{44.7}$$

where $E^{(\delta)}$ is the same as in (44.6). Note that $\frac{(1-\chi(\frac{x-y}{\delta}))}{|x-y|^{2N}} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ since $\frac{(1-\chi(\frac{x-y}{\delta}))}{|x-y|^{2N}} = 0$ for $|x-y| < \delta$. We show that $E^{(\delta)} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Note that $|(-\Delta_\xi)^N a(x,y,\xi)| \le C(1+|\xi|)^{\alpha-2N} \le C(1+|\xi|)^{-(n+1)-2M}$. Therefore, differentiating (44.6) with respect to x and y, we obtain that $E^{(\delta)}$ has M continuous derivatives in x and y. Since M is arbitrary we get that $E^{(\delta)} = (1 - \frac{\chi(x-y)}{\delta})E \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

44.2. Pseudolocal property of ψdo's.

Corollary 44.2 (Pseudolocal property of ψdo). *Assume that $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\psi(x) \in C_0^\infty(\mathbb{R}^n)$, and* $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$. *Let $A(x,D)$ be a ψdo of the form (43.2). Then $\psi(x)A\varphi u$ is an integral operator with $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ kernel.*

Proof: Since $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, there exists $\delta > 0$ such that $\psi(x)E(x,y)\varphi(y) = 0$ when $|x-y| < \delta$. Therefore, $\psi(x)A\varphi$ is an integral operator with $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ kernel, i.e., $\operatorname{ord}\psi A\varphi = -\infty$. Note that $\psi A\varphi = 0$ if A is a differential operator.

Lemma 44.3. *Let $A(x,\xi) \in S^\alpha$ and let $u \in H_s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. We have*

$$\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u.$$

Proof: Recall that $\operatorname{sing\,supp} u$ is the complement of the largest open set where $u \in C^\infty$ (cf. §14). Let $x_0 \notin \operatorname{sing\,supp} u$. Then there exists $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ near x_0 and $\operatorname{supp}\varphi \cap \operatorname{sing\,supp} u = \emptyset$. Let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$, $\psi(x_0) \ne 0$, and $\operatorname{supp}\psi$ is contained in the interior of the set where $\varphi = 1$. Then $\operatorname{supp}\psi \cap \operatorname{supp}(1-\varphi) = \emptyset$. We have:

$$\psi Au = \psi A\varphi u + \psi A(1-\varphi)u. \tag{44.8}$$

Note that $\varphi u \in C_0^\infty(\mathbb{R}^n)$, since $\operatorname{supp}\varphi \cap \operatorname{sing\,supp}\ u = \emptyset$. Then, by Theorem 40.1, $\psi A\varphi u \in C^\infty$. Also $\psi A(1-\varphi)u \in C^\infty$ by Corollary 44.2. Therefore, $\psi Au \in C^\infty$ and $x_0 \notin \operatorname{sing\,supp} Au$ since $\psi(x_0) \ne 0$. □

The following theorem gives a refinement of Lemma 44.3.

Theorem 44.4. *Let $A(x,\xi) \in S^\alpha$ and let $u \in H_s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. We have*

$$WF(Au) \subset WF(u),$$

where $WF(v)$ is the wave front set of v (see §14 for the definition of the wave front set).

Proof: Suppose $(x_0, \xi_0) \notin WF(u)$. There exists $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ in a small neighborhood of x_0, and there exists $\alpha(\xi) \in C^\infty(\mathbb{R}^n \backslash \{0\})$ such that $\alpha(\xi)$ is homogeneous in ξ of degree zero, $\alpha(\xi) = 1$ in a small conic neighborhood of ξ_0, and $\alpha(D)\varphi(x)u \in C^\infty(\mathbb{R}^n)$ (cf. Definition 14.2).

Suppose $\psi(x) \in C_0^\infty(\mathbb{R}^n)$, $\psi(x_0) \neq 0$, and $\operatorname{supp}\psi$ is contained in the interior of $\{x : \varphi(x) = 1\}$. Let $\beta(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and let $\beta(\xi)$ be homogeneous of degree 0, where $\beta(\xi_0) \neq 0$ and $\operatorname{supp}\beta \cap \{|\xi| = 1\}$ is contained in the interior of $\{\xi : \alpha(\xi) = 1\}$.

To prove Theorem 44.4 we need to show that $\beta(D)\psi(x)Au \in C^\infty(\mathbb{R}^n)$. Since $\beta(\xi)$ is not smooth at $\xi = 0$, we replace $\beta(D)$ by $(1 - \chi(D))\beta(D)$, where $\chi(\xi)$ is the cutoff function as above. Note that $\chi(D)w \in C^\infty(\mathbb{R}^n)$ if $w \in H_s, \forall s$. Since $\operatorname{supp}\psi \cap \operatorname{supp}(1 - \varphi(x)) = \emptyset$, we see that $\psi A(1 - \varphi)u \in C^\infty$ by the pseudolocality of $A(x, D)$. Consider now $(1 - \chi(D))\beta(D)\psi A\varphi u$. Applying Theorem 40.2 to $(1 - \chi(D))\beta(D)$ and ψA, we get

$$(1 - \chi(D))\beta(D)\psi A\varphi u = \sum_{|k|=0}^{N} C_k(x, D)\varphi u + T_{\alpha-N-1}\varphi u, \tag{44.9}$$

where $\operatorname{ord} T_{\alpha-N-1} \leq \alpha - N - 1$, and the symbols $C_k(x, \xi)$ contain either $\beta(\xi)$ or the derivatives $\frac{\partial^k \beta(\xi)}{\partial \xi^k}$. Since $\alpha(\xi) = 1$ on $\operatorname{supp}\beta(\xi)$, we have $C(x, \xi) = \sum_{|k|=0}^{N} C_k(x, \xi) = C(x, \xi)\alpha(\xi)$. Therefore

$$(1 - \chi(D))\beta(D)\psi A\varphi u = C\alpha(D)\varphi u + T_{\alpha-N-1}\varphi u.$$

Since $\alpha(D)\varphi u \in C^\infty$, we have $C\alpha(D)\varphi u \in C^\infty$ (cf. Theorem 40.1). Therefore $(1 - \chi(D))\beta(D)\psi A\varphi u \in H_{s-\alpha+N+1}(\mathbb{R}^n)$. Since N is arbitrary, we conclude that $\beta(D)\psi A\varphi u \in C^\infty$. □

44.3. Microlocal regularity.

Theorem 44.5 (Microlocal regularity)**.** *Suppose $A(x, \xi) \in S^m$ and there exists $A_0(x, \xi)$ that is homogeneous of degree m in ξ and such that $A(x, \xi) - A_0(x, \xi)\chi(\xi) \in S^{m-1}$. Suppose $A(x, \xi)$ is microelliptic at (x_0, ξ_0), i.e., $A_0(x_0, \xi_0) \neq 0$. Let $u \in H_s(\mathbb{R}^n)$ be the solution of $Au = f$ and let f be C^∞ at (x_0, ξ_0), i.e., $(x_0, \xi_0) \notin WF(f)$. Then $(x_0, \xi_0) \notin WF(u)$.*

Proof: Let $\varphi(x), \alpha(\xi), \psi(x), \beta(\xi)$ be the same as in the proof of Theorem 44.4. We assume that $A_0(x, \xi) \neq 0$ when $x \in \operatorname{supp}\varphi(x)$, $|\xi| = 1$ and $\xi \in \operatorname{supp}\alpha(\xi)$, and that $\alpha(D)\varphi(x)f \in C^\infty(\mathbb{R}^n)$.

Let $B(x, \xi)$ be the extension of $A(x, \xi)$ to $\mathbb{R}^n \times \mathbb{R}^n$ from a neighborhood of $\operatorname{supp}\varphi(x)\alpha(\xi)(1 - \chi(\xi))$, and let $B(x, \xi) \in S^m$ and $|B(x, \xi)| \geq C(1 + |\xi|)^m$ for $|x|^2 + |\xi|^2 \geq R^2$. As in Remark 42.3, one can construct a ψdo $R^{(N)}(x, D)$, $R^{(N)}(x, \xi) \in S^{-m}$, such that $R^{(N)}(x, D)B(x, D) = I + T_{-N-1}$, where ord $T_{-N-1} \leq -N-1$. Since $Au = f$, we have

$$Bu = f + (B - A)u. \tag{44.10}$$

Applying $R^{(N)}$ to (44.10), we get

$$u = R^{(N)} f + R^{(N)}(B - A))u - T_{-N-1}u. \tag{44.11}$$

Now we apply $(1 - \chi(D))\beta(D)\psi(x)$ to (44.11). Since $R^{(N)}(x, \xi) \in S^{-m}$ and $\alpha(D)\varphi(x)f \in C^\infty$, we conclude from the proof of Theorem 44.4 that $(1 - \chi(D))\beta(D)\psi R^{(N)} f \in C^\infty$. We now apply Theorem 40.2 to the composition C of the pseudodifferential operators $(1 - \chi(D)\beta(D)\psi(x)$, $R^{(N)}(x, D)$, and $B(x, D) - A(x, D)$. We obtain:

$$Cu = \sum_{|k|=0}^{N} C_k(x, D)u + T^{(1)}_{-N-1}u,$$

where ord $T^{(1)}_{-N-1} \leq -N-1$ and $C_k(x, \xi) \in S^{-|k|}$. Note that all $C_k(x, \xi)$ contain either $\psi(x)\beta(\xi)$ or its derivatives with respect to ξ and x. On the other hand, $B(x, \xi) - A(x, \xi) = 0$ on supp $\varphi(x)\alpha(\xi)(1 - \chi(\xi))$ for $|x|^2 + |\xi|^2 \geq R^2$, i.e., $B(x, \xi) - A(x, \xi) = 0$ on a neighborhood of supp $\psi(x)\beta(\xi)(1 - \chi(\xi))$. Therefore $C_k(x, xi) = 0$ for $|x|^2 + |\xi|^2 \geq R^2$ for all k, $0 \leq |k| \leq N$. This implies that $(1 - \chi(D))\psi(x)u \in H_{s+N+1}(\mathbb{R}^n)$ for an arbitrary N. Thus $(1 - \chi(D))\beta(D)\psi(x)u \in C^\infty$. □

Remark 44.1. It follows from (44.3) that a ψdo with symbol S^α, $\alpha < -n$, has a continuous kernel. We shall prove the same result for any operator of order $\alpha < -n$.

Lemma 44.6. *Let T_α be an operator of order $\alpha < -n$. Then the kernel $T_\alpha(x, y)$ of T_α is continuous in (x, y).*

Proof: Take $s = -\frac{n}{2} - \varepsilon$, $\varepsilon > 0$. Since ord $T_\alpha = \alpha$, we have

$$\|v\|_{-\frac{n}{2}-\varepsilon-\alpha} \leq |||T_\alpha|||\; \|u\|_{-\frac{n}{2}-\varepsilon}, \tag{44.12}$$

where $v = T_\alpha u$, $|||T_\alpha|||$ is the operator norm of T_α from $H_{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$ to $H_{-\frac{n}{2}-\varepsilon-\alpha}(\mathbb{R}^n)$. Note that the space $H_{\frac{n}{2}+\varepsilon}$ is dual to $H_{-\frac{n}{2}-\varepsilon}$ with respect to the extension of the L_2-scalar product (cf. §13) and that $H_{\frac{n}{2}+\varepsilon} \supset H_{-\frac{n}{2}-\varepsilon-\alpha}$, since $-\frac{n}{2} - \varepsilon - \alpha > \frac{n}{2} + \varepsilon$ when ε is small. Therefore T_α is a bounded operator from $H_{-\frac{n}{2}-\varepsilon}$ to $H_{\frac{n}{2}+\varepsilon}$.

Let δ_x be the delta-function at x, i.e., $\delta_x(\varphi) = \varphi(x)$. Analogously, $\delta_y(\varphi) = \varphi(y)$. Note that δ_x and δ_y belong to $H_{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$ and depend continuously on x and y, respectively. Let

$$T_\alpha(x, y) = (T_\alpha \delta(y), \delta(x)), \tag{44.13}$$

i.e., $T_\alpha(x,y)$ is the kernel of T_α. Note that $(T_\alpha u_1, u_2)$ is a continuous bilinear form in $H_{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)\times H_{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$. Therefore, $T_\alpha(x,y)$ is continuous in (x,y). Moreover,

$$|T_\alpha(x,y)| \le C\,|||T_\alpha|||\,. \tag{44.14}$$

Example 44.1. Let $E(x,y)$ be the Schwartz kernel of the operator (43.2). We find the wave front set $WF(E(x,y)) \subset \mathbb{R}^{2n}\times\mathbb{R}^{2n}\setminus\{0\}$. It follows from Theorem 44.1 that $\operatorname{sing\,supp} E \subset \{(x,y) : x = y\}$. Fix any point (x_0,x_0) and let $\psi(x,y) \in C_0^\infty(\mathbb{R}^{2n})$, $\psi(x_0,x_0) \neq 0$. Take the Fourier transform of $\psi(x,y)E(x,y)$ in x and in y. We get:

(44.15)
$$F(\psi E) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi(x,y)a(x,y,\theta)e^{-ix\cdot\xi-iy\cdot\eta}e^{i(x-y)\cdot\theta}dxdyd\theta$$
$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\tilde{b}(\xi-\theta,\eta+\theta,\theta)d\theta,$$

where $b(x,y,\theta) = \psi(x,y)a(x,y,\theta)$, and $\tilde{b}(\xi,\eta,\theta)$ is the Fourier transform of $b(x,y,\theta)$ in x and y. To justify (44.15) we introduce $\chi(\varepsilon\theta)$ and take the limit in the distribution sense as $\varepsilon \to 0$ (cf. the proof of Theorem 44.1).

Since $b(x,y,\theta)$ is C_0^∞ in x and in y, we have

$$|\tilde{b}(\xi-\theta,\eta+\theta,\theta)| \le C_N(1+|\xi-\theta|)^{-N}(1+|\eta+\theta|)^{-N}(1+|\theta|)^m, \forall N. \tag{44.16}$$

Make a change of variables $\theta + \eta = \theta'$. Then

$$F(\psi(x,y)E(x,y)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\tilde{b}(\xi+\eta-\theta',\theta',\theta'-\eta)d\theta'.$$

Note that
(44.17)
$$|\tilde{b}(\xi+\eta-\theta',\theta',\theta'-\eta)| \le C_N(1+|\xi+\eta-\theta'|)^{-N}(1+|\theta'|)^{-N}(1+|\theta'|+|\eta|)^m.$$

We show that

$$|F(\psi E)(\xi,\eta)) \le C_N(1+|\xi+\eta|)^{-N}(1+|\eta|)^m, \quad \forall N. \tag{44.18}$$

We have

$$F(\psi E) = I_1 + I_2, \tag{44.19}$$

where

$$I_1 = \int_{|\theta'|<\frac{1}{2}|\xi+\eta|}\tilde{b}(\xi+\eta-\theta',\theta',\theta'-\eta)d\theta' \quad \text{and} \quad I_2 = F(\psi E) - I_1.$$

If $|\theta'| < \frac{1}{2}|\xi+\eta|$, we get $1+|\xi+\eta-\theta'| \ge 1+|\xi+\eta|-|\theta'| \ge \frac{1}{2}(1+|\xi+\eta|)$. Therefore

$$|I_1(\xi,\eta)| \le C_N(1+|\xi+\eta|)^{-N}(1+|\eta|)^m, \ \forall N.$$

If $|\theta'| > \frac{1}{2}|\xi + \eta|$, we get

$$|I_2(\xi, \eta)| \leq C_N(1 + |\xi + \eta|)^{-\frac{N}{2}}(1 + |\eta|)^m, \ \forall N.$$

Therefore (44.18) holds. Let $\omega_1 = \frac{\xi}{\sqrt{|\xi|^2+|\eta|^2}}$, $\omega_2 = \frac{\eta}{\sqrt{|\xi|^2+|\eta|^2}}$, provided $(\xi, \eta) \neq (0, 0)$. Then

$$|F(\psi E)| \leq \frac{C_N(1 + |\eta|)^m}{(1 + |\omega_1 + \omega_2|\sqrt{|\xi|^2 + |\eta|^2})^N}, \ \forall N.$$

Therefore, for any small conic neighborhood of (ω_1, ω_2) such that $\omega_1 + \omega_2 \neq 0$, we have

$$|F(\psi E)| \leq C_N(1 + \sqrt{|\xi|^2 + |\eta|^2})^{-N}, \ \forall N.$$

Thus $WF(E) \subset \{(x, y, \xi, \eta) \in \mathbb{R}^{2n} \setminus (\mathbb{R}^{2n} \setminus \{0\}) : x = y, \xi = -\eta\}$.

45. Change-of-variables formula for ψdo's

Let

$$x = s(\hat{x}) \tag{45.1}$$

be a one-to-one diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$, $s(\hat{x}) = \hat{x}$ for $|\hat{x}| > R$, and $J(\hat{x}) = \det \frac{\partial s(\hat{x})}{\partial \hat{x}} \neq 0$, $\forall \hat{x} \in \mathbb{R}^n$, where $\frac{\partial s(\hat{x})}{\partial \hat{x}}$ is the Jacobian matrix of $x = s(\hat{x})$. Let $A(x, D)$ be a ψdo operator with symbol $A(x, \xi) \in S^\alpha$. We have

$$A(x, D)u = Bu + T_{-\infty}u, \tag{45.2}$$

where

$$Bu = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A(x, \xi)\chi\Big(\frac{x - y}{\delta}\Big)e^{i(x-y)\cdot\xi}u(y)dyd\xi, \tag{45.3}$$

$$T_{-\infty}u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} A(x, \xi)\Big(1 - \chi\Big(\frac{x - y}{\delta}\Big)\Big)e^{i(x-y)\cdot\xi}u(y)dyd\xi. \tag{45.4}$$

It follows from Theorem 44.1 that $T_{-\infty}$ is an integral operator with $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ kernel.

Consider $v(x) = A(x, D)u$, where $u(x) \in C_0^\infty(\mathbb{R}^n)$. Make the change of variables (45.1). Let $\hat{u}(\hat{x}) = u(s(\hat{x}))$ and $\hat{v}(\hat{x}) = v(s(\hat{x}))$ be $u(x)$ and $v(x)$ in the new coordinates and let $\hat{v}(\hat{x}) = \hat{A}\hat{u}$ be the image of $A(x, D)$ in the new coordinates. Then

$$\hat{A}\hat{u} = \hat{B}\hat{u} + \hat{T}_{-\infty}\hat{u},$$

where $\hat{B}$ is the image of B and $\hat{T}_{-\infty}$ is the image of $T_{-\infty}$ in new coordinates. Since $T_{-\infty}$ is an operator with $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ kernel, the operator $\hat{T}_{-\infty}$ is also an integral operator with C^∞ kernel.

We show that $\hat{B}$ is also a ψdo modulo $T_{\alpha-N-1}$, where ord $T_{\alpha-N-1} \leq \alpha - N - 1$, N is arbitrary, and we compute the symbol of $\hat{B}$. Substituting $x = s(\hat{x}), y = s(\hat{y})$ in (45.3), we get

(45.5)
$$Bu = \frac{1}{2\pi^n}\int_{\mathbb{R}^n} A(s(\hat{x}),\xi)\chi\Big(\frac{s(\hat{x})-s(\hat{y})}{\delta}\Big)e^{i(s(\hat{x})-s(\hat{y}))\cdot\xi}u(s(\hat{y}))|J(\hat{y})|d\hat{y}d\xi.$$

Note that

$$s(\hat{x}) - s(\hat{y}) = \int_0^1 \frac{d}{dt}s(\hat{y}+t(\hat{x}-\hat{y}))dt = H(\hat{x},\hat{y})(\hat{x}-\hat{y}), \tag{45.6}$$

where $H(\hat{x},\hat{y})$ is a C^∞ matrix and $H(\hat{x},\hat{x}) = \frac{\partial s(\hat{x})}{\partial \hat{x}}$. Since $\det \frac{\partial s(\hat{x})}{\partial \hat{x}} \neq 0, \forall \hat{x}$, we see that $\det H(\hat{x},\hat{y}) \neq 0$ if $|s(\hat{x}) - s(\hat{y})| \leq \delta$, where δ is small. We have

$$(s(\hat{x})-s(\hat{y}))\cdot\xi = H(\hat{x},\hat{y})(\hat{x}-\hat{y})\cdot\xi = (x-y)\cdot H^T(\hat{x},\hat{y})\xi, \tag{45.7}$$

where $H^T(\hat{x},\hat{y})$ is a matrix transpose to $H(\hat{x},\hat{y})$. We substitute (45.7) into (45.5) and make the change of variables

$$\eta = H^T(\hat{x},\hat{y})\xi. \tag{45.8}$$

In order to justify the change of variables (45.8), one must introduce the cutoff factor $\chi(\varepsilon\xi)$ into (45.5), make the change of variables, and then take the limit as $\varepsilon \to 0$. We get

(45.9)
$$\hat{B}\hat{u} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} A(s(\hat{x}),(H^T(\hat{x},\hat{y}))^{-1}\eta)\chi\Big(\frac{s(\hat{x})-s(\hat{y})}{\delta}\Big)e^{i(\hat{x}-\hat{y})\cdot\eta}\hat{u}(\hat{y}) \\ \times |J(\hat{y})|\,|\det H(\hat{x},\hat{y})|^{-1}d\hat{y}d\eta.$$

Therefore $\hat{B}$ is a ψdo of the form (43.2) with symbol

(45.10)
$$\hat{b}(\hat{x},\hat{y},\eta) = A(s(\hat{x}),(H^T(\hat{x},\hat{y}))^{-1}\eta)\chi(\frac{s(\hat{x})-s(\hat{y})}{\delta})|J(\hat{y})|\,|\det H(\hat{x},\hat{y})|^{-1}.$$

Using Theorem 43.1, we can represent (45.9) in the form

$$\hat{B}\hat{u} = \sum_{|k|=0}^N \hat{B}_k(\hat{x},\hat{D})\hat{u} + \hat{T}_{\alpha-N-1}\hat{u},$$

where

$$\hat{B}_0(\hat{x},\eta) = \hat{b}(\hat{x},\hat{x},\eta) = A\Big(s(\hat{x}),\Big(\Big(\frac{\partial s(\hat{x})}{\partial\hat{x}}\Big)^T\Big)^{-1}\eta\Big),$$

$$B_k(\hat{x},\eta) = \frac{1}{k!}D_{\hat{y}}^k\frac{\partial^k}{\partial\eta^k}b(\hat{x},\hat{y},\eta)|_{\hat{y}=\hat{x}},$$

ord $\hat{T}_{\alpha-N-1} \leq \alpha - N - 1$, and N is arbitrary. $\square$

46. The Cauchy problem for parabolic equations

46.1. Parabolic ψdo's.

Consider a differential equation in $\mathbb{R}^{n+1}_+ = \{t > 0, x \in \mathbb{R}^n\}$ of the form:

$$\frac{\partial u(x,t)}{\partial t} + A(x,t,D)u(x,t) = f,\ t > 0,\ x \in \mathbb{R}^n, \tag{46.1}$$

where $A(x,t,\xi) = A_0(x,t,\xi) + A_1(x,t,\xi),\ A_0(x,t,\xi) = \sum_{|k|=m} a_k(x,t)\xi^k$, $A_1(x,t,\xi) = \sum_{|k|\le m-1} a_k(x,t)\xi^k,\ D = -i\frac{\partial}{\partial x}$.

