Preface

0.1. Prerequisites

A person interested in reading this book should have the following background:

- Algebraic geometry (e.g., [**Har77**]: up to Chapter II, §8 as a minimum, but familiarity with later chapters is also needed at times)—this is not needed so much in our Chapter 1.
- Algebraic number theory (e.g., [**Cas67**, **Frö67**] or [**Lan94**, Part One] or [**Neu99**, Chapters I and II]).
- Some group cohomology (e.g., [**AW67**] or [**Mil13**, Chapter 2]).

0.2. What kind of book this is

The literature on rational points is vast. To write a book on the subject, an author must

1. write thousands of pages to cover all the topics comprehensively, or
2. focus on one aspect of the subject, or
3. write an extended survey serving as an introduction to many topics, with pointers to the literature for those who want to learn more about any particular one.

Our approach is closest to 3, so as to bring newcomers quickly up to speed while also providing more experienced researchers with directions for further exploration.

This book originated as the lecture notes for a semester-long course, taught during spring 2003 at the University of California, Berkeley, and fall

2008 and fall 2013 at the Massachusetts Institute of Technology. But it has grown since then; probably now it is about 50% too large for a single semester, unless students are willing to read much of it outside of class.

0.3. The nominal goal

Many techniques have been used to decide whether a variety over a number field has a rational point. Some generalize Fermat's method of infinite descent, some use quadratic reciprocity, and others appear at first sight to be ad hoc. But over the past few decades, it was discovered that nearly all of these techniques could be understood as applications of just two cohomological obstructions, the étale-Brauer obstruction and the descent obstruction. Moreover, while this book was being written, it was proved that the étale-Brauer obstruction and the descent obstruction are equivalent! The topics in this book build up to an explanation of this "grand unified theory" of obstructions.

0.4. The true goal

Our ulterior motive, however, is to introduce readers to techniques that they are likely to need while researching arithmetic geometry more broadly. Along the way, we mention open problems and applications that are interesting in their own right.

0.5. The content

Chapter 1 introduces fields of special interest to arithmetic geometers, and it discusses properties and invariants (the C_r property, cohomological dimension, and the Brauer group) that control the answers to some arithmetic questions about fields in general. Not all of Chapter 1 is needed in future chapters, but the Brauer group plays a key role later on (in Chapters 6 and 8).

Chapter 2 discusses aspects of varieties with particular attention to the case of ground fields that are not algebraically closed. Ultimately, we aim to treat global fields of positive characteristic as well as number fields, so we do not require our ground field to be perfect. Among other topics, this chapter discusses properties of varieties under base extension (e.g., irreducible vs. geometrically irreducible), the functor of points of a scheme, closed points and their relation to field-valued points, and genus change of curves under field extension. A final section introduces the main questions about rational points that motivate the subject, such as the questions of whether the local-global principle and weak approximation hold.

Chapter 3 begins with morphisms of finite presentation in order to discuss spreading out (e.g., extending a variety over a number field k to a scheme over a ring of S-integers in k). But the heart of the chapter is an extended introduction to smooth and étale morphisms, going beyond the treatment in [**Har77**, III.§10]. This, together with a section on flat morphisms, provides the basis for the definitions of the Grothendieck topologies commonly used for cohomology theories (see Chapter 6). The chapter also includes sections on rational maps and Frobenius morphisms: the latter are used to understand the Weil conjectures in Chapter 7.

The word "descent" has two unrelated meanings in arithmetic geometry. One meaning is as in Fermat's method of infinite descent and its generalizations, in which it is the height of rational points that descends in the course of a proof. The second meaning is that of descent of the ground field: the problem here is to decide whether a variety over a field extension $L \supseteq k$ arises as the base extension of a variety X over k (and to describe all possibilities for X). Chapter 4 studies this problem and its analogue for morphisms, and its generalizations to schemes. It also gives applications to the classification of twists of geometric objects (different k-forms of the same object over a field extension L), and to restriction of scalars, which transforms varieties over a field extension $L \supseteq k$ into varieties over k.

Chapter 5 is a survey on group schemes and algebraic groups over fields. After discussing their general properties, it defines special types of algebraic groups, and it states the classification theorem that decomposes arbitrary smooth algebraic groups into those types. The final section of Chapter 5 introduces torsors, which are needed to define the descent obstruction in Chapter 8. For algebraic groups over global fields, we discuss weak and strong approximation (Section 5.10) and the local-global principle for their torsors (Section 5.12.8).

