Chapter 1

The complex exponential function

This is a very important function!

1.1. The series

For any $z \in \mathbb{C}$, we define:

$$\exp(z) := \sum_{n \geq 0} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots.$$

On the closed disk

$$\overline{D}(0, R) := \{ z \in \mathbb{C} \mid |z| \leq R \},$$

one has $\left| \frac{1}{n!} z^n \right| \leq \frac{1}{n!} R^n$ and we know that the series $\sum_{n \geq 0} \frac{1}{n!} R^n$ converges for any $R > 0$. Therefore, $\exp(z)$ is a normally convergent series of continuous functions on $\overline{D}(0, R)$, and $z \mapsto \exp(z)$ is a continuous function from $\mathbb{C}$ to $\mathbb{C}$.

**Theorem 1.1.** For any $a, b \in \mathbb{C}$, one has $\exp(a + b) = \exp(a) \exp(b)$. 
Proof. We just show the calculation, but this should be justified by arguments from real analysis (absolute convergence implies commutative convergence):

\[
\exp(a + b) = \sum_{n \geq 0} \frac{1}{n!} (a + b)^n \\
= \sum_{n \geq 0} \frac{1}{n!} \sum_{k+l=n} \frac{(k + l)!}{k!l!} a^k b^l \\
= \sum_{n \geq 0} \frac{1}{n!} \frac{(k + l)!}{k!l!} a^k b^l \\
= \sum_{k,l \geq 0} \frac{1}{k!l!} a^k b^l \\
= \exp(a) \exp(b).
\]

\[\square\]

We now give a list of basic, easily proved properties. First, the effect of complex conjugation:

\(\forall z \in \mathbb{C}, \overline{\exp(z)} = \exp(\overline{z}).\)

Since obviously \(\exp(0) = 1\), one obtains from the previous theorem:

\(\forall z \in \mathbb{C}, \exp(z) \in \mathbb{C}^* \text{ and } \exp(-z) = \frac{1}{\exp(z)}.\)

Also, \(z \in \mathbb{R} \Rightarrow \exp(z) \in \mathbb{R}^*\) and then, writing \(\exp(z) = (\exp(z/2))^2\), one sees that \(\exp(z) \in \mathbb{R}_+^*\).

Last, if \(z \in \mathbb{iR}\) (pure imaginary), then \(\overline{z} = -z\), so putting \(w := \exp(z)\), one has \(\overline{w} = w^{-1}\) so that \(|w| = 1\). In other words, \(\exp\) sends \(\mathbb{iR}\) to the unit circle \(U := \{z \in \mathbb{C} \mid |z| = 1\}\).

Summarizing, if \(x := \Re(z)\) and \(y := \Im(z)\), then \(\exp\) sends \(z\) to \(\exp(z) = \exp(x) \exp(iy)\), where \(\exp(x) \in \mathbb{R}_+^*\) and \(\exp(iy) \in U\).

1.2. The function \(\exp\) is \(\mathbb{C}\)-derivable

Lemma 1.2. If \(|z| \leq R\), then \(|\exp(z) - 1 - z| \leq \frac{e^R}{2} |z|^2\).
1.2. The function exp is C-derivable

Proof. \(|\exp(z) - 1 - z| = \frac{z^2}{2} \left(1 + \frac{z}{3} + \frac{z^2}{12} + \cdots\right)\) and \(1 + \frac{z}{3} + \frac{z^2}{12} + \cdots\) \\
\leq 1 + \frac{R}{3} + \frac{R^2}{12} + \cdots \leq e^R.\) □

Theorem 1.3. For any fixed \(z_0 \in \mathbb{C}\):

\[
\lim_{h \to 0} \frac{\exp(z_0 + h) - \exp(z_0)}{h} = \exp(z_0).
\]

Proof. \(\frac{\exp(z_0 + h) - \exp(z_0)}{h} = \exp(z_0) \frac{\exp(h) - 1}{h}\) and, after the lemma, \\
\(\frac{\exp(h) - 1}{h} \to 1\) when \(h \to 0.\) □