We assume that $a_k(x,t) \in C^\infty(\mathbb{R}^{n+1})$, $a_k(x,t) = a_k^\infty$ for $|x| > R$, and $a_k(x,t) = a_k(x,\infty)$ when $|t| > R$. Equation (46.1) is called parabolic if $\operatorname{Re} A_0(x,t,\xi) \ge C|\xi|^m,\ \forall(x,t)$. We study the Cauchy problem, i.e., $u(x,t)$ satisfies (46.1) and the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \tag{46.2}$$

Let $v(x,t) = e^{-t\tau}u(x,t),\ g = e^{-t\tau}f,\ \tau > 0$. Then $v(x,t)$ satisfies the equation:

$$\frac{\partial v(x,t)}{\partial t} + \tau v + A(x,t,D)v(x,t) = g, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{46.3}$$

and

$$v(x,0) = u_0(x). \tag{46.4}$$

We prove the existence and the uniqueness of the solution of the Cauchy problem (46.3), (46.4) provided τ is large.

We introduce the Sobolev spaces that are adapted to the form of the parabolic equation (46.1). Let $\Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ be the space of distributions with finite norm

$$\|u\|^2_{\frac{s}{m},s} = \int_{\mathbb{R}^{n+1}} |\tilde{u}(\xi,\sigma)|^2 |i\sigma + (|\xi|^2+1)^{\frac{m}{2}}|^{\frac{2s}{m}} d\xi d\sigma, \tag{46.5}$$

where

$$\tilde{u}(\xi,\sigma) = \int_{\mathbb{R}^{n+1}} u(x,t)e^{-ix\cdot\xi - i\sigma t}dxdt.$$

As in \$ 22, we denote by $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ the subspace of $\Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ consisting of distributions with supports in $\mathbb{R}^{n+1}_+ = \{t > 0, x \in \mathbb{R}^n\}$ and denote by $\Pi_{\frac{s}{m},s,\tau}(\mathbb{R}^{n+1}_+)$ the space of restrictions of $u \in \Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ to the half-space $t > 0$ with the norm

$$\|u\|^+_{\frac{s}{m},s} = \inf_l \|lu\|_{\frac{s}{m},s}, \tag{46.6}$$

where lu is an arbitrary extension of u to $\mathbb{R}^{n+1}$ (cf. §22).

Denote by $P_{\alpha,m}$ the class of symbols $A(x,t,\xi,\sigma)$ such that A is C^∞ in all variables, $A(x,t,\xi,\sigma) = A(\infty,\xi,\sigma)$ when $|x| \geq R$, $A(x,t,\xi,\sigma) = A(x,\infty,\xi,\sigma)$ when $|t| > R$, and

$$\left|\frac{\partial^{k_1+k_2+k_3+k_4}A(x,t,\xi,\sigma)}{\partial x^{k_1}\partial t^{k_2}\partial\xi^{k_3}\partial\sigma^{k_4}}\right| \leq C_k|\Lambda_m^+|^{\frac{\alpha}{m}-\frac{k_3}{m}-k_4}, \quad \forall k, \tag{46.7}$$

where

$$\Lambda_m^+ = i\sigma + (|\xi|^2+1)^{\frac{m}{2}}. \tag{46.8}$$

Note that if $m = 1$, then $P_{\alpha,m}$ coincides with S^α (cf. §40). Note also that the norm (46.5) has the form $\|(\Lambda_m^+)^{\frac{s}{m}}u\|_0$, where $\|\ \|_0$ is the norm in $L_2(\mathbb{R}^{n+1})$, and Λ_m^+ is the ψdo with symbol $\Lambda_m^+(\xi,\sigma)$.

Analogously to the proof of Theorems 40.1 and 40.2, we have

Lemma 46.1. *A pseudodifferential operator A with symbol $A(x,t,\xi,\sigma) \in P_{\alpha,m}$ is bounded from $\Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ to $\Pi_{\frac{s-\alpha}{m},s-\alpha}(\mathbb{R}^{n+1})$. If $A(x,t,\xi,\sigma) \in P_{\alpha,m}$, $B(x,t,\xi,\sigma) \in P_{\beta,m}$, then*

$$AB = C + T_{\alpha+\beta-1}, \tag{46.9}$$

where $C(x,t,\xi,\sigma) = A(x,t,\xi,\sigma)B(x,t,\xi,\sigma) \in P_{\alpha+\beta,m}$, and

$$\|T_{\alpha+\beta-1}u\|_{\frac{s}{m},s} \leq C\|u\|_{\frac{\alpha+\beta-1+s}{m},\alpha+\beta-1+s} \quad \forall u \in C_0^\infty(\mathbb{R}^{n+1}),\ \forall s. \tag{46.10}$$

The following lemma is crucial for this section:

Lemma 46.2. *Suppose $A(x,t,\xi,\sigma-i\tau) \in P_{\alpha,m}$ is analytic in $z = \sigma - i\tau$ for $\tau > 0$ and continuous for $\tau \geq 0$. Suppose also that $A(x,t,\xi,\sigma-i\tau)$ satisfies estimates (46.7) with σ replaced by $z = \sigma - i\tau, \tau \geq 0$. Then the ψdo $A(x,t,D_x,D_t)$ maps $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ into $\overset{\circ}{\Pi}_{\frac{s-\alpha}{m},s-\alpha}(\mathbb{R}^{n+1})$ for each $s \in \mathbb{R}$.*

Proof: For any $u(x,t) \in C_0^\infty(\mathbb{R}_+^{n+1})$ we have:

$$Au = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} A(x,t,\xi,\sigma)\tilde{u}(\xi,\sigma)e^{ix\cdot\xi+it\sigma}dxdt. \tag{46.11}$$

Since A and $\tilde{u}(\xi,z)$ are analytic in $z = \sigma - i\tau$ for $\tau > 0$ and since $\tilde{u}(\xi,z)$ decays rapidly as $|\operatorname{Re} z| \to \infty$, we can move the line of integration in z using the Cauchy theorem

$$Au = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} A(x,t,\xi,\sigma-i\tau)\tilde{u}(\xi,\sigma-i\tau)e^{ix\cdot\xi+it(\sigma-i\tau)}d\sigma d\xi, \tag{46.12}$$

where $\tau > 0$ is arbitrary. Therefore

$$|(Au)(x,t)| \leq Ce^{t\tau}, \quad \forall\tau > 0, \tag{46.13}$$

since $|\tilde{u}(\xi,\sigma-i\tau)| \leq C_N(1+|\xi|+|\sigma-i\tau|)^{-N}$, $\forall N$, for $\xi \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^1$, $\tau \geq 0$.

Fix any $t_0 < 0$. Taking the limit in (46.13) as $\tau \to +\infty$, we get $(Au)(x,t_0) = 0$. Therefore, $\operatorname{supp} Au \subset \overline{\mathbb{R}^{n+1}_+}$.

It follows from Lemma 46.1 that $Au \in \Pi_{\frac{s-\alpha}{m},s-\alpha}(\mathbb{R}^{n+1})$. Therefore $Au \in \overset{\circ}{\Pi}_{\frac{s-\alpha}{m},s-\alpha}$. Since $C_0^\infty(\mathbb{R}^{n+1}_+)$ is dense in $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$, we have that $Au \in \overset{\circ}{\Pi}_{\frac{s-\alpha}{m},s-\alpha}$ for any $u \in \overset{\circ}{\Pi}_{\frac{s}{m},s}$. □

46.2. The Cauchy problem with zero initial conditions.

We shall denote by $P^+_{\alpha,m}$ the class of symbols in $P_{\alpha,m}$ that are analytic in $z = \sigma - i\tau$ for $\tau > 0$ and satisfy estimates (46.7) with σ replaced by $\sigma - i\tau, \tau \geq 0$.

Theorem 46.3. *Consider the equation* (46.3) *in* $\mathbb{R}^{n+1}$. *Let* s *be arbitrary. If* τ *is sufficiently large, then for any* $g \in \overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ *there exists a unique* $v \in \overset{\circ}{\Pi}_{\frac{s+m}{m},s+m}(\mathbb{R}^{n+1})$ *that solves* (46.3) *in* $\mathbb{R}^{n+1}$.

Proof: Let $R_0(x,t,\xi,\sigma - i\tau) = (i\sigma + \tau + A_0(x,t,\xi))^{-1},\ \tau > 0$. We have $|R_0(x,t,\xi,\sigma - i\tau)| \leq C(|\sigma| + \tau + |\xi|^m)^{-1}$. Moreover, $R_0 \in P^+_{-m,m}$ (cf. (41.1), (41.2)).

It follows from Lemmas 46.1 and 46.2 that the ψdo R_0 maps $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ to $\overset{\circ}{\Pi}_{\frac{s+m}{m},s+m}(\mathbb{R}^{n+1})$ for any s. Note that

$$(i\sigma + \tau + A(x,t,\xi))R_0(x,t,\xi,\sigma - i\tau) = 1 + A_1(x,\xi)(i\sigma + \tau + A_0)^{-1}.$$

We have

$$|A_1(x,\xi)|\ |R_0| \leq \frac{C(1+|\xi|)^{m-1}}{|\sigma| + \tau + |\xi|^m} \leq C_1(|\sigma| + \tau + |\xi|^m)^{-\frac{1}{m}}. \tag{46.14}$$

Therefore, $A_1(x,\xi)R_0 \in P^+_{-1,m}$. Applying (46.9) to the composition of $\frac{\partial}{\partial t} + \tau + A$ and R_0, we get

$$\left(\frac{\partial}{\partial t} + \tau + A\right) R_0 = I + T^{(1)}_{-1}, \tag{46.15}$$

where $T^{(1)}_{-1} = A_1(x,D)R_0 + T_{-1},\ \operatorname{ord} T^{(1)}_{-1} \leq -1$. Moreover, the presence of the parameter τ implies that (cf. (46.14) and (41.4))

$$\|T^{(1)}_{-1}u\|_{\frac{s}{m},s} \leq \frac{C}{\tau^{\frac{1}{m}}}\|u\|_{\frac{s}{m},s}. \tag{46.16}$$

Note that $T^{(1)}_1$ acts in $\overset{\circ}{\Pi}_{\frac{1}{m},s}(\mathbb{R}^{n+1})$ since $((\frac{\partial}{\partial t} + \tau + A)R_0 - I)f = 0$ when $t < 0$ if $f = 0$ when $t < 0$. Therefore $T^{(1)}_1$ has a small norm in $\overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ when $\tau \geq \tau_0$, τ_0 is large and $R = R_0(I + T^{(1)}_{-1})^{-1}$ is the right inverse of

$\frac{\partial}{\partial t} + \tau + A$. Analogously (cf. \$ 41) one can prove that $\frac{\partial}{\partial t} + \tau + A$ has the left inverse when $\tau \geq \tau_0$. □

Theorem 46.4. *Let $\tau \geq \tau_0$, where τ_0 is large. Then for any $f(x,t)$ in $\mathbb{R}^{n+1}$ such that $e^{-t\tau} f \in \mathring{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$, there exists a unique solution of (46.1) in $\mathbb{R}^{n+1}$ such that $e^{-t\tau} u \in \mathring{\Pi}_{\frac{s+m}{m},s+m}(\mathbb{R}^{n+1})$.*

Proof: Let $v(x,t) \in \mathring{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ be the solution of (46.3) in $\mathbb{R}^{n+1}$, where $\tau \geq \tau_0$, $g = e^{-t\tau} f$. We will write $v(x,t,\tau)$ instead of $v(x,t)$ to emphasize the dependence of v on τ. For $\tau = \tau_0$ we have $\frac{\partial v(x,t,\tau_0)}{\partial t} + \tau_0 v(x,t,\tau_0) + Av(x,t,\tau_0) = e^{-t\tau_0} f(x,t)$ in $\mathbb{R}^{n+1}$. Then $e^{-t(\tau-\tau_0)} v(x,t,\tau_0)$ satisfies (46.3) with $g = e^{-t\tau} f$. The uniqueness of the solution of (46.3) in $\mathring{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ implies that $e^{-t(\tau-\tau_0)} v(x,t,\tau_0) = v(x,t,\tau)$. Set $u(x,t) = e^{t\tau_0} v(x,t,\tau_0)$. Then $v(x,t,\tau) = e^{-t\tau} u(x,t)$, and $u(x,t)$ solves (46.1) in $\mathbb{R}^{n+1}$. □

46.3. The Cauchy problem with nonzero initial conditions.

Now we study the Cauchy problem (46.3) with nonzero initial data (46.4). Analogously to the proof of Theorem 13.6 we have:

Lemma 46.5. *Suppose $s > \frac{m}{2}$. Then $\Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ is embedded into the space $C(\mathbb{R}, H_{s-\frac{m}{2}}(\mathbb{R}^n))$ and*

$$\sup_t \|u(\cdot,t)\|_{s-\frac{m}{2}} \leq C\|u\|_{\frac{s}{m},s}. \tag{46.17}$$

Theorem 46.6. *Suppose $\frac{1}{2} < \frac{s}{m} < \frac{3}{2}$ and τ is large. Then for any $u_0(x) \in H_{s-\frac{m}{2}}(\mathbb{R}^n)$ and for any $g \in \Pi_{\frac{s-m}{m},s-m}(\mathbb{R}^{n+1}_+)$ there exists a unique solution $v \in \Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1}_+)$ of (46.3) such that (46.4) holds.*

Proof: Analogously to the proof of Example 13.3, for any $u_0(x) \in H_{s-\frac{m}{2}}(\mathbb{R}^n)$ there exists $v_0(x,t) \in \Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ such that $v_0(x,0) = u_0(x)$ and

$$\|v_0\|_{\frac{s}{m},s} \leq C\|u_0\|_{s-\frac{m}{2}}. \tag{46.18}$$

If $w(x,t) = v(x,t) - v_0(x,t)$ for $t \geq 0$, then

$$w(x,0) = 0 \tag{46.19}$$

and $w(x,t)$ satisfies

$$\left(\frac{\partial}{\partial t} + \tau + A\right) w = g_1 \quad \text{for} \quad t > 0, \tag{46.20}$$

where $g_1 = g - p(\frac{\partial}{\partial t} + \tau + A)v_0 \in \Pi_{\frac{s-m}{m},s-m}(\mathbb{R}^{n+1}_+)$. Here p is the restriction operator to the half-space $\mathbb{R}^{n+1}_+$.

Let $g_+ = g_1$ for $t > 0$, $g_+ = 0$ for $t < 0$. Since $-\frac{1}{2} < \frac{s-m}{m} < \frac{1}{2}$, one can prove (see, for example, Eskin [**E1**], §5) that $g_+ \in \overset{\circ}{\Pi}_{\frac{s-m}{m},s-m}(\mathbb{R}^{n+1})$. Applying Theorem 46.3, we conclude that there exists $w_+ \in \overset{\circ}{\Pi}_{\frac{s}{m},s}(\mathbb{R}^{n+1})$ that satisfies $(\frac{\partial}{\partial t} + \tau + A)w_+ = g_+$ and

$$\|w_+\|_{\frac{s}{m},s} \le C\|g_+\|_{\frac{s-m}{m},s-m}. \tag{46.21}$$

Note that $w_+(x,0) = 0$. Choosing $v(x,t) = w_+(x,t) + v_0(x,t)$ for $t > 0$, we prove Theorem 46.6. □

It is convenient to consider the spaces $\Pi_{\frac{s}{m},s,s'}(\mathbb{R}^{n+1})$ with the norm

$$\|u\|_{\frac{s}{m},s,s'} = \|\Lambda^{s'}(D)u\|_{\frac{s}{m},s}, \tag{46.22}$$

where $\Lambda^{s'}(D)$ is a ψdo with symbol $(1+|\xi|^2)^{\frac{s'}{2}}$, $s' \in \mathbb{R}$. Repeating the proof of Theorem 46.6, we get:

Theorem 46.7. *Let $\tau \ge \tau_0$, τ_0 be large. For any $g \in \Pi_{\frac{s-m}{m},s-m,s'}(\mathbb{R}^{n+1}_+)$ and any $u_0 \in H_{s-\frac{m}{2}+s'}(\mathbb{R}^n)$, $\frac{m}{2} < s < \frac{3m}{2}$, there exists a unique solution $v \in \Pi_{\frac{s}{m},s,s'}(\mathbb{R}^{n+1}_+)$ of the Cauchy problem* (46.3), (46.4) *and*

$$\|v\|^+_{\frac{s}{m},s,s'} \le C\|g\|^+_{\frac{s-m}{m},s-m,s'} + C\|u_0\|_{s-\frac{m}{2}+s'}. \tag{46.23}$$

It follows from (46.23) and (46.17) that

$$\sup_t \|v(\cdot,t)\|_{s'+s-\frac{m}{2}} \le C\|g\|^+_{\frac{s-m}{m},s-m,s'} + C\|u_0\|_{s-\frac{m}{2}+s'}. \tag{46.24}$$

Remark 46.1. As in Theorem 46.4, we see that for τ large there exists a unique solution $u(x,t)$ of the Cauchy problem (46.1), (46.2) such that $e^{-\tau t}u \in \Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1}_+)$ provided $\frac{m}{2} < s < \frac{3m}{2}$, $u_0(x) \in H_{s-\frac{m}{2}}(\mathbb{R}^n)$, and $e^{-\tau t}f \in \Pi_{\frac{s-m}{m},s-m}(\mathbb{R}^{n+1}_+)$.

The following estimate holds:

$$\|e^{-\tau t}u\|^+_{\frac{s}{m},s} \le C\|e^{-\tau t}f\|^+_{\frac{s}{m}-1,s-m} + C\|u_0\|_{s-\frac{m}{2}}, \tag{46.25}$$

where $\|\ \|^+_{\frac{s}{m},s}$ is the norm in $\Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1}_+)$.

Using the norms (46.22), we also have (cf. (46.23))

$$\|e^{-\tau t}u\|^+_{\frac{s}{m},s,s'} \le C\|e^{-\tau t}f\|^+_{\frac{s-m}{m},s-m,s'} + C\|u_0\|_{s-\frac{m}{2}+s'}, \tag{46.26}$$

where $\frac{m}{2} < s < m$, and s' is arbitrary.

Remark 46.2. Consider the Cauchy problem (46.1), (46.2) on a finite time interval $R_T = \mathbb{R}^n \times (0,T)$. Suppose $f \in \Pi_{\frac{s}{m}-1,s-m}(R_T)$. Let f_1 be an extension of $f(x,t)$ such that $e^{-\tau t}f_1 \in \Pi_{\frac{s-m}{m},s-m}(\mathbb{R}^{n+1}_+)$ and $\|e^{-\tau t}f_1\|_{\frac{s}{m}-1,s-m} \le C\|f\|_{\frac{s}{m}-1,s-m,R_T}$.

By Remark 46.1, there exists a unique solution u_1 of the Cauchy problem (46.1), (46.2) in $\mathbb{R}^{n+1}_+$ such that $e^{-\tau t}u_1 \in \Pi_{\frac{s}{m},s}(\mathbb{R}^{n+1}_+)$ and the estimate (46.25) holds.

