is archimedean. Equip k_v and its subset $\mathcal{O}_v$ with the analytic (i.e., v-adic) topology coming from the place.

The **adèle ring** $\mathbf{A} = \mathbf{A}_k$ of k is defined as the restricted product

$$\prod\nolimits'_{v\in\Omega_k}(k_v, \mathcal{O}_v);$$

it is a k-algebra for the diagonal embedding of k, and it is equipped with the unique topology such that

- $\mathbf{A}$ is a topological group under addition,
- the subset $\prod_{v\in\Omega_k}\mathcal{O}_v$ is open, and
- the subspace topology on $\prod_{v\in\Omega_k}\mathcal{O}_v$ agrees with the product topology.

The image of k in $\mathbf{A}$ is discrete, and $\mathbf{A}/k$ is compact.

1.1.4. Other fields. For some other kinds of fields, see Appendix B.

1.2. C_r fields

(References: [**Gre69**], [**Sha72**], [**Pfi95**, Chapter 5])

Definition 1.2.1 ([**Lan52**])**.** Let k be a field, and let $r \in \mathbb{R}_{\geq 0}$. Then k is C_r if and only if every homogeneous form $f(x_1,\ldots,x_n)$ of degree $d > 0$ in n variables with $n > d^r$ has a nontrivial zero in k^n. The adjective **quasi-algebraically closed** is a synonym for C_1.

1.2.1. Norm forms and normic forms.

Definition 1.2.2. Let L be a finite extension of a field k. Let $e_1,\ldots,e_n$ be a k-basis of L. Write $L' = L(x_1,\ldots,x_n)$ and $k' = k(x_1,\ldots,x_n)$, where $x_1,\ldots,x_n$ are indeterminates. If $N_{L'/k'}$ denotes the norm from L' to k', then $N_{L'/k'}(x_1e_1 + \cdots + x_ne_n)$ is called a **norm form** for L over k.

Example 1.2.3. Let $k = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{7})$. The norm form for L over k associated to the basis $1, \sqrt{7}$ is $x_1^2 - 7x_2^2$.

Each norm form for L over k is a degree n homogeneous polynomial in $k[x_1,\ldots,x_n]$, where $n = [L : k]$. Although it depends on the choice of basis, changing the basis changes the norm form only by an invertible k-linear transformation of the variables. The value of the norm form at a point $(b_1,\ldots,b_n) \in k^n$ equals $N_{L/k}(b_1e_1 + \cdots + b_ne_n)$.

Definition 1.2.4. Let k be a field. A homogeneous form $f \in k[x_1,\ldots,x_n]$ is called **normic** if $\deg f = n$ and f has only the trivial zero in k^n.

Any norm form is normic. To construct other normic forms, we introduce some notation. If f and g are homogeneous forms, let $f(g \mid g \mid \cdots \mid g)$ be the homogeneous form obtained by substituting a copy of g for each variable in f, except that a new set of variables is used after each occurrence of $\mid$. If f is of degree d in n variables, and g is of degree e in m variables, then $f(g \mid g \mid \cdots \mid g)$ is of degree de in nm variables. If f and g are normic, then so is $f(g \mid g \mid \cdots \mid g)$.

Lemma 1.2.5. *If k is a field and k is not algebraically closed, then k has normic forms of arbitrarily high degree.*

Proof. Since k is not algebraically closed, it has a finite extension of degree $d > 1$. Let $F_1 = f$ be an associated norm form. For $\ell \geq 2$, let

$$F_\ell = F_{\ell-1}(f \mid f \mid \cdots \mid f).$$

By induction, F_ℓ is normic of degree d^ℓ. □

1.2.2. Systems of forms.

Proposition 1.2.6 (Artin, Lang, Nagata)**.** *Let k be a C_r field, and let $f_1, \ldots, f_s$ be homogeneous forms of the same degree $d > 0$ in n common variables. If $n > sd^r$, then $f_1, \ldots, f_s$ have a nontrivial common zero in k^n.*

Proof. Suppose that k is algebraically closed. Since $n > sd^r \geq s$, the projective dimension theorem [**Har77**, I.7.2] implies that the intersection of the s hypersurfaces $f_i = 0$ in $\mathbb{P}^{n-1}$ contains a point.

Therefore, from now on assume that k is not algebraically closed. Suppose also that the f_i have no nontrivial common zero. We will inductively build forms Φ_m of degree D_m in N_m variables, each having no nontrivial zero, and get a contradiction for large m. By Lemma 1.2.5, we can find a normic form Φ_0 of arbitrarily high degree e (later we will specify how large we need e to be). So $D_0 = N_0 = e$. For $m \geq 1$, define

$$\Phi_m = \Phi_{m-1}(f_1, \ldots, f_s \mid f_1, \ldots, f_s \mid \cdots \mid f_1, \ldots, f_s \mid 0, 0, \ldots, 0),$$

where within each block $f_1, \ldots, f_s$ the same n variables are used, but new variables are used after each $\mid$, and we use as many blocks as possible (namely, $\lfloor N_{m-1}/s \rfloor$ blocks) and pad with zeros to get the right number of arguments to Φ_{m-1}. Thus $D_m = dD_{m-1}$ and $N_m = n\lfloor N_{m-1}/s \rfloor$. By induction on m, the form Φ_m has no nontrivial zero.

By induction, $D_m = d^m e$. If we could ignore the $\lfloor\ \rfloor$, then N_m would be $(n/s)^m e$, and

$$\frac{N_m}{D_m^r} = \left(\frac{n}{sd^r}\right)^m e^{1-r} > 1$$

for sufficiently large m, since $n > sd^r$. But we cannot quite ignore $\lfloor\ \rfloor$, so we choose $\beta \in \mathbb{R}$ with $d^r < \beta < n/s$ and choose the degree e of the normic form Φ_0 so that $n\lfloor x/s \rfloor \geq \beta x$ holds for all $x \geq e$. Then $N_m \geq \beta^m e$ by induction on m, and

$$\frac{N_m}{D_m^r} \geq \left(\frac{\beta}{d^r}\right)^m e^{1-r} > 1$$

for m sufficiently large.

Since k is C_r, the form Φ_m has a nontrivial zero, a contradiction. □

1.2.3. Transition theorems.

Theorem 1.2.7. *Let k be a C_r field, and let L be a field extension of k.*

(i) *If L is algebraic over k, then L is C_r.*

(ii) *If $L = k(t)$, where t is an indeterminate, then L is C_{r+1}.*

(iii) *If* $\operatorname{tr}\deg(L/k) = s$, *then L is C_{r+s}.*

Proof.

(i) Let $f \in L[x_1, \ldots, x_n]$ be a form of degree $d > 0$, where $n > d^r$. Since L is algebraic over k, the coefficients of f generate a finite extension L_0 of k. If we find a nontrivial zero of f over L_0, then the same is a nontrivial zero over L. Thus we reduce to the case where L is a *finite* extension of k.

Choose a basis $e_1, \ldots, e_s$ of L over k. Introduce new variables y_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq s$, and substitute

$$x_i = \sum_{j=1}^{s} y_{ij} e_j$$

for all i into f, so that

$$f(x_1, \ldots, x_n) = F_1 e_1 + \cdots + F_s e_s,$$

where each $F_\ell \in k[\{y_{ij}\}]$ is a form of degree d in ns variables. Since $n > d^r$, we have $ns > sd^r$, so Proposition 1.2.6 implies that the F_ℓ have a nontrivial common zero (y_{ij}) over k. This means that f has a nontrivial zero over L.

(ii) Let $f \in k(t)[x_1, \ldots, x_n]$ be a form of degree $d > 0$, where $n > d^{r+1}$. Multiplying f by a polynomial in $k[t]$ to clear denominators, we may assume that f has coefficients in $k[t]$. Let m be the maximum of the degrees of these coefficients. Choose $s \in \mathbb{Z}_{>0}$ large (later we will say how large), introduce new variables y_{ij} with $1 \leq i \leq n$ and

$0 \le j \le s$, and substitute

$$x_i = \sum_{j=0}^{s} y_{ij} t^j$$

for all i into f, so that

$$f(x_1, \dots, x_n) = F_0 + F_1 t + \cdots + F_{ds+m} t^{ds+m},$$

where each $F_\ell \in k[\{y_{ij}\}]$ is a form of degree d in $n(s+1)$ variables. Because $n > d^{r+1}$,

$$n(s+1) > (ds+m+1)d^r$$

holds for sufficiently large s, and then Proposition 1.2.6 implies that the F_ℓ have a nontrivial common zero (y_{ij}) over k. This means that f has a nontrivial zero over $k[t]$, hence over $k(t)$.

(iii) This follows from (i) and (ii), by induction on s. □

1.2.4. Examples of C_r fields.

(1) A field is C_0 if and only if it is algebraically closed. For a generalization, see Exercise 1.3.

(2) The following special case of Theorem 1.2.7 is known as **Tsen's theorem**: If L is the function field of a curve over an algebraically closed field k (that is, L is a finitely generated extension of k of transcendence degree 1), then L is C_1.

(3) The **Chevalley–Warning theorem** states that finite fields are C_1. This was conjectured by E. Artin and was proved first by Chevalley [**Che36**], who proved more generally that over a finite field $\mathbb{F}_q$, a (not necessarily homogeneous) polynomial f of total degree d in $n > d$ variables with zero constant term has a nontrivial zero. Warning's proof [**War36**] of this proceeded by showing that the total number of zeros, including the trivial zero, was a multiple of $p := \operatorname{char} \mathbb{F}_q$. Ax [**Ax64**] showed moreover that the number of zeros was divisible by q, and in fact divisible by q^b, where $b = \lceil n/d \rceil - 1$ is the largest integer strictly less than n/d. For an improvement in a different direction, observe that Warning's theorem says that a hypersurface X in $\mathbb{P}^{n-1}$ over $\mathbb{F}_q$ defined by a homogeneous form of degree $d < n$ satisfies $\#X(\mathbb{F}_q) \equiv 1 \pmod{p}$; this can be extended to some varieties that are not hypersurfaces, such as smooth projective rationally chain connected varieties [**Esn03**, Corollary 1.3]; see [**Wit10**] for a survey about this and further generalizations.

(4) Lang proved that if k is complete with respect to a discrete valuation having algebraically closed residue field, then k is C_1. More generally, if k is a henselian discrete valuation field with algebraically closed residue field such that the completion $\widehat{k}$ is separable over k, then k is C_1. (See Section B.3 for the definition of henselian.) This applies in particular if k is the maximal unramified extension of a complete discrete valuation field with perfect residue field. For example, the maximal unramified extension $\mathbb{Q}_p^{\mathrm{unr}}$ of $\mathbb{Q}_p$ is C_1. See [**Lan52**] for all these results.

(5) A local field of positive characteristic is C_2; see [**Lan52**, Theorem 8]. More generally, if k is C_r, then $k((t))$ is C_{r+1} [**Gre66**].

1.2.5. Counterexamples. The field $\mathbb{R}$ is not C_r for any r, since for every $n \geq 1$ the equation $x_1^2 + \cdots + x_n^2 = 0$ has no nontrivial solution. The same argument applies to any formally real field.

E. Artin conjectured that nonarchimedean local fields were C_2, the expectation being that if a field k is complete with respect to a discrete valuation with a C_r residue field, then k should be C_{r+1}. That nonarchimedean local fields satisfy the C_2 property *restricted to degree d forms* was proved for $d = 2$ [**Has24**] and $d = 3$ [**Dem50**, **Lew52**]. Also Ax and Kochen [**AK65**] nearly proved that the field $\mathbb{Q}_p$ is C_2: using model theory they showed that for each d, for all primes p *outside a finite set depending on* d, every homogeneous form of degree d in $> d^2$ variables over $\mathbb{Q}_p$ has a nontrivial zero. But then Terjanian [**Ter66**] disproved Artin's conjecture by finding a homogeneous form of degree 4 in 18 variables over $\mathbb{Q}_2$ with no nontrivial zero. Later it was shown that if $[k : \mathbb{Q}_p] < \infty$, then k is not C_r for any r [**AK81**, **Ale85**]. It follows that if k is a number field, then k is not C_r for any r (Exercise 1.8).

1.2.6. Open questions.

Question 1.2.8. Is there a field k and $r \in \mathbb{R}_{\geq 0}$ such that k is C_r but not $C_{\lfloor r \rfloor}$?

Question 1.2.9 (E. Artin)**.** Let $\mathbb{Q}^{\mathrm{ab}}$ be the maximal abelian extension of $\mathbb{Q}$. (The Kronecker–Weber theorem states that $\mathbb{Q}^{\mathrm{ab}}$ is obtained by adjoining all roots of 1 to $\mathbb{Q}$.) Is $\mathbb{Q}^{\mathrm{ab}}$ a C_1 field?

Definition 1.2.10. A field k is called C_r' if whenever one has homogeneous forms $f_1, \ldots, f_s$ in n common variables of degrees $d_1, \ldots, d_s$, respectively, with $n > d_1^r + \cdots + d_s^r$, the forms have a nontrivial common zero in k^n.

Question 1.2.11 ([**Gre69**, p. 21])**.** Is C_r equivalent to C_r'?

By definition, C_r' implies C_r. The converse holds at least for fields k such that for every $d \geq 1$ there exists a homogeneous form of degree d in

d^r variables over k with no nontrivial zero [**Lan52**, Theorem 4]. The C_r' property is studied in more detail in [**Pfi95**, Chapter 5].

Question 1.2.12. What general classes of varieties are guaranteed to have a k-point whenever k is C_1?

Question 1.2.13 (Ax)**.** Is every perfect PAC field C_1? (See Section B.5 for the definition of PAC.)

By [**Kol07a**, Theorem 1], every PAC field of characteristic 0 is C_1, and even C_1'. See [**FJ08**, 21.3.6] for a few other positive partial results toward Question 1.2.13.

1.3. Galois theory

1.3.1. $\mathfrak{G}_k$-sets. Let k be a field. Let $\mathfrak{G}_k$ be the profinite group $\operatorname{Gal}(k_s/k)$. A $\mathfrak{G}_k$-**set** is a set S (with the discrete topology) equipped with a continuous action of $\mathfrak{G}_k$. A morphism of $\mathfrak{G}_k$-sets is a map of sets respecting the $\mathfrak{G}_k$-actions. A $\mathfrak{G}_k$-set is called **finite** if it is finite as a set. For example, if H is an open subgroup of $\mathfrak{G}_k$, then $\mathfrak{G}_k/H$ equipped with the left multiplication action of $\mathfrak{G}_k$ is a finite $\mathfrak{G}_k$-set.

A continuous action of $\mathfrak{G}_k$ on a set S is called **transitive** if $S \neq \emptyset$ and for every $s_1, s_2 \in S$ there exists $g \in \mathfrak{G}_k$ such that $gs_1 = s_2$. In this case, if we fix $s \in S$ and define $H = \operatorname{Stab}_{\mathfrak{G}_k}(s) := \{g \in \mathfrak{G}_k : gs = s\}$, then H is open and the map $\mathfrak{G}_k/H \to S$ sending $gH \in \mathfrak{G}_k/H$ to gs is an isomorphism of $\mathfrak{G}_k$-sets; in particular, S is finite. Every $\mathfrak{G}_k$-set decomposes uniquely as a disjoint union of transitive $\mathfrak{G}_k$-sets, the orbits.

1.3.2. Étale algebras. The problem with field extensions $L \supseteq k$ is that if we change the base by tensoring with a field extension k' of k, the resulting algebra $L \otimes_k k'$ over k' need not be a field. The notion of étale algebra generalizes the notion of finite separable field extension in order to fix this problem.

Definition 1.3.1. A k-algebra L is called **étale** if it satisfies any of the following equivalent conditions:

- L is a direct product of finite separable extensions of k;
- the k_s-algebra $L \otimes_k k_s$ is a finite product of copies of k_s;
- the morphism of schemes $\operatorname{Spec} L \to \operatorname{Spec} k$ is finite and étale in the sense of Section 3.5.8 (see Proposition 3.5.35).

A **morphism** between two étale k-algebras is a homomorphism of k-algebras. If L is an étale k-algebra and k' is any field extension of k, then $L \otimes_k k'$ is an étale k'-algebra.

As mentioned above, Yoneda's lemma implies that $\mathcal{N}$ is unique if it exists. Moreover, if X is a group scheme (see Section 5.1) and $\mathcal{N}$ exists, then the functor of points of $\mathcal{N}$ factors through **Groups**, so $\mathcal{N}$ is a group object in the category of smooth R-schemes; that is, $\mathcal{N}$ is a smooth group scheme over R.

Néron has his name attached to the concept because in 1964 he proved that $\mathcal{N}$ exists when X is an abelian variety over the fraction field of a discrete valuation ring; see Section 5.7.5. Here is a more recent result, in a different direction:

Theorem 3.5.83 (Liu and Tong)**.** *Let S be an integral Dedekind scheme. Let $K = \mathbf{k}(S)$. Let X be a nice curve over K of positive genus. Let $\mathcal{X} \to S$ be the minimal regular proper model of X (see Section* 9.3.1.6*). Then the smooth locus $\mathcal{X}^{\text{smooth}}$ of $\mathcal{X} \to S$ is a Néron model of X.*

Proof. See [**LT16**, Theorem 1.1]. □

3.6. Rational maps

3.6.1. Rational maps and domain of definition.

(Reference: [**EGA I**, §7])

Definition 3.6.1 ([**EGA I**, 7.1.2])**.** Let X and Y be S-schemes. Consider pairs (U, ϕ) in which U is a dense open subscheme of X and $\phi\colon U \to Y$ is an S-morphism. Call pairs (U, ϕ) and (V, ψ) equivalent if ϕ and ψ agree on a dense open subscheme of $U \cap V$. A **rational map** $X \dashrightarrow Y$ is an equivalence class of such pairs. In other words,

$$\{\text{rational maps } X \dashrightarrow Y\} := \varinjlim_{U} \operatorname{Hom}_S(U, Y),$$

where U ranges over dense open subschemes of X ordered by reverse inclusion.