Therefore, exp is derivable with respect to the complex variable: we say that it is C-derivable (we shall change terminology later) and that its C-derivative is itself, which we write \\
\[
\frac{d\exp(z)}{dz} = \exp(z)\) or \(\exp' = \exp.\)

Corollary 1.4. On \(\mathbb{R}\), \(\exp\) restricts to the usual real exponential function; that is, for \(x \in \mathbb{R}\), \(\exp(x) = e^x.\)

Proof. The restricted function \(\exp : \mathbb{R} \to \mathbb{R}\) sends 0 to 1 and it is its own derivative, so it is the usual real exponential function. □

For this reason, for now on, we shall put \(e^z := \exp(z)\) when \(z\) is an arbitrary complex number.

Corollary 1.5. For \(y \in \mathbb{R}\), one has \(\exp(iy) = \cos(y) + i \sin(y).\)

Proof. Put \(f(y) := \exp(iy)\) and \(g(y) := \cos(y) + i \sin(y).\) These functions satisfy \(f(0) = g(0) = 1\) and \(f' = if, g' = ig.\) Therefore the function \(h := f/g,\) which is well defined from \(\mathbb{R}\) to \(\mathbb{C},\) satisfies \(h(0) = 1\) and \(h' = 0,\) so that it is constant equal to 1. □

Note that this implies the famous formula of Euler \(e^{i\pi} = -1.\)

Corollary 1.6. For \(x, y \in \mathbb{R},\) one has \(e^{x+iy} = e^x(\cos y + i \sin y).\)

Corollary 1.7. The exponential map \(\exp : \mathbb{C} \to \mathbb{C}^*\) is surjective.

Proof. Any \(w \in \mathbb{C}^*\) can be written \(w = r(\cos \theta + i \sin \theta), r > 0\) and \(\theta \in \mathbb{R},\) so \(w = \exp(\ln(r) + i\theta).\) □

The reader can find a beautiful proof which does not require any previous knowledge of trigonometric functions in the preliminary chapter of [Rud87].
The exponential viewed as a map $\mathbb{R}^2 \to \mathbb{R}^2$. It will be useful to consider functions $f : \mathbb{C} \to \mathbb{C}$ as functions $\mathbb{R}^2 \to \mathbb{R}^2$, under the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$: $x + iy \leftrightarrow (x, y)$. In this way, $f$ is described by $(x, y) \mapsto F(x, y) = (A(x, y), B(x, y))$, where 

\[
\begin{align*}
A(x, y) &= \Re(f(x + iy)), \\
B(x, y) &= \Im(f(x + iy)).
\end{align*}
\]

In the case where $f$ is the exponential function $\exp$, we easily compute:

\[
\begin{align*}
A(x, y) &= e^x \cos(y), \\
B(x, y) &= e^x \sin(y),
\end{align*}
\]

\[
\implies F(x, y) = (e^x \cos(y), e^x \sin(y)).
\]

We are going to compare the differential of the map $F$ with the $\mathbb{C}$-derivative of the exponential map. We use the terminology of differential calculus [Car97, Spi65]. On the one hand, the differential $dF(x, y)$ is the linear map defined by the relation

\[
F(x + u, y + v) = F(x, y) + dF(x, y)(u, v) + o(u, v),
\]

where $o(u, v)$ is small compared to the norm of $(u, v)$ when $(u, v) \to (0, 0)$. Actually, $dF(x, y)$ can be expressed using partial derivatives:

\[
dF(x, y)(u, v) = \left( \frac{\partial A(x, y)}{\partial x} u + \frac{\partial A(x, y)}{\partial y} v, \frac{\partial B(x, y)}{\partial x} u + \frac{\partial B(x, y)}{\partial y} v \right).
\]