Let f_2 be another extension of $f(x,t)$ from $R_T = \mathbb{R}^n \times (0,T)$ to $\mathbb{R}^{n+1}_+$ such that $\|e^{-\tau t}f_2\|_{\frac{s}{m}-1,s-m} \le C\|f\|_{\frac{s}{m}-1,s-m,R_T}$ and u_2 is the corresponding unique solution of the Cauchy problem (46.1), (46.2). Denote $u_+ = u_1 - u_2$ for $t > 0$, $u_+ = 0$ for $t < 0$, and analogously $f_+ = f_1 - f_2$ for $t > 0$ and $f_+ = 0$ for $t < 0$. Then

$$\left(\frac{\partial}{\partial t} + A\right) u_+ = f_+ \quad \text{in} \quad \mathbb{R}^{n+1}. \tag{46.27}$$

Note that $f_+ = 0$ for $t < T$. Then analogously to the proof of Lemma 46.2 and Theorem 46.3 one can show that $u_+ = 0$ for $t < T$, i.e., the restriction of $u_1(x,t)$ to R_T does not depend on the choice of the extension f_1.

Note that $e^{-\tau t}$ and $e^{\tau t}$ are C^∞ bounded functions on $[0,T]$. Thus $u_1(x,t)$ solves the Cauchy problem (46.1), (46.2) on $\mathbb{R}^n \times (0,T)$ and

$$\begin{aligned}\|u_1\|_{\frac{s}{m},s,R_T} \le C\|e^{-\tau t}u_1\|^+_{\frac{s}{m},s} &\le C\|u_0\|_{s-\frac{m}{2}} + C\|e^{-\tau t}f_1\|^+_{\frac{s}{m}-1,s-m} \\ &\le C\|u_0\|_{s-\frac{m}{2}} + C_1\|f\|_{\frac{s}{m}-1,s-m,R_T}.\end{aligned}$$

47. The heat kernel

47.1. Solving the Cauchy problem by Fourier-Laplace transform.

Consider the Cauchy problem (46.1), (46.2) provided the coefficients of A are independent of t and $\operatorname{Re} A_0(x,\xi) \ge C|\xi|^m$. Let $f = 0$ and let $u(x,t)$ be the solution of

$$\frac{\partial u}{\partial t} + A(x,D)u = 0, \quad t > 0,\ x \in \mathbb{R}^n, \tag{47.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \tag{47.2}$$

Let s_0 be arbitrary. For any $\frac{m}{2} < s < \frac{3m}{2}$, $s' = s_0 - s + \frac{m}{2}$, and for any $u_0(x) \in H_{s_0}(\mathbb{R}^n)$, there exists a unique solution $u(x,t)$ of (47.1), (47.2) such that $e^{-\tau t}u(x,t) \in H_{\frac{s}{m},s,s_0-s+\frac{m}{2}}(\mathbb{R}^{n+1}_+)$ and (cf. (46.23), (46.24))

$$\sup_{t\ge 0}\|e^{-\tau t}u(\cdot,t)\|_{s_0} \le C\|e^{-\tau t}u\|^+_{\frac{s}{m},s,s_0-s+\frac{m}{2}} \le C\|u_0\|_{s_0}. \tag{47.3}$$

Here $\tau \ge \tau_0 > 0$ is large.

Let $u_+ = u(x,t)$ for $t > 0$, $u_+ = 0$ for $t < 0$. Since $e^{-\tau t}u(x,t) \in \Pi_{\frac{s}{m},s,s_0-s+\frac{m}{2}}(\mathbb{R}^{n+1}_+)$, we have that $e^{-\tau t}u(x,t) \in L_2(\mathbb{R}^1_+, H_{\frac{m}{2}+s_0}(\mathbb{R}^n))$, i.e., $e^{-\tau t}u(x,t)$ is an L_2-function of t with values in $H_{\frac{m}{2}+s_0}(\mathbb{R}^n)$. Suppose

$\tilde{u}_+(x, \sigma - i\tau)$, $\tau \geq \tau_0$, is the Fourier-Laplace transform of $u_+(x,t)$ in t only. The Parseval equality gives

$$\int_0^\infty e^{-2t\tau} \|u(\cdot,t)\|^2_{\frac{m}{2}+s_0} dt = \frac{1}{2\pi} \int_{-\infty}^\infty \|\tilde{u}_+(\cdot, \sigma - i\tau)\|^2_{\frac{m}{2}+s_0} d\sigma \tag{47.4}$$

for each $\tau \geq \tau_0$. We have

$$\frac{\partial u_+}{\partial t} + A(x,D)u_+ = \delta(t)u_0(x). \tag{47.5}$$

Performing the Fourier transform in (47.5), we get

$$i(\sigma - i\tau)\hat{u}_+(x, \sigma - i\tau) + A(x,D)\hat{u}_+(x, \sigma - i\tau) = u_0(x).$$

It follows from Lemma 41.3 that for $\tau \geq \tau_0$,

$$\hat{u}(x, \sigma - i\tau) = (A + i(\sigma - i\tau)I)^{-1}u_0(x).$$

Therefore

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty (A + i(\sigma - i\tau)I)^{-1} u_0(x) e^{it(\sigma - i\tau)} d\sigma, \quad \tau \geq \tau_0. \tag{47.6}$$

Note that the integral (47.6) does not depend on τ by the Cauchy integral theorem, and for $\tau = \tau_0$ it is understood as the Fourier transform of a function in $L_2(\mathbb{R}^1, H_{\frac{m}{2}+s_0}(\mathbb{R}^n))$ (cf. Parseval's formula (47.4)).

Applying $(A + i(\sigma - i\tau)I)^{-1}$ to (41.9) from the left, we get

$$(A + \lambda I)^{-1} = R^{(N)} + T^{(1)}_{-m-N-1}, \tag{47.7}$$

where $\lambda = i(\sigma - i\tau)$, $R^{(N)} = \sum_{k=0}^N R_{-k}$, $T^{(1)}_{-(m+N+1)} = -(A + \lambda I)^{-1} T_{-N-1}$. Note that $\operatorname{ord} T^{(1)}_{-(m+N+1)} \leq -(m + N + 1)$ since $\operatorname{ord}(A + \lambda I)^{-1} = -m$. Substituting (47.7) into (47.6), we get

$$u(x,t) = u_1(x,t) + u_2(x,t),$$

where

$$u_1(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty R^{(N)}(x, D, i(\sigma - i\tau)) u_0(x) e^{it(\sigma - i\tau)} d\sigma, \tag{47.8}$$

$$u_2(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty (T^{(1)}_{-(m+N+1)} u_0)(x) e^{it(\sigma - i\tau)} d\sigma, \tag{47.9}$$

and $R^{(N)}(x, \xi, \lambda)$ has the form (41.11).

The integral with respect to σ in (47.8) can be computed easily by residue theory using the Jordan lemma (cf. Proposition 16.1):

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{it(\sigma - i\tau)} d\sigma}{(A_0(x,\xi) + i(\sigma - i\tau))^{k+1}} = \frac{t^k}{k!} e^{-tA_0(x,\xi)}. \tag{47.10}$$

Therefore

$$u_1(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\sum_{k=0}^{N} p_k(x,\xi)\frac{t^k}{k!}e^{-tA_0(x,\xi)}\tilde{u}_0(\xi)e^{ix\cdot\xi}d\xi, \tag{47.11}$$

where $p_0 = 1,\ \deg p_k \le mk - \frac{k}{2},\ k \ge 1$ (cf. (41.11)).

It follows from Lemma 44.6 that $T^{(1)}_{-(m+N+1)}$ is an integral operator with a continuous kernel, $T^{(1)}_{-(m+N+1)}(x,y,\sigma - i\tau)$, when $-m-N-1 < -n$.

47.2. Asymptotics of the heat kernel as $t \to +0$.

Denote by e^{-tA} the solution operator of the Cauchy problem (47.1), (47.2), i.e.,

$$u(x,t) = e^{-tA}u_0(x). \tag{47.12}$$

Let $G(x,y,t)$ be the kernel of e^{-tA}. Note that $G(x,y,t)$ is the solution of the Cauchy problem

$$\begin{gathered}\left(\frac{\partial}{\partial t} + A\right)G(x,y,t) = 0, \quad t > 0,\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^n,\\ G(x,y,0) = \delta(x-y),\end{gathered} \tag{47.13}$$

i.e., $u_0(x) = \delta(x-y) \in H_{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$.

$G(x,y,t)$ is called the heat kernel.

We have from (47.9)–(47.11) that

$$G(x,y,t) = \sum_{k=0}^{N}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} p_k(x,\xi)\frac{t^k}{k!}e^{-tA_0(x,\xi)+i(x-y)\cdot\xi}d\xi + G_N(x,y,t), \tag{47.14}$$

where

$$G_N(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T^{(1)}_{-m-N-1}(x,y,\sigma - i\tau)e^{it(\sigma - i\tau)}d\sigma. \tag{47.15}$$

Note that $T^{(1)}(x,y,\sigma - i\tau)$ is analytic in $\sigma - i\tau$ for $\tau > \tau_0$ since all the other terms in (47.7) are analytic in $z = \sigma - i\tau$. It follows from estimates of the form (44.14), (46.16) that

$$|T^{(1)}_{-m-N-1}(x,y,\sigma)| \le C|\sigma - i\tau|^{-N_1-2}, \tag{47.16}$$

where $-m-N-1 < -n-N_1-2$. Therefore, (47.16) implies that $G_N(x,y,t)$ has N_1 continuous derivatives with respect to t. Also $G_N(x,y,t) = 0$ for $t < 0$ because of the analyticity of $T^{(1)}_{-m-N-1}$. Therefore

$$|G_N(x,y,t)| \le C_N t^{N_1}, \quad t > 0. \tag{47.17}$$

Take $x = y$ in (47.14) and make a change of variables

$$\eta = \frac{\xi}{t^{\frac{1}{m}}}. \tag{47.18}$$

We get

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-tA_0(x,\xi)}d\xi = \frac{c_0(x)}{t^{\frac{n}{m}}}, \tag{47.19}$$

where $c_0(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-A_0(x,\eta)}d\eta$. Making the change of variables (47.18) for $1 \le k \le N$ in (47.14), taking $x = y$, and collecting the terms having the same power of t, we get

$$G(x,x,t) = \frac{1}{t^{\frac{n}{m}}}\left(c_0(x) + \sum_{k=1}^{N_2-1} c_k(x)t^{\frac{k}{m}}\right) + O(t^{\frac{N_2-n}{m}}), \tag{47.20}$$

where the coefficients $c_k(x)$ have an explicit form. Note that N_2 in (47.20) is arbitrary since N in (47.14), (47.15) is arbitrary and we take $N_1 > \frac{N_2-n}{m}$.

If $A_0(x,\eta)$ is even in η, then the integrals of the form

$$\int_{\mathbb{R}^n} \eta_1^{r_1}\eta_2^{r_2}\cdots\eta_n^{r_n}e^{-A_0(x,\eta)}d\eta$$

are equal to zero if the $r_1 + r_2 + \cdots + r_n$ are odd. In this case (see (47.20)) $c_k(x) = 0$ if the k are odd.

Further simplifications occur when $A_0(x,D)$ is a second order elliptic operator, $A_0(x,\xi) = \sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k \ge C|\xi|^2$, i.e., $\operatorname{Im} A_0 = 0$. We have

$$c_0(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\sum_{j,k=1}^n g^{jk}(x)\eta_j\eta_k}d\eta = \frac{1}{(2\sqrt{\pi})^n}\sqrt{g(x)}, \tag{47.21}$$

where $g(x) = (\det[g^{jk}(x)]_{j,k=1}^n)^{-1}$.

To compute (47.21), one can make an orthogonal transformation $\eta = O\zeta$ such that $\sum_{j,k=1}^n g^{jk}(x)\eta_j\eta_k = \sum_{j=1}^n \lambda_j\zeta_j$ and then make the changes of coordinates $\zeta_j = \frac{1}{\sqrt{\lambda_j}}\zeta_j'$, $1 \le j \le n$.

48. The Cauchy problem for strictly hyperbolic equations

Let $H(x,t,\xi,\sigma)$ be a polynomial of degree m,

$$H(x,t,\xi,\sigma) = H_0(x,t,\xi,\sigma) + H_1(x,t,\xi,\sigma),$$

where

$$H_0(x,t,\xi,\sigma) = \sum_{|k|+j=m} a_{jk}(x,t)\xi^k\sigma^j, \quad a_{m0} \ne 0,$$

$$H_1(x,t,\xi,\sigma) = \sum_{|k|+j\le m-1} a_{jk}(x,t)\xi^k\sigma^j.$$

We assume that $a_{jk}(x,t) \in C^\infty(\mathbb{R}^{n+1})$ and $a_{jk}(x,t) = a_{jk}^\infty$ for $|x| > R$, and $a_{jk}(x,t) = a_{jk}(x,\infty)$ for $|t| > R$. The polynomial $H(x,t,\xi,\sigma)$ is called strictly hyperbolic if $H_0(x,t,\xi,\sigma) = 0$ has m real roots $\sigma_k(x,t,\xi)$ such that $\sigma_j(x,t,\xi) \neq \sigma_k(x,t,\xi)$ when $j \neq k$ for all $(x,t) \in \mathbb{R}^{n+1}$ and $\xi \neq 0$. Note that $\sigma_k(x,t,\xi)$ is a homogeneous function of ξ of degree 1 and $\sigma_k(x,t,\xi) \in C^\infty(\mathbb{R}^{n+1} \times (\mathbb{R}^n \setminus \{0\}))$.

Consider the Cauchy problem

$$H(x,t,D_x,D_t)u(x,t) = f(x,t), \quad (x,t) \in \mathbb{R}_+^{n+1} = \{t > 0, x \in \mathbb{R}^n\}, \tag{48.1}$$

where $D_x = -i\frac{\partial}{\partial x}$, $D_t = -i\frac{\partial}{\partial t}$, with the initial conditions

$$\left.\frac{\partial^k u(x,t)}{\partial t^k}\right|_{t=0} = g_k(x), \quad 0 \le k \le m-1. \tag{48.2}$$

Suppose first that $g_k = 0$, $0 \le k \le m-1$. As in the parabolic case, let $v(x,t) = e^{-\tau t}u(x,t)$, $g(x,t) = e^{-\tau t}f(x,t)$. Then $v(x,t)$ satisfies the equation

$$H(x,t,D_x,D_t - i\tau)v(x,t) = g(x,t), \quad (x,t) \in \mathbb{R}_+^{n+1}, \tag{48.3}$$

and

$$\left.\frac{\partial^k v(x,t)}{\partial t^k}\right|_{t=0} = 0, \quad 0 \le k \le m-1. \tag{48.4}$$

Denote by $H_{p,s}(\mathbb{R}^{n+1})$ the Sobolev space with the norm

$$\|v\|_{p,s}^2 = \int_{\mathbb{R}^{n+1}} (1 + |\xi|^2 + \sigma^2)^p (1 + |\xi|^2)^s |\tilde{u}(\xi,\sigma)|^2 d\xi d\sigma, \tag{48.5}$$

where $p \ge 0$ is an integer, $s \in \mathbb{R}$. The spaces $\overset{\circ}{H}_{p,s}(\mathbb{R}^{n+1})$ and $H_{p,s}(\mathbb{R}_+^{n+1})$ are defined as in the parabolic case.

Let $f_+ = f$ for $t > 0$, and $f_+ = 0$ for $t < 0$. Analogously, let $u_+ = u$, $v_+ = v(x,t)$, $g_+ = g(x,t)$ for $t > 0$, and $u_+ = v_+ = g_+ = 0$ for $t < 0$.

Theorem 48.1. *Let $\tau \ge \tau_0$, where τ_0 is large, and let s be arbitrary. Then for any f_+ such that $e^{-\tau t}f_+ \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$ there exists a unique solution u_+ of $H(x,t,D_x,D_t)u_+ = f_+$ in $\mathbb{R}^{n+1}$ such that $e^{-\tau t}u_+ \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$.*

Proof: Assume, for simplicity, that $a_{m0} = 1$. Otherwise, divide (48.1) by a_{m0}. Then

$$H_0(x,t,\xi,\sigma - i\tau) = \prod_{j=1}^m (\sigma - i\tau - \sigma_j(x,t,\xi)) \neq 0$$

if $\tau \neq 0$. We have

$$\frac{\partial H_0(x,t,\xi,\sigma - i\tau)}{\partial \sigma} = \sum_{k=1}^m q_k(x,t,\xi,\sigma - i\tau),$$

where

$$q_k(x,t,\xi,\sigma-i\tau)=\prod_{j\neq k}(\sigma-i\tau-\sigma_j(x,t,\xi)),\quad 1\leq k\leq m. \tag{48.6}$$

Therefore

$$\begin{aligned} H_0\overline{\frac{\partial H_0}{\partial\sigma}} &= \prod_{j=1}^{m}(\sigma-i\tau-\sigma_j)\sum_{k=1}^{m}\prod_{j\neq k}(\sigma+i\tau-\sigma_j) \\ &= \sum_{k=1}^{m}(\sigma-i\tau-\sigma_k)\prod_{j\neq k}((\sigma-\sigma_j)^2+\tau^2)=-i\tau Q_0+Q_1, \end{aligned} \tag{48.7}$$

where

$$Q_0=\sum_{k=1}^{m}q_k\overline{q_k}=\sum_{k=1}^{m}\prod_{j\neq k}((\sigma-\sigma_j)^2+\tau^2),\quad Q_1=\sum_{k=1}^{m}(\sigma-\sigma_k)q_k\overline{q_k}. \tag{48.8}$$

Since $Q_0(x,t,\xi,\sigma,\tau)>0$ for $(\xi,\sigma,\tau)\neq(0,0,0)$, we get by the homogeneity in (ξ,σ,τ),

$$Q_0(x,t,\xi,\sigma,\tau)\geq C_0(|\xi|^2+|\sigma|^2+\tau^2)^{m-1}. \tag{48.9}$$

Note that Q_0 and Q_1 are symmetric polynomials in the roots $\sigma_1,\dots,\sigma_m$. It is a well-known theorem in algebra that any symmetric polynomial in $\sigma_1,\dots,\sigma_n$ can be represented as a polynomial in the coefficients of H_0. Therefore Q_0 and Q_1 are polynomials in (ξ,σ,τ).

48.1. The main estimate.

Lemma 48.2. *For any $v_+\in\overset{\circ}{H}_{m,s}(\mathbb{R}^{n+1})$ the following estimate holds:*

$$\|\Gamma^{m-1}(D_x,D_t,\tau)v_+\|_{0,s}\leq\frac{C}{\tau}\|H(x,t,D_x,D_t-i\tau)v_+\|_{0,s}, \tag{48.10}$$

where $\Gamma^{m-1}(D_x,D_t,\tau)$ is a pseudodifferential operator with symbol

$$\Gamma^{m-1}(\xi,\sigma,\tau)=(|\xi|^2+\sigma^2+\tau^2)^{\frac{m-1}{2}}$$

and $\tau\geq\tau_0$, τ_0 is large.

Proof: Let $\Lambda^s(\xi)=(1+|\xi|^2)^{\frac{s}{2}}$. Since $a_{m0}=1$, we have, using Theorem 40.2:

$$\Lambda^s(D)H(x,t,D_x,D_t-i\tau)=H(x,t,D_x,D_t-i\tau)\Lambda^s(D)+Q_2\Lambda^s(D), \tag{48.11}$$

where $\|Q_2\Lambda^s v_+\|_0\leq C\|\Gamma^{m-1}(D_x,D_t,\tau)v_+\|_{0,s}$.

Consider the L_2 scalar product in $\mathbb{R}^{n+1}$:

$$(H\Lambda^s v_+,H_{0\sigma}\Lambda^s v_+)=(H_{0\sigma}^*H\Lambda^s v_+,\Lambda^s v_+),$$

where $H_{0\sigma}^*$ is formally adjoint to $H_{0\sigma}$. Using the theorems analogous to Theorems 40.2 and 43.2 for the case of differential operators, we get:

$$(48.12)\quad (H\Lambda^s v_+, H_{0\sigma}\Lambda^s v_+) = -i\tau(Q_0\Lambda^s v_+, \Lambda^s v_+) + (Q_1\Lambda^s v_+, \Lambda^s v_+) + (Q_3\Lambda^s v_+, \Lambda^s v_+),$$

where Q_0, Q_1 are differential operators with symbols (48.8) and

$$(48.13)\qquad |(Q_3\Lambda^s v_+, \Lambda^s v_+)| \le C\|\Gamma^{m-1}\Lambda^s v_+\|_0^2.$$

Note that the symbol $Q_1(x,t,\xi,\tau)$ is real-valued. Therefore, $\operatorname{ord}(Q_1 - Q_1^*) \le 2m-2$, and we see that $\operatorname{Im}(Q_1\Lambda^s v_+, \Lambda^s v_+)$ satisfies the estimate of the form (48.13). Thus, taking the imaginary part of (48.12), we get

$$(48.14)\quad -\operatorname{Im}(H\Lambda^s v_+, H_{0\sigma}\Lambda^s v_+) \ge \tau\sum_{k=1}^{m}(Q_0\Lambda^s v_+, \Lambda^s v_+) - C\|\Gamma^{m-1}\Lambda^s v_+\|_0^2.$$

We will show that

$$(48.15)\qquad (Q_0\Lambda^s v_+, \Lambda^s v_+) \ge C\|\Gamma^{m-1}\Lambda^s v_+\|_0^2 - C_1\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2.$$

The inequality (48.15) is called the Gårding inequality. We have

$$(48.16)\quad Q_0(x,t,\xi,\sigma,\tau) = Q_0 - \frac{C_0}{2}(|\xi|^2 + |\sigma|^2 + \tau^2)^{m-1} + \frac{C_0}{2}(|\xi|^2 + |\sigma|^2 + \tau^2)^{m-1},$$

where Q_0 and C_0 are the same as in (48.9).