Definition 3.6.2 ([**EGA I**, 7.2.1])**.** The **domain of definition** of a rational map is the union of the U as (U, ϕ) ranges over the equivalence class. It is an open subscheme of X.

Definition 3.6.2 is useful mainly when X is reduced and Y is separated:

Proposition 3.6.3. *Let W be the domain of definition of a rational map $X \dashrightarrow Y$, where X is reduced and Y is separated. Then there is a unique $\xi\colon W \to Y$ such that (W, ξ) belongs to the equivalence class.*

Proof. If (U, ϕ) and (V, ψ) are equivalent, so ϕ and ψ agree on a dense open subscheme of $U \cap V$, then by Corollary 2.3.23 they agree on all of $U \cap V$. Therefore all the (U, ϕ) can be glued to give (W, ξ). □

Remark 3.6.4. One can drop the hypothesis that X is reduced in Proposition 3.6.3 if one replaces "dense" by the stronger property "scheme-theoretically dense" everywhere in Definition 3.6.1. This leads to the notion of **pseudo-morphism**, a variant of the notion of rational map; see [**EGA IV**$_4$, 20.2.1].

3.6.2. Rational points over a function field.

Proposition 3.6.5. *Let X be an integral k-variety, and let Y be an arbitrary k-variety. Let $K = \mathbf{k}(X)$.*

(a) *The natural map*

$$\{\text{rational maps from } X \text{ to } Y\} \longrightarrow Y(K)$$
$$[\phi\colon U \to Y] \longmapsto (\text{the composition } \operatorname{Spec} K \hookrightarrow U \overset{\phi}{\to} Y)$$

is a bijection.

(b) *If moreover X is a regular curve and Y is proper, then we get bijections*

$$\operatorname{Hom}_k(X, Y) \quad \overset{\sim}{\to} \quad \{\text{rational maps from } X \text{ to } Y\} \quad \overset{\sim}{\to} \quad Y(K).$$

Proof.

(a) Every dense open subscheme of X contains a dense affine open subscheme; i.e., the inverse system $(\operatorname{Spec} A_i)$ of dense affine open subschemes of X is cofinal in the system of all dense open subschemes. Thus we have bijections

$$\begin{aligned}
\{\text{rational maps from } X \text{ to } Y\} &= \varinjlim_U Y(U) && \text{(by definition)}\\
&\simeq \varinjlim Y(A_i) && \text{(by cofinality)}\\
&\simeq Y(\varinjlim A_i) && \text{(by Remark 3.1.11)}\\
&= Y(K) && (\text{since } \varinjlim A_i = K).
\end{aligned}$$

(b) The first bijection comes from the valuative criterion for properness: The map $Y(X) \to Y(K)$ is bijective by Remark 3.2.14. The second bijection was given already in (a). $\square$

3.6.3. Dominant rational maps.

Definition 3.6.6. A rational map $X \dashrightarrow Y$ is **dominant** if and only if for some (or equivalently, for each) representative (U, ϕ), the image $\phi(U)$ is dense in Y.

Corollary 3.6.7 (cf. [**Har77**, Theorem I.4.4]). *The functor*

$$\begin{Bmatrix} \text{integral } k\text{-varieties,} \\ \text{dominant rational maps} \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{finitely generated field extensions of } k, \\ k\text{-algebra homomorphisms} \end{Bmatrix}^{\text{opp}}$$
$$X \longmapsto \mathbf{k}(X)$$

is an equivalence of categories.

Proof. A rational map $X \dashrightarrow Y$ is dominant if and only if it maps the generic point of X to the generic point of Y; thus we have a functor from left to right. Restricting the bijection in Proposition 3.6.5(a) to the dominant rational maps $X \dashrightarrow Y$ shows that the functor is fully faithful. Every finitely generated field extension of k is isomorphic to the function field of an integral k-variety (cf. Proposition 2.2.13); i.e., the functor is essentially surjective. $\square$

Definition 3.6.8. If X is an integral k-variety, the set of birational maps $X \dashrightarrow X$ forms a group $\operatorname{Bir} X$. By Corollary 3.6.7, $\operatorname{Bir} X$ is isomorphic to the group $\operatorname{Aut}(\mathbf{k}(X)/k)$ of automorphisms of the function field over k.

Example 3.6.9. The group $\operatorname{Bir} \mathbb{P}^n$ is also called the **Cremona group**.

Definition 3.6.10. If $\pi \colon X \to Y$ is a dominant rational map between integral k-varieties of the same dimension, then $\mathbf{k}(X)$ may be viewed as a finite extension of $\mathbf{k}(Y)$, and we define the **degree** of π as $\deg \pi := [\mathbf{k}(X) : \mathbf{k}(Y)]$.

3.6.4. Lang–Nishimura theorem. If $\pi \colon X \to Y$ is a morphism of k-varieties and X has a k-point x, then Y has a k-point, namely $\pi(x)$. If π is only a rational map, this argument fails, since π might be undefined at x, but surprisingly the same conclusion can be drawn, under mild hypotheses. The following theorem is due to Lang [**Lan54**] and Nishimura [**Nis55**].

Theorem 3.6.11 (Lang–Nishimura theorem). *Let $X \dashrightarrow Y$ be a rational map between k-varieties, where Y is proper. If X has a smooth k-point, then Y has a k-point.*

Proof. Let x be the given smooth k-point on X. Replacing X by an open neighborhood of x, we may assume that X is integral. Let $n = \dim X$. Proposition 3.5.66 gives the isomorphism in the chain of embeddings

$$\mathscr{O}_{X,x} \hookrightarrow \widehat{\mathscr{O}}_{X,x} \simeq k[[t_1, \ldots, t_n]] \hookrightarrow F := k((t_1))((t_2)) \cdots ((t_n)).$$

Since F is a field (an iterated formal Laurent series field), the fraction field $\operatorname{Frac} \mathscr{O}_{X,x} = \mathbf{k}(X)$ embeds in F. By Proposition 3.6.5(a), the rational map gives an element of $Y(\mathbf{k}(X))$, and hence an element of $Y(F)$. Applying Lemma 3.6.12 n times shows that Y has a k-point. $\square$

Lemma 3.6.12. *Let Y be a proper k-variety. Let L be a field extension of k, and let $L((t))$ be the formal Laurent series field over L. If Y has an $L((t))$-point, then Y has an L-point.*

Proof. By the valuative criterion for properness (Theorem 3.2.12), the element of $Y(L((t)))$ extends to an element of $Y(L[[t]])$, which reduces modulo t to an element of $Y(L)$. $\square$

Remark 3.6.13. The Lang–Nishimura theorem can be explained geometrically as follows. If $\dim_x X > 0$, then one can show that X contains an integral curve C such that

- x is a smooth point of C, and
- C meets the domain of definition of the rational map ϕ.

The valuative criterion for properness shows that $\phi|_C \colon C \dashrightarrow Y$ extends to be defined at x. It maps x to a k-point of Y. (The reason that we did not present the proof this way is that the existence of C is not immediate.)

Remark 3.6.14. For another proof of Theorem 3.6.11, see Exercise 3.11.

Remark 3.6.15. In Theorem 3.6.11 one cannot conclude that Y has a *smooth* k-point.

The Lang–Nishimura theorem implies that the property of having a k-point is a birational invariant of smooth, proper, integral k-varieties:

Corollary 3.6.16. *Let X and Y be smooth, proper, integral k-varieties that are birational to each other. Then X has a k-point if and only if Y has a k-point.*

3.7. Frobenius morphisms

(Reference: [**SGA 5**, XV])

Let p be a prime number. Let X be a scheme of characteristic p, i.e., a scheme with $p\mathscr{O}_X = 0$. Then the **absolute Frobenius morphism** is the morphism of schemes $F_X \colon X \to X$ that is the identity on topological spaces and that induces the pth-power homomorphism $f \mapsto f^p$ on each ring $\mathscr{O}_X(U)$.

Now let S be a scheme of characteristic p, and let X be an S-scheme. Let $X^{(p)}$ be the base extension of X by F_S. Then the universal property of the fiber product gives a morphism $F_{X/S} \colon X \to X^{(p)}$ called the **relative**

Cohomological obstructions to rational points

In 1970, Manin [**Man71**] explained how, for a variety X over a global field k, elements of $\operatorname{Br} X$ could produce obstructions to the local-global principle. Meanwhile, Fermat's method of infinite descent was generalized to show how a torsor under an algebraic group G over X could give rise to an obstruction, by Chevalley and Weil [**CW30**] for finite G, by Colliot-Thélène and Sansuc [**CTS77, CTS80, CTS87**] for commutative G, and by Harari and Skorobogatov [**HS02**] for general G. In this chapter, we will explain these and related obstructions.

8.1. Obstructions from functors

8.1.1. The F-obstruction to the local-global principle. Let k be a global field, and let $\mathbf{A}$ be its adèle ring. Let $F\colon \mathbf{Schemes}_k^{\mathrm{opp}} \to \mathbf{Sets}$ be a functor. For a k-algebra L, write $F(L)$ for $F(\operatorname{Spec} L)$. Let X be a k-variety.

Suppose that $A \in F(X)$. For each k-algebra L, define $\mathrm{ev}_A\colon X(L) \to F(L)$ as follows: Given $x \in X(L)$, the corresponding morphism $\operatorname{Spec} L \xrightarrow{x} X$ induces a map $F(X) \to F(L)$, sending A to some element of $F(L)$ called

$\mathrm{ev}_A(x)$ or $A(x)$. Then the diagram

$$\begin{tikzcd} X(k) \arrow[r, hook] \arrow[d, "\mathrm{ev}_A"'] & X(\mathbf{A}) \arrow[d, "\mathrm{ev}_A"'] \\ F(k) \arrow[r] & F(\mathbf{A}) \end{tikzcd} \tag{8.1.1}$$

commutes. Let $X(\mathbf{A})^A$ be the subset of $X(\mathbf{A})$ consisting of elements whose image in $F(\mathbf{A})$ lies in the image of $F(k) \to F(\mathbf{A})$. Then (8.1.1) shows that $X(k) \subseteq X(\mathbf{A})^A$. In other words, A puts constraints on the locus in $X(\mathbf{A})$ where k-points can lie.

Imposing the constraints for all $A \in F(X)$ yields the subset

$$X(\mathbf{A})^F = X(\mathbf{A})^{F(X)} := \bigcap_{A \in F(X)} X(\mathbf{A})^A,$$

again containing $X(k)$.

Definition 8.1.2. If $X(\mathbf{A}) \neq \emptyset$ but $X(\mathbf{A})^F = \emptyset$, then we say that there is an F-**obstruction to the local-global principle**; in this case $X(k) = \emptyset$.

8.1.2. The F-obstruction to weak approximation. We have $X(\mathbf{A}) \subseteq X(\prod k_v) = \prod X(k_v)$; cf. Exercise 3.4. (If X is proper, then all three sets are equal.) There is a variant of Definition 8.1.2 in which $X(\mathbf{A})$ is replaced by $X(\prod k_v) = \prod X(k_v)$ and $F(\mathbf{A})$ is replaced by $\prod F(k_v)$ in (8.1.1); call the resulting set $X(\prod k_v)^F$.

Definition 8.1.3. If $X(\prod k_v)^F \neq X(\prod k_v)$, then we say that there is an F-**obstruction to weak approximation**. Usually this terminology is used in a context where $X(\prod k_v)^F$ is known to be closed in $X(\prod k_v)$, in which case such an F-obstruction would imply that $X(k)$ is not dense in $X(\prod k_v)$.

8.1.3. Examples. In order for the F-obstruction to be nontrivial, F must be such that $F(k) \to F(\mathbf{A})$ is not surjective. In order for the F-obstruction to be useful, the image of $F(k) \to F(\mathbf{A})$ must be describable in some way. This is so in the following two examples, as will be explained in subsequent sections.

Example 8.1.4. Taking $F = \mathrm{Br}$ defines the **Brauer set** $X(\mathbf{A})^{\mathrm{Br}}$.

Example 8.1.5. Taking $F = \mathrm{H}^1(-, G)$ for an affine algebraic group G over k defines a set $X(\mathbf{A})^{\mathrm{H}^1(X,G)}$.

Remark 8.1.6. To avoid having to understand the Brauer group of a non-noetherian ring like $\mathbf{A}$, in Section 8.2 we will replace $\mathrm{Br}\,\mathbf{A}$ in (8.1.1) by $\bigoplus_v \mathrm{Br}\,k_v$ when defining $X(\mathbf{A})^{\mathrm{Br}}$; in fact, the Brauer–Manin obstruction was originally defined using $\bigoplus_v \operatorname{Br} k_v$. It turns out that the natural homomorphism $\operatorname{Br} \mathbf{A} \to \bigoplus_v \operatorname{Br} k_v$ is an isomorphism [**Čes15**, Theorem 2.13], so the resulting set $X(\mathbf{A})^{\mathrm{Br}}$ is the same. Similarly, we replace $\mathrm{H}^1(\mathbf{A}, G)$ by $\prod_v \mathrm{H}^1(k_v, G)$ in Section 8.4; the natural map $\mathrm{H}^1(\mathbf{A}, G) \to \prod_v \mathrm{H}^1(k_v, G)$ is an injection (a consequence of [**Čes15**, Theorem 2.18]), so again the resulting set $X(\mathbf{A})^{\mathrm{H}^1(X,G)}$ is the same.

Question 8.1.7. Are there other functors that one could use to obtain obstructions?

8.1.4. Functoriality. The proofs of the following three statements are left to the reader as Exercise 8.1.

Proposition 8.1.8. *Let $\pi\colon X' \to X$ be a morphism of k-varieties. Let $x' \in X'(L)$ for some k-algebra L, and let $A \in F(X)$. Then the two ways of evaluating A on x' yield the same result: If we define $x := \pi(x) \in X(L)$ and $A' := \pi^* A \in F(X')$, then $A'(x') = A(x)$ in $F(L)$.*

Corollary 8.1.9. *The assignment $X \mapsto X(\mathbf{A})^F$ is functorial in X.*

Corollary 8.1.10. *Let $\pi\colon X' \to X$ be a morphism of k-varieties. If the map $F(X) \to F(X')$ is surjective, then $X'(\mathbf{A})^F$ is the inverse image of $X(\mathbf{A})^F$ under $X'(\mathbf{A}) \to X(\mathbf{A})$.*

8.2. The Brauer–Manin obstruction

Throughout this section, k is a field, and X is a k-variety.

8.2.1. Evaluation. Let $A \in \operatorname{Br} X$. If L is a k-algebra and $x \in X(L)$, then $\operatorname{Spec} L \xrightarrow{x} X$ induces a homomorphism $\operatorname{Br} X \to \operatorname{Br} L$, which maps A to an element of $\operatorname{Br} L$ that we call $A(x)$; cf. Section 8.1.1.

8.2.2. The Brauer set.

(Reference: [**Sko01**, §5.2])

Now suppose that k is a global field. Fix $A \in \operatorname{Br} X$.

Proposition 8.2.1. *If $(x_v) \in X(\mathbf{A})$, then $A(x_v) = 0$ for almost all v.*

Proof. By Corollary 6.6.11, for some finite set of places S (containing all the archimedean places), we can spread out X to a finite-type $\mathcal{O}_{k,S}$-scheme $\mathcal{X}$ and spread out A to an element $\mathcal{A} \in \operatorname{Br} \mathcal{X}$. Enlarging S if necessary, we may also assume that $x_v \in \mathcal{X}(\mathcal{O}_v)$ for all $v \notin S$. Then $A(x_v)$ comes from an element $\mathcal{A}(x_v) \in \operatorname{Br} \mathcal{O}_v$. But $\operatorname{Br} \mathcal{O}_v = 0$ by Corollary 6.9.3. $\square$

Thus A determines a map

$$\begin{aligned} X(\mathbf{A}) &\longrightarrow \mathbb{Q}/\mathbb{Z} \\ (x_v) &\longmapsto (A,(x_v)) := \sum_v \mathrm{inv}_v(A(x_v)). \end{aligned}$$

Proposition 8.2.2. *If $x \in X(k) \subseteq X(\mathbf{A})$, then $(A,x) = 0$.*

Proof. Use the commutativity of

(8.2.3)
$$\begin{tikzcd} & X(k) \arrow[r, hook] \arrow[d] & X(\mathbf{A}) \arrow[d] & & \\ 0 \arrow[r] & \operatorname{Br} k \arrow[r] & \bigoplus_v \operatorname{Br} k_v \arrow[r, "\sum \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$
□

Remark 8.2.4. Compare (8.2.3) with (8.1.1).

Definition 8.2.5. For $A \in \operatorname{Br} X$, define

$$X(\mathbf{A})^A := \{ (x_v) \in X(\mathbf{A}) : (A,(x_v)) = 0 \}.$$

Also define

$$X(\mathbf{A})^{\mathrm{Br}} := \bigcap_{A \in \operatorname{Br} X} X(\mathbf{A})^A.$$

This agrees with the definition in Example 8.1.4, because of Remark 8.1.6.

Corollary 8.2.6. *We have $X(k) \subseteq X(\mathbf{A})^{\mathrm{Br}}$.*

Proof. This is a restatement of Proposition 8.2.2. □

8.2.3. The Brauer–Manin obstruction to the local-global principle.

Definition 8.2.7. One says that there is a **Brauer–Manin obstruction to the local-global principle** for X if $X(\mathbf{A}) \neq \emptyset$, but $X(\mathbf{A})^{\mathrm{Br}} = \emptyset$.

Definition 8.2.8. For a class of nice varieties X over global fields, one says that **the Brauer–Manin obstruction to the local-global principle is the only one** if the implication

$$X(\mathbf{A})^{\mathrm{Br}} \neq \emptyset \quad \Longrightarrow \quad X(k) \neq \emptyset$$

holds.

See Conjecture 9.2.27 for a setting in which it is conjectured that the Brauer–Manin obstruction to the local-global principle is the only one.

