Chapter 1

Complex Analysis

1.1. Green’s Theorem and the Cauchy-Green Formula

1.1.1. Green’s Theorem.

**Theorem 1.1.1** (Green’s Theorem). Let $D \subset \subset \mathbb{R}^2$ be an open connected set\(^1\) whose boundary $\partial D$ is piecewise smooth and positively oriented.\(^2\) Let $P$ and $Q$ be smooth, complex-valued functions on some neighborhood of the closure $\overline{D}$ of $D$. Then

$$\oint_{\partial D} P\,dx + Q\,dy = \int_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \,dA.$$  

A proof of Green’s Theorem can be found in almost any calculus text.

1.1.2. The Cauchy-Green Formula. Green’s Theorem can be used to prove the Cauchy-Green Integral Formula. To state the latter, we recall Wirtinger’s complex partial derivatives, defined as

$$\frac{\partial f}{\partial z} := \frac{1}{2} \left( \frac{\partial f}{\partial x} - \sqrt{-1} \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial f}{\partial x} + \sqrt{-1} \frac{\partial f}{\partial y} \right).$$

(The notation $A \subset \subset B$ means that item $A$ is defined to be the known item $B$.)

**Theorem 1.1.2.** Let $D \subset \subset \mathbb{C}$ be open and connected with piecewise smooth boundary, and let $f : \overline{D} \to \mathbb{C}$ be a $C^1$-function. Then for all $z \in D$,

$$f(z) = \frac{1}{2\pi \sqrt{-1}} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \,d\zeta - \frac{1}{\pi} \iint_{D} \frac{\partial f}{\partial \bar{z}} \frac{dA(\zeta)}{\zeta - z}.$$  

\[^1\text{The notation } A \subset \subset B \text{ means ‘}A\text{ is relatively compact in }B\text{’.}\]

\[^2\text{The boundary } \partial D \text{ is oriented positively if, when one moves forward along } \partial D, \text{ one finds } D \text{ to one’s left.}\]
**Proof.** Fix $z \in D$. Fix $\varepsilon > 0$ such that $D(z, \varepsilon) := \{ x \in \mathbb{C} ; |x - z| < \varepsilon \} \subseteq D$. Applying Green’s Theorem to the functions

\[ P = \frac{1}{2\pi\sqrt{-1}} f(\zeta) \quad \text{and} \quad Q = \frac{\sqrt{-1}}{2\pi\sqrt{-1}} f(\zeta) \]

and the domain

\[ D_\varepsilon := D - D(z, \varepsilon), \]

we obtain, using the definition of the complex partial derivatives, the formula

\[
\frac{1}{2\pi\sqrt{-1}} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi\sqrt{-1}} \int_{\partial D(z, \varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{\pi} \iint_{D_\varepsilon} \frac{\partial f}{\partial \zeta} A(\zeta) dA(\zeta) \zeta - z.
\]

In polar coordinates centered at $z$, the measure $dA$ is $r dr d\theta$, and since $\zeta - z = re^{i\theta}$, the integrand of the right integral is locally bounded. It follows that, as $\varepsilon \to 0$, the right side converges to

\[
\frac{1}{\pi} \iint_{D} \frac{\partial f}{\partial \zeta} A(\zeta) dA(\zeta) \zeta - z.
\]

On the other hand, since $f$ is differentiable, we have

\[
\int_{\partial D(z, \varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial D(z, \varepsilon)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + 2\pi\sqrt{-1} f(z) \to 2\pi\sqrt{-1} f(z)
\]
as $\varepsilon \to 0$. \qed

## 1.2. Holomorphic functions and Cauchy Formulas

### 1.2.1. The homogeneous Cauchy-Riemann equations.

Recall the definition of a holomorphic function.

**Definition 1.2.1.** A $C^1$ function $f$ is **holomorphic** in a domain $D$ if and only if it satisfies the homogeneous Cauchy-Riemann equations

\[
\frac{\partial f}{\partial \overline{z}} \equiv 0
\]
on $D$.

**Remark.** As is well known, one can relax the condition that $f$ is $C^1$ to assume only that $f$ is a distribution.\(^3\) We will not worry so much about regularity issues in the definition of holomorphic functions, since such issues are often thoroughly treated in a first serious course in complex analysis. Later in this chapter we will establish regularity for harmonic functions, which can be used to prove the regularity of a holomorphic distribution.

\(^3\)It is assumed that the reader is familiar with distributions, though the lack of familiarity with this notion does not pose a significant problem to following this book. References for distributions abound, but perhaps the most thorough reference relevant to the present book is [Hörmander-2003].
1.3. Power series

REMARK. Some readers may wonder why the author refers to this equation in the plural form. The reason is classical; the equation is a complex one and therefore has two components. Moreover, from the point of view of partial differential equations, the pair of equations obtained by taking real and imaginary parts form an elliptic system, and ultimately this ellipticity is responsible for the regularity of the functions in the kernel, though in this setting one can give a direct proof without referring to ellipticity. In any case, we have chosen to keep the plural.

1.2.2. Cauchy’s Theorem and the Cauchy Integral Formula. As a corollary of the Cauchy-Green Formula, we obtain the Cauchy Theorem and the Cauchy Integral Formula for \( C^1 \)-functions.

**Theorem 1.2.2.** Let \( D \subset \subset \mathbb{C} \) be open with piecewise smooth boundary, and let \( f \) be holomorphic on a neighborhood of the closure of \( D \). Then the following hold.

1. (Cauchy’s Theorem)
   \[
   \int_{\partial D} f(z) \, dz = 0.
   \]
2. (Cauchy Integral Formula)
   \[
   f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta.
   \]

**Proof.** The second result is obvious from the Cauchy-Green Formula. For the first result, apply the Cauchy-Green Formula to the function \( z \mapsto (z - \zeta) f(z) \).

1.3. Power series

1.3.1. Series representation. Suppose \( f \) is holomorphic in a neighborhood of the closure of the disk \( \mathbb{D}(0, r) \). For \( z \in \mathbb{D}(0, r) \) and \( |\zeta| = r \) we have the series
   \[
   \frac{1}{\zeta - z} = \frac{1}{\zeta(1 - \frac{z}{\zeta})} = \sum_{j=0}^{\infty} \frac{z^j}{\zeta^{j+1}}.
   \]

Applying the Cauchy Integral Formula, we find that
   \[
   f(z) = \sum_{j=0}^{\infty} a_j z^j,
   \]
where \( a_j = a_j(f, 0) := \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta) \, d\zeta}{\zeta^{n+1}} \).

In particular, \( f \) is given locally by a convergent power series. A similar result, properly scaled and shifted, holds at any point \( z_0 \) in place of the origin:
   \[
   f(z) = \sum_{j=1}^{\infty} a_j (z - z_0)^j,
   \]
where
   \[
   a_j = a_j(f, z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta) \, d\zeta}{(\zeta - z_0)^{n+1}}.
   \]
It follows that $f$ is infinitely complex-differentiable and that
\begin{equation}
\frac{\partial^n f}{\partial z^n} \bigg|_{z=z_0} = \frac{n!}{2\pi \sqrt{-1}} \int_{|\zeta-z_0|=r} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}.
\end{equation}

### 1.3.2. Corollaries.

The fact that $f$ is given by a convergent power series implies the following theorems.