Let

$$(48.17)\qquad B(x,t,\xi,\sigma,\tau) = \sqrt{Q_0 - \frac{C_0}{2}\Gamma^{2(m-1)}(\xi,\sigma,\tau)}.$$

Note that $B > 0$ and $B(x,t,\xi,\sigma,\tau) \in S^{m-1}$. Denote by B^* the adjoint operator to B. By Theorems 40.2 and 43.2 we have

$$(48.18)\qquad Q_0 - \frac{C_0}{2}\Gamma^{2(m-1)} = B^*B + Q_4,$$

where $\operatorname{ord} Q_4 \le 2m-3$, and

$$(48.19)\qquad |(Q_4\Lambda^s v_+, \Lambda^s v_+)| \le C_1\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2.$$

Note that $(B^*B\Lambda^s v_+, \Lambda^s v_+) = \|B\Lambda^s v_+\|^2 \ge 0$. Therefore, (48.18) and (48.19) imply that

$$(48.20)\quad \tau(Q_0\Lambda^s v_+, \Lambda^s v_+) \ge \frac{C_0}{2}\tau\|\Gamma^{m-1}\Lambda^s v_+\|_0^2 - C_1\tau\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2 \ge \left(\frac{C_0}{2}\tau - C_1\right)\|\Gamma^{m-1}\Lambda^s v_+\|_0^2,$$

since $\tau\|\Gamma^{m-\frac{3}{2}}\Lambda^s v_+\|_0^2 \le \|\Gamma^{m-1}\Lambda^s v_+\|_0^2$. Combining (48.14) and (48.20), we obtain

$$\operatorname{Im}(H\Lambda^s v_+, H_{0\sigma}\Lambda^s v_+) \ge C\tau\|\Gamma^{m-1}\Lambda^s v_+\|_0^2.$$

It follows from (48.11) that

$$(\Lambda^s H v_+, H_{0\sigma}\Lambda^s v_+) = (H\Lambda^s v_+, H_{0\sigma}\Lambda^s v_+) + (Q_2\Lambda^s v_+, H_{0\sigma}\Lambda^s v_+).$$

Therefore

$$\begin{aligned} C\tau\|\Gamma^{m-1}\Lambda^s v_+\|_0^2 &\le |(\Lambda^s H v_+, H_{0\sigma}\Lambda^s v_+)| + |(Q_2\Lambda^s v_+, H_{0\sigma}\Lambda^s v_+)| \\ &\le C_1\|\Lambda^s H v_+\|_0 \, \|\Gamma^{m-1}\Lambda^s v_+\|_0 + C_2\|\Gamma^{m-1}\Lambda^s v_+\|_0^2. \end{aligned}$$

Hence (48.10) holds. □

48.2. Uniqueness and parabolic regularization.

Now we prove the uniqueness result.

Lemma 48.3. *Suppose $Hu_+ = 0$ in $\mathbb{R}^{n+1}$, where*

$$e^{-\tau t}u_+ = v_+ \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1}).$$

Then $u_+ = 0$.

Proof: Since $v_+ \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$ and

$$Hv_+ = D_t^m v_+ + \sum_{k=0}^{m-1} b_k(x,t,D_x,\tau)D_t^k v_+ = 0,$$

where $\operatorname{ord} b_k \le m-k$, we see that

$$D_t^m v_+ = -\sum_{k=0}^{m-1} b_k(x,t,D_x,\tau)D_t^k v_+ \in \overset{\circ}{H}_{0,s-1}(\mathbb{R}^{n+1}). \tag{48.21}$$

Therefore $v_+ \in \overset{\circ}{H}_{m,s-1}(\mathbb{R}^{n+1})$. Applying the inequality (48.10) with $Hv_+ = 0$ and s replaced by $s-1$, we obtain that $v_+ = 0$. □

The next step is the proof of the existence result.

Lemma 48.4. *For any $g_+ \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$ there exists a unique solution of (48.3) in $\mathbb{R}^{n+1}$, where $v_+ \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$.*

Proof: We use the method of "parabolic" regularization.

Let $H_\varepsilon(x,t,\xi,\sigma - i\tau) = H(x,t,\xi,\sigma - i\tau - i\varepsilon\Lambda(\xi))$, $\varepsilon > 0$. The operator $H(x,t,D_x,D_t - i\tau - \varepsilon\Lambda(D_x))$ is a "parabolic" ψdo in the sense that $H_\varepsilon = H(x,t,\xi,z - i\varepsilon\Lambda(\xi))$ and the H_ε^{-1} are analytic when $\operatorname{Im} z < 0$ and

$$C_{1\varepsilon}(|\xi| + |z|)^m \ge |H(x,t,\xi,z - i\varepsilon\Lambda(\xi))| \ge C_\varepsilon(|\xi| + |z|)^m \tag{48.22}$$

for all $\operatorname{Im} z > 0$ large (cf. the class $P_{\alpha,m}^+$ in §46).

Repeating the proof of Theorem 46.3, we conclude that for any $g_+ \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$, there exists $v_\varepsilon \in \overset{\circ}{H}_{m,s}(\mathbb{R}^{n+1})$ such that

$$H(x,t,D_x,D_t - i\tau - i\varepsilon\Lambda(D_x))v_\varepsilon = g_+, \quad (x,t) \in \mathbb{R}^{n+1}. \tag{48.23}$$

Note that the proof of the estimate (48.10) can be repeated for the equation (48.23), and we get

$$\|\Gamma^{m-1}\Lambda^s v_\varepsilon\|_0 \le C\|\Lambda^s g_+\|_0, \tag{48.24}$$

where the constant C is independent of ε. Therefore, $\{v_\varepsilon\}$, $\varepsilon > 0$, is bounded in $\overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$. It follows from the weak compactness of a bounded set in $\overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$ that there exists a sequence v_{ε_k} that converges weakly to some $v_+ \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$ (see Rudin [**R**]) . For any $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$, we have

$$(g_+,\varphi) = (H_{\varepsilon_k}v_{\varepsilon_k},\varphi) = (v_{\varepsilon_k}, H_{\varepsilon_k}^*\varphi). \tag{48.25}$$

Passing to the limit as $\varepsilon_k \to 0$, we get $(g_+,\varphi) = (v_+, H^*\varphi)$, i.e., $Hv_+ = g_+$ in $\mathbb{R}^{n+1}$. □

Lemmas 48.3 and 48.4 imply that for $\tau \ge \tau_0$, where $\tau_0 > 0$ is large, and for any f such that $e^{-t\tau}f \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$, there exists a unique solution $v(x,t,\tau) \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$ of (48.3) with $g_+ = e^{-t\tau}f_+$.

Analogously to the proof of Theorem 46.4 one can show that there exists $u_+(x,t)$ solving (48.1) and such that

$$v_+(x,t,\tau) = e^{-t\tau}u_+(x,t) \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1}).$$

Remark 48.1. It follows from Theorem 48.1 that for any $g_+ \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$ there exists a unique $v_+ \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$. Then it follows from (48.21) that $D_t^m v_+ \in \overset{\circ}{H}_{0,s-1}(\mathbb{R}^{n+1})$. Therefore, $v_+ \in \overset{\circ}{H}_{m,s-1}(\mathbb{R}^{n+1})$ and consequently the $\frac{\partial^k v_+(x,t)}{\partial t^k}$ are continuous functions of t with values in $H_{m+s-k-\frac{3}{2}}(\mathbb{R}^n)$, $0 \le k \le m-1$ (cf. Theorem 13.6). Since $v_+ = 0$ for $t < 0$, we see that $\frac{\partial^k v_+(x,0)}{\partial t^k} = 0$, $0 \le k \le m-1$, i.e., $v_+(x,t)$ satisfies the zero initial conditions (48.4). □

One may consider also the Cauchy problem (48.1), (48.2) with nonzero initial conditions. If $g_k(x) \in H_{m-k+s-\frac{1}{2}}(\mathbb{R}^n)$, $0 \le k \le m-1$, there exists $v_0 \in H_{m,s}(\mathbb{R}_+^{n+1})$ such that $\frac{\partial^k v_0(x,+0)}{\partial t^k} = g_k(x)$, $0 \le k \le m-1$, and $v_0 = 0$ for large t (cf. Example 13.3). We look for the solution of (48.1) in the form $u = v_0 + w$, where $w(x,t)$ satisfies

$$H(x,t,D_x,D_t)w(x,t) = f_0, \ t > 0, \quad f_0 = f(x,t) - Hv_0. \tag{48.26}$$

Let f_+ be the extension of f_0 by zero for $t < 0$. Note that $e^{-t\tau} f_+ \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$. By Theorem 48.1, there exists w_+ such that $Hw_+ = f_+$ in $\mathbb{R}^{n+1}$ and $e^{-t\tau} w_+ \in \overset{\circ}{H}_{m-1,s}(\mathbb{R}^{n+1})$. Let $w = w_+$ for $t > 0$. It was shown in Remark 48.1 that $e^{-t\tau} w_+ \in H_{m,s-1}(\mathbb{R}^{n+1}_+)$ and $\frac{\partial^k w(x,0)}{\partial t^k} = 0$, $0 \le k \le m-1$. Therefore, $u = v_0 + w$ satisfies (48.1) for $t > 0$, $e^{-t\tau} u \in H_{m,s-1}(\mathbb{R}^{n+1}_+)$, and $\frac{\partial^k u(x,0)}{\partial t^k} = g_k(x)$, $0 \le k \le m-1$.

We have proven the following result:

Lemma 48.5. *For any $e^{-\tau t} f \in H_{0,s}(\mathbb{R}^{n+1}_+)$ and any $g_k(x) \in H_{m+s-k-\frac{1}{2}}(\mathbb{R}^n)$, $0 \le k \le m-1$, there exists a unique $e^{-\tau t} u(x,t) \in H_{m,s-1}(\mathbb{R}^{n+1}_+)$ such that* (48.1)*,* (48.2) *hold.* □

48.3. The Cauchy problem on a finite time interval.

A modification of the proof of Theorem 48.1 gives an existence and uniqueness theorem in $R_T = \mathbb{R}^n \times (0,T)$ with less restrictive requirements on the smoothness of the initial data than those in Lemma 48.5.

Denote by $[u,v]$ the L_2 scalar product in $\mathbb{R}^n$ for t fixed, and let $[u]_s = [\Lambda^s u]_0$ be the norm in $H_s(\mathbb{R}^n)$. Denote by $C_{m-1,s}$ the space of functions such that $\frac{\partial^k u(x,t)}{\partial t^k}$, $0 \le k \le m-1$, are continuous in $t \in [0,T]$ with values in $H_{m-k-1+s}(\mathbb{R}^n)$. The norm in $C_{m-1,s}$ is

$$\max_{0 \le t \le T} \sum_{k=0}^{m-1} [D_t^k u]_{m-1-k+s}.$$

Also denote by $L_1[(0,T), H_s(\mathbb{R}^n)]$ the space with the norm $\int_0^T [f(x,t)]_s dt$.

Theorem 48.6. *Let s be arbitrary. For any $g_k(x) \in H_{s+m-1-k}(\mathbb{R}^n)$, $0 \le k \le m-1$, and any $f(x,t) \in L^1[(0,T), H_s(\mathbb{R}^n)]$, there exists a unique solution $u(x,t) \in C_{m-1,s}$ of the Cauchy problem* (48.1)*,* (48.2) *in R_T such that*

$$\max_{0 \le t \le T} \sum_{k=0}^{m-1} [D_t^k u(x,t)]_{m-k-1+s} \le C \sum_{k=0}^{m-1} [g_k]_{m-1-k+s} + C \int_0^T [f]_s dt. \tag{48.27}$$

Proof: Take any $u(x,t) \in H_{m,s}(R_T)$. Note that $H_{m,s}(R_T) \subset C_{m-1,s}$. Let $w = e^{-t\tau} u(x,t)$, where τ is large, and let $F(x,t) = H(x,t,D_x,D_t - i\tau)w$, $\varphi_k(x) = \frac{\partial^k w(x,0)}{\partial t^k}$, $0 \le k \le m-1$. Note that $\sigma_k(x,t,\xi)$ have bounded first derivatives in ξ. Therefore, as in (48.12), we have, using Theorem 40.2 with $N = 0$:

$$(48.28)\quad [\Lambda^s H(x,t,D_x,D_t - i\tau)w, H_{0\sigma}\Lambda^s w] = \sum_{k=1}^{m}[(D_t - i\tau - \sigma_k(x,t,D_x))q_k\Lambda^s w, q_k\Lambda^s w] + [Q_2\Lambda^s w, \Lambda^s w],$$

where

$$(48.29)\quad |[Q_2\Lambda^s w, \Lambda^s w]| \le C\sum_{k=0}^{m-1}[\Lambda_\tau^{m-1-k}D_t^k w]_s^2 \stackrel{\text{def}}{=} C|[w]|^2_{m-1,s},$$

and $\Lambda_\tau = (|\xi|^2 + \tau^2 + 1)^{\frac{1}{2}}$. Integrating (48.28) from 0 to t, multiplying by -1 and taking the imaginary part, we get as in (48.14):

$$(48.30)\quad \sum_{k=1}^{m}[q_k(x,t,D_x,D_t - i\tau)\Lambda^s w]_0^2 - \sum_{k=1}^{m}[q_k(x,0,D_x,D_t - i\tau)\Lambda^s w]_0^2 + \tau\int_0^t \sum_{k=1}^{m}[q_k\Lambda^s w]_0^2 dt' \le C\int_0^t |[w]|^2_{m-1,s}dt' + C\int_0^t [F]_s|[w]|_{m-1,s}dt'.$$

We need a variant of the Gårding inequality (cf. (48.15)):

$$(48.31)\quad \sum_{k=1}^{m}[q_k(x,t,D_x,D_t - i\tau)\Lambda^s w]_0^2 \ge C|[w]|^2_{m-1,s} - \frac{C_1}{\tau}|[w]|^2_{m-1,s}.$$

Let

$$(48.32)\quad w_k(x,t) = \Lambda_\tau^{m-1-k}\Lambda^s D_t^k w(x,t), \quad 0 \le k \le m-1.$$

We have

$$q_k(x,t,D_x,D_t - i\tau)\Lambda^s w = \sum_{j=0}^{m-1} a_{kj}(x,t,D_x,\tau)w_j(x,t),$$

where the $a_{kj}(x,t,D_x,\tau)$ are ψdo's of order zero. Note that

$$(48.33)\quad \sum_{k=1}^{m}[q_k\Lambda^s w]_0^2 = \Big[\sum_{j,p=0}^{m-1} a_{jp}^{(1)}w_p, w_j\Big],$$

where $a_{jp}^{(1)} = \sum_{k=1}^m a_{kj}^* a_{kp}$, and the a_{kj}^* are the adjoint operators to a_{kj}.

Denote by $A(x,t,\xi,\tau)$ the matrix with elements

$$\sum_{k=1}^{m}\overline{a_{kj}(x,t,\xi,\tau)}a_{kp}(x,t,\xi,\tau).$$

Fix any (x_0, t_0). Then we have

$$(A(x_0, t_0, D_x, D_t)\vec{w}(x,t), \vec{w}(x,t)) = (Q_0(x_0, t_0, D_x, D_t, \tau)w(x,t), w(x,t)),$$

where $\vec{w} = (w_0, \dots, w_{m-1})$. Therefore (48.9) implies that $A(x,t,\xi,\sigma,\tau)$ is positive definite for any (x,t) and

$$A(x,t,\xi,\tau) \geq C_0 I.$$

Let $B(x,t,\xi,\tau) = (A(x,t,\xi,\tau) - \frac{C_0}{2}I)^{\frac{1}{2}}$ and let $B(x,t,D_x,\tau)$ be the matrix ψdo with symbol $B(x,t,\xi,\tau)$. Using Theorems 40.2, 43.2 with $N = 0$, we get (cf. (48.18))

$$\sum_{j,p=0}^{m-1} [a_{jp}^{(1)} w_p, w_j] = \frac{C_0}{2} \sum_{k=0}^{m-1} [w_k]_0^2 + [B^* B\vec{w}, \vec{w}] + [C_2 \vec{w}, \vec{w}], \tag{48.34}$$

where $\operatorname{ord} C_2 \leq -1$, i.e., $|[C_2\vec{w}, \vec{w}]| \leq C \sum_{k=0}^{m-1} [\Lambda_\tau^{-1} w_k]_0^2$. Therefore

$$|[C_2\vec{w}, \vec{w}]| \leq \frac{C}{\tau} \sum_{k=0}^{m-1} [w_k]_0^2. \tag{48.35}$$

Since $[B^* B\vec{w}, \vec{w}] = [B\vec{w}, B\vec{w}] \geq 0$ and since $\sum_{k=0}^{m-1} [w_k]_0^2 = |[w]|_{m-1,s}^2$ we get (48.31). □

Taking τ large and combining (48.30) and (48.31), we get

$$\begin{aligned} |[w(x,t)]|_{m-1,s}^2 \leq C \sum_{k=1}^{m} [q_k(x,0,D_x,D_t - i\tau)\Lambda^s w]_0^2 \\ + C \int_0^T [F]_s dt \max_{0 \leq t \leq T} |[w]|_{m-1,s}. \end{aligned} \tag{48.36}$$

Take a t_0 that satisfies

$$|[w(x,t_0)]|_{m-1,s}^2 = \max_{0 \leq t \leq T} |[w(x,t)]|_{m-1,s}.$$

Choosing such t_0 in (48.36) and using the inequality

$$\begin{aligned} &\sum_{k=1}^{m} [q_k(x,0,D_x,D_t - i\tau)\Lambda^s w]_s^2 \\ &\quad \leq C \sum_{k=0}^{m-1} [\Lambda_\tau^{m-k-1} \varphi_k]_s^2 \leq C_1 \max_{0 \leq t \leq T} |[w(x,t)]|_{m-1,s}^2, \end{aligned} \tag{48.37}$$

we get, dividing by $\max_{0 \leq t \leq T} |[w(x,t)]|_{m-1,s}$:

$$\max_{0 \leq t \leq T} |[w(x,t)]|_{m-1,s} \leq C \sum_{k=0}^{m-1} [\Lambda_\tau^{m-1-k} \varphi_k]_s + C \int_0^T [F]_s dt. \tag{48.38}$$

We have $w = e^{-t\tau}u(x,t), u(x,t) \in H_{m,s}(R_T)$ is arbitrary. Note that $e^{-t\tau}$ and $e^{t\tau}$ are bounded on $[0,T]$. Estimating $u = e^{t\tau}w$ using (48.38) we get

$$\max_{0\le t\le T}\sum_{k=0}^{m-1}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{m-1-k+s} \le C\sum_{k=0}^{m-1}\left[\frac{\partial^k u(x,0)}{\partial t^k}\right]_{m-1-k+s} + C\int_0^T [H(x,t,D_x,D_t)u]_0 dt. \tag{48.39}$$

□

The proof of the uniqueness for the solution of the Cauchy problem in R_T is the same as in Lemma 48.3: Let $u \in C_{m-1,s}$ be a solution of $H(x,t,D_x,D_t)u = 0$ in R_T, with zero initial conditions. Note that $u \in H_{m-1,s}(R_T)$ since $C_{m-1,s} \subset H_{m-1,s}$. It follows from $Hu = 0$ that $\frac{\partial^m u}{\partial t^m} \in H_{0,s-1}(R_T)$ (cf. (48.21). Therefore $w \in H_{m,s-1}(R_T)$. Applying the estimate (48.39) with s replaced by $s-1$, we get $u = 0$.