8.2.4. Brauer evaluation is locally constant.

Proposition 8.2.9. *Let k be a local field, and let X be a k-variety. Let $A \in \operatorname{Br} X$.*

(a) *The map $X(k) \to \operatorname{Br} k$ sending x to $A(x)$ is locally constant with respect to the analytic topology on $X(k)$.*

(b) *If $k = \mathbb{R}$, the map $X(\mathbb{R}) \to \operatorname{Br} \mathbb{R}$ is constant on each connected component of $X(\mathbb{R})$.*

Proof.

(a) Given $\alpha \in \operatorname{Br} k$, we need to show that $\{x \in X(k) : A(x) = \alpha\}$ is open in $X(k)$. The structure morphism $X \to \operatorname{Spec} k$ induces a homomorphism $\operatorname{Br} k \to \operatorname{Br} X$, which sends α to a "constant" element $\alpha_X \in \operatorname{Br} X$. Replacing A by $A - \alpha_X$ subtracts α from all the values $A(x)$. Thus we may reduce to showing that $\{x \in X(k) : A(x) = 0\}$ is open in $X(k)$.

Let $x_0 \in X(k)$. Consider pairs (Y, y) where Y is an étale X-scheme, Y is affine, and $y \in Y(k)$ maps to $x_0 \in X(k)$. Let R be the direct limit of $\mathscr{O}(Y)$ over the system of such (Y, y). Then R is the henselization of the local ring $\mathscr{O}_{X,x_0}$, so R is a henselian local ring with residue field k (see Section B.3). For each (Y, y), we have morphisms $\operatorname{Spec} k \to \operatorname{Spec} R \to Y \to X$, inducing homomorphisms $\operatorname{Br} X \to \operatorname{Br} Y \to \operatorname{Br} R \to \operatorname{Br} k$, the composition of which sends A to $A(x_0) = 0$. By Remark 6.9.2, the homomorphism $\operatorname{Br} R \to \operatorname{Br} k$ is an isomorphism, so A maps to 0 already in $\operatorname{Br} R$. By Theorem 6.4.3, A maps to 0 in $\operatorname{Br} Y$ for some (Y, y). Let $\pi \colon Y \to X$ be the structure morphism, which is étale. By functoriality as in Proposition 8.1.8, $A(\pi(y)) = 0$ for every $y \in Y(k)$. Since $\pi \colon Y \to X$ is étale, Proposition 3.5.73 shows that $\pi(Y(k))$ is open in $X(k)$, and it contains x_0.

(b) A locally constant map is constant on connected components. $\square$

Remark 8.2.10. The proof of (b) works for every local field k. But if k is nonarchimedean, then each connected component of $X(k)$ is a point. And if $k = \mathbb{C}$, then $\operatorname{Br} k = 0$. So only the case $k = \mathbb{R}$ is interesting.

Corollary 8.2.11. *Let k be a global field. Let X be a k-variety.*

(a) *For any $A \in \operatorname{Br} X$, the map $X(\mathbf{A}) \to \mathbb{Q}/\mathbb{Z}$ sending (x_v) to $(A, (x_v))$ is locally constant.*

(b) *For any $A \in \operatorname{Br} X$, the set $X(\mathbf{A})^A$ is open and closed in $X(\mathbf{A})$.*

(c) *The set $X(\mathbf{A})^{\operatorname{Br}}$ is closed in $X(\mathbf{A})$.*

(d) *Let $\overline{X(k)}$ be the closure of $X(k)$ in $X(\mathbf{A})$. Then $\overline{X(k)} \subseteq X(\mathbf{A})^{\operatorname{Br}}$.*

(e) *If X is proper and $X(\mathbf{A})^{\mathrm{Br}} \neq X(\mathbf{A})$, then weak approximation for X fails. In this case, one says that there is a* ***Brauer–Manin obstruction to weak approximation for*** X.

Proof.

(a) Combine Propositions 8.2.1 and 8.2.9.

(b) A fiber of a locally constant map is open and closed.

(c) The set $X(\mathbf{A})^{\mathrm{Br}}$ is the intersection of the closed sets $X(\mathbf{A})^A$ as A varies.

(d) This follows from (c) and Corollary 8.2.6.

(e) If $\overline{X(k)} \subseteq X(\mathbf{A})^{\mathrm{Br}} \subsetneq X(\mathbf{A})$, then $\overline{X(k)} \neq X(\mathbf{A})$, while $X(\mathbf{A}) = \prod_v X(k_v)$ if X is proper. Thus weak approximation for X fails. $\square$

Remark 8.2.12. Suppose that X is a *proper* variety over a global field k. For a place v of k, what does a locally constant function f on $X(k_v)$ look like?

- Suppose that v is archimedean. Then f is constant on each connected component of $X(k_v)$.
- Suppose that v is nonarchimedean. Let $\mathcal{O}_v$ be the valuation ring, and let $\pi_v \in \mathcal{O}_v$ be a uniformizer. Then $X(k_v) = X(\mathcal{O}_v) = \varprojlim_n X(\mathcal{O}_v/\pi_v^n)$, which is compact, and f factors through the finite set $X(\mathcal{O}_v/\pi_v^n)$ for some n.

Now suppose that $A \in \operatorname{Br} X$. Then the remarks above apply to the evaluation map $X(k_v) \to \operatorname{Br} k_v$ given by A for each v, and this map is 0 for all but finitely many v, by Proposition 8.2.1. Thus the map $X(\mathbf{A}) \to \mathbb{Q}/\mathbb{Z}$ sending (x_v) to $(A, (x_v))$ admits a finite explicit description, in principle. In Section 8.2.5, we will see an example of this.

8.2.5. Example: Iskovskikh's conic bundle with 4 singular fibers.

(References: [**Isk71**], [**Sko01**, Chapter 7])

Let U be the smooth, affine, geometrically integral surface

$$y^2 + z^2 = (3 - x^2)(x^2 - 2)$$

over $\mathbb{Q}$. We will construct a nice $\mathbb{Q}$-surface X containing U as an open subscheme, and then we will show that there is a Brauer–Manin obstruction to the local-global principle for X.

8.2.5.1. *Conic bundles.* The X above will be a conic bundle. Before constructing it, let us discuss conic bundles more generally.

A (possibly degenerate) **conic** over a field k is the zero locus in $\mathbb{P}^2$ of a nonzero degree 2 homogeneous polynomial s in $k[x_0, x_1, x_2]$. It is a **diagonal conic** if s is $ax_0^2 + bx_1^2 + cx_2^2$ for some $a, b, c \in k$ not all zero.

The generalization to conic bundles will be easier if we first re-express the situation over k in a coordinate-free way. If E is the k-vector space with basis x_0, x_1, x_2, then $\mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1, x_2] = \operatorname{Proj} \operatorname{Sym} E =: \mathbb{P}E$, and a degree 2 homogeneous polynomial s is an element of $\operatorname{Sym}^2 E$. We have $E = L_0 \oplus L_1 \oplus L_2$, where $L_i = kx_i$. To say that $s = 0$ is a diagonal conic is to say that $s = s_0 + s_1 + s_2$ for some $s_i \in kx_i^2 = L_i^{\otimes 2}$ not all zero.

If B is a k-scheme, then a **conic bundle** over B is the zero locus of s in $\mathbb{P}\mathscr{E} := \mathbf{Proj} \operatorname{Sym} \mathscr{E}$, where $\mathscr{E}$ is a rank 3 vector bundle on B, and $s \in \Gamma(B, \operatorname{Sym}^2 \mathscr{E})$ is a section vanishing nowhere on B. In the special case where $\mathscr{E} = \mathscr{L}_0 \oplus \mathscr{L}_1 \oplus \mathscr{L}_2$ for some line bundles $\mathscr{L}_i$ on B, and $s = s_0 + s_1 + s_2$ for some $s_i \in \Gamma(B, \mathscr{L}_i^{\otimes 2})$ such that s_0, s_1, s_2 do not simultaneously vanish anywhere on B, the zero locus of s is called a **diagonal conic bundle**.

8.2.5.2. *Châtelet surfaces.* We now specialize further to the following setting:

$$\begin{aligned}
k &: \text{ field of characteristic not } 2, \\
B &:= \mathbb{P}^1_k, \\
\mathscr{L}_0 &:= \mathscr{O}, & s_0 &:= 1, \\
\mathscr{L}_1 &:= \mathscr{O}, & s_1 &:= -a, \\
\mathscr{L}_2 &:= \mathscr{O}(2), & s_2 &:= -F(w, x),
\end{aligned}$$

where $a \in k^\times$, and $F(w, x) \in \Gamma(\mathbb{P}^1_k, \mathscr{O}(4))$ is a separable homogeneous polynomial of degree 4 in the homogeneous coordinates w, x on $B = \mathbb{P}^1$. The result is a nice k-surface X containing the affine surface

$$y^2 - az^2 = f(x)$$

as an open subscheme, where $f(x)$ is the dehomogenization $F(1, x)$. Such a surface X is called a **Châtelet surface**. It has a map to $B = \mathbb{P}^1$, and the fibers of $X \to \mathbb{P}^1$ are conics. In fact, all the fibers of $X \to \mathbb{P}^1$ above points in $\mathbb{P}^1(\overline{k})$ are *nice* conics, except above four points (the zeros of F) where the fiber degenerates to the union of two intersecting lines in $\mathbb{P}^2$.

8.2.5.3. *Iskovskikh's example.* Iskovskikh's surface is the Châtelet surface X over $\mathbb{Q}$ given by the choices $a := -1$ and $f(x) := (3 - x^2)(x^2 - 2) \in \mathbb{Q}[x]$.

Remark 8.2.13. One could choose other nice compactifications X' of the affine surface

$$U \colon y^2 + z^2 = (3 - x^2)(x^2 - 2).$$

For instance, one could let X' be the blowup of X at a closed point of $X-U$. But the question of whether such a compactification has a rational point is independent of the choice, by Corollary 3.6.16.

Let $K = \mathbf{k}(X)$. As explained in Section 1.5.7.4, given two elements $a, b \in K^\times$, one can define a quaternion algebra with class $(a, b) \in (\operatorname{Br} K)[2]$. Let $A = (3-x^2, -1) \in \operatorname{Br} K$. By Proposition 6.6.7(i), we may view $\operatorname{Br} X$ as a subgroup of $\operatorname{Br} K$.

Proposition 8.2.14. *The element $A \in \operatorname{Br} K$ lies in the subgroup $\operatorname{Br} X$.*

Proof. By Theorem 6.8.3, we need only check that A has no residue along any integral divisor on X. Therefore it will suffice to find a Zariski open covering $\{U_i\}$ of X such that A extends to an element of $\operatorname{Br} U_i$ for each i.

To accomplish this, we rewrite A in other ways. Define $B := (x^2-2, -1)$ and $C := (3/x^2 - 1, -1)$ in $\operatorname{Br} K$. Then $A + B = (y^2 + z^2, -1) = 0$ by Proposition 1.5.23 since $y^2 + z^2 = N_{K(\sqrt{-1})/K}(y + z\sqrt{-1})$. Also, $A - C = (x^2, -1) = 0$ since x^2 is a square in K. But A, B, C are all killed by 2, so $A = B = C$.

Let P_{3-x^2} and P_{x^2-2} be the closed points of $\mathbb{P}^1_{\mathbb{Q}}$ given by $3 - x^2 = 0$ and $x^2 - 2 = 0$, respectively. Now $A = (3-x^2, -1)$ represents a quaternion Azumaya algebra on all of X except along integral divisors where $3 - x^2$ or -1 has a zero or pole. Thus A comes from $\operatorname{Br} U_A$, where

$$U_A := X - (\text{fiber above } \infty) - (\text{fiber above } P_{3-x^2}).$$

Similarly, $B \in \operatorname{Br} U_B$, where

$$U_B := X - (\text{fiber above } \infty) - (\text{fiber above } P_{x^2-2}),$$

and $C \in \operatorname{Br} U_C$, where

$$U_C := X - (\text{fiber above } 0) - (\text{fiber above } P_{3-x^2}).$$

Since $U_A \cup U_B \cup U_C = X$ (in fact, $U_B \cup U_C = X$), the element $A = B = C \in \operatorname{Br} K$ belongs to $\operatorname{Br} X$. $\square$

From now on, we consider A as an element of $\operatorname{Br} X$. To evaluate A at a point $P \in X(k)$ for any field $k \supset \mathbb{Q}$, choose one of

$$(3 - x^2, -1), \quad (x^2 - 2, -1), \quad (3/x^2 - 1, -1)$$

such that the rational function of x is defined and nonzero at P, so that A extends to an element of the Brauer group of an open subset U_A, U_B, or U_C containing P, and replace the rational function by its value at P. For example, if $P \in U_A(\mathbb{Q}_p)$ for some $p \le \infty$, then

$$\begin{aligned}\operatorname{inv}_p A(P) &= \operatorname{inv}_p(3 - x(P)^2, -1)\\ &= \begin{cases} 0 & \text{if } 3 - x(P)^2 \in N_{\mathbb{Q}_p(\sqrt{-1})/\mathbb{Q}_p}(\mathbb{Q}_p(\sqrt{-1})^\times),\\ 1/2 & \text{otherwise,}\end{cases}\end{aligned}$$

by Proposition 1.5.23.

Proposition 8.2.15. *We have $X(\mathbf{A}) \neq \emptyset$, but $X(\mathbf{A})^A = \emptyset$. In particular, $X(\mathbb{Q}) = \emptyset$, and there is a Brauer–Manin obstruction to the local-global principle for X.*

Proof. A computation involving Hensel's lemma (Theorem 3.5.63(a)) shows that $X(\mathbf{A}) \neq \emptyset$.

Suppose that $P \in X(\mathbb{Q}_p)$ for some $p \le \infty$. If $p \neq \infty$, let $v_p \colon \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ denote the p-adic valuation. Let $x = x(P) \in \mathbb{Q}_p \cup \{\infty\}$.

Case I: $p \notin \{2, \infty\}$. If $v_p(x) < 0$ (or $x = \infty$), then $3/x^2 - 1 \in \mathbb{Z}_p^\times$. If $v_p(x) \ge 0$, then either $3 - x^2$ or $x^2 - 2$ is in $\mathbb{Z}_p^\times$ because their sum is 1. In either case, $A(P)$ has the form (u_1, u_2) with $u_1, u_2 \in \mathbb{Z}_p^\times$, so $A(P) \in \operatorname{Br} \mathbb{Z}_p$ (this uses $p \neq 2$). But $\operatorname{Br} \mathbb{Z}_p = 0$ by Corollary 6.9.3, so $\operatorname{inv}_p A(P) = 0$.

Case II: $p = \infty$. The leading coefficient of $(3 - x^2)(x^2 - 2)$ is not a sum of squares in $\mathbb{R}$, so any $P \in X(\mathbb{R})$ satisfies $x(P) \neq \infty$. Then $x(P)^2 < 3$ or $x(P)^2 > 2$, so $3 - x(P)^2$ or $x(P)^2 - 2$ is in $\mathbb{R}_{>0} = N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$. Thus $\operatorname{inv}_\infty A(P) = 0$.

Case III: $p = 2$. Let $P \in X(\mathbb{Q}_2)$. Let $x = x(P)$. Then

$$\begin{aligned} v_2(x) > 0 \quad &\Longrightarrow \quad 3 - x^2 \equiv 3 \equiv -1 \pmod 4\\ v_2(x) = 0 \quad &\Longrightarrow \quad x^2 - 2 \equiv -1 \pmod 4\\ v_2(x) < 0 \quad &\Longrightarrow \quad 3/x^2 - 1 \equiv -1 \pmod 4. \end{aligned}$$

But an element of $\mathbb{Z}_2$ that is $-1 \bmod 4$ is not of the form $a^2 + b^2$ with $a, b \in \mathbb{Q}_2$, so it is not a norm from $\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2$. Thus $\operatorname{inv}_2 A(P) = 1/2$.

Cases I, II, III imply that if $(P_p) \in X(\mathbf{A})$, then $(A, (P_p)) = 1/2 \neq 0$. Thus $X(\mathbf{A})^A = \emptyset$. □

Remark 8.2.16. Iskovskikh's original proof that $X(\mathbb{Q}) = \emptyset$ used only ad hoc methods based on quadratic reciprocity. Ironically, according to [**CTPS16**, §1], Iskovskikh's intention was to produce an example that the Brauer–Manin obstruction could not explain! It was only a few years later that it was realized that the Brauer–Manin obstruction *could* explain it, as above.

Remark 8.2.17. Theorem B of [**CTSSD87a**, **CTSSD87b**] shows that for *any* Châtelet surface over a number field, the Brauer–Manin obstruction to the local-global principle is the only one, and even better, the Brauer–Manin obstruction to weak approximation is the only one; that is, $X(k)$ is dense in $X(\mathbf{A})^{\mathrm{Br}}$. These results were generalized in [**Sal90**, **CT90**, **SS91**] to conic bundle surfaces over $\mathbb{P}^1$ with at most five degenerate fibers. Moreover, Schinzel's hypothesis on prime values of polynomials would imply the same when the number of degenerate fibers is arbitrary, and more generally for "generalized Severi–Brauer bundles over $\mathbb{P}^1$" [**CTSD94**, Theorem 4.2]. A key ingredient in these works is the *fibration method*; for an introduction, see [**CT92**, §3] and [**CT98**, §2], and for examples of the further development of this method, see [**Har94**, **Har97**, **Lia14**, **HW16**].

8.2.6. Effectivity. Let X be a nice variety over a global field k. One can imagine the following procedure for attempting to decide whether X has a k-point:

- by day, search for k-points;
- by night, search for a finite set of Azumaya $\mathscr{O}_X$-algebras that obstructs k-points.

If the Brauer–Manin obstruction to the local-global principle is the only one for X, then this procedure terminates successfully. See [**Poo06**, Remark 5.3] for more details.

Under additional assumptions on X, one can give more reasonable algorithms and even compute a kind of finite description of $X(\mathbf{A})^{\mathrm{Br}}$; see [**KT08**, **KT11**].