Therefore, it is described by the Jacobian matrix:

\[
JF(x, y) = \left( \begin{array}{cc} \frac{\partial A(x, y)}{\partial x} & \frac{\partial A(x, y)}{\partial y} \\ \frac{\partial B(x, y)}{\partial x} & \frac{\partial B(x, y)}{\partial y} \end{array} \right).
\]

On the side of the complex function $f := \exp$, putting $z := x + iy$ and $h := u + iv$, we write:

\[
f(z + h) = f(z) + hf'(z) + o(h), \quad \text{that is,} \quad \exp(z + h) = \exp(z) + h \exp(z) + o(h).
\]

Here, the linear part is $f'(z)h = \exp(z)h$, so we draw the conclusion that (under our correspondence of $\mathbb{C}$ with $\mathbb{R}^2$):

\[
hf'(z) \longleftrightarrow dF(x, y)(u, v),
\]

that is, comparing real and imaginary parts:

\[
\begin{align*}
\frac{\partial A(x, y)}{\partial x} u + \frac{\partial A(x, y)}{\partial y} v &= \Re(f'(z))u - \Im(f'(z))v, \\
\frac{\partial B(x, y)}{\partial x} u + \frac{\partial B(x, y)}{\partial y} v &= \Im(f'(z))u + \Re(f'(z))v.
\end{align*}
\]
Since this must be true for all \( u, v \), we conclude that:

\[
JF(x, y) = \begin{pmatrix}
\frac{\partial A(x, y)}{\partial x} & \frac{\partial A(x, y)}{\partial y} \\
\frac{\partial B(x, y)}{\partial x} & \frac{\partial B(x, y)}{\partial y}
\end{pmatrix} = \begin{pmatrix}
\Re(f'(z)) & -\Im(f'(z)) \\
\Im(f'(z)) & \Re(f'(z))
\end{pmatrix}.
\]

As a consequence, the Jacobian determinant \( \det JF(x, y) \) is equal to \( |f'(z)|^2 \) and thus vanishes if, and only if, \( f'(z) = 0 \); in the case of the exponential function, it vanishes nowhere.

**Exercise 1.8.** Verify these formulas when \( A(x, y) = e^x \cos(y) \), \( B(x, y) = e^x \sin(y) \) and \( f'(z) = \exp(x + iy) \).

### 1.3. The exponential function as a covering map

From equation \( e^{x+iy} = e^x (\cos y + i \sin y) \), one sees that \( e^z = 1 \iff z \in 2i\pi \mathbb{Z} \), i.e., \( \exists k \in \mathbb{Z} : z = 2i\pi k \). It follows that \( e^{z_1} = e^{z_2} \iff e^{z_2 - z_1} = 1 \iff z_2 - z_1 \in 2i\pi \mathbb{Z} \), i.e., \( \exists k \in \mathbb{Z} : z_2 = z_1 + 2i\pi k \). We shall write this relation: \( z_2 \equiv z_1 \pmod{2i\pi} \) or for short \( z_2 \equiv z_1 \pmod{2i\pi} \).

**Theorem 1.9.** The map \( \exp : \mathbb{C} \to \mathbb{C}^* \) is a covering map. That is, for any \( w \in \mathbb{C}^* \), there is a neighborhood \( V \subset \mathbb{C}^* \) of \( w \) such that \( \exp^{-1}(V) = \bigsqcup U_k \) (disjoint union), where each \( U_k \subset \mathbb{C} \) is an open set and \( \exp : U_k \to V \) is a homeomorphism (a bicontinuous bijection).