To prove the existence, choose any $F \in H_{0,s+1}(R_T)$. Note that $F \in L^1[(0,T), H_s(\mathbb{R}^n)]$ since $(\int_0^T [F]_s dt)^2 \le T\|F\|_{0,s}^2$. Choose also any $\varphi_k \in H_{m-k+\frac{1}{2}+s}(\mathbb{R}^n)$, $0 \le k \le m-1$. It follows from Lemma 48.5 that there exists $w \in H_{m,s}(R_T)$ such that $Hw = F$, $\frac{\partial^k w(x,0)}{\partial t^k} = \varphi_k(x)$, $0 \le k \le m-1$. Note that the estimate (48.39) holds for $w(x,t)$ since $w \in H_{m,s}(R_T)$. Take a sequence $F_n \in H_{0,s+1}(R_T)$ that converges to $f(x,t)$ in the norm of $L^1[(0,T), H_s(\mathbb{R}^n)]$. Also assume that $\varphi_k^{(n)}(x) \in H_{m-k+\frac{1}{2}+s}(\mathbb{R}^n)$ is convergent to $g_k \in H_{m-1-k+s}(\mathbb{R}^n)$ in $H_{m-k-1+s}(\mathbb{R}^n)$, $0 \le k \le m-1$. Let $w_n(x) \in H_{m,s}(R_T)$ be the solution of the Cauchy problem $Hw_n = F_n$ in R_T, $\frac{\partial^k w_n(x,0)}{\partial t^k} = \varphi_k^{(n)}(x)$, $0 \le k \le m-1$. It follows from (48.39) that w_n converges in $C_{m-1,s}$ to the solution $u(x,t) \in C_{m-1,s}$ of the Cauchy problem (48.1), (48.2) in R_T.

48.4. Strictly hyperbolic equations of second order.

Let

$$H(x,t,\xi,\sigma) = g^{00}(x,t)\sigma^2 + 2\sum_{j=1}^n g^{j0}(x,t)\xi_j\sigma + \sum_{j,k=1}^n g^{jk}(x,t)\xi_j\xi_k + b_0(x,t)\sigma + \sum_{j=1}^n b_j(x,t)\xi_j + c(x,t) \tag{48.40}$$

be a strictly hyperbolic polynomial of second order. Assume that $g^{00}(x,t) = 1$. Let

$$\sigma_\pm(x,t,\xi) = -a(x,t,\xi) \pm \sqrt{b(x,t,\xi)}$$

be the roots of $H_0(x, t, \xi, \sigma) = 0$, where

$$a = \sum_{j=1}^{n} g^{jo}(x,t)\xi_j, \quad b = a^2 - \sum_{j,k=1}^{n} g^{jk}(x,t)\xi_j\xi_k.$$

The strict hyperbolicity means that $\sigma_+ - \sigma_- = 2\sqrt{b(x, t\xi)} > 0$ for all x, t, and $\xi \neq 0$. In the notation of (48.7), (48.8) we have

$$\begin{aligned} Q_0(x,t,\xi,\sigma,\tau) &= (\sigma - \sigma_+)^2 + \tau^2 + (\sigma - \sigma_-)^2 + \tau^2 \\ &= (\sigma + a - \sqrt{b})^2 + \tau^2 + (\sigma + a + \sqrt{b})^2 + \tau^2 = 2[(\sigma + a)^2 + \tau^2 + b] \\ &= 2\Big[\Big(\sigma + \sum_{j=1}^{n} g^{j0}\xi_j\Big)^2 + \Big(\sum_{j=1}^{n} g^{j0}\xi_j\Big)^2 - \sum_{j,k=1}^{n} g^{jk}\xi_j\xi_+\tau^2\Big]. \end{aligned}$$

The proof of Theorem 48.6 works, obviously, in the case $m = 2$. In this subsection we present another proof of the estimate (48.27).

Consider the Cauchy problem:

$$H\left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u(x,t) = f(x,t), \quad x \in \mathbb{R}^n, \ 0 < t < T, \tag{48.41}$$

$$u(x,0) = g_0(x), \quad \frac{\partial u(x,0)}{\partial t} = g_1(x), \ x \in \mathbb{R}^n. \tag{48.42}$$

Assume, for simplicity, that all coefficients in (48.40) are real-valued. We will look for a real-valued solutions $u(x,t)$ of the Cauchy problem (48.41), (48.42).

Let $w(x,t) \in H_2(R_T)$ be arbitrary. Consider the identity

(48.43)

$$\left[H\left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) w(x,t), \frac{\partial w}{\partial t} + \sum_{j=0}^{n} g^{j0} \frac{\partial w}{\partial x_j}\right] = \left[F(x,t), \frac{\partial w}{\partial t} + \sum_{j=1}^{n} g^{j0}\frac{\partial w}{\partial x_j}\right],$$

where $F(x,t) = H(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})w$ and $[u, v]$ is the the scalar product in $L_2(\mathbb{R}^n)$. We have

$$\begin{aligned} &\left[\frac{\partial^2 w}{\partial t^2} + 2\sum_{j=1}^{n} g^{j0}(x,t)\frac{\partial^2 w}{\partial x_j \partial t}, \frac{\partial w}{\partial t} + \sum_{j=1}^{n} g^{j0}\frac{\partial w}{\partial x_j}\right] \\ &\quad = \frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t} + \sum_{j=1}^{n} g^{j0}\frac{\partial w}{\partial x_j}\right]_0^2 + \frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{j=1}^{n} g^{j0}\frac{\partial w}{\partial x_j}\right]_0^2 \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(g^{j0}\left(\frac{\partial w}{\partial t}\right)^2\right)dx + R_1, \end{aligned} \tag{48.44}$$

where

$$|R_1| \le C\left([w]_1^2 + \left[\frac{\partial w}{\partial t}\right]_0^2\right), \tag{48.45}$$

and $[w]_s$ is the norm in $H_s(\mathbb{R}^n)$. Note that the third term in the right hand side of (48.44) is equal to zero. Analogously, after the integration by parts with respect to x_j, $1 \le j \le n$, we get

$$\begin{aligned}\left[\sum_{j,k=1}^n g^{jk}\frac{\partial^2 w}{\partial x_j \partial x_k}, \frac{\partial w}{\partial t}\right] &= -\left[\sum_{j,k=1}^n g^{jk}\frac{\partial w}{\partial x_k}, \frac{\partial^2 w}{\partial x_j \partial t}\right] + R_2 \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{j,k=1}^n g^{jk}\frac{\partial w}{\partial x_k}, \frac{\partial w}{\partial x_j}\right] + R_3,\end{aligned} \tag{48.46}$$

where R_2, R_3 satisfy estimates of the form (48.45).

Finally,

$$\begin{aligned}&\left[\sum_{j,k=1}^n g^{jk}\frac{\partial^2 w}{\partial x_j \partial x_k}, \sum_{p=1}^n g^{p0}\frac{\partial w}{\partial x_p}\right] \\ &= -\int_{\mathbb{R}^n}\sum_{p=1}^n \frac{\partial}{\partial x_p}\left(g^{p0}\sum_{j,k=1}^n g^{jk}\frac{\partial w}{\partial x_j}\,\frac{\partial w}{\partial x_k}\right)dx + R_4 = R_4,\end{aligned} \tag{48.47}$$

because the integral in the right hand side of (48.47) is equal to zero. Note that R_4 has an estimate of the form (48.45).

Substituting (48.44), (48.46), (48.47) in (48.43), we get

$$\frac{d}{dt}E_1(t) \le C|[w]|_1^2 + C[F]_0|[w]|_1, \tag{48.48}$$

where

$$\begin{aligned}E_1(t) = \frac{1}{2}\int_{\mathbb{R}^n}\Bigg\{\left(\frac{\partial w}{\partial t} + \sum_{j=1}^n g^{j0}\frac{\partial w}{\partial x_j}\right)^2 + \left(\sum_{j=1}^n g^{j0}\frac{\partial w}{\partial x_j}\right)^2 \\ - \sum_{j,k=1}^n g^{jk}\frac{\partial w}{\partial x_j}\frac{\partial w}{\partial x_k}\Bigg\}dx,\end{aligned} \tag{48.49}$$

$$|[w]|_1^2 = [w]_1^2 + \left[\frac{\partial w}{\partial t}\right]_0^2.$$

Note that

$$C_2|[w]|_1 \le E_1(t) \le C_1|[w]|_1, \quad \forall t. \tag{48.50}$$

Using (48.50) and the Gronwall's inequality, we get

$$|[w(\cdot,t)]|_1^2 \le C\int_0^t [F(\cdot,t')]_0|[w(\cdot,t')]|_1 dt' + C|[w(\cdot,0)]|_1^2. \tag{48.51}$$

Now we can continue the proof of the estimate (48.17), using (48.51) instead of (48.36), and we get (cf. (48.38))

$$\max_{0\le t\le T}|[w(\cdot,t)]|_1 \le C[g_0]_1 + C[g_1]_0 + C\int_0^T [F(\cdot,t)]_0 dt. \tag{48.52}$$

□

Having estimate (48.52) one can complete the proof of the uniqueness and existence theorem for the Cauchy problem (48.41)), (48.42) as in Subsection 48.3.

49. Domain of dependence

We shall study now the domain of dependence and the domain of influence of the solutions to hyperbolic equations. We give here only an estimate of the domains of influence and dependence.

Denote by $K_{a,-t_0}$ the half-cone

$$K_{a,-t_0} = \{|x|^2 < a^2(t+t_0)^2,\ t > -t_0\}. \tag{49.1}$$

Theorem 49.1. *Let $f_+(x,t)$ and $u_+(x,t)$ be the same as in Theorem 48.1. Suppose $\operatorname{supp} f_+(x,t) \subset \overline{K}_{a_0,0}$, $a_0 > 0$. There exists $a > a_0$ such that $\operatorname{supp} u_+(x,t) \subset K_{a,-t_0}$, where $t_0 > 0$.*

Proof: We have

$$\operatorname{supp} f_+ \subset K_{a_0,0} \subset \overline{K}_{a,-t_0},$$

for any $a > a_0, t_0 > 0$. We shall define a one-to-one map

$$(y, y_0) = s(x,t) \tag{49.2}$$

of $K_{a,-t_0}$ onto the half-space $y_0 > -t_0,\ y \in \mathbb{R}^n$ such that $s(x,t) \in C^\infty$ when $(x,t) \ne (0,-t_0)$ and the Jacobian matrix $\frac{\partial(y,y_0)}{\partial(x,t)}$ is close to I.

To describe explicitly the map (49.2), we choose spherical coordinates (r,ψ,ω) in $K_{a,-t_0}$, where

$$\omega = \frac{x}{|x|} \in S^{n-1}, \quad r = \sqrt{|x|^2 + (t+t_0)^2},$$

$$\cos\psi = \frac{t+t_0}{\sqrt{|x|^2+(t+t_0)^2}}, \quad 0 \le \psi \le \frac{\pi}{2} - \alpha, \quad a = \cot\alpha.$$

Analogously, we introduce spherical coordinates in the half-space $y_0 > -t_0$, $y \in \mathbb{R}^n$:

$$\omega' = \frac{y}{|y|}, \quad r' = \sqrt{|y|^2 + (y_0 + t_0)^2},$$
$$\cos\psi' = \frac{y_0 + t_0}{\sqrt{|y|^2 + (y_0 + t_0)^2}}, \quad 0 \le \psi \le \frac{\pi}{2}.$$

The change of coordinates (49.2) has the following form in spherical coordinates:

$$\omega' = \omega, \quad r' = r, \quad \psi' = \frac{1}{1 - \frac{2\alpha}{\pi}}\psi.$$

In the Cartesian coordinates we have

$$y = x, \; y_0 + t_0 = \sqrt{|x|^2 + (t + t_0)^2}\cos\frac{\psi}{1 - \frac{2\alpha}{\pi}}, \tag{49.3}$$

where $\psi = \cos^{-1}\frac{t+t_0}{\sqrt{|x|^2+(t+t_0)^2}}$.

Let $\chi(x_1) \in C_0^\infty(\mathbb{R}^1)$, $\chi(x_1) = 1$ for $|x_1| < 1$, $\chi(x_1) = 0$ for $|x_1| > 2$, and $|\chi(x_1)| \le 1$. Instead of (49.3) we consider the following change of variables $(y, y_0) = s_N(x, t)$:

$$y = x, \quad y_0 + t_0 = \sqrt{|x|^2 + (t + t_0)^2}\cos\frac{\psi}{1 - \frac{2\alpha}{\pi}\chi(\frac{|x|}{N})}. \tag{49.4}$$

Then (49.4) coincides with (49.3) if $|x| < N$, and (49.4) is the identity map if $|x| > 2N$. Note that $\frac{\partial s_N(x,t)}{\partial(x,t)}$ is close to the identity when α is small.

Make the change of variables (49.4). Then $H(x, t, D_x, D_t)$ has the form $H_N(y, y_0, D_y, D_{y_0}) = H(s_N^{-1}(y, y_0), ((\frac{\partial s_N^{-1}(y,y_0)}{\partial(y,y_0)})^T)^{-1}(D_y, D_{y_0}))$ (cf. §45). We assume that $H(x, t, D_x, D_t)$ is strictly hyperbolic with respect to D_t, i.e., $H_0(x, t, \xi, \sigma) = 0$ has m distinct real roots $\sigma_k(x, t, \xi)$, $1 \le k \le m$, for all $x, t, \xi \ne 0$. By the implicit function theorem, $H_N(y, y_0, D_y, D_{y_0})$ will also be strictly hyperbolic with respect to D_{y_0}, since the Jacobian matrix $\frac{\partial s_N(x,t)}{\partial(x,t)}$ is close to the identity.

Consider the equation

$$H_N(y, y_0, D_y, D_{y_0})\hat{w}_N(y, y_0) = \hat{f}_+(y, y_0) \tag{49.5}$$

in the strip $\{-t_0 + \varepsilon < y_0 < T, \; y \in \mathbb{R}^n\}$, where $\hat{f}_+(y, y_0)$ is the image of $f_+(x, t) \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$ under the map (49.4). We look for

$$\hat{w}_N(y, y_0) \in H_{m,s-1}(\mathbb{R}^n \times (-t_0 + \varepsilon, T))$$

with zero initial conditions when $t = -t_0 + \varepsilon$. It follows from Theorem 48.6 that such $\hat{w}_N(y, y_0)$ exists and is unique.

Denote by $\Sigma_{-t_0+\varepsilon}^{(N)}$ and $\Sigma_T^{(N)}$ the images of the planes $y_0 = -t_0 + \varepsilon$ and $y_0 = T$ under the map $s_N^{-1}(y, y_0)$. Let $\Omega_{-t_0+\varepsilon,T}^{(N)}$ be the domain between $\Sigma_{-t_0+\varepsilon}^{(N)}$ and $\Sigma_T^{(N)}$. Denote by $w_N(x,t)$ the image of $\hat{w}_N(y, y_0)$ restricted to the strip $-t_0 + \varepsilon < y_0 < T,\ y \in \mathbb{R}^n$. Then $w_N(x,t)$ satisfies the equation $Hw_N = f_+(x,t)$ in $\Omega_{-t_0+\varepsilon,T}^{(N)}$ and has zero Cauchy data on $\Sigma_{-t_0+\varepsilon}^{(N)}$. Let $w_N^+(x,t)$ be the extension of $w_N(x,t)$ by zero below $\Sigma_{-t_0+\varepsilon}^{(N)}$. Then

$$H(x,t,D_x,D_t)w_N^+(x,t) = f_+(x,t) \quad \text{for} \quad (x,t) \in \Omega_{-\infty,T}^{(N)}, \tag{49.6}$$

where $(x,t) \in \Omega_{-\infty,T}^{(N)}$ if (x,t) is below $\Sigma_T^{(N)}$.

Compare $w_N^+(x,t)$ with $u_+(x,t)$ that solves $H(x,t,D_x,D_t)u_+ = f_+(x,t)$ in $\mathbb{R}^{n+1}$. Let T_1 be such that the plane $t = T_1$ is below $\Sigma_T^{(N)}$, i.e., $t = T_1$ is contained in $\Omega_{-\infty,T}^{(N)}$. Then by the uniqueness part of Theorem 48.1 we have that $u_+ = w_N^+(x,t)$ for $t < T_1$. Therefore $u_+(x,t) = 0$ if $t < T_1$ and $(x,t) \in \Omega_{-\infty,-t_0+\varepsilon}^{(N)}$. Taking the limit as $N \to \infty$, we see that $u_+(x,t) = 0$ for $|x|^2 > a^2(t + t_0 - \varepsilon)^2,\ t < T_1$, since $\Sigma_{-t_0+\varepsilon}^{(N)}$ coincides with $\{(x,t) : |x|^2 = a^2(t + t_0 - \varepsilon)^2\}$ for $|x| < N$. Here $a = \cot\alpha$. Since we can choose T (and consequently T_1) arbitrarily large, and since $\varepsilon > 0$ is arbitrary, we prove that $u_+ = 0$ for $a^2|x|^2 > |t + t_0|^2,\ \forall t > -t_0$. $\square$

Corollary 49.2 (Domain of influence)**.** *Let $G(x,y,t,t')$ be the forward Green function, i.e.,*

$$\begin{aligned} &G(x,y,t,t') = 0 \quad \textit{for} \quad t < t',\ x \in \mathbb{R}^n, \\ &H(x,t,D_x,D_t)G(x,y,t,t') = 0 \quad \textit{for} \quad t > t',\ x \in \mathbb{R}^n, \\ &\frac{\partial^k G(x,y,t,t')}{\partial t^k}\Big|_{t=t'+0} = 0, \quad 0 \le k \le m-2, \\ &\frac{\partial^{m-1} G(x,y,t,t')}{\partial t^{m-1}}\Big|_{t=t'+0} = \delta(x-y), \end{aligned} \tag{49.7}$$

where $t' + 0$ means the limit as $t \to t',\ t > t'$.

We use Theorem 49.1 to show that $\operatorname{supp} G(x,y,t,t') \subset \overline{K}_{a,y,t'}$, where $K_{a,y,t'}$ is the half-cone $\{(x,t) : |x-y|^2 < a^2(t-t')^2, t \ge t'\}$, and $a > 0$ is large.

Let $h_+(x,t) = \frac{(t-t')^{m-1}}{(m-1)!}\delta(x-y)$ for $t \ge t',\ h_+(x,t) = 0$ for $t < t'$, and let

$$G^{(1)}(x,t) = G(x,y,t,t') - h_+(x,t) \quad \text{for} \quad t \ge t'. \tag{49.8}$$

We have

$$HG^{(1)} = f_+, \quad t > t',$$

where $f_+ = -Hh_+$. Note that $G^{(1)}$ has zero initial data if $t = t'$.

Let $G_+^{(1)} = G^{(1)}$ for $t \geq t'$, $G_+^{(1)} = 0$ for $t < t'$. Note that $\operatorname{supp} f_+ = \{(x,t) : x = y,\ t \geq t'\}$ and $e^{-t\tau} f_+ \in \overset{\circ}{H}_{0,s}(\mathbb{R}^{n+1})$, where $s < -\frac{n}{2} - m$, $\tau > 0$. By Theorem 48.1, there exists $G_+^{(1)}$, $e^{-t\tau} G_+^{(1)} \in \overset{\circ}{H}_{m,s-1}(\mathbb{R}^{n+1})$, that satisfies $HG_+^{(1)} = f_+$ in $\mathbb{R}^{n+1}$. Therefore, we have proven that $G(x,y,t,t') = G_+^{(1)} + h_+(x,t)$ is the forward Green function. To prove that $\operatorname{supp} G(x,y,t,t') \subset K_{a,y,t'}$, it is enough to prove that $\operatorname{supp} G_+^{(1)} \subset \overline{K}_{a,y,t'}$. Consider the cone $K_{a,y,t'-t_0}$, where $t_0 > 0$ is arbitrary. Then $K_{a,y,t'-t_0}$ contains the support of f_+ for $a > 0$ and each $t_0 > 0$. By Theorem 49.1, $\overline{K}_{a,y,t'-t_0} \supset \operatorname{supp} G_+^{(1)}(x,t)$ when $a > 0$ is large, and since t_0 is arbitrary, we see that $\overline{K}_{a,y,t'} \supset \operatorname{supp} G_+^{(1)}(x,t)$. Therefore the domain of influence of the point (y,t') is contained in $\overline{K}_{a,y,t'}$. □

Consider the Cauchy problem $Hu_+ = f_+$ with zero initial conditions when $t = 0$, where $\operatorname{supp} f_+ \subset D$, $\overline{D} \subset \mathbb{R}_+^{n+1}$ is a bounded domain, and f_+ is smooth. We have $u_+ = \int_D G(x,y,t,t') f_+(y,t') dy dt'$.

This shows that that the domain of influence of $\overline{D}$ is contained in the closure of $\Delta = \bigcup_{(y,t') \in \overline{D}} \overline{K}_{a,y,t'}$. If D is a bounded domain then the intersection $\overline{\Delta} \cap \{t = t_0\}$ is bounded for each t_0. This means that the speed of propagation of disturbances originated in D is finite.

Corollary 49.3 (Domain of dependence). *Denote by $G^*(x,y,t,t')$ the backward Green function, i.e.,*

$$
\begin{aligned}
&G^*(x,y,t,t') = 0 \quad \textit{for} \quad t > t',\ x \in \mathbb{R}^n, \\
&H^*(x,t,D_x,D_t) G^*(x,y,t,t') = 0 \quad \textit{for} \quad t < t',\ x \in \mathbb{R}^n, \\
&\left. \frac{\partial^k G^*(x,y,t,t')}{\partial t^k} \right|_{t = t'-0} = 0,\ 0 \leq k \leq m-2, \\
&\left. \frac{\partial^{m-1} G^*(x,y,t,t')}{\partial t^{m-1}} \right|_{t = t'-0} = \delta(x-y),
\end{aligned}
\tag{49.9}
$$

where $t' - 0$ means the limit as $t \to t'$, $t < t'$.

Here H^* is the operator formally adjoint to H.