8.3. An example of descent

Suppose (as in [**Fly00**, §6]) that we want to find the rational solutions to

$$y^2 = (x^2+1)(x^4+1). \tag{8.3.1}$$

Write $x = X/Z$, where X, Z are integers with gcd 1. Then $y = Y/Z^3$ for some integer Y with $\gcd(Y, Z) = 1$. We get

$$Y^2 = (X^2+Z^2)(X^4+Z^4).$$

If a prime p divides both X^2+Z^2 and X^4+Z^4, then

$$\begin{aligned} Z^2 &\equiv -X^2 \pmod{p}, \\ Z^4 &\equiv -X^4 \pmod{p}, \end{aligned}$$

so

$$2Z^4 = (Z^2)^2 + Z^4 \equiv (-X^2)^2 + (-X^4) = 0 \pmod{p},$$

and similarly

$$2X^4 = (X^2)^2 + X^4 \equiv (-Z^2)^2 + (-Z^4) = 0 \pmod{p}.$$

But $\gcd(X, Z) = 1$, so this forces $p = 2$. (Alternatively, the resultant of the homogeneous forms $X^2 + Z^2$ and $X^4 + Z^4$ is 4, so the only prime p modulo which these forms have a common nontrivial zero is $p = 2$.)

Each odd prime p divides at most one of $X^2 + Z^2$ and $X^4 + Z^4$, but the product $(X^2 + Z^2)(X^4 + Z^4)$ is a square, so the exponent of p in each must be even. In other words,

$$X^4 + Z^4 = cW^2$$

for some $c \in \{\pm 1, \pm 2\}$. Since X, Z are not both zero, the left-hand side is positive, so $c > 0$. Thus $c \in \{1, 2\}$.

Dividing by Z^4 and setting $w = W/Z^2$, we obtain a rational point on one of the following smooth curves:

$$\begin{aligned} Y_1\colon &\quad w^2 = x^4 + 1, \\ Y_2\colon &\quad 2w^2 = x^4 + 1. \end{aligned}$$

Each curve Y_c is of geometric genus g where $2g + 2 = 4$; i.e., $g = 1$. The point $(x, w) = (0, 1)$ belongs to $Y_1(\mathbb{Q})$, and $(1, 1)$ belongs to $Y_2(\mathbb{Q})$, so both Y_1 and Y_2 are open subsets of elliptic curves.

One can show that Y_1 and Y_2 are birational to the curves

$$\begin{aligned} \text{32A2}\colon &\quad y^2 = x^3 - x, \\ \text{64A1}\colon &\quad y^2 = x^3 - 4x, \end{aligned}$$

where the labels are as in [**Cre97**]. A "2-descent" (or a glance at [**Cre97**, Table 1]!) shows that both elliptic curves have rank 0. One also can compute that their torsion subgroups are isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus the nice models of Y_1 and Y_2 have four rational points each. It follows that rational points on Y_1 satisfy $x = 0$ (there are two more rational points at infinity), and rational points on Y_2 satisfy $x \in \{\pm 1\}$. So (8.3.1) has six solutions, namely,

$$(0, 1), (0, -1), (1, 2), (1, -2), (-1, 2), (-1, -2).$$

8.3.1. Explanation. We are asked to find $U(\mathbb{Q})$, where U is the smooth affine curve

$$y^2 = (x^2 + 1)(x^4 + 1)$$

in $\mathbb{A}^2_{\mathbb{Q}}$. Let X be the nice genus 2 curve over $\mathbb{Q}$ containing U as an open subscheme; explicitly, $X = \operatorname{Proj} k[x, y, z]/(y^2 - (x^2 + z^2)(x^4 + z^4))$, where $\deg x = \deg z = 1$ and $\deg y = 3$. This description shows also that $X - U$ consists of two rational points. In particular, finding $U(\mathbb{Q})$ is equivalent to finding $X(\mathbb{Q})$, and the latter is finite by Faltings's theorem (Theorem 2.6.8).

Let Z be the nice curve over $\mathbb{Q}$ birational to the curve in (x, y, w)-space defined by the system

$$\begin{aligned} y^2 &= (x^2+1)(x^4+1), \\ w^2 &= x^4+1, \end{aligned}$$

so $\mathbf{k}(Z) = \mathbb{Q}(x, \sqrt{x^2+1}, \sqrt{x^4+1})$. For $c \in \mathbb{Q}^\times$, let Z_c be the twist of Z that is birational to the curve

$$\begin{aligned} y^2 &= (x^2+1)(x^4+1), \\ cw^2 &= x^4+1. \end{aligned}$$

For each c, there is a degree 2 morphism

$$\begin{aligned} Z_c &\longrightarrow X \\ (x, y, w) &\longmapsto (x, y). \end{aligned}$$

The argument of the previous section can be reinterpreted as follows:

- Each point in $X(\mathbb{Q})$ is the image of $f_c \colon Z_c(\mathbb{Q}) \to X(\mathbb{Q})$ for some $c \in \mathbb{Q}^\times$.
- Up to multiplying c by $\mathbb{Q}^{\times 2}$, there are only finitely many $c \in \mathbb{Q}^\times$ for which Z_c has $\mathbb{Q}_p$-points for all $p \le \infty$. Moreover, such a finite set of c's can be computed effectively.

The finite set of c's turned out to be $\{1, 2\}$. Thus the problem of determining $X(\mathbb{Q})$ was reduced to the problem of determining $Z_c(\mathbb{Q})$ for $c \in \{1, 2\}$.

If Y_c is the nice genus 1 curve birational to

$$cy^2 = x^4 + 1,$$

then we have a morphism

$$\begin{aligned} \pi_c \colon Z_c &\longrightarrow Y_c \\ (x, y, w) &\longmapsto (x, w). \end{aligned}$$

Fortunately, for $c \in \{1, 2\}$, the curve Y_c is an elliptic curve of rank 0, so $Y_c(\mathbb{Q}) = Y_c(\mathbb{Q})_{\mathrm{tors}}$ is a computable finite set. We determine the $\mathbb{Q}$-points in the 0-dimensional preimage $\pi_c^{-1}(Y_c(\mathbb{Q})) \subset Z_c$; this gives $Z_c(\mathbb{Q})$. Finally we compute $X(\mathbb{Q}) = \bigcup_{c \in \{1,2\}} f_c(Z_c(\mathbb{Q}))$.

Remark 8.3.2. The elliptic curve

$$E\colon \quad y^2 = (t+1)(t^2+1)$$

is dominated by X, by the morphism

$$\begin{aligned} \phi\colon X &\longrightarrow E \\ (x, y) &\longmapsto (x^2, y). \end{aligned}$$

Unfortunately, the approach of computing $E(\mathbb{Q})$ and then computing $\phi^{-1}(P)$ for each $P \in E(\mathbb{Q})$ cannot be carried out directly, since $E(\mathbb{Q})$ is infinite, of rank 1. Moreover, one can show that the Jacobian J of X is isogenous to $E \times E$, so $\operatorname{rk} J(\mathbb{Q}) = 2$ is not less than $g(X) = 2$, so Chabauty's method (see [**Ser97**, §5.1] or [**MP12**]) cannot be applied directly to X. On the other hand, X has two independent maps to E, so the Demyanenko–Manin method [**Ser97**, §5.2] could be applied to determine $X(\mathbb{Q})$.

8.3.2. Galois covering. One of the key points is the argument was that there are only finitely many c such that Z_c has $\mathbb{Q}_p$-points for all $p \leq \infty$. What makes this work is the fact that $Z \to X$ is a Galois covering.

Let us first explain why $f\colon Z \to X$ is étale. Over the affine open subset V_1 of $U \subseteq X$ where $x^4 + 1$ is nonvanishing, the open subset $f^{-1}V_1 \subseteq Z$ is obtained by adjoining $\sqrt{x^4+1}$ to the affine coordinate ring; this is an étale extension. Similarly, over the affine open subset V_2 of U where $x^2 + 1$ is nonvanishing, $f^{-1}V_2$ is obtained by adjoining $\sqrt{x^2+1}$. Since V_1 and V_2 cover U, it follows that f is étale above U. A similar argument shows that f is étale above the other affine open piece U' of X. Thus $f\colon Z \to X$ is étale.

Remark 8.3.3. The argument that f is étale is a special case of the proof of Abhyankar's lemma [**SGA 1**, X.3.6]. It is analogous to the proof that the field $\mathbb{Q}(\sqrt{15}, \sqrt{3}) = \mathbb{Q}(\sqrt{15}, \sqrt{5})$ is an everywhere unramified extension of $\mathbb{Q}(\sqrt{15})$.

In fact, the following shows that $Z \to X$ is a Galois covering with Galois group $\mathbb{Z}/2\mathbb{Z}$:

Proposition 8.3.4. *Let $Z \to X$ be an étale morphism between nice k-curves. If $\mathbf{k}(Z)/\mathbf{k}(X)$ is a Galois extension of field with Galois group G, then $Z \to X$ is a Galois covering with Galois group G.*

Proof. By the equivalence of categories between curves and function fields, the left G-action on $\mathbf{k}(Z)$ induces a right G-action on Z considered as an X-scheme. Since $\mathbf{k}(Z)/\mathbf{k}(X)$ is Galois, the X-morphism

$$\psi\colon Z \times G \longrightarrow Z \times_X Z$$

is an isomorphism above the generic point of X. By spreading out (Theorem 3.2.1(iv)), ψ gives an isomorphism from an open dense subscheme of $Z \times G$ to an open dense subscheme of $Z \times_X Z$. Both $Z \times G$ and $Z \times_X Z$ are smooth, proper, and 1-dimensional over k, so any birational maps between their components are isomorphisms. $\square$

8.4. Descent

(Reference: [**Sko01**, §5.3])

In our example, Z was a $\mathbb{Z}/2\mathbb{Z}$-torsor over X. We now generalize by replacing $\mathbb{Z}/2\mathbb{Z}$ by an arbitrary smooth affine algebraic group G over k. When we speak of a G-torsor over X, we mean a right fppf G_X-torsor over X, where G_X is the base extension. Throughout the rest of Chapter 8, all cohomology is fppf cohomology, and we use $\mathrm{H}^1(X, G)$ as an abbreviation for the pointed set $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, G)$ (which is a group if G is commutative). By Theorem 6.5.10(i), isomorphism classes of G-torsors over X are in bijection with $\mathrm{H}^1(X, G)$.

8.4.1. Evaluation. Let k be a field. Let X be a k-variety. Let G be a smooth algebraic group over k. Let $Z \xrightarrow{f} X$ be an G-torsor over X, and let ζ be its class in $\mathrm{H}^1(X, G)$. If $x \in X(k)$, then the fiber $Z_x \to \{x\}$ is a G-torsor over k, and its class in $\mathrm{H}^1(k, G)$ will be denoted $\zeta(x)$. Equivalently, x determines a morphism in cohomology mapping ζ to $\zeta(x)$:

$$\begin{aligned} x\colon \operatorname{Spec} k &\longrightarrow X \\ \mathrm{H}^1(k, G) &\longleftarrow \mathrm{H}^1(X, G) \\ \zeta(x) &\longleftarrow\!\shortmid \zeta. \end{aligned}$$

Thus the torsor $Z \to X$ gives rise to an "evaluation" map

$$\begin{aligned} X(k) &\longrightarrow \mathrm{H}^1(k, G) \\ x &\longmapsto \zeta(x). \end{aligned}$$

In other words, $Z \to X$ can be thought of as a family of torsors parameterized by X, and $\zeta(x)$ gives the class of the fiber above x.

8.4.2. The fibers of the evaluation map. We may partition $X(k)$ according to the class of the fiber above each rational point:

$$X(k) = \coprod_{\tau \in \mathrm{H}^1(k,G)} \{\, x \in X(k) : \zeta(x) = \tau \,\}.$$

The following key theorem reinterprets the right-hand side.

Theorem 8.4.1. *Let k be a field. Let X be a k-variety. Let G be a smooth affine algebraic group. Suppose that $f\colon Z \to X$ is a G-torsor over X, and let $\zeta \in \mathrm{H}^1(X, G)$ be its class. For each $\tau \in \mathrm{H}^1(k, G)$, let $f^\tau\colon Z^\tau \to X$ be the twisted torsor constructed in Example* 6.5.12. *Then*

$$\{\, x \in X(k) : \zeta(x) = \tau \,\} \quad = \quad f^\tau(Z^\tau(k)).$$

In particular,

$$X(k) = \coprod_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(k)).$$

Proof. For each $x \in X(k)$, we have the following equivalences:

$$
\begin{aligned}
& x \in f^\tau(Z^\tau(k)) \\
\iff\ & \text{the fiber } Z^\tau_x \text{ is a trivial } \mathcal{G}^\tau\text{-torsor over } k \quad \text{(Proposition 5.12.14)} \\
\iff\ & Z_x \overset{G}{\times} T^{-1} \text{ is a trivial } \mathcal{G}^\tau\text{-torsor over } k \\
\iff\ & Z_x \simeq T \text{ as } G\text{-torsor} \\
& \quad \text{(by taking the contracted product with } T \text{ on the right)} \\
\iff\ & \zeta(x) = \tau.
\end{aligned}
$$

□

8.4.3. The evaluation map over a local field.

Proposition 8.4.2. *Let k be a local field. Let X be a proper k-variety. Let F be a finite étale algebraic group over k. Let $f\colon Z \to X$ be an F-torsor over X. Then the image of $X(k) \to \mathrm{H}^1(k, F)$ is finite.*

Proof. For each $x \in X(k)$, the fiber $f^{-1}(x)$ is $\operatorname{Spec} L$ for some étale k-algebra L. By Krasner's lemma (Proposition 3.5.74), there exists an open neighborhood U of x in $X(k)$ such that for $u \in U$, the fiber $f^{-1}(u)$ is isomorphic to $f^{-1}(x)$ as a k-scheme. In other words, if $\mathrm{H}^1(k, F)$ is given the discrete topology, then the evaluation map $X(k) \to \mathrm{H}^1(k, F)$ is continuous.

On the other hand, X is proper, so Proposition 2.6.1(i) shows that $X(k)$ is compact. Thus the image of $X(k) \to \mathrm{H}^1(k, F)$ is compact and hence finite. □

Remark 8.4.3. If $\operatorname{char} k = 0$, then the whole set $\mathrm{H}^1(k, F)$ is finite, by Theorem 5.12.24(a).

8.4.4. The Selmer set. Return to the notation of Theorem 8.4.1, but assume moreover that k is a global field. For each place v of k, the inclusion $k \hookrightarrow k_v$ induces a homomorphism of fppf cohomology groups $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k_v, G)$. (Equivalently, it is the restriction homomorphism of Galois cohomology associated with the inclusion of $\mathrm{Gal}(k_v^s/k_v)$ as a decomposition group in $\mathrm{Gal}(k_s/k)$.) If $\tau \in \mathrm{H}^1(k, G)$, let $\tau_v \in \mathrm{H}^1(k_v, G)$ be its image.

Definition 8.4.4. The **Selmer set** is the following subset of $\mathrm{H}^1(k, G)$:

$$\mathrm{Sel}_Z(k, G) := \left\{ \tau \in \mathrm{H}^1(k, G) : \tau_v \in \operatorname{im}\big(X(k_v) \to \mathrm{H}^1(k_v, G)\big) \text{ for all } v \in \Omega_k \right\}.$$

Remark 8.4.5. This terminology and notation is compatible with the notion of the Selmer group, in the case where $f\colon Z \to X$ is an isogeny between abelian varieties, viewed as a torsor under $G := \ker f$. For instance, if $f\colon E \to E$ is the multiplication-by-2 map on an elliptic curve over a number field, then $\mathrm{Sel}_E(k, E[2]) \subseteq \mathrm{H}^1(k, E[2])$ is the 2-Selmer group defined in [**Sil92**, X.§4].

By Theorem 8.4.1 applied over each k_v, we have

$$\begin{aligned}\mathrm{Sel}_Z(k,G) &= \{\, \tau \in \mathrm{H}^1(k,G) : Z^\tau(k_v) \neq \emptyset \text{ for all } v \in \Omega_k \,\} \\ &\supseteq \{\, \tau \in \mathrm{H}^1(k,G) : Z^\tau(k) \neq \emptyset \,\}.\end{aligned}$$

In particular,

$$X(k) \quad = \coprod_{\tau \in \mathrm{Sel}_Z(k,G)} f^\tau(Z^\tau(k)).$$

Theorem 8.4.6. *If X is a proper variety over a global field k, then $\mathrm{Sel}_Z(k,G)$ is finite.*

Proof. Let F be the component group of G. For a suitable finite nonempty subset $S \subseteq \Omega_k$ containing the archimedean places, Theorem 3.2.1 lets us spread out G to a smooth finite-type separated group scheme $\mathcal{G}$ over $\mathcal{O}_{k,S}$, spread out X to a proper scheme $\mathcal{X}$ over $\mathcal{O}_{k,S}$, and spread out Z to a $\mathcal{G}$-torsor over $\mathcal{X}$. Let $\tau \in \mathrm{H}^1(k,G)$. For $v \notin S$, the commutative diagram

$$\begin{tikzcd} & \mathrm{H}^1(k,G) \arrow[d] & \tau \arrow[d, mapsto] \\ X(k_v) \arrow[r] \arrow[d, equal, "\substack{\text{valuative criterion} \\ \text{for properness}}"'] & \mathrm{H}^1(k_v,G) & \tau_v \\ \mathcal{X}(\mathcal{O}_v) \arrow[r] & \mathrm{H}^1(\mathcal{O}_v,\mathcal{G}) \arrow[u] & \end{tikzcd}$$

shows that if τ_v comes from $X(k_v)$, then τ_v also comes from $\mathrm{H}^1(\mathcal{O}_v,\mathcal{G})$. Thus $\mathrm{Sel}_Z(k,G)$ is contained in $\mathrm{H}^1_S(k,\mathcal{G})$. Moreover, for each $v \in S$, the image of $X(k_v) \to \mathrm{H}^1(k_v,F)$ is finite by Proposition 8.4.2, so the image of $\mathrm{Sel}_Z(k,G)$ in $\prod_{v\in S} \mathrm{H}^1(k_v,F)$ is finite. The preceding two sentences combined with Theorem 6.5.13(a) show that $\mathrm{Sel}_Z(k,G)$ is finite. □

Remark 8.4.7. One can show that $\mathrm{Sel}_Z(k,G)$ is not only finite, but also effectively computable, even if one does not know $X(k)$. This makes it potentially useful for the determination of $X(k)$.

