Preface

The purpose of this book is to provide a thorough introduction to the theory of separable algebras over commutative rings. After introducing the general theory of separable algebras and establishing their basic properties, many of the fundamental roles played by separable algebras are then studied. In particular, rigorous introductions and treatments of Azumaya algebras, the henselization of local rings, and the Galois theory of commutative rings are presented. Interwoven throughout these applications is the essential role played by étale algebras.

Recall that an extension of fields F/k is separable if every element of F is a root of a separable polynomial over k, hence a root of a polynomial with no repeated roots. For an algebra A over a commutative ring R, separability is not defined element-wise because in general there are not enough separable elements in A. This is the subject of Exercise 12.7.12. To generalize the definition of separability to algebras over commutative rings, it is necessary to raise the level of abstraction. The definition is not in terms of elements of A, but instead is based on a certain module structure of the ring A over the enveloping algebra $A \otimes_R A^o$ which is induced by the multiplication map $x \otimes y \mapsto xy$. Naturally, the definitions agree for a finite extension of fields. Therefore we could say that the theory of separable algebras has as its essence the study of separable polynomials. Keeping this in mind, the reader of this book can perhaps make some sense out of the many abstract structures that arise.

Almost from the start we see that this subject matter has strong ties to all areas of algebra, including Algebraic Number Theory, Ring Theory, Commutative Algebra and Algebraic Geometry. These fundamental connections are present even in the definition of separability itself. As alluded to above, A is defined to be a separable R-algebra if A is a projective module over $A \otimes_R A^o$. Before Auslander and Goldman applied it to algebras over commutative rings, this module theoretic definition had been applied by Hochschild and others to study separable algebras over fields. This definition is in fact the noncommutative analog of one of the equivalent conditions used by Grothendieck in SGA 1 (see [**Gro71**, Proposition I.3.1, p. 2]), namely, that the diagonal morphism is an open immersion. While not obvious at first, the connection between the two ideas is the following. When A is commutative, the multiplication map is a homomorphism of R-algebras $A \otimes_R A \to A$ which in the language of Algebraic Geometry defines the so-called diagonal morphism $\operatorname{Spec} A \to \operatorname{Spec} A \otimes_R A$. If A is commutative and finitely generated as an R-algebra, then one can show A is a projective module over $A \otimes_R A$ if and only if the diagonal morphism is an open immersion. This is the subject of Exercise 8.1.31.

In the chapters which follow, many important attributes of separable algebras are studied. It is shown that separability is preserved under a change of base ring. The tensor product of two separable algebras is a separable algebra. Separability is transitive: if A is separable over S, and S is separable over R, then A is separable over R. Any localization of R is separable over R and a homomorphic image of R is separable over R. In the language of Algebraic Geometry, this means an open immersion is separable, and a closed immersion is separable. Over a field k, the separable algebras turn out to be finite direct sums of matrix algebras over finite dimensional k-division algebras and the center of each division algebra appearing is a finite separable extension field of k. More generally, for any commutative ring R, A is separable over R if and only if A is separable over its center and its center is separable over R. This fact shows that the study of separability can be split into two parts: algebras which are commutative, and algebras which are central. Following the example set by Grothendieck in his papers on the Brauer group ([**Gro68a**], [**Gro68b**], [**Gro68c**]), central separable R-algebras are called Azumaya algebras. Over a field, an algebra is Azumaya if and only if it is central simple.

Many examples of separable algebras are included along the way. Among the first examples of a noncommutative separable R-algebra which is exhibited is the ring of n-by-n matrices over R, which is denoted $M_n(R)$. The importance of this example is illustrated by the fact that every Azumaya R-algebra A of constant rank is a twisted form of matrices for a faithfully flat extension. By this we mean that if A is an Azumaya R-algebra of constant rank n^2, then there is a faithfully flat R-algebra S such that upon extension of the ring of scalars to S, $A \otimes_R S$ is isomorphic to the matrix algebra $M_n(S)$. The trivial commutative separable extension of R of rank n is R^n, the direct sum of n copies of R. Every commutative separable R-algebra

S which is a finitely generated projective R-module of constant rank n is a twisted form of R^n. In other words, there is a faithfully flat R-algebra T such that $T \otimes_R S$ is isomorphic to T^n. An R-algebra is said to be étale if it is commutative, separable, flat, and finitely presented as an R-algebra. Chapter 9 is devoted to the study of étale algebras. The henselization and strict henselization of a local ring are constructed in Chapter 10. A central theme of SGA 4 ([**AGV72a**], [**AGV72b**], [**AGV73**]) is that faithfully flat étale R-algebras are the counterpart in Algebraic Geometry of covering spaces in the analytic topology and the strictly henselian local rings play the role of the stalks at the points. The examples mentioned above show that Azumaya R-algebras of constant rank are those R-algebras which become isomorphic to a matrix algebra upon restriction to a suitably refined étale covering and commutative separable finitely generated projective algebras of constant rank are those which locally (for the étale topology) are isomorphic to a trivial covering.