The proof of the existence of the backward Green function is the same as for the forward Green function. Let $K_{a,y,t'}^- = \{(x,t) : |x-y|^2 < a^2(t-t')^2,\ t < t'\}$ is the backward half-cone. Analogously to the proof of Theorem 49.1 and Corollary 49.2, we show that $\operatorname{supp} G^*(x,y,t,t')$ is contained in $\overline{K}_{a,y,t'}^-$. Therefore the domain of dependence of the point (y,t') is contained in $\overline{K}_{a,y,t'}^-$. Let $Hu_+ = f_+$, where $f_+ = u_+ = 0$ for $t < 0$, f_+ is smooth, and

$\operatorname{supp} f_+ \subset D \subset \mathbb{R}^{n+1}_+$. Since $H^*G^* = \delta(x-y)\delta(t-t')$, we get $(f_+, G^*) = (Hu_+, G^*) = (u_+, H^*G^*) = u_+(y,t')$, where (u,v) is the extension of L_2 scalar product in $\mathbb{R}^{n+1}$. Therefore, $u_+(y,t') = \int_D \overline{G^*(x,y,t,t')} f_+(x,t)dxdt$, i.e., $u_+(y,t')$ depends only on the values of $f_+(x,t)$ in $\bar{D} \cap \bar{K}^-_{a,y,t'}$.

50. Propagation of singularities

50.1. The null-bicharacteristics.

Let $y = (x, x_0),\ \eta = (\xi, \xi_0)$, where $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,\ x_0 \in \mathbb{R},\ \xi_0 \in \mathbb{R}$. Let $P_0(y,\eta)$ be a symbol homogeneous in η, $\deg_\eta P_0(y,\eta) = m$. We assume that $P_0(y,\eta) \in C^\infty$ when $\eta \neq 0$, $P_0(y,\eta)$ is real-valued, and $\frac{\partial P_0(y,\eta)}{\partial \eta} \neq 0$ when $P_0(y,\eta) = 0,\ \eta \neq 0$.

Such symbols P_0 will be called the symbols of real principal type.

The bicharacteristics of $P_0(y,\eta)$ are the solutions of the system

(50.1)
$$\begin{aligned} \frac{d\hat{y}}{ds} &= \frac{\partial P_0(\hat{y}(s),\hat{\eta}(s))}{\partial \eta}, \quad \hat{y}(0) = y, \\ \frac{d\hat{\eta}}{ds} &= -\frac{\partial P_0(\hat{y}(s),\hat{\eta}(s))}{\partial y}, \quad \hat{\eta}(0) = \eta,\ s \in \mathbb{R}. \end{aligned}$$

Note that

(50.2)
$$\frac{d}{ds} P_0(\hat{y}(s),\hat{\eta}(s)) = \frac{\partial P_0(\hat{y}(s),\hat{\eta}(s))}{\partial y}\frac{d\hat{y}}{ds} + \frac{\partial P_0(\hat{y}(s),\hat{\eta}(s))}{\partial \eta}\frac{d\hat{\eta}}{ds} = 0,$$

where $\hat{y}(s), \hat{\eta}(s)$ is the solution of (50.1), i.e., $P_0(\hat{y}(s),\hat{\eta}(s)) = \text{const}$ along a bicharacteristic. If the initial value (y,η) of (50.1) is such that $P_0(y,\eta) = 0$, then $P_0(\hat{y}(s),\hat{\eta}(s)) = 0$ for all s.

In this case the bicharacteristic is called the null-bicharacteristic.

We write the solution of (50.1) in the form:

(50.3)
$$\begin{aligned} \hat{y} &= \hat{y}(s,y,\eta), \\ \hat{\eta} &= \hat{\eta}(s,y,\eta). \end{aligned}$$

50.2. Operators of real principal type.

Let

(50.4)
$$P(y, D_y) = \sum_{k=0}^{N} P_k(y, D_y)(1 - \chi(D_y)) + T_{m-N-1},$$

where $P_k(y,\eta)$ are homogeneous and smooth when $\eta \neq 0$, $\deg_\eta P_k(y,\eta) = m-k$, $P_0(y,\eta)$ is a symbol of real principal type, $\operatorname{ord} T_{m-N-1} \leq m-N-1$, N is arbitrary, $\chi(\eta) \in C_0^\infty(\mathbb{R}^{n+1})$, $\chi(\eta) = 1$ for $|\eta| < 1$, and $P(y,\eta) = P(\infty,\eta)$ for large $|x|$.

We call $P(y, D_y)$ of the form (50.4) a ψdo of real principal type.

The following theorem holds.

Theorem 50.1. *Let $u(y) \in H_s(\mathbb{R}^{n+1})$ be the solution of $P(y, D_y)u = f$ in $\mathbb{R}^{n+1}$ for some $s \in \mathbb{R}$. Assume that $P(y, D_y)$ is a ψdo of real principal type. If $P(y^{(0)}, \eta^{(0)}) = 0$, $(y^{(0)}, \eta^{(0)}) \notin WF(u)$ and if the null-bicharacteristic $\hat{y} = \hat{y}(s, y^{(0)}, \eta^{(0)})$, $\hat{\eta} = \hat{\eta}(s, y^{(0)}, \eta^{(0)})$, $\hat{y}(0, y^{(0)}, \eta^{(0)}) = y^{(0)}$, $\hat{\eta}(0, y^{(0)}, \eta^{(0)}) = \eta^{(0)}$ does not belong to $WF(f)$ for any $s \in [0, L]$, then this null-bicharacteristic does not belong to $WF(u)$ for $s \in [0, L]$.*

We denote by $\gamma(y, \eta)$ the null-bicharacteristic given by (50.1).

Proof: Let Λ^s be a ψdo with the symbol $(1 + |\xi|^2 + |\xi_0|^2)^{\frac{s}{2}}$. If $s < 0$, we apply Λ^s to $Pu = f$. Using Theorem 40.2, we get $\hat{P}\hat{u} = \hat{f}$, where $\hat{f} = \Lambda^s f$, $\hat{u} = \Lambda^s u$, and $\hat{P}$ is again a ψdo of the form (50.4) with the same principal part. Note that $\hat{u} \in H_0(\mathbb{R}^n)$ and $\gamma(y^{(0)}, \eta^{(0)})$ does not belong to $WF(\hat{f})$ since Λ^s is an elliptic operator (see §44). Therefore, without loss of generality we may assume that $u \in H_s(\mathbb{R}^{n+1})$, where $s \geq 0$.

Let $P_0(y^{(0)}, \eta^{(0)}) = 0$, $\eta^{(0)} \neq 0$. Since $P_0(y, \eta)$ is of real principal type, there exists ξ_j such that $\frac{\partial P_0(y^{(0)}, \eta^{(0)})}{\partial \xi_j} \neq 0$, $0 \leq j \leq n$. Assume, for the definiteness, that $\frac{\partial P_0(y^{(0)}, \eta^{(0)})}{\partial \xi_0} > 0$, where $y^{(0)} = (x^{(0)}, x_0^{(0)})$, $\eta^{(0)} = (\xi^{(0)}, \xi_0^{(0)})$. Note that $\xi^{(0)} \neq 0$. Otherwise we have that $P_0(y^{(0)}, 0, \xi_0) = 0$ for any $\xi_0 = \tau \xi_0^{(0)}, \tau > 0$, and hence $\frac{\partial P_0(y^{(0)}, 0, \xi_0^{(0)})}{\partial \xi_0} = 0$.

By the implicit function theorem, there exists a conic neighborhood U_0 of $(y^{(0)}, \eta^{(0)})$ such that

$$P_0(y, \eta) = q_0(y, \eta)(\xi_0 - \lambda_0(y, \xi)), \tag{50.5}$$

where $q_0(y, \eta) \neq 0$ in U_0, $\lambda_0(y, \xi)$ is real-valued, $\xi \neq 0$ in U_0 since $\xi^{(0)} \neq 0$, $\deg_\eta q_0(y, \eta) = m - 1$, $\deg_\xi \lambda_0(y, \xi) = 1$, and q_0 and λ_0 are C^∞ in U_0.

For simplicity of notation assume that $x_0^{(0)} = 0$. Denote by U_1 a neighborhood of $(x^{(0)}, 0, \eta^{(0)})$ in $U_0 \cap \{x_0 = 0\}$.

We often write $\hat{y}(s) = (\hat{x}(s), \hat{x}_0(s))$, $\hat{\eta}(s) = (\hat{\xi}(s), \hat{\xi}_0(s))$ instead of $\hat{y}(s, y, \eta)$, $\hat{\eta}(s, y, \eta)$. Since $\hat{y}(s), \hat{\eta}(s)$ is a null-bicharacteristic and $q_0 \neq 0$ in U_0, we have

$$\hat{\xi}_0(s) - \lambda_0(\hat{x}(s), \hat{x}_0(s), \hat{\xi}(s)) = 0. \tag{50.6}$$

Consider the null-bicharacteristics in U_0 with initial conditions on U_1, i.e., $\hat{x}(0) = x$, $\hat{x}_0(0) = 0$, $\hat{\eta}(0) = \eta$, $(x, 0, \eta) \in U_1$.

Since $\frac{\partial P_0}{\partial \xi_0} > 0$ in U_0, we can take

$$\hat{x}_0 = \int_0^s \frac{\partial P_0(\hat{y}(s'), \hat{\eta}(s'))}{\partial \xi_0} ds' \tag{50.7}$$

as a parameter on $\gamma(x, 0, \eta)$. Note that

$$(50.8)\quad \frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial \xi_0} = \frac{\partial q_0(\hat{y}(s), \hat{\eta}(s))}{\partial \xi_0} \times \left(\hat{\xi}_0(s) - \lambda_0(\hat{x}(s), \hat{x}_0(s), \hat{\xi}(s))\right) + q_0(\hat{y}(s), \hat{\eta}(s)) = q_0(\hat{y}(s), \hat{\eta}(s)).$$

Analogously,

$$(50.9)\quad \frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial \xi} = q_0(\hat{y}(s), \hat{\eta}(s)) \left(-\frac{\partial \lambda_0(\hat{y}(s), \hat{\xi}(s))}{\partial \xi} \right),$$

$$(50.10)\quad \frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial y} = q_0(\hat{y}(s), \hat{\eta}(s)) \left(-\frac{\partial \lambda_0(\hat{y}(s), \hat{\xi}(s))}{\partial y} \right).$$

For simplicity of notation we write x_0 instead of $\hat{x}_0$. Therefore, the system (50.1) takes the following equivalent form when x_0 is a parameter:

$$(50.11)\quad \begin{aligned} \frac{d\hat{x}(x_0)}{dx_0} &= -\frac{\partial \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))}{\partial \xi}, \quad \hat{x}(0) = x, \\ \frac{d\hat{\xi}(x_0)}{dx_0} &= \frac{\partial \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))}{\partial x}, \quad \hat{\xi}(0)) = \xi, \end{aligned}$$

and

$$(50.12)\quad \frac{d\hat{\xi}_0(x_0)}{dx_0} = \frac{\partial \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))}{\partial x_0}, \quad \hat{\xi}_0(0) = \xi_0.$$

Taking into account (50.11) that the solution of (50.12) is

$$\hat{\xi}_0(x_0) = \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0))$$

and the system (50.11) supplemented with the equations $x_0 = x_0$ and $\hat{\xi}(x_0) - \lambda_0(\hat{x}(x_0), x_0, \hat{\xi}(x_0)) = 0$ is the equation for the null-bicharacteristics of $P_0(y, D_y)$ and $D_{x_0} - \lambda_0(y, D_x)$.

50.3. Propagation of singularities for operators of real principal type.

Let $V(-3\varepsilon, L + 3\varepsilon)$ be the union of all null-bicharacteristics starting at $(x, 0, \eta) \in U_1$ and having $-3\varepsilon < x_0 < L + 3\varepsilon$. We assume that U_1 and L are such that $V(-3\varepsilon, L + 3\varepsilon) \subset U_0$ and $WF(f) \cap \overline{V(-3\varepsilon, L + 3\varepsilon)} = \emptyset$.

Let $\chi_{2\varepsilon}(x_0) \in C_0^\infty(-3\varepsilon, L + 3\varepsilon)$ and $\chi_{2\varepsilon} = 1$ for $-2\varepsilon < x_0 < L + 2\varepsilon$ and let $\chi_\varepsilon(x_0) \in C_0^\infty(-2\varepsilon, L + 2\varepsilon)$, $\chi_\varepsilon(x_0) = 1$ for $-\varepsilon < x_0 < L + \varepsilon$.

Lemma 50.2. *There exists $\varphi(y, \eta) \in S^0$ such that $\varphi(y^{(0)}, \eta^{(0)}) \neq 0$,*

$$\operatorname{supp} \varphi(y, \eta) \subset V(-3\varepsilon, L + 3\varepsilon),$$

and

(50.13) $$\varphi(y,D_y)P(y,D_y)u = P(y,D_y)\varphi(y,D_y)u + T_{m-N-1}u + T_\varepsilon u,$$

where $T_\varepsilon u = 0$ *for* $-\varepsilon < x_0 < L+\varepsilon$, $\operatorname{ord} T_{m-N-1} \le m-N-1$, *and* N *is arbitrary.*

Note that T_ε will play no role, since later we will consider the restriction of (50.13) to the interval $(-\varepsilon, L+\varepsilon)$. Note also that $\varphi(y,\eta)$ depends on N, but we did not indicate this for simplicity of notation.

Proof of Lemma 50.2. We are looking for $\varphi(y,\eta)$ in the form:

(50.14) $$\varphi(y,\eta) = \sum_{k=0}^{N} \varphi_{-k}(y,\eta)\left(1-\chi\left(\frac{\eta}{R}\right)\right)\chi_\varepsilon(x_0),$$

where the φ_{-k} are homogeneous in η of degree $-k$, $\operatorname{supp}\varphi_{-k}\chi_\varepsilon(x_0) \subset V(-2\varepsilon, L+2\varepsilon)$, $\chi(\eta) \in C_0^\infty(\mathbb{R}^n)$, $\chi(\eta) = 1$ for $|\eta| < 1$, and R is large. Compute the commutator $P\varphi - \varphi P$. Using Theorem 40.2 and equating to zero symbols of the same degree of homogeneity in $\eta = (\xi, \xi_0)$, we get:

$$\varphi_0(y,\eta)P_0(y,\eta) = P_0(y,\eta)\varphi_0(y,\eta),$$

(50.15) $$\sum_{j=0}^{n} \frac{\partial\varphi_0(y,\eta)}{\partial\xi_j}\frac{\partial P_0(y,\eta)}{\partial x_j} - \sum_{j=0}^{n} \frac{\partial\varphi_0(y,\eta)}{\partial x_j}\frac{\partial P_0(y,\eta)}{\partial \xi_j} = 0,$$

(50.16)
$$\sum_{j=0}^{n}\left(\frac{\partial\varphi_{-k}(y,\eta)}{\partial\xi_j}\frac{\partial P_0(y,\eta)}{\partial x_j} - \frac{\partial\varphi_{-k}(y,\eta)}{\partial x_j}\frac{\partial P_0(y,\eta)}{\partial \xi_j}\right) = b_k(y,\eta), 1 \le k \le N,$$

where $\deg_\eta b_k(y,\eta) = m-k-1$, and $b_k(y,\eta)$ depends on $\varphi_0, \varphi_{-1}, \dots, \varphi_{-k+1}$.

The left hand side of (50.15) is called the Poisson bracket of φ_0 and P_0 (cf. Corollary 40.3). Substituting (50.1) or (50.11), (50.12) into (50.15), we get

(50.17) $$\frac{d}{dx_0}\varphi_0(\hat{x}(x_0,x,\eta), x_0, \hat{\eta}(x_0,x,\eta)) = 0.$$

Let $\psi(x,\eta) \in C_0^\infty(U_1)$, $\psi(x^{(0)}, \eta^{(0)}) \neq 0$, $\deg_\eta \psi(x,\eta) = 0$. We assume that the $\varphi_{-k}(x,x_0,\eta)$, $0 \le k \le n$, satisfy the following initial conditions for $x_0 = 0$:

(50.18) $$\begin{aligned} &\varphi_0(x,0,\eta) = \psi(x,\eta), \\ &\varphi_{-k}(x,0,\eta) = 0, \quad 1 \le k \le N. \end{aligned}$$

Then

$$\varphi_0(\hat{x}(x_0,x,\eta), x_0, \hat{\eta}(x_0,x,\eta)) = \psi(x,\eta)$$

for $-2\varepsilon \le x_0 \le L + 2\varepsilon$, where $\hat{x}, \hat{\eta}$ are the solutions of (50.11), (50.12), $\hat{x}(0) = x,\ \hat{\eta}(0) = \eta$. Let

$$x = x(x_0, \hat{x}, \hat{\eta}), \qquad \eta = \eta(x_0, \hat{x}, \hat{\eta}) \tag{50.19}$$

be the inverse to (50.11), (50.12). Then

$$\varphi_0(\hat{x}, x_0, \hat{\eta}) = \psi(x(x_0, \hat{x}, \hat{\eta}), \eta(x_0, \hat{x}, \hat{\eta})), \quad (\hat{x}, x_0, \hat{\eta}) \in V(-2\varepsilon, L + 2\varepsilon). \tag{50.20}$$

Using x_0 as a parameter, we get from (50.16)

$$\frac{d}{dx_0}\varphi_{-k}(\hat{x}(x_0), x_0, \hat{\eta}(x_0)) = b_k(\hat{x}(x_0), x_0, \hat{\eta}(x_0)) \left(\frac{\partial P_0(\hat{x}(x_0), x_0, \hat{\eta}(x_0))}{\partial \xi_0}\right)^{-1}.$$

Integrating with respect to x_0 from 0 to x_0 and using the initial conditions (50.18) and the inverse map (50.19), we obtain, analogously to (50.20), that

$$\operatorname{supp} \varphi_{-k}\chi_\varepsilon(x_0) \subset V(-2\varepsilon, L + 2\varepsilon).$$

Note that $\varphi(x, x_0, \eta) \ne 0$ on $\gamma(x^{(0)}, 0, \eta^{(0)})$ since $\varphi_0 \ne 0$, $\deg_\eta \varphi_{-k} \le -k$, $k \ge 1$, $\chi(\frac{\eta}{R}) = 1$ for $|\eta| < R$, where R can be taken large enough.

Applying $\varphi(y, D_y)$ to $P(y, D_y)u = f$ and using (50.15), (50.16), we get

$$P(y, D_y)\varphi(y, D_y)u = \varphi(y, D_y)f - T_{m-N-1}u - T_\varepsilon u. \tag{50.21}$$

The contribution to $T_\varepsilon u$ comes from the terms containing derivatives of $\chi_\varepsilon(x_0)$ when we compute the commutator $P\varphi - \varphi P$. Note that $\varphi(y, D_y)f \in C^\infty$ since $\operatorname{supp}\varphi \subset V(-2\varepsilon, L + 2\varepsilon)$ and therefore $\operatorname{supp}\varphi \cap WF(f) = \emptyset$. $\square$

Lemma 50.3. *There exists $Q_N(y, \eta) \in S^{m-1}$ such that after applying $Q_N(y, D_y)$ to* (50.21), *we get*

$$(D_{x_0} - \lambda_0(y, D_x) + a_N(y, D_x))\varphi u = -T_\varepsilon^{(N)}u - T_{-N-1}^{(N)}u + Q_N\varphi f, \tag{50.22}$$

where $a_N(y, \xi) \in S^0$, $T_\varepsilon^{(N)}u = 0$ for $-\varepsilon < x_0 < L + \varepsilon$, $\operatorname{ord} T_{-N-1}^{(N)} \le -N - 1$, and N is arbitrary.

Proof: Let $\varphi^{(1)}(y, \eta) \in C^\infty(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}))$, $\deg_\eta \varphi^{(1)} = 0$ and let $\operatorname{supp}\varphi^{(1)}$ be contained in U_0. We assume that $\varphi^{(1)} = 1$ in a conic neighborhood of $\operatorname{supp}\varphi_{-k}(y, \eta)$, $0 \le k \le N$. Therefore, $\operatorname{supp}(1 - \varphi^{(1)}) \cap \operatorname{supp}\varphi_{-k} = \emptyset$, $0 \le k \le N$.

Let Q_1 be a ψdo with symbol $\varphi^{(1)}(y, \eta)q_0^{-1}(y, \eta)(1 - \chi(\eta))\chi_{2\varepsilon}(x_0)$. Applying $Q_1(y, D_y)$ to (50.21) and using Theorem 40.2, we get

$$\begin{aligned}(D_{x_0} - \lambda_0(y, D_x))\varphi(y, D_y)u \\ = C_0(y, D_y)\varphi u - T_{-N-1}^{(1)}u - T_\varepsilon^{(1)}u + Q_1\varphi f,\end{aligned} \tag{50.23}$$

where $\operatorname{ord} T_{-N-1}^{(1)} \le -N - 1$, $C_0(y, \xi) \in S^0$, and $T_\varepsilon^{(1)} = 0$ for $-\varepsilon \le x_0 \le L + \varepsilon$. Since $\varphi f \in C^\infty$, we conclude that $g = Q_1\varphi f \in C^\infty(\mathbb{R}^{n+1})$. In

(50.23), we have used that the composition of a ψdo with symbol $(\xi_0-\lambda_0)(1-\varphi^{(1)}(y,\eta)\chi_{2\varepsilon}(x_0))$ and of $\varphi(y,D_y)$ is an operator of order $-\infty$, and that $Q_1T_\varepsilon = T_\varepsilon^{(1)} + T'_{-N-1}$, where $\operatorname{ord} T'_{-N-1} \le -N-1$.