Corollary 8.4.8. *There exists a finite separable extension k' of k such that $X(k) \subseteq f(Z(k'))$.*

Proof. For each $\tau \in \mathrm{H}^1(k,G)$, there exists a finite separable extension k' such that the image of τ in $\mathrm{H}^1(k',G)$ is trivial. By taking a compositum, one can find a k' that works simultaneously for all $\tau \in \mathrm{Sel}_Z(k,G)$. Extending the base from k to k' makes $Z^\tau \xrightarrow{f^\tau} X$ isomorphic to $Z \xrightarrow{f} X$. □

8.4.5. The weak Mordell–Weil theorem. The Mordell–Weil theorem states that for any abelian variety A over a global field k, the abelian group $A(k)$ is finitely generated. The following weaker statement is proved along the way to proving the Mordell–Weil theorem:

Theorem 8.4.9 (Weak Mordell–Weil theorem)**.** *Let A be an abelian variety over a global field k, and let m be a positive integer not divisible by* $\operatorname{char} k$. *Then $A(k)/mA(k)$ is finite.*

Proof of Theorem 8.4.9. By Proposition 5.7.4, the multiplication-by-m map $A \stackrel{m}{\to} A$ is étale, so it is locally surjective in the étale topology. Thus we get an exact sequence of sheaves on $(\operatorname{Spec} k)_{\mathrm{et}}$

$$0 \to A[m] \to A \stackrel{m}{\to} A \to 0$$

(or equivalently of $\mathfrak{G}_k$-modules), where $A[m]$ is the kernel of $A \stackrel{m}{\to} A$. Taking cohomology gives

$$A(k) \xrightarrow{m} A(k) \longrightarrow \mathrm{H}^1(k, A[m]).$$

On the other hand, we may view $[m]\colon A \to A$ as a torsor under the smooth affine algebraic group $A[m]$, and hence we get an evaluation map

$$\begin{aligned} A(k) &\longrightarrow \mathrm{H}^1(k, A[m]) \\ a &\longmapsto \text{class of the torsor } [m]^{-1}(a). \end{aligned}$$

Its image is contained in the Selmer set, which is finite by Theorem 8.4.6.

One checks that the two maps $A(k) \to \mathrm{H}^1(k, A[m])$ coincide. Comparing images shows that $A(k)/mA(k)$ is isomorphic to the image of the evaluation map, and we proved already that the latter image is finite. □

8.4.6. Application of descent to failure of strong approximation. We will use an integral point analogue of descent to prove a theorem of Minchev [**Min89**, Theorem 1] on the failure of strong approximation. Minchev worked over number fields, but with a little more work we can generalize to global fields.

Theorem 8.4.10. *Let k be a global field. Let S be a finite set of places of k. Let $f\colon Y \to X$ be a finite étale morphism of geometrically integral k-varieties. If $X(\mathbf{A}^S) \neq \emptyset$ and f is not an isomorphism, then the image of the inclusion $X(k) \to X(\mathbf{A}^S)$ is not dense; that is, X does not satisfy strong approximation with respect to S.*

Proof. Let $n = \dim X = \dim Y$. Let $d = \deg f > 1$. Use Theorem 3.2.1 to enlarge S so that f spreads out to a finite étale morphism $F\colon \mathcal{Y} \to \mathcal{X}$ of separated $\mathcal{O}_{k,S}$-schemes such that $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{k,S}$ and $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_{k,S}$

have geometrically integral fibers and $\mathcal{X}(\mathcal{O}_v) \neq \emptyset$ for $v \notin S$. For any nonarchimedean $v \in S$, as x_v varies over the compact set $\mathcal{X}(\mathcal{O}_v)$, there are only finitely many possibilities for the finite étale $\mathcal{O}_v$-scheme $F^{-1}(x_v)$, by Krasner's lemma (Proposition 3.5.74). Therefore, as x varies over $\mathcal{X}(\mathcal{O}_{k,S})$, the finite étale $\mathcal{O}_{k,S}$-scheme $F^{-1}(x)$ has bounded degree and bounded ramification over S, so there are only finitely many possibilities for $F^{-1}(x)$. In particular, there exists an infinite set T of nonarchimedean $v \notin S$ such that v splits in $F^{-1}(x)$ for every $x \in \mathcal{X}(\mathcal{O}_{k,S})$.

Let $\mathcal{X}'$ be the smooth locus of $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{k,S}$, and let $\mathcal{Y}' = F^{-1}\mathcal{X}'$. For $v \notin S$, let $\mathbb{F}_v$ be the residue field, and let $q_v = \#\mathbb{F}_v$. By Theorem 7.7.1(ii), $\#\mathcal{X}'(\mathbb{F}_v)$ and $\#\mathcal{Y}'(\mathbb{F}_v)$ are both $q_v^n + O(q_v^{n-1/2})$ as $q_v \to \infty$. In particular, we may choose $w \in T$ such that $\#\mathcal{Y}'(\mathbb{F}_w) < d \cdot \#\mathcal{X}'(\mathbb{F}_w)$. Thus there exists a point $\bar{a}_w \in \mathcal{X}'(\mathbb{F}_w)$ that does not split in $\mathcal{Y}'$. By Hensel's lemma (Theorem 3.5.63(a)), $\bar{a}_w$ lifts to some $a_w \in \mathcal{X}'(\mathcal{O}_w)$. By Krasner's lemma (Proposition 3.5.74), the set $U_w := \{u_w \in \mathcal{X}'(\mathcal{O}_w) : F^{-1}(u_w) \simeq F^{-1}(a_w)\}$ is an open neighborhood of a_w in $X(k_w)$. Let U be the nonempty open set $U_w \times \prod_{v \notin S \cup \{w\}} \mathcal{X}(\mathcal{O}_{k,S})$ of $X(\mathbf{A}^S)$. If $x \in X(k) \cap U$, then $x \in \mathcal{X}'(\mathcal{O}_{k,S})$, so the definition of T implies that w splits in $F^{-1}(x)$, but the definition of U_w implies that w does not split in $F^{-1}(x)$. Thus $X(k) \cap U = \emptyset$, so $X(k)$ is not dense in $X(\mathbf{A}^S)$. □

Corollary 8.4.11 (cf. [**Min89**, Theorem 1]). *Let k be a global field. Let S be a finite set of places of k. Let X be a normal geometrically integral k-variety. If $X(\mathbf{A}^S) \neq \emptyset$ and X_{k_s} is not algebraically simply connected, then X does not satisfy strong approximation with respect to S.*

Proof. If $X(k)$ is empty, strong approximation fails by definition. If $X(k)$ is nonempty, apply Lemma 3.5.57 to obtain a nontrivial geometrically integral finite étale cover $Y \to X$, and apply Theorem 8.4.10. □

Remark 8.4.12. Corollary 8.4.11 can fail if X is not normal. For example, if X is a nodal cubic curve in $\mathbb{P}^2$ over a global field such that the tangent lines to the branches at the node have irrational slope, then X is not algebraically simply connected, but X satisfies strong approximation with respect to any finite S, because there is a dominant morphism $\mathbb{P}^1 \twoheadrightarrow X$.

8.4.7. The descent obstruction to the local-global principle. Let k be a global field. Let X be a k-variety. One can show that there is an injection $X(\mathbf{A}) \hookrightarrow \prod_v X(k_v)$, so an element of $X(\mathbf{A})$ will be written as a sequence (x_v) indexed by the places v of k. The set $X(k)$ embeds diagonally into $X(\mathbf{A})$.

A torsor $Z \xrightarrow{f} X$ under a smooth affine algebraic group G over k restricts the locations in $X(\mathbf{A})$ where rational points can lie. Namely, the commutativity of

$$
\begin{tikzcd}
X(k) \arrow[r, hook] \arrow[d] & X(\mathbf{A}) \arrow[d] \\
\mathrm{H}^1(k,G) \arrow[r] & \prod_v \mathrm{H}^1(k_v,G)
\end{tikzcd}
\tag{8.4.13}
$$

(cf. (8.1.1)) shows that $X(k)$ is contained in the subset $X(\mathbf{A})^f \subseteq X(\mathbf{A})$ consisting of points of $X(\mathbf{A})$ whose image in $\prod_v \mathrm{H}^1(k_v,G)$ comes from $\mathrm{H}^1(k,G)$. One can show also that

$$X(\mathbf{A})^f = \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbf{A})),$$

and that $X(\mathbf{A})^f$ is closed in $X(\mathbf{A})$ if X is proper; see Exercise 8.7. Moreover, one can replace $\mathrm{H}^1(k,G)$ by its subset $\mathrm{Sel}_Z(k,G)$ in either of the two descriptions of $X(\mathbf{A})^f$ above. The condition $X(\mathbf{A})^f = \emptyset$ is equivalent to $\mathrm{Sel}_Z(k,G) = \emptyset$.

One can constrain the possible locations of rational points further by using *many* torsors:

$$
\begin{aligned}
X(\mathbf{A})^{\mathrm{H}^1(X,G)} &:= \bigcap_{\text{all } G\text{-torsors } f\colon Z \to X} X(\mathbf{A})^f, \\
X(\mathbf{A})^{\mathrm{descent}} &:= \bigcap_{\text{all smooth affine } G} X(\mathbf{A})^{\mathrm{H}^1(X,G)}.
\end{aligned}
$$

Then

$$X(k) \subseteq X(\mathbf{A})^{\mathrm{descent}} \subseteq X(\mathbf{A}).$$

Recall that one says that the local-global principle holds for X if and only if the implication

$$X(\mathbf{A}) \neq \emptyset \quad \Longrightarrow \quad X(k) \neq \emptyset$$

holds.

Definition 8.4.14. One says that there is a **descent obstruction to the local-global principle** if $X(\mathbf{A}) \neq \emptyset$ but $X(\mathbf{A})^{\mathrm{descent}} = \emptyset$.

Sometimes we wish to study the adelic subset cut out by torsors under a subset of the possible smooth affine algebraic groups. In particular, we define

$$X(\mathbf{A})^{\mathrm{et}} := \bigcap_{\text{finite étale } G} X(\mathbf{A})^{\mathrm{H}^1(X,G)},$$
$$X(\mathbf{A})^{\mathrm{conn}} := \bigcap_{\text{smooth connected affine } G} X(\mathbf{A})^{\mathrm{H}^1(X,G)},$$
$$X(\mathbf{A})^{\mathrm{PGL}} := \bigcap_{n \geq 1} X(\mathbf{A})^{\mathrm{H}^1(X,\mathrm{PGL}_n)}.$$

8.5. Comparing the descent and Brauer–Manin obstructions

8.5.1. Descent is stronger than Brauer–Manin.

(Reference: [**Sko01**, Proposition 5.3.4])

Proposition 8.5.3 below shows that the Brauer–Manin obstruction is equivalent to the special case of the descent obstruction using only PGL_n-torsors for all n.

Recall from Section 6.6.4 that for any scheme X, we have a map of sets

$$\mathrm{H}^1(X, \mathrm{PGL}_n) \longrightarrow (\operatorname{Br} X)[n]. \tag{8.5.1}$$

(We used Theorem 6.6.17(ii) to know that the image is killed by n.)

Lemma 8.5.2. *Let k be a global field. Let X be a k-variety. Let $Z \xrightarrow{f} X$ be a PGL_n-torsor for some $n \geq 1$. Its class in $\mathrm{H}^1(X, \mathrm{PGL}_n)$ is mapped by* (8.5.1) *to some $A \in \operatorname{Br} X$. Then $X(\mathbf{A})^f = X(\mathbf{A})^A$.*

Proof. Let $(x_v) \in X(\mathbf{A})$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(X, \mathrm{PGL}_n) \arrow[r] \arrow[d, "(x_v)"'] & (\operatorname{Br} X)[n] \arrow[d, "(x_v)"] \\
\prod_v \mathrm{H}^1(k_v, \mathrm{PGL}_n) \arrow[r, "\sim"] & \prod_v (\operatorname{Br} k_v)[n] \\
\mathrm{H}^1(k, \mathrm{PGL}_n) \arrow[r, "\sim"] \arrow[u, "\mathrm{res}_1"] & (\operatorname{Br} k)[n] \arrow[u, "\mathrm{res}_2"']
\end{tikzcd}$$

in which the downward maps are evaluation at (x_v), the upward maps $\mathrm{res}_1, \mathrm{res}_2$ are restriction maps induced by $k \to k_v$, and the horizontal maps are given by (8.5.1). The lower two horizontal maps are bijections by Remark 1.5.18.

The middle horizontal bijection identifies $\mathrm{im}(\mathrm{res}_1)$ with $\mathrm{im}(\mathrm{res}_2)$, so the class of f in $\mathrm{H}^1(X, \mathrm{PGL}_n)$ maps down into $\mathrm{im}(\mathrm{res}_1)$ if and only if the element

$A \in (\operatorname{Br} X)[n]$ maps down into $\operatorname{im}(\mathrm{res}_2)$. In other words, $(x_v) \in X(\mathbf{A})^f$ if and only if $(x_v) \in X(\mathbf{A})^A$. □

Proposition 8.5.3. *Let k be a global field. Let X be a regular quasi-projective k-variety. Then*

$$X(\mathbf{A})^{\mathrm{descent}} \subseteq X(\mathbf{A})^{\mathrm{PGL}} = X(\mathbf{A})^{\mathrm{Br}}.$$

Proof. By Corollary 6.6.19, every $A \in \operatorname{Br} X$ is in the image of (8.5.1) for some n. So intersecting the equality of Lemma 8.5.2 over all PGL_n-torsors over X yields $X(\mathbf{A})^{\mathrm{PGL}} = X(\mathbf{A})^{\mathrm{Br}}$. The inclusion $X(\mathbf{A})^{\mathrm{descent}} \subseteq X(\mathbf{A})^{\mathrm{PGL}}$ holds by definition since each PGL_n is a smooth affine algebraic group. □

8.5.2. The étale-Brauer set.

(References: [**Poo10**, **Dem09**, **Sko09**])

Let k be a global field. Let X be a k-variety. Let G be a smooth affine algebraic group. Recall that if $Z \xrightarrow{f} X$ is a G-torsor, the determination of $X(k)$ can be reduced to the determination of $Z^\tau(k)$ for various twists Z^τ of Z:

$$X(k) = \coprod_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(k)) \subseteq \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbf{A})).$$

We can produce a possibly better "upper bound" on $X(k)$ by replacing $Z^\tau(\mathbf{A})$ by $Z^\tau(\mathbf{A})^{\mathrm{Br}}$. If we do so for every G-torsor for every finite étale group scheme G, we are led to define the **étale-Brauer set**

$$X(\mathbf{A})^{\mathrm{et,Br}} := \bigcap_{\substack{\text{finite étale } G \\ \text{all } G\text{-torsors } f\colon Z \to X}} \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbf{A})^{\mathrm{Br}}),$$

which is the upper bound on $X(k)$ obtained from applying the Brauer–Manin obstruction to étale covers. A priori, the subset

$$X(\mathbf{A})^{\mathrm{et,descent}} := \bigcap_{\substack{\text{finite étale } G \\ \text{all } G\text{-torsors } f\colon Z \to X}} \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbf{A})^{\mathrm{descent}})$$

could be even smaller.

8.5.3. Étale-Brauer equals descent.

(References: [**Dem09**, **Sko09**])

The proof of the following theorem combines work of Demarche, Harari, Skorobogatov, and Stoll.

Theorem 8.5.4. *Let k be a number field. Let X be a nice k-variety. Then*

$$X(\mathbf{A})^{\mathrm{et,Br}} = X(\mathbf{A})^{\mathrm{et,descent}} = X(\mathbf{A})^{\mathrm{descent}}.$$

Sketch of proof. It suffices to prove

$$X(\mathbf{A})^{\text{descent}} \subseteq X(\mathbf{A})^{\text{et,descent}} \subseteq X(\mathbf{A})^{\text{et,Br}} \subseteq X(\mathbf{A})^{\text{descent}}.$$

The first inclusion is [**Sko09**, Theorem 1.1], which generalizes [**Sto07**, Proposition 5.17] (a statement that we would write as $X(\mathbf{A})^{\text{et}} = X(\mathbf{A})^{\text{et,et}}$). The idea in both results is, roughly speaking, to show that if $Y \to X$ is an torsor under a finite étale group scheme, and $Z \to Y$ is a torsor under a smooth affine algebraic group, then $Z \to X$ is dominated by some torsor under an even larger smooth affine algebraic group over X; this is analogous to the fact that a Galois extension of a Galois extension of a field k is contained in some even larger Galois extension of k.

The second inclusion is deduced by applying Proposition 8.5.3 to the étale covers of X.