**Theorem 1.3.1.** *The uniform limit of holomorphic functions is holomorphic.*

**Theorem 1.3.2 (Identity Theorem).** *If two holomorphic functions are defined on an open connected set $D \subset \mathbb{C}$ and they agree on an open subset of $D$, then they agree on all of $D$.***

We also have the following fundamental definition.

**Definition 1.3.3 (Order).** Let $f$ be holomorphic in a neighborhood of a point $x \in \mathbb{C}$. We say that $f$ has a zero of order $n$ at $x$ if $n$ is the smallest integer such that
\[
\left. \frac{\partial^n f}{\partial z^n} \right|_{z=z_0} \neq 0.
\]

We write $n = \text{Ord}_x f$.

### 1.3.3. Cauchy Estimates.

Estimating (1.2) gives us the following result.

**Proposition 1.3.4 (Cauchy Estimates).** *Let $f : \mathbb{D}(0,R) \to \mathbb{C}$ be a holomorphic function. Then
\[
\left| \frac{\partial^n f}{\partial z^n} \right|_{z=0} \leq \frac{n!}{R^n} \sup_{\mathbb{D}(0,R)} |f|.
\]

As a corollary, we have Liouville’s Theorem.

**Corollary 1.3.5 (Liouville).** *Any bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

**Proof.** Let $C := \sup_{\mathbb{C}} |f|$. For any $p \in \mathbb{C}$ and $R > 0$, $|f'(p)| \leq CR^{-1}$. Thus $f' \equiv 0$. $\square$

### 1.4. Isolated singularities of holomorphic functions

#### 1.4.1. Laurent series.

Let $A$ be an annulus centered at 0 with radii $R > r \geq 0$, and let $f$ be holomorphic on $A$. Choose $R'$ and $r'$ such that $R > R' > r' > r$. Then by the Cauchy Integral Formula,
\[
f(z) = \frac{1}{2\pi \sqrt{-1}} \int_{|\zeta|=R'} \frac{f(\zeta)}{\zeta-z} d\zeta - \frac{1}{2\pi \sqrt{-1}} \int_{|\zeta|=r'} \frac{f(\zeta)}{\zeta-z} d\zeta, \quad r' < |z| < R'.
\]
In the first integral, \(|z| < |\zeta| = R\), while in the second \(r' = |\zeta| < |z|\). If we write

\[ \zeta - z = \zeta \left( 1 - \frac{z}{\zeta} \right) \quad \text{and} \quad \zeta - z = -z \left( 1 - \frac{\zeta}{z} \right) \]

in the first integral and in the second integral, respectively, then using the fact that

\((1 - r)^{-1} = 1 + r + r^2 + \ldots\)

we find

\begin{equation}
(1.3) \quad f(z) = \sum_{n \in \mathbb{Z}} a_n z^n,
\end{equation}

where

\begin{equation}
(1.4) \quad a_n = \frac{1}{2\pi \sqrt{-1}} \int_{|z|=\rho} \frac{f(z)}{z^{n+1}} dz,
\end{equation}

for some \(\rho \in (r, R)\). Since \(f(z)/z^n\) is analytic in \(A\), the latter integral is independent of \(\rho\).

**Definition 1.4.1.** The series (1.3) is called the *Laurent series* of \(f\). The number

\[ \text{Res}_0(f) := a_{-1} \]

is called the *residue* of \(f\) at 0.

More generally, if \(c \in \mathbb{C}\) and \(g\) is holomorphic in a punctured neighborhood of \(c\) (i.e., a neighborhood of \(c\) from which \(c\) is then removed), we define

\[ \text{Res}_c(g) := \text{Res}_0(f), \quad \text{where} \quad f(z) = g(z + c). \]

Using the Laurent series of \(f\) at each singularity of \(f\), one obtains the following theorem.

**Theorem 1.4.2 (Residue Theorem).** If \(z_1, \ldots, z_k \in D\) are distinct and \(f : \overline{D} - \{z_1, \ldots, z_k\} \to \mathbb{C}\) is holomorphic, then

\[ \frac{1}{2\pi \sqrt{-1}} \int_{\partial D} f(z) dz = \text{Res}_{z_1}(f) + \ldots + \text{Res}_{z_k}(f). \]

### 1.4.2. Singularities of holomorphic functions at a puncture.

**Definition 1.4.3.** Let \(f\) be holomorphic in a punctured neighborhood of a point \(z \in \mathbb{C}\).

1. We say that \(f\) has a *pole of order* \(n \geq 1\) if the Laurent series \(\sum a_j (\zeta - z)^j\) of \(f\) at \(z\) has the property that \(a_{-n} \neq 0\) but \(a_{-k} = 0\) for \(k > n\).

2. If \(a_j = 0\) for all \(j < 0\) at \(z\), i.e., the Laurent series of \(f\) at \(z\) is holomorphic across \(z\), we say that \(z\) is a *removable singularity* of \(f\).

3. If the Laurent series of \(f\) has infinitely many negative coefficients, we say that \(f\) has an *essential singularity* at \(z\).
(4) If \( f \) does not have an essential singularity at \( z \), we define
\[
\text{Ord}_z(f) := k
\]
where \( k \) is the largest integer such that \( a_\ell = 0 \) for each \( \ell < k \).

**Remark.** Note that if \( f \) has a pole of order \( k > 0 \) then \( \text{Ord}_x(f) = -k \).

**Definition 1.4.4 (Meromorphic function).** Let \( D \subset \mathbb{C} \).

1. We say that \( f \) is meromorphic at a point \( p \in D \) if there is a punctured neighborhood \( U \) of \( p \) in \( D \) such that \( f \) is holomorphic on \( U \) and \( p \) is not an essential singularity for \( f \).
2. We say that \( f \) is meromorphic on \( D \) if \( f \) is meromorphic at all points of \( D \).

We have the following characterization of singularities.

**Theorem 1.4.5.** Let \( U \) be a neighborhood of \( p \in \mathbb{C} \) and let \( f : U - \{p\} \to \mathbb{C} \) be holomorphic.

1. (Riemann Removable Singularities Theorem) If \( f \) is bounded in a neighborhood of \( p \) then \( f \) extends holomorphically to \( U \). In particular, the limit \( \lim_{z \to p} f(z) \) exists.
2. If \( \lim_{z \to p} |f(z)| = +\infty \), then \( f \) has a pole at \( z \).