**Proof.** Choose a particular \( z_0 \in \mathbb{C} \) such that \( \exp(z_0) = w \). Choose an open neighborhood \( U_0 \) of \( z_0 \) such that, for any \( z', z'' \in U_0 \), one has \( |\Im z'' - \Im z'| < 2\pi \). Then \( \exp \) bijectively maps \( U_0 \) to \( V := \exp(U_0) \). Moreover, one has \( \exp^{-1}(V) = \bigsqcup U_k \), where \( k \) runs in \( \mathbb{Z} \) and the \( U_k = U_0 + 2i\pi k \) are pairwise disjoint open sets. It remains to show that \( V \) is an open set. The most generalizable way is to use the local inversion theorem \([\text{Car97}, \text{Spi65}]\), since the Jacobian determinant vanishes nowhere. Another way is to choose an open set as in exercise 2 to this chapter. \( \Box \)

The fact that \( \exp \) is a covering map is a very important topological property and it has many consequences.

**Corollary 1.10** (Path lifting property). Let \( a < b \) in \( \mathbb{R} \) and let \( \gamma : [a, b] \to \mathbb{C}^* \) be a continuous path with origin \( \gamma(a) = w_0 \in \mathbb{C}^* \). Let \( z_0 \in \mathbb{C} \) be such that \( \exp(z_0) = w_0 \). Then there exists a unique lifting, i.e., a continuous path \( \overline{\gamma} : [a, b] \to \mathbb{C}^* \) such that \( \forall t \in [a, b] \), \( \exp(\overline{\gamma}(t)) = \gamma(t) \) and subject to the initial condition \( \overline{\gamma}(a) = z_0 \).
Corollary 1.11 (Index of a loop with respect to a point). Let $\gamma : [a, b] \to \mathbb{C}^*$ be a continuous loop, that is, $\gamma(a) = \gamma(b) = w_0 \in \mathbb{C}^*$. Then, for any lifting $\overline{\gamma}$ of $\gamma$, one has $\overline{\gamma}(b) - \overline{\gamma}(a) = 2\pi n$ for some $n \in \mathbb{Z}$. The number $n$ is the same for all the liftings and it depends only on the loop $\gamma$: it is the index of $\gamma$ around 0, written $I(0, \gamma)$.

Actually, another property of covering maps (the “homotopy lifting property”; see for instance [Ful95, Proposition 11.8]) allows one to conclude that $I(0, \gamma)$ does not change if $\gamma$ is continuously deformed within $\mathbb{C}^*$: it only depends on the “homotopy class” of $\gamma$ (see the last exercise at the end of this chapter).

Example 1.12. If $\gamma(t) = e^{nit}$ on $[0, 2\pi]$, then all liftings of $\gamma$ have the form $\overline{\gamma}(t) = nit + 2i\pi k$ for some $k \in \mathbb{Z}$ and one finds $I(0, \gamma) = n$.

1.4. The exponential of a matrix

For a complex vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$, we define $\|X\|_{\infty} := \max_{1 \leq i \leq n} (|x_i|)$. Then, for a complex square matrix $A = (a_{i,j})_{1 \leq i,j \leq n} \in \text{Mat}_n(\mathbb{C})$, define the
subordinate norm:¹

\[ \|A\|_\infty := \sup_{X \in \mathbb{C}^n, X \neq 0} \frac{\|AX\|_\infty}{\|X\|_\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{i,j}|. \]

Then, for the identity matrix, \( \|I_n\|_\infty = 1 \); and, for a product, \( \|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty \). It follows easily that \( \frac{1}{k!}A^k \) converges absolutely for any \( A \in \text{Mat}_n(\mathbb{C}) \). It actually converges normally on all compacts and therefore defines a continuous map \( \exp : \text{Mat}_n(\mathbb{C}) \to \text{Mat}_n(\mathbb{C}), A \mapsto \sum_{k \geq 0} \frac{1}{k!}A^k \). We shall also write for short \( e^A := \exp(A) \). In the case \( n = 1 \), the notation is consistent.

**Examples 1.13.** (i) For a diagonal matrix \( A := \text{Diag}(\lambda_1, \ldots, \lambda_n) \), one has \( \frac{1}{k!}A^k = \text{Diag}(\lambda_1^k/k!, \ldots, \lambda_n^k/k!) \), so that \( \exp(A) = \text{Diag}(e^{\lambda_1}, \ldots, e^{\lambda_n}) \).