One of the goals of this book is to consolidate the most popularly accepted resources on this subject. It includes almost all of the results that are found in the following standard references: 1. The monograph of DeMeyer and Ingraham on separable algebras ([**DI71**]); 2. The monograph of Knus and Ojanguren on faithfully flat descent and Azumaya algebras ([**KO74b**]); 3. The monograph of Orzech and Small on the Brauer group ([**OS75**]); 4. Auslander and Goldman's original papers on the Brauer group and maximal orders ([**AG60a**]; [**AG60b**]); 5. Raynaud's monograph on henselian local rings ([**Ray70**]). Moreover, it includes almost all of those results contained in Saltman's "Lectures on Division Algebras" ([**Sal99**]) which are primarily about separable algebras.

A serious attempt has been made to include many examples and there is an emphasis on computations. Many nontrivial examples of rings are exhibited both in the text and in the exercises for which the computations of the Brauer group, the Picard group, and other important invariants are completely carried out. I apologize that there are not more. I owe Harley Flanders a debt of gratitude for impressing upon me the importance of computing examples. In the words of Hermann Weyl ([**Wey97**, preface to the first edition]), "the special problems in all their complexity constitute the stock and core of mathematics; and to master their difficulties requires on the whole the harder labor". To compute the Brauer group, the Picard group, or one of the other invariants of a ring emphasized in this book we quickly learn that the trivial examples are few and far between. Even for the most common rings the computations can be very difficult. Then we learn that for those interesting, nontrivial rings for which the computations can be completely carried out, one fact is certain — a lot of machinery is required. Even the computation of the Brauer group of the ring of rational integers $\mathbb{Z}$ requires a lot of work. The original proof does not appear until page 95 in Grothendieck's Brauer group papers and in this book, the proof does not come until the last chapter (Theorem 14.3.8).

In any book of this type, there is always a balance to be achieved between which theorems are stated and proved, which are stated without proof, and which are merely cited (or tacitly assumed). Throughout, our benchmark has been to place higher emphasis on those results which have direct applications to separable algebras. As standard references on general algebra, the books by Dummit and Foote ([**DF04**]), Hungerford ([**Hun80**]), Bourbaki ([**Bou89a**]), and Herstein ([**Her75**]) are recommended (in that order). Rotman's book [**Rot79**] is recommended as a standard reference for those results from homological algebra that are not proved here. For commutative algebra, the books by Atiyah and Macdonald ([**AM69**]), Matsumura ([**Mat80**]), Zariski and Samuel ([**ZS75a**] and [**ZS75b**]), and Bourbaki ([**Bou89b**]) are recommended (in that order).

Chapters 1, 2 and 3 consist of a review of much of the background results on rings, modules, and commutative algebra which are needed for the rest of the book. Proofs are given for many of these results. However, most are stated without proof for reference and ease of exposition in the rest of the text. Whenever a proof is omitted, a citation is provided for the reader. Chapter 3 includes a quick proof of the noncommutative version of Nakayama's Lemma, the Artin-Wedderburn Theorem, and some applications which will be necessary later on. Therefore, Chapters 4 through 14 contain the bulk of the subject matter. For instance, the definition of separable algebras together with proofs of their basic properties are presented in Chapter 4.

Chapter 5 covers the background material from homological algebra that we require. There is an entire section on group cohomology. The Amitsur complex is introduced and applied to prove the basic theorems on faithfully flat descent. The first properties of the Hochschild cohomology groups are derived. Amitsur cohomology and its first properties are presented. Included in this treatment are the pointed cohomology sets in degrees 0 and 1 for noncommutative coefficient groups. For example, it is shown that in degree one, Amitsur cohomology classifies the twisted forms of a module.

The main purpose of Chapter 6 is to introduce and prove the fundamental properties of lattices over integral domains. Included is a method for the construction of reflexive modules in terms of locally free modules which is based on a theorem of B. Auslander. The Weil divisor class group is defined and Nagata's Theorem is proved. For a noetherian normal integral domain R, the class group is identified with the group of reflexive fractional ideals of R in the quotient field, modulo principal ideals.

Chapter 7 begins with the definition and first properties of Azumaya algebras. After proving the commutator theorems, we define the Brauer group. Over a field, Azumaya algebras and central simple algebras are equivalent and the Brauer group parametrizes the finite dimensional central division algebras. The Picard group of invertible bimodules is defined. This leads to the proof of the Skolem-Noether Theorem. As an application of Hochschild cohomology we prove that Azumaya algebras can be lifted modulo an ideal that is contained in the nil radical.

The topics in Chapter 8 are mostly from Commutative Algebra. From our perspective, the motivation here is to derive more tests for separability. In particular, separability criteria are achieved by applications of Hochschild cohomology, derivations, and Kähler differentials. Consequently, a jacobian criterion, a local ring criterion, and a residue field criterion (all for separability) are then proved. Differential crossed product algebras, which are a type of cyclic crossed product for purely inseparable radical extensions, are introduced. When the ground field is infinite, a version of Emmy Noether's Normalization Lemma is proved which allows us to construct the underlying polynomial ring in such a way that it contains a separating transcendence base. Lastly, useful differential and jacobian criteria for regularity are derived.