Chapter 6 is an introduction to étale cohomology and its variants such as fppf cohomology. These cohomology theories are applied to generalize from torsors of algebraic groups over a field to torsors of group schemes over an arbitrary base scheme, and we prove finiteness results for torsors over a global field that are unramified at all but finitely many places. Another application is to generalize the cohomological definition of the Brauer group of a field to Grothendieck's definition of the cohomological Brauer group of a scheme. We end by discussing tools for computing these Brauer groups (the Hochschild–Serre spectral sequence, and residue homomorphisms) and give many examples since these will be needed to understand the Brauer–Manin obstruction in Section 8.2.

Although not needed for the main story of obstructions to rational points, we include in Chapter 7 the motivating application of étale cohomology, namely the Weil conjectures on varieties over finite fields. There we also discuss related issues, such as the étale cohomology classes of algebraic cycles and the Tate conjecture.

Chapter 8 defines the cohomological obstructions to the local-global principle and weak approximation for a variety X over a global field; these are expressed as subsets of the set $X(\mathbf{A})$ of adelic points that constrain where k-points may lie. First is the Brauer–Manin obstruction, coming from elements of the Brauer group of X. Next is the descent obstruction coming from torsors of algebraic groups; this is motivated by an example of a genus 2 curve in which the algebraic group is simply a finite group. Next we define hybrids of these two obstructions and compare their strengths for constraining k-points. Finally, we explain why all these obstructions are still not enough to decide whether a variety has a k-point.

Chapter 9 is a survey of the geometry and arithmetic of higher-dimensional varieties, with special attention paid to surfaces. It begins with the crude classification given by Kodaira dimension, and it compares the properties of being rational, unirational, rationally connected, and so on. Next we give the classification of surfaces over an arbitrary ground field, and we discuss the arithmetic of del Pezzo surfaces in some detail since these serve as excellent examples for the techniques presented earlier in the book. We end by discussing very briefly what is proved and conjectured for curves of genus > 1 and more generally for varieties of general type. The reasons for not exploring this in greater detail are first, that it would require a few hundred more pages to develop the required theory of height functions and diophantine approximation, and second, that several books on these topics exist already (we cite some of them).

A few appendices serve various purposes. Appendix A discusses some set theory that is implicitly used when discussing sheafification, for instance. Appendix B defines certain interesting classes of fields that did not make it into Chapter 1. Appendix C contains reference tables with lists of adjectives that can be applied to morphisms, varieties, or algebraic groups; the tables indicate where to find definitions, and propositions about their preservation under base extension, descent, and so on.

0.6. Anything new in this book?

Almost all of the theorems in this book existed previously in the published literature in some form, but in many places we have tried to make proofs more readable and to organize topics so as to form a coherent exposition. There are a few new results: For example, the finiteness of Selmer sets (see Theorems 6.5.13 and 8.4.6) and Minchev's theorem on the failure of strong approximation (Corollary 8.4.11) were previously known only over number fields, whereas we generalize them to all global fields; these generalizations require extra arguments in the function field case because of the failure of Hermite's finiteness theorem on extensions of bounded degree unramified outside a fixed set of places. A few smaller innovations include an improved proof of Theorem 9.3.1(b)(ii) stating that proper birational morphisms between smooth surfaces factor into blowups at *separable* points, the use of the Lang–Nishimura theorem to avoid general position arguments in the proof of Lemma 9.4.18 on degree 6 del Pezzo surfaces, and the k-rationality of degree 5 del Pezzo surfaces over even the smallest of finite fields (Theorem 9.4.29).

The book whose content overlaps the most with ours is probably [**Sko01**]. That book also discusses torsors and the Brauer–Manin and descent obstructions, and it is written by a leading expert. Our book can serve as preparation for reading that one, since ours includes more background material (on algebraic groups, on étale and fppf cohomology, etc.), while [**Sko01**] goes further in other directions, proving theorems on the Brauer–Manin obstruction for conic bundle surfaces and for homogeneous spaces of simply connected algebraic groups, for instance.