(ii) If \( A \) is an upper triangular matrix with diagonal \( D := \text{Diag}(\lambda_1, \ldots, \lambda_n) \), then \( \frac{1}{k!}D^k \) is an upper triangular matrix with diagonal \( \frac{1}{k!}D^k \), so that \( \exp(A) \) is an upper triangular matrix with diagonal \( \exp(D) = \text{Diag}(e^{\lambda_1}, \ldots, e^{\lambda_n}) \). Similar relations hold for lower triangular matrices.

(iii) Take \( A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then \( A^2 = I_2 \), so that \( \exp(A) = aI_2 + bA = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \), where \( a = \sum_{k \geq 0} \frac{1}{(2k)!} \) and \( b = \sum_{k \geq 0} \frac{1}{(2k+1)!} \).

**Exercise 1.14.** Can you recognize the values of \( a \) and \( b \)?

The same kind of calculations as for the exponential map give the rules:

\[ \exp(0_n) = I_n; \quad \exp(\overline{A}) = \exp(A), \]

and

\[ AB = BA \implies \exp(A + B) = \exp(A) \exp(B) = \exp(B) \exp(A). \]

**Remark 1.15.** The condition \( AB = BA \) is required to use the Newton binomial formula. If we take for instance \( A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), then \( AB \neq BA \). We have \( A^2 = B^2 = 0 \), so that \( \exp(A) = I_2 + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

¹We shall use the same notation for a norm in the complex space \( V \) and for the corresponding subordinate norm in \( \text{End}(V) \) (that is, \( \text{Mat}_n(\mathbb{C}) \) whenever \( V = \mathbb{C}^n \)).
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and \( \exp(B) = I_2 + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \), thus \( \exp(A) \exp(B) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \). On the other hand, \( A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and the previous example gave the value of \( \exp(A + B) \), which was clearly different.

It follows from the previous rules that \( \exp(-A) = (\exp(A))^{-1} \) so that \( \exp \) actually sends \( \text{Mat}_n(\mathbb{C}) \) to \( \text{GL}_n(\mathbb{C}) \). Now there are rules more specific to matrices. For the transpose, using the fact that \( \text{tr}(A^k) = (\text{tr}A)^k \), and also the continuity of \( A \mapsto \text{tr}(A) \) (this is required to go to the limit in the infinite sum), we see that \( \exp(\text{tr}A) = \text{tr}(\exp(A)) \). Last, if \( P \in \text{GL}_n(\mathbb{C}) \), from the relation \( (PAP^{-1})^n = PA^nP^{-1} \) (and also from the continuity of \( A \mapsto PAP^{-1} \)), we deduce the very useful equality:

\[
P \exp(A) P^{-1} = \exp(PAP^{-1}).
\]

Now any complex matrix \( A \) is conjugate to an upper triangular matrix \( T \) having the eigenvalues of \( A \) on the diagonal; using the examples above, one concludes that if \( A \) has eigenvalues \( \lambda_1, \ldots, \lambda_n \), then \( \exp(A) \) has eigenvalues \( e^{\lambda_1}, \ldots, e^{\lambda_n} \) with the corresponding multiplicities:

\[
\text{Sp}(e^A) = e^{\text{Sp}(A)}.
\]

Note that this implies in particular the formula:

\[
\det(e^A) = e^{\text{Tr}A}.
\]

**Example 1.16.** Let \( A := \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix} \). Then \( A \) is diagonalizable with spectrum \( \text{Sp}(A) = \{i\pi, -i\pi\} \). Thus, \( \exp(A) \) is diagonalizable with spectrum \( \{-1, -1\} \). Therefore, \( \exp(A) = -I_2 \).