Chapter 9 is a deeper investigation into the properties of smooth and étale algebras. Every algebra which is étale in a neighborhood is locally isomorphic to a standard étale algebra. In addition to proving the fundamental properties of étale algebras, we show that formally étale implies étale. There is an entire section devoted to the construction of radical extensions that ramify only along a reduced effective divisor.

The henselization and strict henselization of a local ring are constructed in Chapter 10. The existence of a faithfully flat étale splitting ring for an Azumaya algebra is demonstrated. We show that an Azumaya algebra of constant rank over a commutative ring is a form of matrices for a faithfully flat étale covering, an important characterization with many applications. Together with Artin's Refinement Theorem and the Skolem-Noether Theorem, this allows us to show that up to isomorphism the Azumaya R-algebras of constant rank n^2 are classified by the pointed set of Čech cohomology $\check{\mathrm{H}}^1_{\mathrm{et}}(R, \mathrm{PGL}_n)$. From here, the embedding of the Brauer group into the second étale Čech cohomology group is constructed.

Chapter 11 is a deeper study of Azumaya algebras. Viewing an Azumaya algebra as a form of matrices allows us to associate to an element of the algebra a number of invariants that are typically associated to a matrix. In particular, every element has a characteristic polynomial, a norm, and a trace. A proof due to D. Saltman is presented showing that the Brauer group is torsion. Sufficient conditions for a maximal order in a central simple algebra to be an Azumaya algebra are derived. The subgroup of the Brauer group containing algebras that are locally split is described. Following papers by Knus, Ojanguren and Saltman, Brauer groups in characteristic p are studied. Ojanguren's example of a nontrivial locally trivial Azumaya algebra is exhibited. A key step in the proof involves a construction of the rank three reflexive module that is not projective which is based on the local to global theorem of B. Auslander.

Chapter 12 is an introduction to Galois theory for commutative rings. Included are the theorems on Galois descent, the Fundamental Theorem of Galois Theory, the Embedding Theorem, the separable closure, and the Fundamental Theorem of Infinite Galois Theory. There is an entire section devoted to cyclic Galois extensions. In the Kummer context, the short exact sequence classifying the group of cyclic Galois extensions of degree n is derived.

Probably the most useful and practical method for constructing Azumaya algebras is the so-called crossed product. Chapter 13 includes treatments of the usual crossed product algebra, the usual cyclic crossed product algebra, the generalized crossed product algebra of Kanzaki, and the generalized cyclic crossed product algebra. The Brauer group is one of the most important arithmetic invariants of any commutative ring. For instance, if L/K is a finite Galois extension of fields with group G, then any central simple K-algebra split by L is Brauer equivalent to a crossed product algebra. The Crossed Product Theorem says there is an isomorphism between the Galois cohomology group $\mathrm{H}^2(G, L^*)$ with coefficients in the group of invertible elements of L, and the relative Brauer group $\mathrm{B}(L/K)$. For a Galois extension of commutative rings S/R, the crossed product map $\mathrm{H}^2(G, S^*) \to \mathrm{B}(S/R)$ is in general not one-to-one or onto. The precise description of the kernel and cokernel of this map is the so-called Chase, Harrison and Rosenberg seven term exact sequence of Galois cohomology. When S is a factorial noetherian integral domain, the crossed product map is an isomorphism. Therefore, it is no surprise that computations involving the groups in the seven term cohomology sequence rely on a good knowledge of the class group of the covering ring S. Nontrivial examples are exhibited for which all of the terms in the seven term exact sequence of Galois cohomology are computed (see, for example, Example 13.6.13, Example 13.6.15, and Exercise 13.6.16).

In recent years there has been a renewed interest in the computation of the Brauer group of algebraic varieties. Varieties of low dimension (algebraic curves and surfaces) play important roles because in many cases the computations can be completely carried out. For examples the reader is referred to [**Bri13**], [**CTW12**], [**CV15**], [**IS15**], [**vG05**], and their respective bibliographies. Not only is it important to compute the Brauer group, but there is also a desire to construct Azumaya algebras representing any nontrivial Brauer classes. A strong motivation is the role played by the Brauer group in the so-called Brauer-Manin obstruction to the Hasse Principle. This connection was originally drawn by Manin in [**Man71**]. Usually these examples involve computations of nontrivial Azumaya algebras that are split by a finite extension of the ground field. Such computations generally rely on a good knowledge of the class group of Weil divisors of the covering ring. It is this strong relationship between the class group of Weil divisors and the Brauer group that emphasizes the importance of the subject matter of Chapter 6. Exercise 13.6.18 contains an example that explores this phenomenon.

Chapter 14 contains some additional topics that did not seem to fit anywhere else. There is an entire section on the important corestriction map. A Mayer-Vietoris Sequence for the Brauer Group is derived. It is applied to compute the Brauer group of an affine algebraic curve and the Brauer group of a subring of a global field. An elementary example of a commutative ring whose Brauer group is cyclic of order n is exhibited.

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