0.7. Standard notation

Following Bourbaki, define

$$\begin{aligned}
\mathbb{N} &:= \text{the set of natural numbers} = \{0, 1, 2, \dots\},\\
\mathbb{Z} &:= \text{the ring of integers} = \{\dots, -2, -1, 0, 1, 2, \dots\},\\
\mathbb{Q} &:= \text{the field of rational numbers} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0 \right\},\\
\mathbb{R} &:= \text{the field of real numbers},\\
\mathbb{C} &:= \text{the field of complex numbers} = \{\, a + bi : a, b \in \mathbb{R} \,\}, \text{ where } i = \sqrt{-1},\\
\mathbb{F}_q &:= \text{the finite field of } q \text{ elements},\\
\mathbb{Z}_p &:= \text{the ring of } p\text{-adic integers} = \varprojlim \mathbb{Z}/p^n\mathbb{Z},\\
\mathbb{Q}_p &:= \text{the field of } p\text{-adic numbers} = \text{the fraction field of } \mathbb{Z}_p.
\end{aligned}$$

The cardinality of a set S is denoted $\#S$ or sometimes $|S|$. If $(A_i)_{i \in I}$ is a collection of sets, and for all but finitely many $i \in I$ a subset $B_i \subseteq A_i$ is specified, then the **restricted product** $\prod'_{i \in I}(A_i, B_i)$ is the set of $(a_i) \in \prod_{i \in I} A_i$ such that $a_i \in B_i$ for all but finitely many i (with no condition being placed at the i for which B_i is undefined).

If $a, b \in \mathbb{Z}$, then $a \mid b$ means that a divides b, that is, that there exists $k \in \mathbb{Z}$ such that $b = ka$. Similarly, $a \nmid b$ means that a does not divide b. Define $\mathbb{Z}_{\geq 1} := \{\, n \in \mathbb{Z} : n \geq 1 \,\}$, and so on.

Rings are associative and have a 1 by definition [**Poo14**]. Suppose that R is a ring. Let $R^\times$ denote the unit group of R. Let $R[t_1, \dots, t_n]$ denote the ring of polynomials in $t_1, \dots, t_n$ with coefficients in R. Let $R[[t_1, \dots, t_n]]$ denote the ring of formal power series in $t_1, \dots, t_n$ with coefficients in R. The ring $R((t)) := R[[t]][t^{-1}]$ is called the ring of formal Laurent series in t with coefficients in R; its elements can be written as formal sums $\sum_{n \in \mathbb{Z}} a_n t^n$, where $a_n \in R$ for all n and $a_n = 0$ for sufficiently negative n. If R is an integral domain, then $\operatorname{Frac} R$ denotes its fraction field.

Suppose that k is a field. The characteristic of k is denoted $\operatorname{char} k$. The **rational function field** $k(t_1, \dots, t_n)$ is $\operatorname{Frac} k[t_1, \dots, t_n]$. The ring $k((t))$ defined above is a field, isomorphic to $\operatorname{Frac} k[[t]]$. Given an extension of fields L/k, a **transcendence basis** for L/k is a subset $S \subset L$ such that S is algebraically independent over k and L is algebraic over $k(S)$; such an S always exists, and $\#S$ is determined by L/k and is called the **transcendence degree** $\operatorname{tr deg}(L/k)$.

Suppose that R is a ring and $n \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{M}_n(R)$ denotes the R-algebra of $n \times n$ matrices with coefficients in R, and we define the group $\mathrm{GL}_n(R) := \mathrm{M}_n(R)^\times$. If R is commutative, a matrix $A \in \mathrm{M}_n(R)$ belongs to $\mathrm{GL}_n(R)$ if and only if its determinant $\det(A)$ is in $R^\times$.

If $\mathcal{A}$ is a category, then $\mathcal{A}^{\mathrm{opp}}$ denotes the opposite category, with the same objects but with morphisms reversed. We can avoid dealing with an anti-equivalence of categories $\mathcal{A} \to \mathcal{B}$ by rewriting it as an equivalence of categories $\mathcal{A}^{\mathrm{opp}} \to \mathcal{B}$. Let **Sets** be the category whose objects are sets and whose morphisms are functions. Let **Groups** denote the category of groups in which the morphisms are the homomorphisms. Let **Ab** denote the category of abelian groups; this is a full subcategory of **Groups**, where "full" means that for $A, B \in \mathbf{Ab}$, the definition of $\operatorname{Hom}(A, B)$ in **Ab** agrees with the definition of $\operatorname{Hom}(A, B)$ in **Groups**. We work in a fixed universe so that the objects in each category form a *set* (instead of a class); see Appendix A. From now on, we will usually not mention the universe.

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