**Proof.** We may assume \( p = 0 \).

1. Suppose \( |f| \leq M \) in a small disk centered at the origin and let \( \sum_{n \in \mathbb{Z}} a_n z^n \) be the Laurent series of \( f \) at 0. By (1.4), \( |a_n| \leq 2\pi M e^{-n} \) for all sufficiently small \( \varepsilon \). For \( n < 0 \) this means that \( a_n = 0 \).
2. Choose a disk \( D \) centered at the origin such that \( f \) does not vanish on \( \overline{D} \). Then the function \( g(z) = 1/f(z) \) is holomorphic and bounded on \( D - \{0\} \). By part (1), \( g \) extends to a holomorphic function on \( D \), and since it has limit zero at 0, the power series expansion on \( g \) shows there is an integer \( k \geq 1 \) such that \( g(z) = z^k h(z) \) for some holomorphic function \( h \) satisfying \( h(0) \neq 0 \). Thus near the origin, \( f(z) = z^{-k} \varphi(z) \) for some holomorphic function \( \varphi \) satisfying \( \varphi(0) \neq 0 \).

**Remark** (Weierstrass-Casorati Theorem). If \( f \) is unbounded on \( U - \{p\} \) and \( |f(q)| \) is not arbitrarily large for all points \( q \) sufficiently near \( p \), then according to Theorem 1.4.5, \( f \) has an essential singularity at \( p \). In this case we can say more: the image of \( U \) is dense in \( \mathbb{C} \). Indeed, if not, then there is a disk \( D \subset \mathbb{C} \cup \{\infty\} \) such that \( f(U - \{p\}) \subset D \). We can then choose constants \( a, b, c, e \in \mathbb{C} \) such that
\[
g(z) = \frac{a + bf(z)}{c + ef(z)} \in D(0, 1) \quad \text{for all} \ z \in U.
\]
Then \( g \) extends to \( p \), contradicting that \( f \) has an essential singularity.
1.4. Isolated singularities of holomorphic functions

Proposition 1.4.6. If $D$ is a disk and $g : D \to \mathbb{C} - \{0\}$ is holomorphic then $g = e^{h}$ for some holomorphic function $h$ on $D$.

Proof. Fix $p \in D$. Since $g'(z)/g(z)$ is holomorphic on $D$, we can define

$$H(z) = \int_{p}^{z} \frac{g'(\zeta)}{g(\zeta)} d\zeta,$$

where the integral is over some path connecting $p$ to $z$. By Cauchy’s Theorem, the integral is independent of the path connecting $p$ to $z$. We therefore find that $H'(z) = g'(z)/g(z)$, and hence

$$\frac{d}{dz} \left( \frac{g(z)}{e^{H(z)}} \right) = \frac{g'(z)}{e^{H(z)}} - \frac{g(z)H'(z)}{e^{H(z)}} = 0.$$

Thus $g = ce^{H} = e^{h}$. □

Corollary 1.4.7 (Normal Form Theorem). If $f$ is holomorphic near $p$ and $k = \text{Ord}_{p}(f)$ then there is a neighborhood $U$ of $p$ and an injective holomorphic function $g : U \to \mathbb{C}$ such that $f(g^{-1}(w)) = w^{k}$.

Proof. We may assume $p = 0$. By power series, we can write $f(z) = z^{k}G(z)$ with $G(0) \neq 0$. According to Proposition 1.4.6, $f(z) = z^{k}e^{h(z)}$ for some holomorphic $h$. Then $f(z) = (g(z))^{k}$ with $g(z) = ze^{h(z)}$. Since $g'(0) \neq 0$, $g$ is injective by the Inverse Function Theorem. □

Remark. If $|a|$ is sufficiently small but non-zero, then the equation $f(z) = a$ has exactly $k$ distinct solutions $p_{1}, ..., p_{k}$ in $U$, namely $p_{j} = g^{-1}(|a|^{1/k}\omega_{k}^{j})$, where $\omega_{k}$ is a $k$th root of unity. For this reason, we sometimes call the number $k$ the multiplicity of $f$ at $p$. □

The next theorem is an easy and important consequence of Corollary 1.4.7.

Theorem 1.4.8. Let $U \subset \mathbb{C}$ be open.

1. If $f : U \to V$ is injective holomorphic then so is $f^{-1} : f(U) \to U$.
2. If $f : U \to \mathbb{C}$ is a non-constant holomorphic function, then $f(U)$ is open.
3. The zero set of a non-constant holomorphic function is discrete.

Let $f$ be holomorphic in a punctured neighborhood of $0$ and assume $0$ is not an essential singularity. In this case, there is an integer $k$ and a holomorphic function $g$ such that $f(z) = z^{k}g(z)$ and $g(0) \neq 0$. Therefore

$$\frac{f'(z)}{f(z)} = k + \frac{g'(z)}{g(z)}.$$

Modification of this argument to points other than the origin yields the following theorem.
THEOREM 1.4.9 (Argument Principle). Let \( D \subset \mathbb{C} \) be an open connected set with piecewise smooth boundary, and let \( f \) be a meromorphic function on a neighborhood of \( D \) having no zeros or poles on \( \partial D \). Then
\[
\frac{1}{2\pi \sqrt{-1}} \int_{\partial D} \frac{f'(z)}{f(z)} \, dz = \sum_{z \in D} \text{Ord}_z(f).
\]

1.5. The Maximum Principle

THEOREM 1.5.1 (Maximum Principle). If \( f : \{ |z| < r \} \to \mathbb{C} \) is holomorphic and there is a local maximum for \( |f| \) at some \( z_0 \) with \( |z_0| < r \), then \( f \) is constant.

**Proof.** We can assume \( f(z_0) \neq 0 \). For \( \rho > 0 \) small, we find from the Cauchy Formula that
\[
f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{\sqrt{-1}\theta}) \, d\theta
\]
for any \( \rho < \text{dist}(z_0, \{ |z| \geq r \}) \). Rearranging and taking real parts, we have
\[
\int_0^{2\pi} \text{Re} \left( 1 - \frac{f(z_0 + \rho e^{\sqrt{-1}\theta})}{f(z_0)} \right) \, d\theta = 0.
\]

By hypothesis, the integrand is continuous and non-negative, and therefore \( \text{Re} \ f \) is constant. By the Cauchy-Riemann equations, \( f \) is constant. \( \square \)

From the Maximum Principle we obtain the well-known Schwarz Lemma.

THEOREM 1.5.2 (Schwarz Lemma). Let \( f : \mathbb{D} \to \mathbb{D} \) be holomorphic and suppose \( f(0) = 0 \). Then for all \( z \in \mathbb{D} \), \( |f(z)| \leq |z| \). In particular, \( |f'(0)| \leq 1 \). Equality holds in the first estimate for some \( z \in \mathbb{D} - \{0\} \) or in the second estimate if and only if \( f(z) = e^{\sqrt{-1}\theta} z \) for some constant \( \theta \in \mathbb{R} \).

**Proof.** Fix \( r \in (0, 1) \). Let \( g_r(z) := \frac{f(rz)}{rz} \) for \( z \neq 0 \) and \( g_r(0) := f'(0) \). Then \( g_r \) is holomorphic on \( \{ |z| < 1/r \} \), and thus we have
\[
|g_r(z)| \leq \max_{|\zeta|=1} |g_r(\zeta)| = \max_{|\zeta|=1} |f(r\zeta)|/r \leq 1/r.
\]

Thus \( |f(rz)| \leq |z| \). Letting \( r \to 1 \) gives the estimates.