By the Taylor theorem,

$$C_0(y,\xi,\xi_0) = C_0(y,\xi,\lambda_0(y,\xi)) + C_{-1}^{(1)}(y,\xi,\xi_0)(\xi_0-\lambda_0(y,\xi)). \tag{50.24}$$

Denote $\lambda_{-1}(y,\xi) = C_0(y,\xi,\lambda_0(y,\xi))$. We can rewrite (50.24) in the form

$$\begin{aligned} C_0(y,\xi,\xi_0) = \lambda_{-1}(y,\xi) &+ C_{-1}^{(1)}(y,\xi,\xi_0)(\xi_0-\lambda_0(y,\xi)-\lambda_{-1}(y,\xi)) \\ &+ C_{-1}^{(1)}(y,\xi,\xi_0)\lambda_{-1}(y,\xi). \end{aligned} \tag{50.25}$$

Since $\xi \ne 0$ on $\gamma(x^{(0)},0,\xi^{(0)},\xi_0^{(0)})$ and $\operatorname{supp}\varphi^{(1)}$ is contained in a small conic neighborhood of γ, we have that $|\xi|^{-1} \le C(|\xi|+|\xi_0|)^{-1}$ on $\operatorname{supp}\varphi^{(1)}$. Therefore, since $C_0(y,\xi,\xi_0)$ contains $\varphi^{(1)}$ or derivatives of $\varphi^{(1)}$, we have that $C_0(y,\xi,\xi_0) = 0$ when $\frac{|\xi|}{|\xi|+|\xi_0|}$ is small. Hence $C_{-1}^{(1)}(y,\xi,\xi_0) \in S^{-1}$ and $\lambda_{-1}(y,\xi) \in S^0(\mathbb{R}^{n+1}\times\mathbb{R}^n)$.

Applying Theorem 40.2 to (50.25), we get

$$\begin{aligned} C_0(y,D_y)\varphi u &= \lambda_{-1}(y,D_x)\varphi u \\ &\quad + C_{-1}^{(1)}(y,D_y)(D_{x_0}-\lambda_0(y,D_x)-\lambda_{-1}(y,D_x))\varphi u \\ &\quad + C_{-1}^{(2)}(y,D_y)\varphi u + T_{-N-1}^{(2)}u + T_\varepsilon^{(2)}\varphi u, \end{aligned} \tag{50.26}$$

where $\operatorname{ord} T_{-N-1}^{(2)} \le -N-1$, $C_{-1}^{(2)}(y,\eta) \in S^{-1}$. Substituting (50.26) into (50.23), we obtain:

$$\begin{aligned} (D_{x_0}-(\lambda_0+\lambda_{-1}))\varphi u &- C_{-1}^{(1)}(y,D_y)(D_{x_0}-(\lambda_0+\lambda_{-1}))\varphi u \\ &- C_{-1}^{(2)}(y,D_y)\varphi u = -T_\varepsilon^{(3)}u - T_{-N-1}^{(3)}u + Q_1\varphi f, \end{aligned} \tag{50.27}$$

Since $C_{-1}^{(1)} \in S^{-1}$, there exists $Q_2 = I + Q_2'$, $Q_2'(y,\eta) \in S^{-1}$ such that

$$Q_2(I - C_{-1}^{(1)}) = I + T_{-N-1}^{(3)} \tag{50.28}$$

(cf. §42). Multiplying (50.27) by Q_2, we get

$$\begin{aligned} (D_{x_0} &- (\lambda_0(y,D_x)+\lambda_{-1}(y,D_x)))\varphi u \\ &= C_{-1}^{(3)}(y,D_y) - T_\varepsilon^{(4)} - T_{-N-1}^{(4)}\varphi u + Q_2Q_1\varphi f, \end{aligned} \tag{50.29}$$

where $C_{-1}^{(3)}(y,\eta) \in S^{-1}$.

Now again by the Taylor formula (cf. (50.24)) we have:

$$C_{-1}^{(3)}(y,\xi,\xi_0) = C_{-1}^{(3)}(y,\xi,\lambda_0+\lambda_{-1}) + C_{-2}^{(3)}(y,\xi,\xi_0)(\xi_0-\lambda_0-\lambda_1)$$
$$= \lambda_{-2}(y,\xi) + C_{-2}^{(3)}(y,\xi,\xi_0)(\xi_0-\lambda_0-\lambda_{-1}-\lambda_{-2}) + C_{-2}^{(3)}(y,\xi)\lambda_{-2}(y,\xi), \tag{50.30}$$

where $C_{-2}^{(3)}(y,\eta) \in S^{-2}$, $\lambda_{-2}(y,\xi) = C_{-1}^{(3)}(y,\xi,\lambda_0+\lambda_1)$. Repeating (50.27) with $C_0, C_{-1}^{(1)}$ replaced by $C_{-1}^{(3)}, C_{-2}^{(3)}$, we get, as in (50.27), (50.29), that

$$(D_{x_0} - (\lambda_0(y,D_x) + \lambda_{-1}(y,D_x) + \lambda_{-2}(y,D_x)))\varphi u$$
$$- C_{-2}^{(3)}(y,D_y)(D_{x_0} - (\lambda_0(y,D_x) + \lambda_{-1}(y,D_x) + \lambda_{-2}(y,D_x)))\varphi u$$
$$= C_{-2}^{(4)}(y,D_y)\varphi u - T_\varepsilon^{(5)}\varphi u - T_{-N-1}^{(5)}\varphi u + Q_2Q_1\varphi f, \tag{50.31}$$

where $C_{-2}^{(4)} \in S^{-2}$.

Let $Q_3(y,D_y) = I + Q_3'(y,D_y)$ be such that $Q_3(I - C_{-2}^{(3)}) = I + T_{-N-1}$. Multiplying (50.31) by $Q_3(y,D_y)$, we get, as in (50.29):

$$(D_{x_0} - (\lambda_0 + \lambda_{-1} + \lambda_{-2}))\varphi u = C_{-2}^{(5)}\varphi u - T_\varepsilon^{(6)}u - T_{-N-1}^{(6)}u + Q_3Q_2Q_1\varphi f,$$

where $C_{-2}^{(5)}(y,\eta) \in S^{-2}$.

After N such steps we get (50.22) with $a_N = -\sum_{k=1}^N \lambda_{-k}(y,D_x)$. □

Now we can conclude the proof of Theorem 50.1.

Take the restriction of (50.22) to $R_L = \mathbb{R}^n \times (0,L)$. Then $T_\varepsilon^{(N)}u = 0$. We assumed that $(x^{(0)},0,\eta^{(0)}) \notin WF(u)$. Therefore

$$\varphi u|_{x_0=0} = h \in C^\infty. \tag{50.32}$$

Also the right hand side of (50.22) belongs to $H_{s+N+1}(R_L)$. By Theorem 48.6, there exists a unique solution of the Cauchy problem (50.22), (50.32) belonging to $H_{0,s+N+1}(R_L)$. Here $H_{0,s}(R_L)$ is the same as in §48. Since, by assumption, $s \geq 0$, we have $\varphi u \in H_s(R_L) \subset H_{0,s}(R_L)$. Therefore, by the uniqueness part of Theorem 48.6, $\varphi u \in H_{0,s+N+1}(R_L)$.

In view of (50.22), $D_{x_0}\varphi u \in H_{0,s+N}(R_L)$. Differentiating (50.22) with respect to x_0, we obtain $D_{x_0}^2\varphi u \in H_{0,s+N-1}(R_L)$. Since N is arbitrary, we conclude, repeatedly differentiating (50.22) with respect to x_0, that $\varphi u \in H_N(R_L)$ and therefore $\varphi u \in C^\infty$. □

Remark 50.1. We show that the solution of (50.11) exists for all $-\infty < x_0 < +\infty$, provided that $\lambda_0(x,x_0,\xi) = \lambda(\infty,x_0,\xi)$ for large $|x|$ and $\lambda(x,x_0,\xi) = \lambda(x,\infty,\xi)$ for large $|x_0|$. Since $\deg_\xi \lambda_0(x,x_0,\xi) = 1$, we have that $|\frac{\partial\lambda_0(x,x_0,\xi)}{\partial\xi}| \leq C$ for all $(x,x_0) \in \mathbb{R}^{n+1}, \xi \neq 0$. Therefore

$$|\hat{x}(x_0) - x| \leq C|x_0 - x_0^{(0)}|. \tag{50.33}$$

Considering the dot product $\hat{\xi}(x_0) \cdot \frac{d\hat{\xi}(x_0)}{dx_0}$, we get

$$\left|\hat{\xi}(x_0) \cdot \frac{d\hat{\xi}_0}{dx_0}\right| = \left|\hat{\xi}(x_0) \cdot \frac{\partial \lambda_0}{\partial x}\right| \leq C|\hat{\xi}(x_0)|^2.$$

Therefore $\frac{1}{2}\frac{d}{dx_0}|\hat{\xi}(x_0)|^2 \leq C|\hat{\xi}(x_0)|^2$, and we obtain

$$|\hat{\xi}(x_0)|^2 \leq |\xi^{(0)}|^2 \exp(2C|x_0 - x_0^{(0)}|). \tag{50.34}$$

Estimates (50.33), (50.34) imply that the solution of (50.11), (50.12) exists on $(-\infty, +\infty)$. □

We prove that the solution of (50.1) also exists for all $\infty < s < +\infty$. Note that $|\frac{\partial P_0(y,\eta)}{\partial y}| \neq 0$ for all (y,η), $\eta \neq 0$. For $P_0(y,\eta) = 0$ this follows from the definition of the symbol of real principal type. For $P_0(y,\eta) \neq 0$ this follows from Euler's formula $mP_0(y,\eta) = \eta \cdot \frac{\partial P_0(y,\eta)}{\partial \eta}$.

Introduce a new parameter in (50.1):

$$\tau = \int_0^s \left|\frac{\partial P_0(\hat{y}(s), \hat{\eta}(s))}{\partial \eta}\right| ds.$$

When τ is a parameter, the system (50.1) takes the form:

$$\frac{d\hat{y}}{d\tau} = \frac{\frac{\partial P_0(\hat{y},\hat{\eta})}{\partial \eta}}{|\frac{\partial P_0(\hat{y},\hat{\eta})}{\partial \eta}|}, \qquad \frac{d\hat{\eta}}{d\tau} = \frac{\frac{\partial P_0(\hat{y},\hat{\eta})}{\partial y}}{|\frac{\partial P_0(\hat{y},\hat{\eta})}{\partial \eta}|}. \tag{50.35}$$

The right hand sides in (50.35) have the same degree of homogeneity in $\hat{\eta}$ as the right hand sides of (50.11). Therefore we can repeat without changes the proof of the existence of a global solution for the system (50.11). □

Using Remark 50.1 we get the global version of Theorem 50.1:

Theorem 50.4. *Let $P(x,D)u = f$, where $P(x,D)$ is a ψdo of real principal type. Let $\gamma(s) = \{\hat{y} = y(s),\ \hat{\eta} = \eta(s),\ -\infty < s < +\infty\}$ be the null-bicharacteristic* (50.1). *Suppose $WF(f) \cap \gamma(s) = \emptyset$ for all $s \in \mathbb{R}$. Then either $WF(u) \cap \gamma(s) = \emptyset$ for all $s \in \mathbb{R}$, or $\gamma(s) \in WF(u)$ for all $s \in \mathbb{R}$.*

Proof: Suppose $\gamma(0) \notin WF(u)$. Then, by Theorem 50.1, there exists a conic neighborhood $U_0 \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ of $(\hat{y}(0), \hat{\eta}(0))$ such that $(U_0 \cap \gamma) \cap WF(u) = \emptyset$. Since $P_0(y,\eta)$ is homogeneous in η and $P_0(y,\eta) = P_0(\infty,\eta)$ for $|y|$ large, there exists a finite cover of $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ by conic neighborhoods $U_j, 1 \leq j \leq N$, such that (50.5) holds. Using Theorem 50.1 and the connectedness of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, we get that $(\gamma \cap U_j) \cap WF(u) = \emptyset$ for all $j,\ 1 \leq j \leq N$. Therefore $\gamma \cap WF(u) = \emptyset$ for all $s \in \mathbb{R}$.

Suppose now that $\gamma(s_1) = (\hat{y}(s_1), \hat{\eta}(s_1)) \in WF(u)$. Then all $\gamma(s) = (y(s), \eta(s)) \in WF(u)$ because if some point $\gamma(s_2) = (\hat{y}(s_2), \hat{\eta}(s_2)) \notin WF(u)$, then, as proven above, $\gamma(s) \notin W(u)$ for all $s \in \mathbb{R}$. Therefore, either

$\gamma(s),\ -\infty < s < +\infty$, is contained in $WF(u)$, or the intersection of $\gamma(s)$ with $WF(u)$ is empty. □

50.4. Propagation of singularities in the case of a hyperbolic Cauchy problem.

Let $H(x,t,D_x,D_t)$ be a strictly hyperbolic operator. We describe the wave front set $WF(u)$ of the Cauchy problem (48.1), (48.2) in terms of $WF(f)$ and the wave front sets of initial conditions $g_k(x),\ 0 \le k \le m-1$. Since $H_0(x,t,\xi,\sigma)$ has m distinct real roots $\sigma = \lambda_k(x,x_0,\xi),\ 1 \le k \le m$, there are m null-bicharacteristics passing through an arbitrary point $(x^{(0)},t_0,\xi^{(0)}) \in \mathbb{R}^{n+1} \times (\mathbb{R}^n \setminus \{0\})$ with $\xi_0^{(k)} = \lambda_k(x^{(0)},t_0,\xi^{(0)}),\ 1 \le k \le m$. Consider first the Cauchy problem (48.1) with zero initial conditions, i.e., $Hu_+ = f_+$ in $\mathbb{R}^{n+1},\ e^{-\tau t}u_+ \in \overset{\circ}{H}_{m,s}(\mathbb{R}^{n+1})$ for some $s \in \mathbb{R}$ and $\tau > 0$, and $u_+ = 0$ for $t < 0$.

If $(x^{(0)},t_0,\xi^{(0)},\xi_0^{(0)}) \in WF(f_+),\ t_0 \ge 0$, then, obviously,

$$(x^{(0)},t_0,\xi^{(0)},\xi_0^{(0)}) \in WF(u_+),$$

since $WF(f_+) \subset WF(u_+)$.

If $\xi_0^{(0)} \neq \lambda_k(x^{(0)},t_0,\xi^{(0)}),\ 1 \le k \le m$, i.e., H_0 is microelliptic at $(x^{(0)},t_0,\xi^{(0)},\xi_0^{(0)})$, then there is no propagation of singularities (Theorem 44.5). If $\xi_0^{(0)} = \lambda_{k_0}(x^{(0)},t_0,\xi^{(0)})$ for some $1 \le k_0 \le m$, then the singularity may propagate or not (cf. Example 50.1 below).

Now consider the case of the Cauchy problem (48.1), (48.2). Assume that $u \in H_{m,s}(\mathbb{R}^{n+1}_+)$ for some $s \in \mathbb{R}$ and let $u_+ = u$ for $t \ge 0,\ u_+ = 0$ for $t < 0$. Then $u_+(x,t)$ satisfies the following equation in $\mathbb{R}^{n+1}$:

$$H(x,t,D_x,D_t)u_+ = f_+ + \sum_{k=0}^{m-1} b_k(x,0,D)g_k(x)\delta^{(m-1-k)}(t), \tag{50.36}$$

where $\operatorname{ord} b_k \le k,\ f_+ = f$ for $t > 0$, and $f_+ = 0$ for $t < 0$.

Suppose $(x^{(0)},\xi^{(0)}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ belongs to $WF(g_{k_0}(x))$ for some $k_0,\ 0 \le k_0 \le m-1$. Then $(x^{(0)},0,\xi^{(0)},\sigma) \in WF(g_{k_0}(x)\delta^{(m-1-k)}(t))$, where σ is arbitrary. In particular, $(x^{(0)},0,\xi^{(0)},\lambda_j(x^{(0)},0,\xi^{(0)}))$ belongs to $WF(g_{k_0}(x)\delta^{(m-1-k)}(t))$ for each $1 \le j \le m$.

There are m null-bicharacteristics $\gamma_{jk_0}(x^{(0)},0,\xi^{(0)},\lambda_j(x^{(0)},0,\xi^{(0)}))$ that start at $(x^{(0)},0,\xi^{(0)},\lambda_j(x^{(0)},0,\xi^{(0)})),\ 1 \le j \le m$.

The following theorem holds:

Theorem 50.5. *Let $u(x,t)$ be the solution of the Cauchy problem* (48.1), (48.2). *Assume, for simplicity, that $f(x,t)$ is smooth for $0 \le t \le \delta$. Let* *the wave front sets* $WF(f_+) \subset \mathbb{R}_+^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ *and* $WF(g_k(x)) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, $0 \le k \le m-1$, *be given. Denote by* $\Sigma(f, g_0, \dots, g_{m-1}) \subset \mathbb{R}_+^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ *the set of points satisfying the following conditions:*

a) $\Sigma \supset WF(f_+) \cap \{t > 0\}$.

b) *If* $(x^{(0)}, t_0, \xi^{(0)}, \sigma_0) \in WF(f_+)$, $t_0 \ge 0$, $\xi^{(0)} \ne 0$, *and* $\sigma_0 = \lambda_{k_0}(x^{(0)}, t_0, \xi^{(0)})$ *for some* k_0, $1 \le k_0 \le m$, *then the forward null-bicharacteristic* $\gamma_{k_0}^+(x^{(0)}, t_0, \xi^{(0)}, \sigma_0)$, $t \ge t_0$, *belongs to* Σ.

c) *If* $(x^{(0)}, \xi^{(0)}) \in WF(g_k)$, $0 \le k \le m-1$, *then the union of* m *forward null-bicharacteristics* $\gamma_{jk}^+(x^{(0)}, 0, \xi^{(0)}, \lambda_j(x^{(0)}, 0, \xi^{(0)}))$, $t \ge 0$, $1 \le j \le m$, *belongs to* Σ.

Then $WF(u) \cap \{t > 0\}$ *is contained in* Σ.

Proof: Let $\mathcal{P}_0 = (x^{(0)}, t_0, \xi^{(0)}, \sigma_0) \notin \Sigma$, where $t_0 > 0$. If $\xi^{(0)} = 0$ or $\sigma_0 \ne \lambda_k(x^{(0)}, t_0, \xi^{(0)})$, $1 \le k \le m$, then H_0 is microelliptic at $\mathcal{P}_0$. Since $\mathcal{P}_0 \notin \Sigma$, we have that $\mathcal{P} \notin WF(f_+)$ and therefore $\mathcal{P} \notin WF(u)$. Suppose $\xi^{(0)} \ne 0$ and $\sigma_0 = \lambda_{j_0}(x^{(0)}, t_0, \xi^{(0)})$ for some $1 \le j_0 \le m$. Denote by γ_- the backward null-bicharacteristic starting at $(x^{(0)}, t_0, \xi^{(0)}, \sigma_0)$ provided that $\sigma_0 = \lambda_{j_0}$. Let $(x^{(1)}, 0, \xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)}))$ be the point on γ_- where $t = 0$. Since $\mathcal{P} \notin \Sigma$, we have that $(x^{(1)}, \xi^{(1)}) \notin WF(g_k(x))$ for $0 \le k \le m-1$.

We show that $(x^{(1)}, 0, \xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)})) \notin WF(b_k g_k(x) \delta^{(m-k-1)}(t))$ for all $0 \le k \le m-1$.

Let $\beta(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$, $\beta(x^{(1)}, 0) \ne 0$, and $\beta(x,t)$ is zero outside of a small neighborhood of $(x^{(1)}, 0)$. Then $F(\beta b_k g_k \delta^{(m-k-1)}(t)) = \sum_{p=0}^{m-k-1} \sigma^p \beta_p(\xi)$, where $|\beta_p(\xi)| \le C_N (1 + |\xi|)^{-N}$, $\forall N$, in a conic neighborhood of $\xi^{(1)}$, since $(x^{(1)}, \xi^{(1)}) \notin WF(g_k), k = 1, \dots, m$. Therefore $|\sum_{p=0}^{m-k-1} \sigma^p \beta_p(\xi)| \le C(1 + |\xi| + |\sigma|)^{-N}$, $\forall N$, in a conic neighborhood of $(\xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)}))$ since $|\xi| \ge C(|\xi| + |\sigma|)$ in this neighborhood. This proves that

$$(x^{(1)}, 0, \xi^{(1)}, \lambda_{j_0}(x^{(1)}, 0, \xi^{(1)})) \notin WF(b_k g_k(x) \delta^{(m-k-1)}(t))$$

for all $0 \le k \le m-1$. Therefore, the wave front set of the right hand side of (50.36) does not intersect γ_- for $-\infty < t \le t_0$. Since $u_+ = 0$ for $t < 0$, we get from Theorem 50.1 that $WF(u_+) \cap \gamma_- = \emptyset$ for $t \le t_0$, i.e., $\mathcal{P}_0 \notin WF(u)$. □

Remark 50.2. Consider the Cauchy problem (48.1), (48.2) assuming that $f(x,t) \in C^\infty(\overline{\mathbb{R}_+^{n+1}})$. Then, by the partial hypoellipticity (cf. Proposition 7.6), the solution $u(x,t)$ is a smooth function of $t \ge 0$ with the distribution values in $\mathcal{D}'(\mathbb{R}^n)$. We describe the wave front set of $u(x,t) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ for a fixed $t > 0$. It follows from Theorems 50.1 and 50.5 that if

$(x^{(0)}, t_0, \xi^{(0)}, \sigma^{(0)}) \in WF(u),\ t_0 > 0$, then $\xi^{(0)} \neq 0$ and $\sigma_0 = \lambda_{j0}(x^{(0)}, t_0, \xi^{(0)})$ for some $1 \leq j_0 \leq m$.