The third inclusion is the main result of [**Dem09**], which generalizes the equality $X(\mathbf{A})^{\text{conn}} = X(\mathbf{A})^{\text{Br}}$ of [**Har02**, Théorème 2, 2., and Remarque 4]. (The latter already is striking in that it implies that the torsors under all smooth connected affine algebraic groups give no more information than the torsors under all the groups PGL_n.) ☐

8.5.4. Iterated descent obstruction. In the hope of obtaining an obstruction beyond the descent obstruction one might define

$$X(\mathbf{A})^{\text{descent,descent}} := \bigcap_{\substack{\text{all smooth affine } G \\ \text{all } G\text{-torsors } f\colon Z \to X}} \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbf{A})^{\text{descent}})$$

and similarly $X(\mathbf{A})^{\text{descent,descent,descent}}$, and so on. But Cao, answering a question of the author, proved the following:

Theorem 8.5.5 ([**Cao17**, Corollaire 1.2])**.** *For any smooth quasi-projective geometrically integral variety X over a number field,*

$$X(\mathbf{A})^{\text{descent,descent}} = X(\mathbf{A})^{\text{descent}}.$$

Corollary 8.5.6. *For any smooth quasi-projective geometrically integral variety X over a number field,*

$$X(\mathbf{A})^{\text{descent}} = X(\mathbf{A})^{\text{descent,descent}} = X(\mathbf{A})^{\text{descent,descent,descent}} = \cdots.$$

Proof. Use induction on the number of descents! Apply the inductive hypothesis to all the torsors Z^τ over X. ☐

8.6. Insufficiency of the obstructions

8.6.1. A bielliptic surface.

(Reference: [**Sko99**])

Skorobogatov proved that the Brauer–Manin obstruction is insufficient to explain all counterexamples to the local-global principle:

Theorem 8.6.1 ([**Sko99**])**.** *There exists a nice $\mathbb{Q}$-variety X such that $X(\mathbf{A})^{\mathrm{Br}} \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$.*

The proof is involved, so we only outline it. First, we describe the kind of variety used.

Definition 8.6.2. A **bielliptic surface** over an algebraically closed field k is a surface isomorphic to $(E_1 \times E_2)/G$ for some elliptic curves E_1 and E_2 and some finite group scheme G such that G is a subgroup scheme of E_1 acting by translations on E_1 and G acts on E_2 so that the quotient E_2/G is isomorphic to $\mathbb{P}^1$. (Since G acts freely on E_1, it acts freely on $E_1 \times E_2$; i.e., $E_1 \times E_2 \to (E_1 \times E_2)/G$ is G-torsor.) A surface over an arbitrary field k is called bielliptic if $X_{\overline{k}}$ is bielliptic.

Warning 8.6.3. Some authors use the term **hyperelliptic surface** to mean bielliptic surface, but these surfaces have nothing to do with hyperelliptic curves.

Skorobogatov's example was a bielliptic surface $X := Y/G$, where Y was a product of two genus 1 curves over $\mathbb{Q}$, and G was a group generated by a fixed-point free automorphism of order 2 of Y. Explicitly, his X was birational to the affine surface defined by

$$(x^2+1)y^2 = (x^2+2)z^2 = 3(t^4 - 54t^2 - 117t - 243).$$

To show that $X(\mathbb{Q}) = \emptyset$, Skorobogatov proved $X(\mathbf{A})^{\mathrm{et,Br}} = \emptyset$, by applying the Brauer–Manin obstruction to the étale cover $Y \to X$ and its twists.

Remark 8.6.4. Because $X(\mathbf{A})^{\mathrm{et,Br}} = X(\mathbf{A})^{\mathrm{descent}}$, the nonexistence of rational points must also be explained by a descent obstruction. In fact, it can be explained by the obstruction from a single torsor under a noncommutative finite étale group scheme [**HS02**, §5.1].

8.6.2. A quadric bundle over a curve.

(References: [**Poo10**], [**CTPS16**])

We next construct an "even worse" example:

Theorem 8.6.5 ([**Poo10**])**.** *There exists a nice $\mathbb{Q}$-variety X such that $X(\mathbf{A})^{\mathrm{et,Br}} \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$.*

Combined with Theorem 8.5.4, this shows that even the descent obstruction is not enough to explain all counterexamples to the local-global principle. In the original proof of Theorem 8.6.5, X was a Châtelet surface bundle over a curve of positive genus. We will present a simpler variant, based on [**CTPS16**, §3.1], using quadrics instead of Châtelet surfaces. In this section, all varieties are over $\mathbb{Q}$.

Start with a nice curve C such that $C(\mathbb{Q})$ consists of a single point c. (For example, C could be the elliptic curve $y^2 = x^3 - 3$, named 972B1 in [**Cre97**].) Let $f\colon C \to \mathbb{P}^1$ be a morphism that is étale at c (for instance, take f corresponding to a uniformizing parameter at c). Compose with an automorphism of $\mathbb{P}^1$ to assume that $f(c) = \infty$. Let U be a connected open neighborhood of c in $C(\mathbb{R})$. By the implicit function theorem, $f(U)$ contains an open neighborhood of ∞ in $\mathbb{P}^1(\mathbb{R})$. Compose f with a translation automorphism of $\mathbb{P}^1$ to assume that $1 \in f(U)$ and that f is étale above $0, 1 \in \mathbb{P}^1$.

Next we construct a quadric bundle $Y \to \mathbb{P}^1$. View $\mathbb{P}^1$ as the result of gluing $\mathbb{A}^1_t := \operatorname{Spec}\mathbb{Q}[t]$ and $\mathbb{A}^1_T := \operatorname{Spec}\mathbb{Q}[T]$ using $t = 1/T$. In $\mathbb{P}^4 \times \mathbb{A}^1_t$, define the closed subscheme

$$Y^{(t)}\colon \quad t(t-1)x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

Similarly, in $\mathbb{P}^4 \times \mathbb{A}^1_T$, define the closed subscheme

$$Y^{(T)}\colon \quad (1-T)X_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

Glue $Y^{(t)} \to \mathbb{A}^1_t$ and $Y^{(T)} \to \mathbb{A}^1_T$ using $t = 1/T$ and $x_0 = T/X_0$ to obtain $Y \to \mathbb{P}^1$. Alternatively, if $\mathscr{E}$ denotes the rank 5 vector bundle $\mathscr{O}(1) \oplus \mathscr{O}^{\oplus 4}$ on $\mathbb{P}^1$, then Y is the zero locus in $\mathbb{P}\mathscr{E} := \mathbf{Proj}\operatorname{Sym}\mathscr{E}$ of a section of $\operatorname{Sym}^2 \mathscr{E}$; in particular, Y is projective over $\mathbb{Q}$. A calculation shows that $Y^{(t)}$ and $Y^{(T)}$ are smooth over $\mathbb{Q}$, so Y is smooth over $\mathbb{Q}$. Thus Y is a family of 3-dimensional quadrics over the base $\mathbb{P}^1$, with two degenerate fibers, above 0 and 1. For each $t \in \mathbb{P}^1$, let Y_t denote the fiber above t. In particular, the locus in $Y^{(T)}$ above $T = 0$ is the fiber

$$Y_\infty\colon \quad X_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

a smooth quadric in $\mathbb{P}^4$. See Figure 6.

Let $\pi\colon X \to C$ be the base extension of $Y \to \mathbb{P}^1$ by f:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & Y \arrow[d] \\ C \arrow[r, "f"] & \mathbb{P}^1. \end{tikzcd}$$

Proposition 8.6.6. *The $\mathbb{Q}$-variety X is nice.*

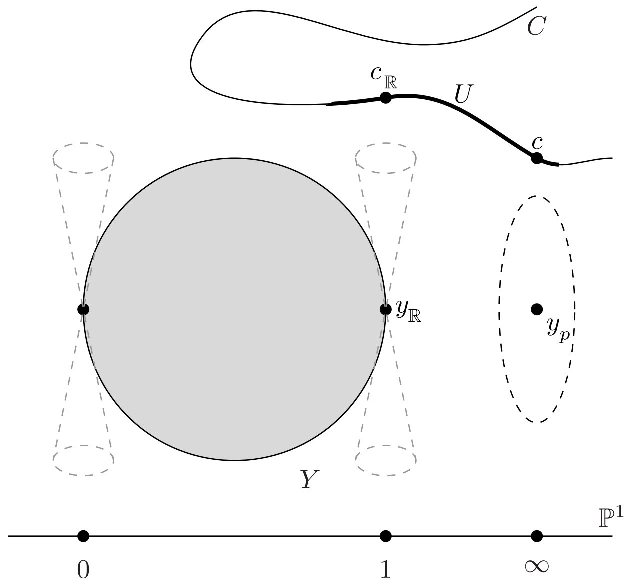

Figure 6. Real points of the varieties C and Y over $\mathbb{P}^1$ are shown in solid black and gray. The open subset U of $C(\mathbb{R})$ is shown as a thicker curve. The dotted lines indicate some fibers of $Y \to \mathbb{P}^1$ with imaginary points.

Proof. Since $Y \to \mathbb{P}^1$ is projective with geometrically integral fibers, the same is true of $X \to C$; in particular, X is a projective and geometrically integral $\mathbb{Q}$-variety. The morphism $Y \to \mathbb{P}^1$ is smooth above all points except $0, 1$, so $X \to C$ is smooth above all points of C outside those above $0, 1 \in \mathbb{P}^1$; since C is smooth over $\mathbb{Q}$, this implies that X is smooth over $\mathbb{Q}$ outside the points above $0, 1 \in \mathbb{P}^1$. Similarly, $C \to \mathbb{P}^1$ is smooth above $0, 1$, so $X \to Y$ is smooth at the points above $0, 1$; since Y is smooth over $\mathbb{Q}$, this implies that X is smooth over $\mathbb{Q}$ also at the points above $0, 1 \in \mathbb{P}^1$. Thus X is nice. $\square$

Proposition 8.6.7. *We have* $X(\mathbb{Q}) = \emptyset$.

Proof. The sole point of $C(\mathbb{Q})$ maps to $\infty \in \mathbb{P}^1$, but Y_∞ has no $\mathbb{Q}$-points. $\square$

As a warm-up to proving that $X(\mathbf{A})^{\text{et,Br}} \neq \emptyset$, we prove that $X(\mathbf{A})^{\text{Br}} \neq \emptyset$.

For each finite prime p, any quadratic form over $\mathbb{Q}_p$ of rank ≥ 5 has a nontrivial zero [**Ser73**, IV.2.2, Theorem 6], so we may choose $y_p \in Y_\infty(\mathbb{Q}_p)$ and let $x_p = (y_p, c) \in X(\mathbb{Q}_p)$. Let $y_{\mathbb{R}}$ be the unique point in $Y_1(\mathbb{R})$, let $c_{\mathbb{R}} \in U \subseteq C(\mathbb{R})$ be such that $f(c_{\mathbb{R}}) = 1 \in \mathbb{P}^1(\mathbb{R})$, and let $x_{\mathbb{R}} = (y_{\mathbb{R}}, c_{\mathbb{R}}) \in X(\mathbb{R})$ (we use the subscript $\mathbb{R}$ for the archimedean place to avoid confusion with the point $\infty \in \mathbb{P}^1$). Together, these define $x = (x_v) \in X(\mathbf{A})$.

Proposition 8.6.8. *We have* $x \in X(\mathbf{A})^{\text{Br}}$.

Proof. The adeles $\pi(x)$ and c agree except for their archimedean parts $c_{\mathbb{R}}$ and c, which lie in the same connected component of $C(\mathbb{R})$. By this and Proposition 8.2.9, any $A \in \operatorname{Br} C$ takes the same value at $\pi(x)$ as at $c \in C(\mathbb{Q})$; by Proposition 8.2.2, that value is 0. Thus $\pi(x) \in C(\mathbf{A})^{\mathrm{Br}}$. Also, the homomorphism $\operatorname{Br} C \to \operatorname{Br} X$ is surjective by Proposition 6.9.15. Corollary 8.1.10 then implies $x \in X(\mathbf{A})^{\mathrm{Br}}$. $\square$

To generalize Proposition 8.6.8 to prove that $x \in X(\mathbf{A})^{\mathrm{et,Br}}$, we must understand the category $\mathbf{FEt}(X)$ of finite étale covers of X.

Lemma 8.6.9. *The morphism $X \to C$ induces an equivalence of categories $\mathbf{FEt}(C) \to \mathbf{FEt}(X)$.*

Proof. This follows (by [**SGA 1**, IX.6.8]) from the fact that each geometric fiber of $X \to C$ (a smooth 3-dimensional quadric or a cone over a smooth 2-dimensional quadric) is algebraically simply connected. $\square$

Proposition 8.6.10. *We have $x \in X(\mathbf{A})^{\mathrm{et,Br}}$.*

Proof. Suppose that G is a finite étale group scheme over $\mathbb{Q}$, and $X' \to X$ is a G-torsor. We must show that one of the twists of $X' \to X$ has an adelic point not obstructed by the Brauer group. By Lemma 8.6.9, $X' \to X$ is the base extension of a G-torsor $C' \to C$. We may replace C' by a twist to assume that c lifts to some $c'' \in C'(\mathbb{Q})$. Let C'' be the irreducible component of C' containing c''. The fiber product $X'' := X \times_C C''$ fits in a diagram

$$\begin{tikzcd} X'' \arrow[r, hook] \arrow[d, "\pi''"'] & X' \arrow[r, "G\text{-torsor}"] \arrow[d, "\pi'"'] & X \arrow[r] \arrow[d, "\pi"'] & Y \arrow[d] \\ C'' \arrow[r, hook] & C' \arrow[r, "G\text{-torsor}"] & C \arrow[r, "f"] & \mathbb{P}^1. \end{tikzcd}$$

Since $C'' \to C$ is finite étale, C'' is smooth and projective; moreover, C'' is integral and has a $\mathbb{Q}$-point, so C'' is a nice curve. Similarly, X'' is smooth and projective, and $X'' \to C''$ has geometrically integral fibers (just like $Y \to \mathbb{P}^1$), so X'' is nice too.

We claim that x lifts to a point $x'' \in X''(\mathbf{A})$. For each finite prime p, let $x''_p = (x_p, c'') \in X''(\mathbb{Q}_p)$. Since U is algebraically simply connected, the inverse image of U in $C''(\mathbb{R})$ is a disjoint union of copies of U; let U'' be the copy containing c'', let $c''_{\mathbb{R}} \in U''$ be the point mapping to $c_{\mathbb{R}} \in U$, and let $x''_{\mathbb{R}} = (x_{\mathbb{R}}, c''_{\mathbb{R}}) \in X''(\mathbb{R})$. Thus we have $x'' \in X''(\mathbf{A})$ mapping to $x \in X(\mathbf{A})$.

The same proof as for Proposition 8.6.8 shows that $x'' \in X''(\mathbf{A})^{\mathrm{Br}}$, so $X'(\mathbf{A})^{\mathrm{Br}}$ is nonempty. This argument applies to all finite étale torsors over X, so $X(\mathbf{A})^{\mathrm{et,Br}}$ is nonempty. $\square$

This completes the proof of Theorem 8.6.5.

8.6.3. Hypersurfaces and complete intersections.

(Reference: [**PV04**])

Definition 8.6.11. A scheme-theoretic intersection $X = H_1 \cap \cdots \cap H_r$ of hypersurfaces $H_i \subset \mathbb{P}^n$ is called a **complete intersection** if $\dim X = n - r$.

In particular, any hypersurface in $\mathbb{P}^n$ is a complete intersection.

Theorem 8.6.12. *Let k be a number field. If X is a smooth complete intersection in some $\mathbb{P}^n_k$ and $\dim X \geq 3$, then the descent obstruction and Brauer–Manin obstruction for X are vacuous; that is,*

$$X(\mathbf{A})^{\text{descent}} = X(\mathbf{A})^{\text{Br}} = X(\mathbf{A}).$$

Sketch of proof. By Theorem 8.5.4, it suffices to prove $X(\mathbf{A})^{\text{et,Br}} = X(\mathbf{A})$. This follows immediately from the following two claims:

(i) The variety $X_{\overline{k}}$ is algebraically simply connected (Definition 3.5.45).

(ii) The homomorphism $\operatorname{Br} k \to \operatorname{Br} X$ is an isomorphism.

Part (i) follows from the weak Lefschetz theorem, which says that the homomorphism of fundamental groups $\pi_1(X(\mathbb{C}), x) \to \pi_1(\mathbb{P}^n(\mathbb{C}), x)$ is an isomorphism (here an embedding $\overline{k} \hookrightarrow \mathbb{C}$ is chosen and $x \in X(\mathbb{C})$) [**Mil63**, Theorem 7.4]. For the proof of (ii), see [**PV04**, Proposition A.1]. $\square$

Heuristics suggest that most smooth hypersurfaces $X \subseteq \mathbb{P}^n_{\mathbb{Q}}$ of degree $d > n + 1 = \dim X + 2$ have no rational points. On the other hand, a positive fraction of such hypersurfaces have $\mathbb{Q}_p$-points for all $p \leq \infty$ [**PV04**, Theorem 3.6]. Thus one expects many counterexamples to the local-global principle among such hypersurfaces. But there is no smooth hypersurface of dimension ≥ 3 for which the local-global principle has been *proved* to fail! The reason we are unable to prove anything in this setting is that our only available tools, the descent and Brauer–Manin obstructions, give no information.

We need some new obstructions!

Remark 8.6.13. The Brauer–Manin obstruction does yield counterexamples to the local-global principle for some *2-dimensional* hypersurfaces, such as some cubic surfaces; see Section 9.4.9.

Remark 8.6.14. There are some *conditional* counterexamples among hypersurfaces of higher dimension. For instance, Lang's conjecture [**Lan74**, (1.3)] that $V(\mathbb{Q})$ is finite for every nice hyperbolic $\mathbb{Q}$-variety V implies the existence of nice hypersurfaces in $\mathbb{P}^4$ that violate the local-global principle; see [**SW95**, **Poo01**]. (A smooth variety V over a subfield of $\mathbb{C}$ is **(Brody) hyperbolic** if every holomorphic map $\mathbb{C} \to V(\mathbb{C})$ is constant.)

Exercises

8.1. Prove Proposition 8.1.8, Corollary 8.1.9, and Corollary 8.1.10.

8.2. Let k be a global field. Let X be a proper k-variety such that $X(\mathbf{A}) \neq \emptyset$. Let $A \in \operatorname{Br} X$. Suppose that there exists a place w such that the evaluation map $X(k_w) \to \operatorname{Br} k_w$ given by A is not constant. Prove that weak approximation for X fails.