If equality holds at any point \( z_0 \in \mathbb{D} \), i.e., \( g_1(z_0) = 1 \), then for some \( r \in (|z_0|, 1) \), \( g_r(z_0/r) = 1 \). By the Maximum Principle \( g_r \) is constant. Thus \( |f(rz)| = |rz| \) holds for \( |z| < 1/r \). It follows that \( f(z) = e^{\sqrt{-1}\theta} z \) for some constant \( \theta \in \mathbb{R} \). The proof is complete. \( \square \)
1.6. Compactness theorems

1.6.1. Montel’s Theorem.

THEOREM 1.6.1 (Montel). Let $U \subset \mathbb{C}$ be an open set. Let $K_1 \subset K_2 \subset \ldots \subset U$ be a sequence of compact sets such that any compact subset $K \subset U$ is contained in some $K_j$. Let $M_1 \leq M_2 \leq \ldots$. Then the set of holomorphic functions

$$\mathcal{B}_{K,M} := \{ f : U \to \mathbb{C} ; \sup_{K_j} |f| \leq M_j, j = 1, 2, \ldots \}$$

is compact in $\mathcal{O}(U)$ in the compact-open topology.

Proof. By the Cauchy Estimates, the family $\mathcal{B}_{K,M}$ is uniformly bounded and equicontinuous on each open subset $V \subset U$. By the Arzela-Ascoli Theorem, every sequence in $\mathcal{B}_{K,M}$ has a uniformly convergent subsequence. By a diagonal argument, every sequence in $\mathcal{B}_{K,M}$ has a uniformly convergent subsequence. An application of Theorem 1.3.1 completes the proof. □

COROLLARY 1.6.2. Let $U \subset \mathbb{C}$ be an open set such that $\mathbb{C} - U$ has interior points. Then the set

$$\mathcal{F} := \{ f \in \mathcal{O}(D) ; f(D) \subset U \}$$

is compact in $\mathcal{O}(D)$ in the compact-open topology.

Proof. Let $p \in \mathbb{C} - U$ be an interior point. The transformation $f \mapsto (f - p)^{-1}$ maps $\mathcal{F}$ to a family of uniformly bounded holomorphic functions. An application of Montel’s Theorem completes the proof. □

1.6.2. Köbe’s Compactness Theorem. A fundamental result in complex analysis is the following theorem of P. Köbe.

THEOREM 1.6.3 (Köbe). The collection $\mathcal{I}$ of all injective holomorphic functions $f : D \to \mathbb{C}$ such that $f(0) = 0$ and $f'(0) = 1$ is compact in $\mathcal{O}(D)$ in the compact-open topology.

Proof. Fix a sequence $\{ f_n \} \subset \mathcal{I}$. Let

$$R_n := \sup \{ R > 0 ; D(0, R) \subset f_n(D) \}.$$

Since $f_n^{-1} \mid_{D(0,R)} : D(0,R) \to D$ is holomorphic for any $D(0,R) \subset f(D)$, the Schwarz Lemma implies that $R_n \leq 1$. Choose a point $x_n \in \partial D(0, R_n) - f_n(D)$ and let $g_n := f_n/x_n$. Then $D \subset g_n(D) \not\subset 1$.

Now, $g_n(D)$ is simply connected, so there is a holomorphic branch $\psi$ of $\sqrt{z - 1}$ such that $\psi(0) = \sqrt{-1}$. Then $h_n := \psi \circ g_n$ satisfies $h_n^2 = g_n - 1$.

We claim that $h_n(D) \cap (-h_n(D)) = \emptyset$. Indeed, if $w = h_n(z)$ and $-w = h_n(z')$, then $g_n(-w) = g_n(w)$ and by injectivity $w = -w = 0$. But then $g_n(z) = 1$, which is impossible.
Since \( \mathbb{D} \subset g_n(\mathbb{D}) \), we have \( U := \psi(\mathbb{D}) \subset h_n(\mathbb{D}) \), and thus \( (-U) \cap h_n(\mathbb{D}) = \emptyset \). By Corollary 1.6.2, \( h_n \) has a convergent subsequence. Thus, since \( |x_n| = R_n \leq 1 \), \( f_n = x_n(1 + h_n^2) \) has a convergent subsequence. Let \( f \) be the limit of this subsequence.

Let \( a \in f(\mathbb{D}) \). By the argument principle, for any \( z \) with \( f(z) \neq a \),

\[
\ell := \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} \frac{f'(z)dz}{f(z) - a}
\]

is an integer, which evidently is \( \geq 1 \) for \( r \) sufficiently close to 1. Since \( f_n \) is arbitrarily close to \( f \) on \( |z| \leq r \), the injectivity of \( f_n \) implies that

\[
1 = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} \frac{f_n'(z)dz}{f_n(z) - a} = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} \frac{f'(z)dz}{f(z) - a} = \ell.
\]

Thus \( f \) is injective, and the proof is complete. \( \square \)

**Remark.** Clearly for each \( f \in \mathcal{S} \), \( f(\mathbb{D}) \) contains a disk centered at the origin. Since \( \mathcal{S} \) is compact, the radius of this disk is bounded below. For many applications, the fact that the radius is bounded below suffices. Nevertheless, Köbe conjectured that the smallest possible radius is \( 1/4 \). Köbe knew that \( 1/4 \) was sharp, realized for the following function:

\[
f(z) = \frac{1}{4} \left(1 - \left(\frac{z - 1}{z + 1}\right)^2\right).
\]

Köbe’s conjecture was proved by Bieberbach, and though we will not need it, we will give a proof here. To this end, let \( f \in \mathcal{S} \). Suppose \( w \in \mathbb{C} - f(\mathbb{D}) \). Then, with

\[
g(z) := \frac{wf(z)}{w - f(z)},
\]

\( g \in \mathcal{S} \). Note that \( g(0) = 0 \), \( g'(0) = 1 \), and \( g''(0) = f''(0) + 2w^{-1} \). Now, since \( \mathcal{S} \) is compact,

\[
D_2 := \sup\{|f''(0)| : f \in \mathcal{S}\}
\]

is finite. We therefore have the estimate

\[
D_2 \geq |g''(0)| \geq 2|w|^{-1} - |f''(0)| \geq 2|w|^{-1} - D_2,
\]

so that \( |w| \geq D_2^{-1} \).

Observe that if \( f \in \mathcal{S} \) then the functions

\[
\check{f}(z) = (f(z^{-1}))^{-1} \quad \text{and} \quad \check{f}(z) := \frac{1}{\sqrt{f(z^{-2})}},
\]

where we take the branch of \( \sqrt{\cdot} \) sending \(-1\) to \( \sqrt{-1} \), are both injective on the complement of the unit disk and have a Laurent series of the form

\[
z + b_1z^{-1} + b_2z^{-2} + \cdots.
\]
Let us denote by $\Sigma$ the set of univalent maps of $\mathbb{C} - \mathbb{D}$ whose Laurent series has the form (1.5). For any $g \in \Sigma$ and any $r > 1$,

$$
0 \leq \text{Area inside}(g(\partial \mathbb{D}(0, r))) = \frac{1}{2\sqrt{-1}} \int_{g(\partial \mathbb{D}(0, r))} \bar{w}dw = \frac{1}{2\sqrt{-1}} \int_{|z|=r} \frac{g(z)g'(z)dz}{2\sqrt{-1}}.
$$

As the reader can check,

$$
\lim_{r \to 1} \frac{1}{2\sqrt{-1}} \int_{|z|=r} \frac{g(z)g'(z)dz}{2\sqrt{-1}} = \pi \left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right).
$$

We therefore see that $|b_1| \leq 1$.