1.5. Application to differential equations

Let \( A \in \text{Mat}_n(\mathbb{C}) \) be fixed. Then \( z \mapsto e^{zA} \) is a \( \mathbb{C} \)-derivable function from \( \mathbb{C} \) to the complex linear space \( \text{Mat}_n(\mathbb{C}) \); this simply means that each coefficient is a \( \mathbb{C} \)-derivable function from \( \mathbb{C} \) to itself. Derivating our matrix-valued function coefficientwise, we find:

\[
\frac{d}{dz} e^{zA} = Ae^{zA} = e^{zA} A.
\]

Indeed, \( \frac{e^{(z+h)A} - e^{zA}}{h} = e^{zA} \frac{e^{hA} - I_n}{h} = e^{hA} \frac{e^{hA} - I_n}{h} e^{zA} \) and \( \frac{e^{hA} - I_n}{h} = A + \frac{h}{2} A^2 + \cdots \).

\[2\]Most of the time, it is convenient to consider the spectrum as a multiset, that is, a set whose elements have multiplicities. In the same way, the roots of a polynomial make up a multiset.
Now consider the vectorial differential equation:

\[ \frac{d}{dz} X(z) = AX(z), \]

where \( X : \mathbb{C} \to \mathbb{C}^n \) is looked at as a \( \mathbb{C} \)-derivable vector-valued function, and again derivation is performed coefficientwise. We solve this by change of unknown function: \( X(z) = e^{zA}Y(z) \). Then, applying Leibniz’ rule for derivation, \((fg)' = f'g + fg'\) (it works the same for \( \mathbb{C} \)-derivation), we find:

\[ X' = AX \implies e^{zA}Y' + Ae^{zA}Y = Ae^{zA}Y \implies e^{zA}Y' = 0 \implies Y' = 0. \]

Therefore, \( Y(z) \) is a constant function. (Again, we admit a property of \( \mathbb{C} \)-derivation: that \( f' = 0 \implies f \text{ constant.} \)) If we now fix \( z_0 \in \mathbb{C} \), \( X_0 \in \mathbb{C}^n \) and we address the Cauchy problem:

\[
\begin{cases}
\frac{d}{dz} X(z) = AX(z), \\
X(z_0) = X_0,
\end{cases}
\]

we see that the unique solution is \( X(z) := e^{(z-z_0)A}X_0 \).

An important theoretical consequence is the following. Call \( \text{Sol}(A) \) the set of solutions of \( \frac{d}{dz} X(z) = AX(z) \). This is obviously a complex linear space. What we proved is that the map \( X \mapsto X(z_0) \) from \( \text{Sol}(A) \) to \( \mathbb{C}^n \), which is obviously linear, is also bijective. Therefore, it is an isomorphism of \( \text{Sol}(A) \) with \( \mathbb{C}^n \). This is a very particular case of the Cauchy theorem for complex differential equations that we shall encounter in Section 7.3.

**Example 1.17.** To solve the linear homogeneous second-order scalar equation (with constant coefficients) \( f'' + pf' + qf = 0 \) \((p, q \in \mathbb{C})\), we introduce the vector-valued function \( X(z) := \begin{pmatrix} f(z) \\ f'(z) \end{pmatrix} \) and find that our scalar equation is actually equivalent to the vector equation:

\[ X' = AX, \text{ where } A := \begin{pmatrix} 0 & 1 \\ -q & p \end{pmatrix}. \]

Therefore, the solution will be searched in the form \( X(z) := e^{(z-z_0)A}X_0 \), where \( z_0 \) may be chosen at will or else imposed by initial conditions.
Exercises

(1) For \( z \in \mathbb{C} \), define
\[
\cos(z) := \frac{\exp(iz) + \exp(-iz)}{2} \quad \text{and} \quad \sin(z) := \frac{\exp(iz) - \exp(-iz)}{2i},
\]
so that \( \cos \) is an even function, \( \sin \) is an odd function and \( \exp(z) = \cos(z) + i \sin(z) \). Translate the property of Theorem 1.1 into properties of \( \cos \) and \( \sin \).