8.3. (Brauer–Manin obstruction for a degree 4 del Pezzo surface) Let X be the smooth surface defined by

$$uv = x^2 - 5y^2,$$
$$(u+v)(u+2v) = x^2 - 5z^2$$

in $\mathbb{P}^4_{\mathbb{Q}}$. (This example is from [**BSD75**, §4].) Let $K = \mathbf{k}(X)$.

(a) Prove that $X(\mathbf{A}) \neq \emptyset$. (*Suggestion*: Let Y be the smooth genus 1 curve obtained by intersecting X with the hyperplane $x = 0$. Spread out Y to a smooth proper scheme over $\mathbb{Z}[S^{-1}]$ for some finite set of places S. For $p \notin S$, use the Hasse–Weil bound or Lang's theorem on H^1 over finite fields to show that Y has an $\mathbb{F}_p$-point, and deduce that Y has a $\mathbb{Q}_p$-point.)

(b) Let A be the class of the quaternion algebra $\left(5, \frac{u+v}{u}\right)$ in $\operatorname{Br} K$. Find other representations of A to show that $A \in \operatorname{Br} X$. (*Hint*: Why does it suffice to find representations on open subsets that cover the codimension 1 points of X?)

(c) Prove that if $P = (u : v : x : y : z) \in X(\mathbb{Q}_p)$ for some $p \leq \infty$, then

$$\operatorname{inv}_p A(P) = \begin{cases} 0 & \text{if } p \neq 5, \\ 1/2 & \text{if } p = 5. \end{cases}$$

(*Hint*: If $5 \in \mathbb{Q}_p^{\times 2}$, what can be said about the image of A in $\operatorname{Br} X_{\mathbb{Q}_p}$?)

(d) Deduce that $X(\mathbf{A})^{\operatorname{Br}} = \emptyset$, so $X(\mathbb{Q}) = \emptyset$.

8.4. Let S be a finite set of places of a number field k, containing all the archimedean places. Let $\mathcal{O}_{k,S}$ be the ring of S-integers. Let $\mathcal{G}$ be a finite étale group scheme over $\mathcal{O}_{k,S}$. Prove that $\mathrm{H}^1(\mathcal{O}_{k,S}, \mathcal{G})$ is finite.

8.5. (Integral descent) Let $\mathcal{O}_{k,S}$ and $\mathcal{G}$ be as above. Let X be a finite-type separated $\mathcal{O}_{k,S}$-scheme, and let $Z \to X$ be a $\mathcal{G}$-torsor. For each $\tau \in \mathrm{H}^1(\mathcal{O}_{k,S}, \mathcal{G})$, define a twisted torsor $f^\tau \colon Z^\tau \to X$ such that

$$X(\mathcal{O}_{k,S}) = \coprod_{\tau \in \mathrm{H}^1(\mathcal{O}_{k,S}, \mathcal{G})} f^\tau(Z^\tau(\mathcal{O}_{k,S})).$$

8.6. Let $\mathcal{O}_{k,S}$ be as above. Let $\mathcal{U}$ be an "affine curve of genus 1 over $\mathcal{O}_{k,S}$", by which we mean a smooth, separated, finite-type $\mathcal{O}_{k,S}$-scheme whose generic fiber is an affine open subset U of a nice k-curve E of genus 1. Show that Faltings's theorem implies that $\mathcal{U}(\mathcal{O}_{k,S})$ is finite. (*Hints*: Show that you may enlarge S and/or extend k as needed. Find a sequence of Galois coverings $U'' \to U' \to U$, where $U' = X' - F'$ with X' a nice genus 1 curve and $F' \subseteq X'$ a closed subscheme with $\#F(\overline{k}) \geq 4$, and U'' is an affine open subset of a ramified covering $X'' \to X'$ branched only over F'.)

8.7. Let k be a number field. Let X be a k-variety. Let G be a smooth affine algebraic group over k. Let $Z \xrightarrow{f} X$ be a G-torsor.

(a) Prove that for each place v, the set $f(Z(k_v))$ is open in $X(k_v)$. (*Hint*: Proposition 3.5.73(ii).)

(b) Prove that for each place v, the evaluation map
$$X(k_v) \to \mathrm{H}^1(k_v, G)$$
associated to f is continuous (for the v-adic topology on $X(k_v)$ and the discrete topology on $\mathrm{H}^1(k_v, G)$).

(c) Prove that for each place v, the set $f(Z(k_v))$ is closed in $X(k_v)$.

(d) Use results from the proof of Theorem 6.5.13 to prove that
$$f(Z(\mathbf{A})) = X(\mathbf{A}) \cap \prod_v f(Z(k_v))$$
as subsets of $\prod_v X(k_v)$.

(e) Prove that $f(Z(\mathbf{A}))$ is closed in $X(\mathbf{A})$.

(f) Prove that for each $\tau \in \mathrm{H}^1(k, G)$,
$$\{\, (x_v) \in X(\mathbf{A}) : x_v \text{ maps to } \tau_v \in \mathrm{H}^1(k_v, G) \text{ for all } v \,\} = f^\tau(Z^\tau(\mathbf{A})),$$
where τ_v denotes the image of τ in $\mathrm{H}^1(k_v, G)$.

(g) Prove that $X(\mathbf{A})^f = \bigcup_{\tau \in \mathrm{H}^1(k,G)} f^\tau(Z^\tau(\mathbf{A}))$.

(h) Prove that if X is proper, then $X(\mathbf{A})^f$ is closed in $X(\mathbf{A})$.

(b) *A ruled surface over k is birational to a conic bundle over a nice k-curve.*

Remark 9.3.22. The arithmetic of del Pezzo surfaces will be discussed in detail in Section 9.4. For the arithmetic of conic bundles, see [**Sko01**, Chapter 7].

Corollary 9.3.23. *Let k be a separably closed field.*

(a) *The relatively minimal rational surfaces over k are $\mathbb{P}^2$ and the Hirzebruch surfaces F_n for $n \in \{0\} \cup \{2, 3, \ldots\}$.*

(b) *The relatively minimal ruled surfaces over k with base of positive genus are the surfaces $\mathbb{P}\mathscr{E} \to Y$, where Y is a nice k-curve of positive genus and $\mathscr{E}$ is a rank 2 vector bundle on Y.*

Proof. In Theorem 9.3.20, we are in case (ii) or (iii). If (ii), then $X_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$, so $X \simeq \mathbb{P}^2$ by Remark 4.5.9.

If (iii), then the conic bundle X corresponds to an element of

$$\mathrm{H}^1(Y, \mathrm{PGL}_2) \hookrightarrow \mathrm{H}^2(Y, \mathbb{G}_m) = \operatorname{Br} Y,$$

but the latter is trivial by Theorem 6.9.7. Thus $X \simeq \mathbb{P}\mathscr{E}$ for some rank 2 vector bundle $\mathscr{E}$ on Y. Finally, if Y itself is a conic, then $Y \simeq \mathbb{P}^1$ (Remark 4.5.9 again), and the classification of vector bundles on $\mathbb{P}^1$ shows that $\mathscr{E} \simeq \mathscr{O} \oplus \mathscr{O}(n)$ for some $n \geq 0$. Finally, $\mathbb{P}\mathscr{E}$ is relatively minimal if and only if $(\mathbb{P}\mathscr{E})_{\overline{k}}$ is relatively minimal, which holds if and only $n \neq 1$. □

Proposition 9.3.24 ([**Wei56**])**.** *Let k be a finite field $\mathbb{F}_q$. Let X be a nice rational surface over k. Then*

$$\#X(k) = q^2 + (\operatorname{tr} \operatorname{Frob}_q \mid \operatorname{Pic} X_{\overline{k}})q + 1,$$

and X has a k-point.

Proof. Apply Proposition 9.2.6 to $X_{\overline{k}}$ and then use the Lefschetz trace formula (7.5.18) to obtain the formula. Since $\operatorname{tr} \operatorname{Frob}_q \mid \operatorname{Pic} X_{\overline{k}} \in \mathbb{Z}$, we obtain $\#X(k) \equiv 1 \pmod{q}$, so $X(k) \neq \emptyset$. □

Remark 9.3.25. As mentioned in (3) in Section 1.2.4, the final conclusion of Proposition 9.3.24 generalizes to rationally chain connected nice varieties [**Esn03**, Corollary 1.3].

9.4. Del Pezzo surfaces

(Reference: [**Kol96**, III.3])

Recall from Section 9.2.5 that a Fano variety is a nice variety for which $-K$ (an anticanonical divisor) is ample.

Definition 9.4.1. A **del Pezzo surface** is a (nice) Fano variety of dimension 2.

Let X be a del Pezzo surface. According to the general definition for Fano varieties in Section 9.2.5, the degree of X is the positive integer $d := (-K).(-K) = K.K$. It then turns out that $\dim \mathrm{H}^0(X, -K) = d+1$ [**Kol96**, Corollary III.2.3.5.2], and that $-K$ is very ample when $d \geq 3$ [**Kol96**, Proposition III.3.4.3]. Thus, if $d \geq 3$, then $|-K|$ embeds X as a degree d surface in $\mathbb{P}^d$.

9.4.1. Del Pezzo surfaces over a separably closed field.

Lemma 9.4.2. *Let k be a separably closed field. Let X be a del Pezzo surface over k. If C is a closed integral curve on X with $C.C < 0$, then C is a (-1)-curve with constant field k.*

Proof. Since $-K$ is ample, $C.(-K) > 0$. Theorem 9.3.3(iv)⇒(v) implies that C is a (-1)-curve with constant field k. □

Definition 9.4.3. Let $0 \leq r \leq 8$. Points $P_1, \ldots, P_r \in \mathbb{P}^2(k)$ are in **general position** if they are distinct and *none* of the following hold:

(i) Three of the P_i lie on a line.
(ii) Six of the P_i lie on a conic.
(iii) Eight of the P_i lie on a singular cubic, with one of these eight points at the singularity.

Theorem 9.4.4 (Classification of del Pezzo surfaces)**.** *Let k be a separably closed field. Let X be a del Pezzo surface over k. Then exactly one of the following holds:*

- $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$*; then* $\deg X = 8$.
- *There exists r with $0 \leq r \leq 8$ such that X is the blowup of $\mathbb{P}^2$ at r k-points in general position; then* $\deg X = 9 - r \in \{1, 2, \ldots, 9\}$.

Proof. Let $X \to Y$ be a proper birational morphism to a relatively minimal surface Y. By Corollary 9.3.23, $Y \simeq \mathbb{P}^2$ or $Y \simeq F_n$ for some $n \in \{0\} \cup \mathbb{Z}_{\geq 2}$. A section of $F_n \to \mathbb{P}^1$ has self-intersection $-n$ [**Har77**, Proposition V.2.9], and its strict transform in X would have self-intersection at least as negative, which contradicts Lemma 9.4.2 if $n \geq 2$. Thus $Y \simeq \mathbb{P}^2$ or $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$. By Theorem 9.3.1(b), X is obtained from Y by iteratively blowing up k-points. The blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a k-point is isomorphic to the blowup of $\mathbb{P}^2$ at two k-points (Example 9.3.11), so we need only consider blowups of $\mathbb{P}^2$. If we ever blow up a point on an exceptional curve from a previous blowup, the strict transform C of that exceptional curve in X would satisfy $C.C < -1$, contradicting Lemma 9.4.2. Thus X is the blowup of $\mathbb{P}^2$ at a finite subset

$\{P_1, \ldots, P_r\}$ of $X(k)$. Since $-K$ is ample, $0 < (-K).(-K) = K.K = 9 - r$ (the last equality follows from [**Har77**, Proposition V.3.3]), so $r \leq 8$. If three of the P_i were on a line, the strict transform C of that line would satisfy $C.C \leq 1 - 3 \leq -2$, contradicting Lemma 9.4.2. The other restrictions on the P_i are similarly derived; see [**Dem80**, Théorème 1(i)⇔(iii)]. □

Remark 9.4.5. If X is as in Theorem 9.4.4, then $\operatorname{Pic} X \simeq \mathbb{Z}^{10-d}$: this is true when X is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, and blowing up a k-point adds a new factor of $\mathbb{Z}$ [**Har77**, Proposition V.3.2]. One can also describe the canonical class and the intersection pairing on $\operatorname{Pic} X$ explicitly with respect to a suitable basis [**Man86**, Theorem 23.8].

Remark 9.4.6. If $r \geq 9$ and X is the blowup of $\mathbb{P}^2$ at any r points, then $K.K = 9 - r \leq 0$, so X cannot be a del Pezzo surface. This shows that the property of being a del Pezzo surface is not invariant under birational maps.

Proposition 9.4.7 (Exceptional curves on a del Pezzo surface)**.** *Let k be a separably closed field. Let $X \to \mathbb{P}^2$ be the blowup of points $x_1, \ldots, x_r$ in general position, where $0 \leq r \leq 8$. Then the exceptional curves are the fibers above the x_i together with the strict transforms of the following curves in $\mathbb{P}^2$:*

(i) *a line through 2 of the x_i;*
(ii) *a conic through 5 of the x_i;*
(iii) *a cubic passing through 7 of the x_i, such that one of them is a double point (on the cubic);*
(iv) *a quartic passing through 8 of the x_i, such that three of them are double points;*
(v) *a quintic passing through 8 of the x_i, such that six of them are double points; and*
(vi) *a sextic passing through 8 of the x_i, such that seven of them are double points and one of them is a triple point.*

Proof. See [**Man86**, Theorem 26.2]. □

9.4.2. Del Pezzo surfaces over an arbitrary field. The proof of the following will be scattered over the next few subsections within Section 9.4.

Theorem 9.4.8. *Let k be a field. Let X be a del Pezzo surface over k of degree $d \geq 5$.*

(i) *If $d = 7$ or 5, then X has a k-point.*
(ii) *If $\dim k \leq 1$, then X has a k-point.*
(iii) *If X has a k-point, then X is birational to $\mathbb{P}^2_k$.*
(iv) *If X has a k-point and k is infinite, then $X(k)$ is Zariski dense in X.*

(v) *The homomorphism* $\operatorname{Br} k \to \operatorname{Br} X$ *is surjective.*

(vi) *If* k *is a global field, then* X *satisfies the local-global principle.*

(vii) *If* k *is a global field, then* X *satisfies weak approximation.*

Part (iii) implies (iv). To prove (v), list all finite groups G acting on $\mathbb{Z}^{10-d}$ that could be the image of the $\mathfrak{G}_k$-action on $\operatorname{Pic} X_{k_s}$ respecting the intersection pairing, check that $\mathrm{H}^1(\mathfrak{G}_k, \operatorname{Pic} X_{k_s}) = 0$ in each case, and apply Corollaries 6.7.8 and 6.9.11. Parts (vi) and (iii) imply (vii).

Remark 9.4.9. In the case where k is finite, Proposition 9.3.24 proves (ii) even without the restriction $d \geq 5$.

Remark 9.4.10. When $d \leq 4$, most parts of Theorem 9.4.8 can fail. Part (i) holds for $d = 1$ but can fail for $d = 2, 3, 4$. Part (ii) can fail for $d = 2, 3, 4$ [**CTM04**]. The birational invariant $\mathrm{H}^1(\mathfrak{G}_k, \operatorname{Pic} X_{k_s})$ (see Exercise 9.6) is 0 for $\mathbb{P}^2_k$, but it can be nonzero for a degree d del Pezzo surface over a global field if $d \leq 4$; thus (iii) can fail for each $d \leq 4$. For such an example, Corollaries 6.7.8 and 6.9.11 shows that $\operatorname{Br} k \to \operatorname{Br} X$ will fail to be surjective; that is, (v) fails. This means that there is potentially a Brauer–Manin obstruction to the local-global principle and/or weak approximation, and in fact part (vi) can fail for $d = 2, 3, 4$ and part (vii) can fail for $d = 1, 2, 3, 4$, as will be discussed below.

Remark 9.4.11 (Unirationality of del Pezzo surfaces)**.** It may be that every del Pezzo surface over k with a k-point is k-*unirational*. This has already been proved for all $d \geq 3$ [**Seg43**; **Seg51**; **Man86**, Theorems 29.4 and 30.1; **Kol02**, Theorem 1.1; **Pie12**, Proposition 5.19; **Kne15**, Theorem 2.1], and also under additional hypotheses for $d = 2$ and $d = 1$ [**Man86**, Theorems 29.4; **STVA14**, Theorems 1.1 and 3.2; **FvL16**, Theorem 1.1; **KM17**, Corollary 36]. For each surface for which this holds, part (iv) of Theorem 9.4.8 holds too.

9.4.3. Degree 9**.** Then $X_{k_s} \simeq \mathbb{P}^2_{k_s}$, so X is a Severi–Brauer surface. In particular:

- If X has a k-point, then $X \simeq \mathbb{P}^2_k$ (Proposition 4.5.10).
- If $\dim k \leq 1$, then $X \simeq \mathbb{P}^2_k$.
- If k is a global field, then X satisfies the local-global principle (Theorem 4.5.11).

9.4.4. Degree 8**.**

Proposition 9.4.12. *Let* X *be a degree* 8 *del Pezzo surface over a field* k*. Then exactly one of the following holds:*

(1) *There is a degree 2 étale extension $L \supseteq k$ and a nice conic C over L such that X is isomorphic to the restriction of scalars* $\operatorname{Res}_{L/k} C$. (*In the split case $L = k \times k$, this means simply that X is a product of two nice conics over k.*)

(2) *X is the blowup of $\mathbb{P}^2_k$ at a k-point.*

Proof. By Theorem 9.4.4, either $X_{k_s} \simeq (\mathbb{P}^1 \times \mathbb{P}^1)_{k_s}$ or X_{k_s} is the blowup of $\mathbb{P}^2_{k_s}$ at a k_s-point.