Finally, the reader can check that $\tilde{f}(z) = z - \frac{f''(0)}{4}z^{-1} + \ldots$. Our bound for $|b_1|$ on $\Sigma$ thus shows $D_2 \leq 4$. We have therefore proved the following result.

**Theorem 1.6.4 (Köbe $\frac{1}{4}$-Theorem).** For all $f \in \mathcal{S}$, $f(\mathbb{D}) \supset \mathbb{D}(0, \frac{1}{4})$, and the number $\frac{1}{4}$ is sharp.

### 1.7. Harmonic functions

#### 1.7.1. Laplacian and harmonic

The Laplace operator (or Laplacian) $\Delta$ is defined in a plane domain by

$$
\Delta u = u_{xx} + u_{yy} = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}.
$$

**Definition 1.7.1.** A function $u \in \mathcal{C}^2(\Omega)$ is said to be harmonic if $\Delta u \equiv 0$, in which case we write $u \in H(\Omega)$.

**Remark.** As we will see, the requirement that $H(\Omega) \subset \mathcal{C}^2(\Omega)$ can be substantially reduced.

**Remark.** Since $\Delta$ is a real operator, $f = u + \sqrt{-1}v \in H(\Omega)$ if and only if $u, v \in H(\Omega)$.

#### 1.7.2. Harmonic conjugate

From the formula $\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$, one sees that every holomorphic function is harmonic, as is the complex conjugate of a holomorphic function. On a simply connected domain, the converse holds, i.e., every harmonic function is the real part of a holomorphic function, which may be seen as follows. Let

$$
d^c := \frac{1}{2\sqrt{-1}} (\partial - \bar{\partial}) = \frac{1}{2} (-dx \otimes \partial_y + dy \otimes \partial_x).
$$

Then for a function $h$ one has

$$
dd^c h = \sqrt{-1}\partial \bar{\partial} h = 2 (\Delta h) dx \wedge dy.
$$
(A review of exterior calculus, in the setting of Riemann surfaces, is presented in Chapter 5.) Suppose \( u \) is a real-valued harmonic function on a simply connected domain \( \Omega \subset \mathbb{C} \). Then \( d^c u \) is a closed 1-form. It essentially follows from Green’s Theorem and the simple connectivity of \( \Omega \) that, given \( z_o \in \Omega \) and a curve \( \gamma_{z_0,z} : [0, 1] \to \Omega \) satisfying \( \gamma_{z_0,z}(0) = z_o \) and \( \gamma_{z_0,z}(1) = z \), the function

\[
v(z) := 2 \int_{\gamma_{z_0,z}} d^c u
\]

is well defined independent of the choice of \( \gamma_{z_0,z} \). (If we change \( z_o \), the function \( v(z) \) changes by an additive constant.)

We claim that the function \( f = u + \sqrt{-1} v \) is holomorphic. Indeed, since \( v \) is independent of the curve \( \gamma_{z_0,z} \), by choosing the \( z \)-end of \( \gamma_{z_0,z} \) to be horizontal or vertical, one obtains from the Fundamental Theorem of Calculus that

\[
v_x = -u_y \quad \text{and} \quad v_y = u_x.
\]

These equations mean precisely that \( \bar{\partial} f = 0 \), and our claim is proved.

### 1.7.3. Mean values on circles.

**Proposition 1.7.2 (Mean Value Property).** If \( u \in H(\Omega) \) and \( D(p, r) \subset \Omega \) then

\[
u(p) = \frac{1}{2\pi} \int_0^{2\pi} u(p + re^{\sqrt{-1} \theta}) d\theta.
\]

**Proof.** Let

\[
A_r(u) := \frac{1}{2\pi} \int_0^{2\pi} u(p + re^{\sqrt{-1} \theta}) d\theta.
\]

By continuity, \( \lim_{r \to 0} A_r(u) = u(p) \). By the divergence form of Green’s Theorem,

\[
\frac{dA_r(u)}{dr} = \frac{1}{2\pi r} \int_{|z-p|<r} \Delta u dxdy = 0.
\]

The proof is complete. \( \square \)

**Corollary 1.7.3.** Let \( \psi \) be an integrable, compactly supported, radially symmetric function (i.e., \( \psi(z) = \psi(|z|) \) for all \( z \in \mathbb{C} \) such that \( \int_{\mathbb{C}} \psi dA = 1 \). Then

\[
u(x) = \int_{\mathbb{C}} \psi(z) u(z + x) dA(z)
\]

for all \( u \in H(\Omega) \) and all \( x \in \Omega \) such that \( x + \text{Supp}(\psi) \subset \Omega \).

### 1.7.4. The Maximum Principle.

The proof of Theorem 1.5.1 for holomorphic functions goes through for harmonic functions.

**Corollary 1.7.4.** Let \( \Omega \subset \mathbb{C} \) be open and connected. If \( u \in H(\Omega) \) has a local maximum in \( \Omega \) then \( u \) is constant.

**Remark.** Since \( u \in H(\Omega) \Rightarrow -u \in H(\Omega) \), harmonic functions also satisfy a minimum principle. \( \diamond \)
1.7. Harmonic functions

1.7.5. The Poisson Formula. If $f \in \mathcal{O}(\overline{D})$ then by the Cauchy Formula

$$f(z) = \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi\sqrt{-1}}, \quad z \in D.$$  

On the other hand, if $w \in \mathbb{C} - \overline{D}$, then

$$0 = \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - w} \frac{d\zeta}{2\pi\sqrt{-1}}.$$  

If we now take $w = 1/\overline{z}$, then by using repeatedly that on the circle $|\zeta| = 1$, $\overline{\zeta} = 1/\zeta$ we obtain

$$f(z) = \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi\sqrt{-1}} - \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - \overline{z}} \frac{d\zeta}{2\pi\sqrt{-1}}.$$

Now, on any circle centered at the origin, $d\zeta/\zeta$ is pure imaginary, since $0 = d\log|\zeta|^2 = \frac{d\zeta}{\zeta} + \frac{d\overline{\zeta}}{\overline{\zeta}}$. Thus, by taking real parts of our calculation above, we obtain the following result.

**Theorem 1.7.5 (Poisson Integral Formula).** If $u \in H(\overline{D})$, then for all $z \in D$ one has the integral representation

$$u(z) = \int_{|\zeta|=1} u(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} \frac{d\zeta}{2\pi\sqrt{-1}}.$$  

**Corollary 1.7.6.** Let $\{u_n\} \subset H(\overline{D})$ be a sequence of harmonic functions that converges uniformly on compact sets of $\overline{D}$ to a function $u$. Then $u$ is harmonic.