Denote by $W(t_0) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ the set consisting of all $(x^{(0)}, \xi^{(0)})$ such that $(x^{(0)}, t_0, \xi^{(0)}, \sigma_0) \in WF(u)$ for some $\sigma_0 \in \mathbb{R}^1$.

Lemma 50.6. *The wave front set $WF(u(\cdot, t_0))$ is contained in $W(t_0)$, $t_0 > 0$.*

Proof. Suppose $(x^{(0)}, \xi^{(0)}) \notin W(t_0)$, i.e., $(x^{(0)}, t_0, \xi^{(0)}, \sigma) \notin WF(u)$ for all $\sigma \in \mathbb{R}$. Note that $(x^{(0)}, t_0, 0, \pm 1) \notin WF(u)$ since $H_0(x^{(0)}, t_0, 0, \pm 1) \neq 0$ and $f \in C^\infty$ for $t > 0$. Then there exists $\varphi(x) \in C_0^\infty(\mathbb{R}^n),\ \varphi(x_0) \neq 0,\ \alpha(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\}),\ \deg_\xi \alpha(\xi) = 0,\ \alpha(\xi^{(0)}) \neq 0$, and a small $\delta > 0$ such that $\alpha(D_x)\varphi(x)\chi(\frac{t - t_0}{\delta})u \in C^\infty(\mathbb{R}^{n+1})$. In particular $\alpha(D_x)\varphi(x)u(x, t_0) \in C^\infty(\mathbb{R}^n)$, i.e., $(x^{(0)}, \xi^{(0)}) \notin WF(u(\cdot, t_0))$. Therefore $WF(u(\cdot, t_0))$ is contained in $W(t_0)$. □

When $m \geq 2$, it may happen that $WF(u(., t_0)) \neq W(t_0)$. For example, let $\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x, t) = 0$ in $\mathbb{R}^{n+1}$, $u(x, t_0) = 0$, $\frac{\partial u(x,t_0)}{\partial t} = \delta(x)$. Then $WF(u(\cdot, t_0)) = \emptyset$ and $(0, t_0, \xi^{(0)}, \pm|\xi^{(0)}|) \in WF(u)$ for any $\xi^{(0)} \neq 0$, i.e., $(0, \xi^{(0)}) \in W(t_0)$. However, in the case $m = 1$, i.e., when $H(x, t, D_x, D_t) = D_t - \lambda(x, t, D_x)$, we have $WF(u(\cdot, t_0)) = W(t_0)$. To prove this take any $(x^{(0)}, \xi^{(0)}) \notin WF(u(\cdot, t_0))$. Then there exist $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ and $\alpha(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\psi(x^{(0)}) \neq 0,\ \alpha(\xi^{(0)}) \neq 0,\ \alpha(\xi)$ is homogeneous of degree zero, and

$$\psi(x)\alpha(D_x)(1 - \chi(D_x))u(x, t_0) \in C^\infty(\mathbb{R}^n).$$

When $m = 1$, one can take the solution $\varphi(x, t, \xi)$ of (50.15) independent of σ. Therefore, taking $\varphi|_{t=t_0} = \psi(x)\alpha(\xi)(1 - \chi(\xi))$, we get by Theorem 50.1 that $\chi(\frac{t-t_0}{\delta})\varphi(x, t, D_x)u \in C^\infty(\mathbb{R}^{n+1})$. Thus, $(x^{(0)}, t_0, \xi^{(0)}, \sigma) \notin WF(u)$ for any $\sigma \in \mathbb{R}$, i.e., $(x^{(0)}, \xi^{(0)}) \notin W(t_0)$. Hence, $W(t_0) \subset WF(u(\cdot, t_0))$. Combining this with Lemma 50.6, we get $WF(u(\cdot, t_0)) = W(t_0)$ for $m = 1$.

Example 50.1. Consider the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = f(x, t), \quad t > 0,\ x \in \mathbb{R}^n, \qquad u(x, 0) = 0,$$

where $f(x, t) = h(t)\delta(x),\ h(t) \in C_0^\infty(\mathbb{R}_+^1),\ \operatorname{supp} h(t) = [1, 2] \cup [3, 4]$, and $\int_1^2 h(t)dt \neq 0,\ \int_1^2 h(t)dt + \int_3^4 h(t)dt = 0$. Then $u(x, t) = \delta(x)\int_{-\infty}^t h(\tau)d\tau$ is the solution of the Cauchy problem. We have $WF(f) = \{0, t, \xi, 0\},\ \forall \xi \neq 0$ and $\forall t \in [1, 2] \cup [3, 4]$, and $WF(u) = \{(0, t, \xi, 0)\}$, $\forall \xi \neq 0$ and $\forall t \in [1, 4]$. Note that here the set Σ defined in Theorem 50.5 is equal to $\{0, t, \xi, 0\},\ \ \forall \xi \neq 0$ and $\forall t \in [1, +\infty)$, i.e., $WF(u) \neq \Sigma$.

51. Problems

1. Consider a boundary value problem in the half-space $\mathbb{R}^n_+$:

$$\sum_{k=1}^{n} \frac{\partial^2 u(x', x_n)}{\partial x_k^2} - u(x', x_n) = 0, \quad x_n > 0, \tag{51.1}$$

$$-\frac{\partial u(x, x_n)}{\partial x_n} + \sum_{k=1}^{n-1} b_k(x') \, \frac{\partial u(x', x_n)}{\partial x_k} + \lambda u \Big|_{x_n=0} = g(x'), \tag{51.2}$$

where $x' = (x_1, \ldots, x_{n-1})$ and $b_k(x') \in C_0^\infty(\mathbb{R}^{n-1})$ are real-valued functions, $1 \le k \le n-1, \ \lambda > 0$.

a) Reduce the boundary value problem (51.1), (51.2) to a pseudodifferential equation $A(x', D')v(x') = g(x')$ in $\mathbb{R}^{n-1}$, where $v(x') = u(x', 0)$.

b) Find the adjoint operator $A^*(x', D')$.

c) Show that $A(x', D')v = g$ has a unique solution in $H_s(\mathbb{R}^{n-1})$, $s \in \mathbb{R}^1$, for any $g \in H_{s-1}(\mathbb{R}^{n-1})$, assuming that $\lambda > 0$ is sufficiently large.

d) Find the first two terms of the parametrix to $A(x', D')$.

2. Denote by L^m the class of symbols $A(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left| \frac{\partial^{p+k} A(x, \xi)}{\partial x^p \partial \xi^k} \right| \le C_{pk}(1 + |\xi| + |x|)^{m-|p|-|k|}, \quad \forall p, k. \tag{51.3}$$

An example of a symbol of class L^m is the symbol of the harmonic oscillator $-\frac{d^2}{dx_1^2} + x_1^2$.

a) Let $E(x, y)$ be the Schwartz kernel of $A(x, D)$ (cf. §44), where $A(x, \xi) \in L^{-N}$ and $N > 0$ is large. Prove that

$$\left| \frac{\partial^{p_1+p_2} E(x, y)}{\partial x^{p_1} \partial y^{p_2}} \right| \le C_N(1 + |x|)^{-N_1}(1 + |y|)^{-N_1}, \tag{51.4}$$

$0 \le |p_1| \le N_1, 0 \le |p_2| \le N_1$, where N_1 is large if N is large.

b) Prove that operator $A(x, D)$ is compact in $L_2(\mathbb{R}^2)$ if $A(x, \xi) \in L^{-\delta}, \delta > 0$.

c) We say that $A(x, \xi) \in L^m$ is elliptic if $|A(x, \xi)| \ge C(|x| + |\xi|)^m$ for $|x|^2 + |\xi|^2 \ge R^2, \ R > 0$. Prove that an elliptic operator with symbol $A(x, \xi) \in L^0$ is Fredholm in $L_2(\mathbb{R}^n)$.

d) Let $A(x, \xi) \in L^m$ be elliptic. Construct a parametrix to $A(x, D)$, i.e., a ψdo $R_N(x, D), R_N(x, \xi) \in L^{-m}$ such that $A(x, D)R_N(x, D) = I + T_{-N}$, where the kernel of T_{-N} satisfies estimates (51.4).

3. Suppose $A(x,\xi) = \sum_{k=0}^{2m} A_k(x,\xi)$ is a polynomial in x and in ξ. Let $\zeta = (x,\xi)$ and let $A_k(x,\xi) = \sum_{|p|=k} a_{pk}\zeta^p$, $0 \le k \le 2m$. Suppose $A_{2m}(x,\xi) = \sum_{|p|=2m} a_{p0}\zeta^p > 0$ when $\zeta = (x,\xi) \ne (0,0)$. Let $G(x,y,t)$ be the heat kernel for $A(x,D)$, i.e.,

$$\frac{\partial G}{\partial t} + A(x,D)G = 0 \quad \text{for} \quad t > 0, \tag{51.5}$$

$$G(x,y,0) = \delta(x-y). \tag{51.6}$$

Find the principal term of the asymptotics of the heat trace $\int_{\mathbb{R}^n} G(x,x,t)dx$ as $t \to 0$. Consider the particular case where $m = 1$, i.e., $A(x,\xi) = Q(\zeta,\zeta)$, where $Q(\zeta,\zeta)$ is a positive definite quadratic form in ζ.

4. Parabolic equations of higher order in D_t. Let $P_0(x,t,\xi,\sigma)$ be a quasi-homogeneous polynomial in (ξ,σ):

$$P_0(x,t,\alpha\xi,\alpha^b\sigma) = \alpha^m P_0(x,t,\xi,\sigma), \quad \forall\alpha > 0,$$

where $b > 1$, $r = \frac{m}{b}$ is an integer and r is the degree of P_0 in σ.

Such a P_0 is called a parabolic symbol if

$$|P_0(x,t,\xi,\sigma - i\tau)| \ge C(|\xi|^m + |\sigma|^{\frac{m}{b}} + \tau^{\frac{m}{b}}) \tag{51.7}$$

for all $\tau \ge 0$.

An equivalent definition of a parabolic symbol is that all the roots $\sigma = \mu_j(x,t,\xi)$, $1 \le j \le \frac{m}{b}$ of $P_0(x,t,\xi,\sigma) = 0$ have an estimate

$$\operatorname{Im}\mu_j(x,t,\xi) \ge C|\xi|^b, \ 1 \le j \le r, \quad r = \frac{m}{b}.$$

An example of parabolic symbol with $r > 1$ is $(i\sigma + |\xi|^2)^2$.

Consider the Cauchy problem

$$P(x,t,D_x,D_t)u(x,t) = f(x,t), \ \ t > 0, \ x \in \mathbb{R}^n, \tag{51.8}$$

$$\frac{\partial^k u(x,0)}{\partial t^k} = g_k(x), \ x \in \mathbb{R}^n, \ 0 \le k \le r-1, \tag{51.9}$$

where $P(x,t,\xi,\sigma) = P_0(x,t,\xi,\sigma) + P_1(x,t,\xi,\sigma)$, P_0 is parabolic, $r = \frac{m}{b}$, and $|P_1(x,t,\xi,\sigma - i\tau)| \le C(1 + |\xi| + |\sigma - i\tau|^{\frac{1}{b}})^{m-1}$.

Extend the results of §46 to this class of parabolic equations of order $r \ge 2$.

5. Elliptic pseudodifferential systems. Let $A(x,D)u(x) = f(x)$ be an $m \times m$ system of pseudodifferential equations where

$$u(x) = (u_1(x),\dots,u_m(x)), \ f = (f_1,\dots,f_m(x)), \quad x \in \mathbb{R}^n,$$

$$A(x,D) = [a_{jk}(x,D)]_{j,k=1}^m, \ a_{jk}(x,\xi) \in S^r(\mathbb{R}^n \times \mathbb{R}^n).$$

Such an $A(x,D)$ is called elliptic in $\mathbb{R}^n$ if

$$|\det A(x,\xi)| \geq C(1+|\xi|)^{mr} \quad \text{when } |x|^2+|\xi|^2 \geq R^2.$$

Prove that for each $s \in \mathbb{R}$, $A(x,D)$ is a Fredholm operator from $H_s(\mathbb{R}^n)$ to $H_{s-r}(\mathbb{R}^n)$.

6. Symmetric first order hyperbolic system. Consider the Cauchy problem for the first order symmetric hyperbolic system

$$D_t u(x,t) + A(x,t,D_x)u(x,t) = f(x,t), \quad x \in \mathbb{R}^n, \ t \in [0,T], \tag{51.10}$$

$$u(x,0) = g(x), \quad x \in \mathbb{R}^n, \tag{51.11}$$

where $A(x,t,\xi) = [a_{ij}(x,t,\xi)]_{i,j=1}^m$, $a_{ij} \in S^1(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ and

$$A^*(x,t,\xi) - A(x,t,\xi) = A_1 \in S^0. \tag{51.12}$$

a) Assuming that $u(x,t) \in H_{1,s}(\mathbb{R}^n \times (0,T))$, $g \in H_s(\mathbb{R}^n)$ and $f \in L^1((0,T), H_s(\mathbb{R}^n)$, prove the estimate

$$\max_{0 \leq t \leq T} [u(\cdot,t)]_s \leq C[g]_s + C\int_0^T [f(\cdot,t')]_s dt' \tag{51.13}$$

b) Prove the uniqueness for the Cauchy problem (51.10), (51.11) in $H_{0,s}(\mathbb{R}^n)$, $\forall s \in \mathbb{R}$.

c) Prove the existence of the solution of (51.10), (51.11) in $C((0,T), H_s(\mathbb{R}^n))$ using the Friedrichs regularization (see Taylor [**T2**]):
Let $A_\varepsilon(x,t,D_x) = A(x,t,D_x)\chi(\varepsilon D_x)$, $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$, and $\chi(\xi) = 1$ for $|\xi| \leq 1$. Prove the existence of u_ε and the estimate (51.13) with A replaced by A_ε. Then use the weak compactness to pass to the limit as $\varepsilon \to 0$.

7. Strictly hyperbolic system of the first order. Consider the Cauchy problem

$$D_t u(x,t) + L(x,t,D_x)u = f(x,t), \quad x \in \mathbb{R}^n, \ t \in [0,T], \tag{51.14}$$

$$u(x,0) = g(x), \quad x \in \mathbb{R}^n. \tag{51.15}$$

The system (51.14) is called strictly hyperbolic if $L(x,t,\xi) \in S^1$, $L(x,t,\xi) = L_0(x,t,\xi) + L_1(x,t,\xi)$ for $|\xi| \geq 1$, where $L_0(x,t,\xi)$ is homogeneous in ξ of order one, $L_1(x,t,\xi) \in S^0$, and $L_0(x,t,\xi)$ has m distinct real eigenvalues $\lambda_j(x,t,\xi)$, $1 \leq j \leq m$, for all $(x,t) \in \mathbb{R}^{n+1}$, $\xi \neq 0$.

a) Show that there exists $B_0(x,t,\xi)$, homogeneous of degree zero, $\det B_0(x,\xi) \neq 0, \forall (x,t)$, $\xi \neq 0$, such that $L_0(x,\xi) = B_0^{-1} D_0 B_0$, where $D_0 = [\lambda_j(x,t,\xi)\delta_{jk}]_{j,k=1}^m$ is a diagonal matrix. Prove

that $Q_0(x,t,\xi) = B_0^*(x,t,\xi)B_0(x,t,\xi)$ has the following property:
$Q_0(x,t,\xi)L_0(x,t,\xi) = L_0^*(x,t,\xi)Q_0(x,t,\xi)$, i.e., Q_0L_0 is a symmetric matrix.

b) Using the Gårding inequality (cf. §48), prove that

(51.16) $$Q_0(x,t,D_x)(1-\chi(D_x)) + C\Lambda^{-1}$$

is a positive definite operator when $C > 0$ is large, $\Lambda^{-1}(\xi) = (1+|\xi|^2)^{-\frac{1}{2}}$.

c) Use (51.16) to reduce the study of the Cauchy problem (51.14), (51.15) for the strictly hyperbolic system to the case of symmetric hyperbolic system (51.10), (51.11).

d) Consider the strictly hyperbolic equation of order m (cf. §48):

(51.17) $$H(x,t,D_x,D_t)u = f.$$

Setting $v_j = D_t^j\Lambda^{m-j}u, 1 \le j \le m$, reduce (51.17) to the strictly hyperbolic system of the first order.

8. Beals-Fefferman classes of ψdo's (see Beals [**B**] and Beals-Fefferman [**BF**]). Let $\Phi(x,\xi), \varphi(x,\xi)$ be C^∞ functions with the following properties:

a) $c \le \Phi(x,\xi) \le C(1+|\xi|),\ C \ge |\varphi(x,\xi)| \ge C(1+|\xi|)^{\varepsilon-1}, \varepsilon > 0.$

b) $\Phi\varphi \ge C$.

c) $\frac{\Phi(x,\xi)}{\varphi(x,\xi)} \sim \frac{\Phi(y,\eta)}{\varphi(y,\eta)}$ if $|\xi| \sim |\eta|$, where $A \sim B$ means $c_1 \le \frac{A}{B} \le c_2$.

d) $\left|\frac{\partial^{p+q}\Phi(x,\xi)}{\partial x^p\partial\xi^q}\right| \le C_{pq}\Phi^{1-|q|}\varphi^{-|p|}, \left|\frac{\partial^{p+q}\varphi(x,\xi)}{\partial x^p\partial\xi^q}\right| \le C_{pq}\Phi^{-|q|}\varphi^{1-|p|}$, $\forall p,q$.

We say that the symbol $A(x,\xi)$ belongs to $S^{M,m}_{\Phi,\varphi}$ if $A(x,\xi) \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and $\left|\frac{\partial^{p+q}A(x,\xi)}{\partial x^p\partial\xi^q}\right| \le C_{pq}\Phi^{M-|q|}\varphi^{m-|p|},\ \forall p,q$. A particular case of the class $S^{M,m}_{\Phi,\varphi}$ is Hörmander's class $S^m_{\rho,\delta}$ of symbols $A(x,\xi) \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ satisfying

$$\left|\frac{\partial^{p+q}A(x,\xi)}{\partial x^p\partial\xi^q}\right| \le C_{pq}(1+|\xi|)^{m-\rho|q|+\delta|p|},$$

where $0 \le \delta < \rho \le 1$. Here $\Phi = (1+|\xi|^2)^{\frac{\rho}{2}},\ \varphi = (1+|\xi|^2)^{-\frac{\delta}{2}}$. The following theorem is proven in Beals-Fefferman [**BF**]:

Theorem 51.1. *If $A(x,\xi) \in S^{0,0}_{\Phi,\varphi}$, then*

(51.18) $$\|A(x,D)u\|_0 \le C\|u\|_0,\quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

Prove that Theorems 40.2 and 43.2 hold for $A(x,\xi) \in S^{M_1,m_1}_{\Phi,\varphi}$, $B(x,\xi) \in S^{M_2,m_2}_{\Phi,\varphi}$

9. The sharp Gårding inequality (see Beals [**B**] and Beals-Fefferman [**BF**]). Let $A(x,\xi) \in S^m$ and $A(x,\xi) \geq C > 0,\ m > 0$. Then

$$\mathrm{Re}(A(x,D)u,u) \geq -C\|u\|^2_{\frac{m-1}{2}}, \forall u \in C_0^\infty(\mathbb{R}^n). \tag{51.19}$$

One way to prove the sharp Gårding inequality is to use the Beals-Fefferman calculus.

Suppose $m = 1$. Define $\Phi(x,\xi) = (1+|\xi|^2)^{\frac{1}{4}}\sqrt{A(x,\xi)}$, $\varphi(x,\xi) = (1+|\xi|^2)^{-\frac{1}{4}}\sqrt{A(x,\xi)}$.

a) Prove that $\Phi(x,\xi)$ and $\varphi(x,\xi)$ satisfy a)–d) and that $A(x,\xi) \in S^{1,1}_{\Phi,\varphi}$ and $\sqrt{A(x,\xi)} \in S^{\frac{1}{2},\frac{1}{2}}_{\Phi,\varphi}$.

b) Complete the proof of (51.19).