(2) Let \( a, b > 0 \) and \( U := \{ z \in \mathbb{C} \mid -a < \Re(z) < a \text{ and } -b < \Im(z) < b \} \) (thus, an open rectangle under the identification of \( \mathbb{C} \) with \( \mathbb{R}^2 \)). Assuming \( b < \pi \), describe the image \( V := \exp(U) \subset \mathbb{C}^* \) and define an inverse map \( V \to U \).

(3) If in Corollary 1.10 one chooses another \( z_0' \in \mathbb{C} \) such that \( \exp(z_0') = w_0 \), one gets another lifting \( \overline{\gamma}' : [a, b] \to \mathbb{C}^* \) such that \( \forall t \in [a, b], \exp(\overline{\gamma}'(t)) = \gamma(t) \) and subject to the initial condition \( \overline{\gamma}'(a) = z_0' \). Show that there is some constant \( k \in \mathbb{Z} \) such that \( \forall t \in [a, b], \overline{\gamma}'(t) = \overline{\gamma}(t) + 2i\pi k \).

(4) (i) Compute \( \exp\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \) in two ways: by direct calculation as in Example 1.13 (iii); by diagonalization as in Example 1.16.

(ii) Deduce from this the value of \( \exp\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \).

(5) In Example 1.17 compute \( e^{(z-z_0)A} \) and solve the problem with initial conditions \( f(0) = a, f'(0) = b \). There will be a discussion according to whether \( p^2 - 4q = 0 \) or \( \neq 0 \).

(6) This exercise aims at introducing you to the concept of homotopy, which shall play a crucial role in all of the course. Ideally, you should read Chapters 11 and 12 of [Ful95] or content yourself with what [Ahl78, Car63, Rud87] say about homotopies and coverings. However, it might be even better to try to figure out by yourself the fundamental properties.

Let \( \Omega \) be a domain in \( \mathbb{C} \) (although the notion is much more general). Two continuous paths \( \gamma_1, \gamma_2 : [0; 1] \to \Omega \) with same origin \( \gamma_1(0) = \gamma_2(0) = x \) and same extremity \( \gamma_1(1) = \gamma_2(1) = y \) are said to be homotopic if there is a continuous map (called a homotopy from \( \gamma_1 \) to \( \gamma_2 \))
Exercise 1.2. Homotopic paths

\[ H : [0; 1] \times [0; 1] \to \Omega \] such that:
\[ \forall s \in [0; 1], H(s, 0) = \gamma_1(s), \]
\[ \forall s \in [0; 1], H(s, 1) = \gamma_2(s), \]
\[ \forall t \in [0; 1], H(0, t) = x, \]
\[ \forall t \in [0; 1], H(1, t) = y. \]

(i) Show that this is an equivalence relation among paths from \( x \) to \( y \).

(ii) Define the composition of a path \( \gamma \) from \( x \) to \( y \) with a path \( \gamma' \) from \( y \) to \( z \) as the path \( \gamma.\gamma' \) from \( x \) to \( z \) given by the formula:
\[ \gamma.\gamma'(s) = \begin{cases} 
\gamma(2s) & \text{if } 0 \leq s \leq 1/2, \\
\gamma'(2s - 1) & \text{if } 1/2 \leq s \leq 1.
\end{cases} \]

Show that this is not associative but that (when everything is defined) \( (\gamma.\gamma').\gamma'' \) and \( \gamma.(\gamma'.\gamma'') \) are homotopic.

(iii) More generally, show that homotopy is compatible with composition (you will have to formulate this precisely first).

(iv) Prove the homotopy lifting property.

(v) Extend all this to paths \([a; b] \to \Omega\).

Remark 1.18. We use paths with good differential properties many times, for instance such that \( \gamma' \) exists and is piecewise continuous. A fundamental theorem of differential topology says that when two such paths are homotopic, a homotopy \( H \) can be chosen such that all intermediate paths \( \gamma_t : s \mapsto H(s, t) \) share the same good differential properties. More on this in the footnote at the bottom of page 61.