The proof is left to the reader.

**Remark (Dirichlet Problem for the unit disk).** Let $f : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. The function $u$ defined by

$$u(z) := \int_{|\zeta|=1} f(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} \frac{d\zeta}{2\pi\sqrt{-1}}, \quad |z| < 1$$

and $u(z) = f(z)$ for $|z| = 1$ is a continuous solution of the Dirichlet Problem

$$\begin{aligned}
\Delta u &= 0 \quad \text{in } \mathbb{D} \\
u &= f \quad \text{on } \partial\mathbb{D}
\end{aligned}$$
for the unit disk. By the Maximum Principle, this Dirichlet Problem has a unique solution, evidently given by (1.6). We shall return to the Dirichlet Problem in due course.

1.7.6. Regularity of harmonic functions. If we interpret the Laplacian classically, we must require that harmonic functions be a priori $C^2$. However, even if we interpret the definition in the sense of distributions, harmonic functions are still smooth—a fact sometimes known as Weyl’s Lemma, which we now prove.

Let $\chi : [0, 1) \to [0, \infty)$ be a smooth function with compact support. Define $\psi(z) := \chi(|z|) / \int_{\mathbb{C}} \chi(|z|)$. Then $\psi$ is radially symmetric and has integral 1. Now let $u$ be a harmonic distribution, i.e., $u(\Delta \varphi) = 0$ for all smooth compactly supported functions $\varphi$. Define the function

$$\tilde{u}(x) := u * \psi(x) = u(\psi(x - \cdot)).$$

Then $u$ is smooth, and since $\Delta_x \psi(x - y) = \Delta_y \psi(x - y)$, we have

$$\Delta \tilde{u}(x) = u(\Delta_x \psi(x - \cdot)) = u(\Delta \psi(x - \cdot)) = 0.$$ 

It follows that $\tilde{u}$ is harmonic. But then

$$\int \tilde{u}(x) \varphi(x) dA(x) = u \left( \int \psi(x - \cdot) \varphi(x) dA(x) \right)$$

$$= u \left( \int \varphi(z + y) \psi(z) dA(z) \right)$$

$$= \int u(\varphi(z + \cdot)) \psi(z) dA(z).$$

But $\Delta_z u(\varphi(z + \cdot)) = u(\Delta_z \varphi(z + \cdot)) = u(\Delta \varphi(z + \cdot)) = 0$, and so $u(\varphi(z + \cdot))$ is also harmonic. Since $\psi$ is radial, we have

$$\int u(\varphi(z + \cdot)) \psi(z) dA(z) = u(\varphi).$$

It follows that $u = \tilde{u}$ is smooth and harmonic. \qed

1.8. Subharmonic functions

For the rest of this section, let $\Omega$ be an open subset of $\mathbb{C}$.

1.8.1. Subharmonic.

DEFINITION 1.8.1. A function $u : \Omega \to [-\infty, \infty]$ is upper semi-continuous if for every $s \in (-\infty, +\infty]$ the set $u^{-1}[-\infty, s]$ is open. \qed

Equivalently, $u$ is upper semi-continuous if

$$\limsup_{\zeta \to z} u(\zeta) \leq u(z).$$
1.8. Subharmonic functions

Note that every upper semi-continuous function \( u \) is Lebesgue measurable. In fact, for any measurable subset \( U \subset \mathbb{C} \)
\[
\int_U f \, dA := \inf \left\{ \int_U f \, dA \mid f \in C_0^\infty(U) \text{ and } f \geq u \right\}.
\]

If \( X \) is a Hausdorff space and \( f : X \to \mathbb{R} \), the upper regularization of \( f \) is the function
\[
f^*(x) := \lim \sup_{y \to x} f(y).
\]
It follows that \( f^* \) is upper semi-continuous, \( f^* \geq f \), and that if \( g \geq f \) is upper semi-continuous then \( g \geq f^* \). In particular, \( f \) is upper semi-continuous if and only if \( f^* = f \).

**Definition 1.8.2.** A function \( u : \Omega \to (-\infty, +\infty) \) is subharmonic (written \( u \in \text{SH}(\Omega) \)) if

(i) \( u \) is upper semi-continuous, and

(ii) if \( K \subset \subset \Omega \) and \( h \in H(\text{interior}(K)) \cap C^0(K) \) satisfy \( u|_{\partial K} \leq h|_{\partial K} \), then \( u \leq h \).

1.8.2. Basic properties.

**Proposition 1.8.3.** Let \( \Omega \subset \subset \mathbb{C} \) be an open set.

(1) If \( u \in \text{SH}(\Omega) \) and \( c > 0 \) is a constant, then \( cu \in \text{SH}(\Omega) \).

(2) If \( \{u_\alpha \mid \alpha \in A\} \subset \text{SH}(\Omega) \) and \( u := \sup_\alpha u_\alpha \) is finite and upper semi-continuous, then \( u \in \text{SH}(\Omega) \).

(3) If \( u_1 \geq u_2 \geq \ldots \) is a sequence of subharmonic functions, then \( u = \lim u_j \) is subharmonic.

**Proof.** Statements (1) and (2) are trivial. To show statement (3), note first that since \( \{u_j\} \) is a decreasing sequence of functions, \( u^{-1}[-\infty, s) = \bigcup_j u_j^{-1}[-\infty, s) \) and so \( u \) is upper semi-continuous. Let \( K \subset \subset \Omega \) and let \( h \) be a continuous function on \( K \) that is harmonic on the interior of \( K \) and majorizes \( u \) on the boundary of \( K \). Fix \( \varepsilon > 0 \) and let \( A_j := \{ z \in \text{bdry } K \mid u_j(z) \geq h(z) + \varepsilon \} \). Then \( A_j \) is closed (hence compact) and \( A_{j+1} \subset A_j \). Since \( \bigcap_j A_j = \emptyset \), it follows that for \( j_0 \) large enough, \( A_j = \emptyset \) for all \( j \geq j_0 \). Thus \( u_j \leq h + \varepsilon \) in \( K \) for all sufficiently large \( j \), and hence \( u \leq h \) in \( K \).

**Remark.** In statement (2) of Proposition 1.8.3 one can drop the requirement that \( u \) is upper semi-continuous, but then the conclusion is that \( u^* \) is subharmonic.

1.8.3. Subharmonic again. There are several useful characterizations of subharmonic functions. We collect some of the most useful characterizations in the following theorem.
THEOREM 1.8.4. Let \( u : \Omega \to [\infty, \infty) \) be upper semi-continuous. The following are equivalent.

1. \( u \in \text{SH}(\Omega) \).
2. If \( f \) is a holomorphic polynomial and \( D \subset \Omega \) is a disk with \( u \leq \text{Re} f \) on \( \partial D \), then \( u \leq \text{Re} f \) on \( D \).
3. If \( \delta > 0, z \in \Omega \) is of distance more than \( \delta \) from the boundary of \( \Omega \), and \( d\mu \) is a positive measure on \([0, \delta]\), then
   \[
   (\mu \text{SMV}) \quad u(z) \leq \frac{1}{2\pi} \int_0^\delta \int_0^{2\pi} u(z + re^{\sqrt{-1}\theta})d\mu(r)d\theta.
   \]
4. For each \( \delta > 0 \) and \( z \in \Omega \) of distance more than \( \delta \) from the boundary of \( \Omega \), there exists a positive measure \( d\mu \) on \([0, \delta]\) that is not supported on \( \{0\} \), such that \((\mu \text{SMV})\) holds.

**Proof.** Clearly statement (2) follows from statement (1), and statement (4) follows from statement (3). We will show that statement (2) \( \Rightarrow \) statement (3) and statement (4) \( \Rightarrow \) statement (1).

Assuming that statement (2) holds, let \( z \in \Omega \) have distance at least \( \delta \) from the boundary of \( \Omega \), let \( 0 < r < \delta \), and let \( \varphi \) be a continuous function on \( \partial \mathbb{D}(z, r) \) majorizing \( u \) there. By approximating \( \varphi \) by a finite part of its Fourier series, we may assume \( \varphi \) is a trigonometric polynomial, which we can extend into \( \mathbb{D}(z, r) \) (by replacing \( e^{\sqrt{-1}\theta} \) by \( z \) and \( e^{-\sqrt{-1}\theta} \) by \( \bar{z} \)) as a harmonic polynomial, hence the real part \( h \) of a holomorphic polynomial. Thus \( u \leq h \) on \( \mathbb{D}(z, r) \), and we have,

\[
  u(z) \leq h(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{\sqrt{-1}\theta})d\theta.
\]

Since we can approximate \( u \) from above, in \( L^1_{\text{loc}} \), by continuous functions, we see that

\[
  u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{\sqrt{-1}\theta})d\theta,
\]

and integration over \([0, \delta]\) with respect to \( d\mu \) shows that statement (3) holds.

Assuming that statement (4) holds, let \( K \subset \subset \Omega \) and suppose

\[
  h \in H(\text{interior} K) \cap \mathcal{C}^0(\overline{K})
\]

is such that \( h \geq u \) on \( \partial \text{dry} K \). Let \( v = u - h \) and set \( M := \sup_K v \). Being an upper semi-continuous function, \( v \) achieves its maximum on \( K \). Assuming, for the sake of contradiction, that \( M > 0 \), we find that there exists a non-empty compact \( F \subset K \) such that \( v = M \) on \( F \). Let \( z_0 \) be the point in \( F \) which is of minimal distance \( \delta \) to the boundary of \( K \). Note that \( \delta > 0 \) because \( v \) is non-positive on the
boundary of \( K \). Since \( v \) is upper semi-continuous, \( v < M \) on an open subset of \( \mathbb{D}(z_0, \delta) \). Thus

\[
M = v(z_0) \leq \frac{1}{2\pi} \int_0^\delta d\mu(r) \int_{\delta}^{2\pi} v(z_0 + re^{\theta})d\mu(r)d\theta < M,
\]

and this is the desired contradiction. \( \square \)

**Corollary 1.8.5.**

1. The sum of two subharmonic functions is subharmonic.

2. Subharmonicity is a local property: \( u \in SH(\Omega) \) if and only if every point \( p \in \Omega \) has a neighborhood \( U \) such that \( u|_U \) is subharmonic.

3. If \( f \in \mathcal{O}(\Omega) \), then \( \log |f| \) is subharmonic (and, in fact, harmonic away from the zeros of \( f \)).

4. If \( \varphi : \mathbb{R} \to \mathbb{R} \) is a convex increasing function then with

\[
\varphi(-\infty) := \lim_{x \to -\infty} \varphi(x),
\]

\( \varphi \circ u \) is subharmonic whenever \( u \) is.

5. If \( u_1 \) and \( u_2 \) are subharmonic functions then \( \log(e^{u_1} + e^{u_2}) \) is subharmonic.

**Proof.** Statements (1) and (2) follow from statements (4) and (2) of Theorem 1.8.4 respectively, while statement (3) follows from statement (2) of Theorem 1.8.4 and the Maximum Principle. Statement (4) follows from Jensen’s inequality. To see statement (5), let \( D \subset \Omega \) be a closed disc, and let \( f \) be a holomorphic polynomial such that \( \log(e^{u_1} + e^{u_2}) \leq \Re f \) on \( \partial D \). Then \( (e^{u_1} + e^{u_2})e^{-\Re f} \leq 1 \) on \( \partial D \). Since \( u_j - \Re f \) is subharmonic and the exponential function is convex, statement (4) implies that \( e^{u_j - \Re f} \) is subharmonic for \( j = 1, 2 \). Hence by statement (1), \( (e^{u_1} + e^{u_2})e^{-\Re f} \) is subharmonic, and therefore \( e^{u_1} + e^{u_2} \leq e^{\Re f} \) on \( D \). Then statement (5) follows from statement (2). \( \square \)

**1.8.4. Local integrability.** Another important corollary of Theorem 1.8.4 is the following result.

**Theorem 1.8.6.** If \( \Omega \subset \mathbb{C} \) is connected and \( u \in SH(\Omega) \), then either \( u \equiv -\infty \) or \( u \in L^1_{\text{loc}}(\Omega) \).

**Proof.** Let

\[
X := \{ z \in \Omega \mid \text{there exists } r > 0 \text{ with } D(z, r) \subset \subset \Omega \text{ and } u \in L^1(D(z, r)) \}.
\]

Then \( X \) is clearly open, and by the upper semi-continuity (hence boundedness from above) of \( u \) and statement (3) of Theorem 1.8.4, each \( z \) with \( u(z) > -\infty \) belongs to \( X \). We claim also that \( X \) is closed. Indeed, if \( p \in X \) then there exist points \( z \) arbitrarily close to \( p \) such that \( u(z) > -\infty \). One of these points is the center of a disc that contains \( p \) and is contained in \( \Omega \), and again by statement (3) of
Theorem 1.8.4. \( u \) is integrable on a neighborhood of \( p \). Thus, since \( X \) is both open and closed and since \( \Omega \) is connected, the theorem is proved. \( \square \)

1.8.5. **Regularity.** Finally, we come to the most important characterization of subharmonicity.

**THEOREM 1.8.7.** Let \( u \) be a distribution. Then \( u \in SH(\Omega) \) if and only if, in the sense of distributions,

\[ \Delta u \geq 0. \]

To prove this result, we need the following two lemmas.

**LEMMA 1.8.8.** Let \( u \in C^\infty(\Omega) \). Then \( u \) is subharmonic if and only if

\[ \Delta u \geq 0. \]

**Proof.** By Theorem 1.8.4, it suffices to show that the sub-mean value property holds precisely when \( \Delta u \geq 0 \). Let us compute the Taylor series of \( u \) to order 2 near a given point \( p \). With \((x, y)\) as real coordinates in \( \mathbb{C} \), we have

\[
\begin{align*}
 u(p + re^{\sqrt{-1} \theta}) &= u(p) + r (u_x(p) \cos \theta + u_y(p) \sin \theta) \\
 &+ \frac{r^2}{2} \left( u_{xx}(p) \cos^2 \theta + u_{yy}(p) \sin^2 \theta + u_{xy}(p) \sin 2 \theta \right) + O(r^3),
\end{align*}
\]

so that

\[
\int_0^{2\pi} u(p + re^{\sqrt{-1} \theta}) \frac{d\theta}{2\pi} - u(p) = \frac{r^2}{4} \Delta u + O(r^3).
\]

Thus, if the mean value property holds, then \( \Delta u \geq 0 \).

Conversely, if \( \Delta u \geq 0 \), then with

\[
f(r) = \int_0^{2\pi} u(p + re^{\sqrt{-1} \theta}) \frac{d\theta}{2\pi}
\]

we have, by the divergence form of Green’s Theorem,

\[
f'(r) = \frac{1}{2\pi r} \int_{\partial D_r(0)} \Delta u \, dA \geq 0.
\]

Since \( u \) is continuous,

\[
\lim_{r \to 0} f(r) = u(p).
\]

Thus the sub-mean value property is established, and the proof is complete. \( \square \)

**LEMMA 1.8.9.** Let \( u \in C^\infty(\Omega) \) be subharmonic, and let \( \psi \in C^\infty_0(\mathbb{D}) \) be a function such that \( \psi \geq 0 \) and \( \psi(x) = \psi(e^{\sqrt{-1} \theta} x) \) for all real \( \theta \). Write

\[
\psi^\varepsilon(z) = \varepsilon^{-2} \psi(\varepsilon^{-1} z) \quad \text{and} \quad u_\varepsilon = u \ast \psi^\varepsilon.
\]

Then \( u_\varepsilon \) is subharmonic and decreases with \( \varepsilon \).
Proof. First, note that \( \Delta (u \ast \psi^\varepsilon) = (\Delta u) \ast \psi^\varepsilon \) and the latter is non-negative because this is the case for both \( \Delta u \) and \( \psi \). Next,

\[
u^\varepsilon(z) = \int_D u(z - \varepsilon \zeta) \psi(\zeta) dA(\zeta) = \int_0^1 r\psi(r)dr \int_0^{2\pi} u(z - \varepsilon re^{-i\theta})d\theta,
\]

and since the mean value integral

\[
\int_0^{2\pi} u(z - \varepsilon e^{i\theta})d\theta
\]

is an increasing function of \( \varepsilon \), the lemma is proved. \( \square \)

Proof of Theorem 1.8.7. Suppose first that \( \Delta u \geq 0 \) in the sense of distributions. Let \( u^\varepsilon \) be as in Lemma 1.8.9. We claim that \( u^\varepsilon \) is a smooth family of subharmonic functions decreasing with \( \varepsilon \). Indeed, \( u^\varepsilon \) is clearly smooth, and

\[
\Delta u^\varepsilon(z) = u(\Delta \psi^\varepsilon(z - \cdot)) \geq 0 \text{ because } \Delta u \text{ is positive in the sense of distributions and } \psi \geq 0.
\]

To see that \( u^\varepsilon \) is decreasing, we use a “double smoothing trick”: let \( \varphi \in C^\infty_0(D) \) be non-negative. Then \( u \ast \varphi^\delta \) is smooth and subharmonic (as we just argued), and hence \( u^\varepsilon \ast \varphi^\delta = u \ast \varphi^\delta \ast \psi^\varepsilon \) is a subharmonic family decreasing with \( \varepsilon \). Letting \( \delta \rightarrow 0 \) shows that \( u^\varepsilon \) is decreasing. It now follows from statement (3) of Proposition 1.8.3 that \( u \) is subharmonic.

Conversely, suppose \( u \) is subharmonic. Define \( u^\varepsilon = u \ast \psi^\varepsilon \) with \( \psi \) as above. Then, using the notation

\[
\int_D f d\mu := \frac{1}{\mu(D)} \int_D f d\mu,
\]

one has

\[
\int_{D(z,r)} u^\varepsilon(\zeta)dA(\zeta) = \int_C \left( \int_{D(\varepsilon r)} u(\zeta - \varepsilon \eta)dA(\zeta) \right) \psi(\eta)dA(\eta)
\geq \int_C u(z - \varepsilon \eta) \psi(\eta)dA(\eta) = u^\varepsilon(z),
\]

and therefore \( u^\varepsilon \) is subharmonic by Theorem 1.8.4. It follows from Lemma 1.8.8 that \( \Delta u^\varepsilon \geq 0 \), and therefore if \( h \geq 0 \) has compact support then

\[
\int u \Delta h = \lim \int u^\varepsilon \Delta h = \lim \int \Delta u^\varepsilon h \geq 0.
\]

Thus \( \Delta u \geq 0 \) in the sense of distributions, as claimed. The proof is finished. \( \square \)

1.9. Exercises

1.1. Prove the formula (1.1) from Green’s Theorem.

1.2. Prove that if \( f \in \mathcal{O}(\mathbb{C}) \) and \( |f(z)|^2 \leq C(1 + |z|^2)^N \) then \( f \) is a polynomial in \( z \), of degree at most \( N \).

1.3. Prove that if \( f \in \mathcal{O}(\mathbb{C}) \) and \( \int_C |f|^2 dA < +\infty \) then \( f \equiv 0 \).
1.4. Prove that if $f$ is holomorphic on the punctured unit disk $\mathbb{D} - \{0\}$ and $\int_{\mathbb{D} - \{0\}} |f|^2 dA < +\infty$ then $f \in \mathcal{O}(\mathbb{D})$.

1.5. Let $\mathcal{F} \subset \mathcal{O}(\mathbb{C})$ be a family of entire functions such for each $R > 0$ there is a constant $C_R$ such that

$$\int_{\mathbb{D}(0,R)} |f|^2 dA \leq C_R \quad \text{for all } f \in \mathcal{F}.$$ Show that every sequence in $\mathcal{F}$ has a convergent subsequence.

1.6. Find a harmonic function in the punctured unit disk that is not the real part of a holomorphic function.

1.7. Let $u$ be a subharmonic function in the unit disk. The goal of this exercise is to show that the function $\psi : [0, \varepsilon) \to \mathbb{R} \cup [-\infty)$ defined by

$$\psi(r) := \frac{1}{2\pi} \int_0^{2\pi} u(e^r e^{\sqrt{-1} \theta}) d\theta$$ is convex and increasing.

(a) Show that $r \to \psi(r)$ is increasing.

(b) Show that if $I, J \subset \mathbb{R}$ are intervals and $v$ is a subharmonic function on $U = I + \sqrt{-1} J$ that depends only on $\text{Im } z$ for all $z \in U$ then $v$ is convex.

(c) Show that the function

$$\Psi(\zeta) := \frac{1}{2\pi} \int_0^{2\pi} u(e^{\zeta + \sqrt{-1} \theta}) d\theta$$ is upper semi-continuous and has the mean value property.

(d) Show that $\psi$ is convex.