(1) Suppose that $X_{k_s} \simeq (\mathbb{P}^1 \times \mathbb{P}^1)_{k_s}$; i.e., X is a twist of $\mathbb{P}^1 \times \mathbb{P}^1$. To understand the twists, we need to compute $\operatorname{Aut}((\mathbb{P}^1 \times \mathbb{P}^1)_{k_s})$. First, $\operatorname{Aut} \mathbb{P}^1_{k_s} \simeq \mathrm{PGL}_2(k_s)$ (see [**Har77**, Example II.7.1.1]). Let $A \leq \operatorname{Aut}((\mathbb{P}^1 \times \mathbb{P}^1)_{k_s})$ be the subgroup generated by the action of $\mathrm{PGL}_2(k_s)$ on each factor and the involution that interchanges the two factors. Let S and I be the kernel and image of the homomorphism

$$\operatorname{Aut}((\mathbb{P}^1 \times \mathbb{P}^1)_{k_s}) \to \operatorname{Aut}(\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)_{k_s})$$

describing the action of automorphisms on the Picard group, which is $\mathbb{Z} \times \mathbb{Z}$ (see [**Har77**, Example II.6.6.1 and Corollary II.6.16]). We have a commutative diagram

(9.4.13)
$$\begin{tikzcd} 1 \arrow[r] & \mathrm{PGL}_2(k_s) \times \mathrm{PGL}_2(k_s) \arrow[r, hook] \arrow[d] & A \arrow[r] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & S \arrow[r] & \operatorname{Aut}((\mathbb{P}^1 \times \mathbb{P}^1)_{k_s}) \arrow[r] & I \arrow[r] & 1 \end{tikzcd}$$

with exact rows. Any automorphism in S induces linear automorphisms of the spaces of global sections of $\mathscr{O}(1,0)$ and $\mathscr{O}(0,1)$, and hence is given by an element of $\mathrm{PGL}_2(k_s) \times \mathrm{PGL}_2(k_s)$. In other words, the left vertical homomorphism is an isomorphism. On the other hand, an automorphism of $(\mathbb{P}^1 \times \mathbb{P}^1)_{k_s}$ acts on the Picard group $\mathbb{Z} \times \mathbb{Z}$ so as to preserve the ample cone, which is the first quadrant, so it can only be the identity or the coordinate-interchanging involution of $\mathbb{Z} \times \mathbb{Z}$. In other words, the right vertical homomorphism is an isomorphism. Thus the middle vertical homomorphism is an isomorphism too.

Taking cohomology of either of the now-identified rows of (9.4.13) yields a map of pointed sets

$$\mathrm{H}^1(k, \operatorname{Aut}((\mathbb{P}^1 \times \mathbb{P}^1)_{k_s}) \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z}).$$

An element of the latter corresponds to a degree 2 étale extension $L \supseteq k$, and its preimage in $\mathrm{H}^1(k, \operatorname{Aut}((\mathbb{P}^1 \times \mathbb{P}^1)_{k_s}))$ is in bijection with

$$\mathrm{H}^1(k, \text{the } L/k\text{-twist of } \mathrm{PGL}_2 \times \mathrm{PGL}_2) \simeq \mathrm{H}^1(L, \mathrm{PGL}_2),$$

the isomorphism arising from a nonabelian analogue of Shapiro's lemma. The latter set $\mathrm{H}^1(L, \mathrm{PGL}_2)$ parameterizes twists of $\mathbb{P}^1$ over L, i.e., conics over L. Thus twists of $\mathbb{P}^1 \times \mathbb{P}^1$ are parameterized by pairs (L, C) where L is a degree 2 étale extension of k and C is a nice conic over L. By writing out explicit 1-cocycles, one can verify that the twist corresponding to (L, C) is the restriction of scalars $\operatorname{Res}_{L/k} C$.

(2) There is a unique exceptional curve on X_{k_s}. It must be Galois invariant, so it descends to a genus 0 curve E over k. Blow down E to get a morphism $X \to Y$, where Y is a Severi–Brauer surface over k. The image of E is a k-point on Y, so $Y \simeq \mathbb{P}^2_k$. Thus X is the blowup of $\mathbb{P}^2_k$ at a k-point. □

Corollary 9.4.14. *If* $\dim k \le 1$, *then any degree* 8 *del Pezzo surface* X *over* k *has a* k*-point.*

Proof. It suffices to consider case (1) of Proposition 9.4.12. Since $\dim k \le 1$, we have $\operatorname{Br} L = 0$. Since C is a 1-dimensional Severi–Brauer variety over L, it has an L-point. Finally, $X(k) = C(L)$. □

Corollary 9.4.15. *Let* X *be a degree* 8 *del Pezzo surface over a field* k. *If* X *has a* k*-point, then* X *is birational to* $\mathbb{P}^2_k$.

Proof. In case (2) of Proposition 9.4.12, X is a blowup of $\mathbb{P}^2_k$, and hence birational to $\mathbb{P}^2_k$. In case (1), X has the form $\operatorname{Res}_{L/k} C$; if X has a k-point, then C has an L-point, so $C \simeq \mathbb{P}^1_L$, which is birational to $\mathbb{A}^1_L$, so X is birational to $\operatorname{Res}_{L/k} \mathbb{A}^1_L \simeq \mathbb{A}^2_k$, which is birational to $\mathbb{P}^2_k$. □

Corollary 9.4.16. *A degree* 8 *del Pezzo surface over a global field* k *satisfies the local-global principle.*

Proof. If $X = \operatorname{Res}_{L/k} C$, apply the local-global principle to C over L. If X is the blowup of $\mathbb{P}^2_k$ at a k-point, then X has a k-point already. □

9.4.5. Degree 7.

Proposition 9.4.17. *A degree* 7 *del Pezzo surface* X *is* $\mathbb{P}^2_k$ *blown up at either two* k*-points or at a closed point whose residue field is separable of degree* 2 *over* k.

Proof. There are three exceptional curves on X_{k_s}, arranged in a chain, say E_1, E_2, E_3 in order. The middle one E_2 is Galois-stable, so E_2 descends to a nice genus 0 curve over k. Blowing down E_2 yields a nice surface Y with a k-point. Blowing down E_1 and E_3 together instead yields a Severi–Brauer variety Z over k. Since Z is birational to Y, it has a k-point too, so $Z \simeq \mathbb{P}^2_k$. The image of $E_1 \cup E_3$ in $Z \simeq \mathbb{P}^2_k$ is what must be blown up to recover X. $\square$

9.4.6. Degree 6.

Lemma 9.4.18. *Let X be a degree 6 del Pezzo surface over a field k. If there exist separable extensions K and L with $[K : k] = 2$ and $[L : k] = 3$ such that X has a K-point and an L-point, then X has a k-point.*

Proof. Consider the anticanonical embedding $X \subseteq \mathbb{P}^6$. If the K-point or L-point is defined over k, we are done. Otherwise the conjugates of the two points give five geometric points on X. If these five points are sufficiently generic on X, then the 4-dimensional linear subspace of $\mathbb{P}^6$ passing through them intersects X in a 0-cycle of degree 6, of which five points are accounted for, and the remaining point is $\mathfrak{G}_k$-stable, hence a k-point. One can remove the genericity hypothesis by invoking the Lang–Nishimura theorem (Theorem 3.6.11): the construction above defines a rational map $\operatorname{Sym}^2 X \times \operatorname{Sym}^3 X \dashrightarrow X$, and the hypothesis supplies a k-point on the smooth source, so the target has a k-point. $\square$

There are six exceptional curves on X_{k_s}, forming a hexagon. Label them $E_1, \ldots, E_6$ in order around the hexagon.

Proposition 9.4.19. *Let X be a degree 6 del Pezzo surface over a field k. If either $\dim k \le 1$, or k is a global field and $X(\mathbf{A}) \neq \emptyset$, then X has a k-point.*

Proof. (This is based on [**CT72**].) Since the action of $\mathfrak{G}_k$ on $\{E_1, \ldots, E_6\}$ respects intersections, it preserves the partition $\{\{E_1, E_3, E_5\}, \{E_2, E_4, E_6\}\}$. The stabilizer in $\mathfrak{G}_k$ of $\{E_1, E_3, E_5\}$ is G_K for some separable extension K of degree 1 or 2. Blowing down E_1, E_3, E_5 simultaneously on X_K yields a degree 9 del Pezzo surface Y. If $\dim k \le 1$, then $\operatorname{Br} K = 0$, so $Y \simeq \mathbb{P}^2_K$, so Y has a K-point. If k is a global field and $X(\mathbf{A}) \neq \emptyset$, then $X(\mathbf{A}_K) \neq \emptyset$, so $Y(\mathbf{A}_K) \neq \emptyset$, so $Y \simeq \mathbb{P}^2_K$ by Theorem 4.5.11 (the local-global principle for Severi–Brauer varieties). In either case, Y has a K-point, and X is birational to Y, so X has a K-point.

The same argument using the partition $\{\{E_1, E_4\}, \{E_2, E_5\}, \{E_3, E_6\}\}$ shows that X has an L-point for some separable extension L of degree 1 or 3. If either K or L has degree 1, then X has a k-point already. Otherwise Lemma 9.4.18 shows that X has a k-point. $\square$

Sketch of alternative proof. Let $U = X - \bigcup_{i=1}^6 E_i$. Then U_{k_s} is $\mathbb{P}^2_{k_s}$ with three lines deleted; in other words $U_{k_s} \simeq \mathbb{G}_m^2$. One can prove that in general, if U is a variety over a field k and $U_{k_s} \simeq \mathbb{G}_m^n$ for some $n \in \mathbb{N}$, then U is a torsor under a torus T. If $\dim k \leq 1$, then Theorem 5.12.19(b) shows that U has a k-point. Now suppose that k is a global field and $X(\mathbf{A}) \neq \emptyset$. For every v, Proposition 3.5.75 shows that $X(k_v)$ is Zariski dense in X, so U has a k_v-point. By Theorem 5.12.32, U has a k-point. Hence X has a k-point. □

Proposition 9.4.20. *Let X be a degree 6 del Pezzo surface over a field k. If X has a k-point, then X is birational to $\mathbb{P}^2_k$.*

Proof. Let $x \in X(k)$.

Case 1: The point x lies on a unique exceptional curve E_i. Then E_i is defined over k and may be blown down, so we reduce to the case of a degree 7 del Pezzo surface.

Case 2: The point x lies on the intersection of two exceptional curves. Suppose that $x \in E_1 \cap E_2$. Then $E_3 \cup E_6$ is $\mathfrak{G}_k$-stable. Blowing down E_3 and E_6 simultaneously, we reduce to the case of a degree 8 del Pezzo surface.

Case 3: The point x does not lie on any exceptional curve. Make the variety *more* complicated by blowing up x! This yields a degree 5 del Pezzo surface Y. Let D be the exceptional divisor for this blowup. Let P be the dual graph of the ten exceptional curves, so P has one vertex for each exceptional curve, and one edge for each intersecting pair of exceptional curves. Then P turns out to be the Petersen graph; this shows that there are three exceptional curves on Y meeting D, and they are disjoint. Blowing them down let us reduce to the case of a degree 8 del Pezzo surface. □

9.4.7. Degree 5. Recall the fine moduli space $M_{0,5}$ of Example 2.3.9.

Lemma 9.4.21. *Let $\mathcal{X}$ be the blowup of $\mathbb{P}^2$ at the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, and $(1:1:1)$. Then there is an open immersion $M_{0,5} \hookrightarrow \mathcal{X}$, and the S_5-action on $M_{0,5}$ extends to an S_5-action on $\mathcal{X}$. Moreover, $S_5 \to \operatorname{Aut} \mathcal{X}$ is an isomorphism.*

Proof. Let $\mathcal{X}'$ be the complement in $\mathcal{X}$ of the ten exceptional curves, so $\mathcal{X}'$ is the complement in $\mathbb{P}^2$ of the $\binom{4}{2}$ lines through the four blown-up points. Then $\mathcal{X}'$ and $M_{0,5}$ are the same open subvariety of $\mathbb{A}^2$! By symmetry, the transposition $(0,1,\infty,x,y) \mapsto (0,1,\infty,y,x)$ of $M_{0,5}$ extends to an automorphism of $\mathcal{X}$. A calculation shows that the 5-cycle

$$(0,1,\infty,x,y) \longmapsto (y,0,1,\infty,x) \sim \left(0,1,\infty,\frac{1}{y},\frac{x-y}{y(x-1)}\right)$$

also extends. These generate S_5, so S_5 acts faithfully on $\mathcal{X}$. The dual graph of the set of the exceptional curves is the Petersen graph P. We have

$$S_5 \hookrightarrow \operatorname{Aut} \mathcal{X} \to \operatorname{Aut} P \simeq S_5. \tag{9.4.22}$$

The homomorphism $\operatorname{Aut}\mathcal{X} \to \operatorname{Aut}P$ is injective since an automorphism preserving each of the ten exceptional curves would act on the blowdown $\mathbb{P}^2$ and would fix the four blown-up points, forcing it to be the identity. Thus all the homomorphisms in (9.4.22) are isomorphisms. $\square$

Remark 9.4.23. One can show that $\mathcal{X} \supset \mathcal{X}'$ is isomorphic to the compactification $\overline{M}_{0,5} \supset M_{0,5}$ of Example 2.3.12. This explains why the S_5-action on $M_{0,5}$ extends to $\mathcal{X}$.

Lemma 9.4.24. *Every degree* 5 *del Pezzo surface over a field k is dominated by the Grassmannian* $\mathrm{Gr}(2,5)$.

Proof. Let $\mathcal{X}$ be the degree 5 del Pezzo surface in Lemma 9.4.21. The group S_5 acts on k^5 by permuting the coordinates, so it acts on the Grassmannian $\mathrm{Gr}(2,5)$ defined in Example 9.2.3. Given a general point $[W] \in \mathrm{Gr}(2,5)$, so W is a 2-dimensional subspace of k^5, the intersections of W with the five coordinate hyperplanes are five distinct lines in W, and projectivizing yields a point of $M_{0,5}$. This defines the first of the two S_5-equivariant rational maps

$$\mathrm{Gr}(2,5) \dashrightarrow M_{0,5} \hookrightarrow \mathcal{X}, \tag{9.4.25}$$

and the second map is the open immersion of Lemma 9.4.21. The first map is dominant since given five distinct points in $\mathbb{P}^1$, or equivalently five distinct lines in a 2-dimensional space W, one can choose linear functionals $\lambda_1, \dots, \lambda_5 \colon W \to k$ cutting out these lines, and the image of the linear map $(\lambda_1, \dots, \lambda_5) \colon W \to k^5$ represents a preimage in $\mathrm{Gr}(2,5)$.

We now twist. Over a *separably closed* field, there is only one degree 5 del Pezzo surface; see Exercise 9.8. Thus, over k, any other one is a twist X of $\mathcal{X}$ by a cocycle ξ representing a class in $\mathrm{H}^1(k, \operatorname{Aut}\mathcal{X}_{k_s}) = \mathrm{H}^1(k, S_5)$. Twist k^5 and (9.4.25) by ξ to obtain a degree 5 étale k-algebra L and

$$\mathrm{Gr}(2,L) \dashrightarrow M_{0,5}^{(L)} \hookrightarrow X, \tag{9.4.26}$$

where $M_{0,5}^{(L)}$ is as in Example 2.3.11. In $\mathrm{Gr}(2,L)$, the space L is just another 5-dimensional vector space, so $\mathrm{Gr}(2,L) \simeq \mathrm{Gr}(2,5)$. Thus $\mathrm{Gr}(2,5)$ dominates X. $\square$

Remark 9.4.27. Let X be a degree 5 del Pezzo surface. Let $X' \subset X$ be the complement of the ten exceptional curves. The proof of Lemma 9.4.24 shows that $X' \subset X$ is isomorphic to $M_{0,5}^{(L)} \subset \overline{M}_{0,5}^{(L)}$ for some degree 5 étale k-algebra L, where $\overline{M}_{0,5}^{(L)}$ is as in 2.3.12.

Corollary 9.4.28. *Every degree* 5 *del Pezzo surface has a k-point.*

Proof. Combine Lemma 9.4.24 with the Lang–Nishimura theorem (Theorem 3.6.11). $\square$

The literature contains several different proofs of Corollary 9.4.28; see [**Enr97**] (not quite complete), [**SD72**], [**SB92**], [**Sko93**], [**Kol96**, Exercise III.3.13], and [**Has09**, Exercise 3.1.4]. The proof we gave is closest to that in [**Sko93**].

Lemma 9.4.24 implies that a degree 5 del Pezzo surface X over k is k-unirational, but even more is true: X is *k-rational*—this was first proved by Manin [**Man66**, Theorem 3.15] (at least for perfect k), assuming Enriques's claim that $X(k) \neq \emptyset$).

Theorem 9.4.29. *Every degree* 5 *del Pezzo surface over a field k is birational to $\mathbb{P}^2_k$.*

Proof. Let X be the del Pezzo surface. Let $X' \subset X$ be the complement of the exceptional curves in X.

First suppose that X' has a k-point x. Then the blowup of X at x is a degree 4 del Pezzo surface Y. There are 16 exceptional curves on Y (over k_s): the strict transforms of the ten exceptional curves on X, the preimage of x, and five more. Moreover, those last five curves are skew, as can be checked over $\overline{k}$: if Y is the blowup of $\mathbb{P}^2$ at points $x_1, \ldots, x_5$ in general position, and X is the blowup of $\mathbb{P}^2$ at $x_1, \ldots, x_4$, then the five curves are the strict transforms of the conic through $x_1, \ldots, x_5$ and of the four lines connecting each of $x_1, \ldots, x_4$ to x_5. Blowing down this $\mathfrak{G}_k$-stable set of five skew lines on Y yields a degree 9 del Pezzo surface Z with a k-point, so $Z \simeq \mathbb{P}^2_k$.

Now suppose that X' has no k-points. The pair $X' \subset X$ is isomorphic to $M_{0,5}^{(L)} \subset \overline{M}_{0,5}^{(L)}$ for some degree 5 étale k-algebra L. Then $M_{0,5}^{(L)}$ has no k-point, so there is no closed immersion $\operatorname{Spec} L \hookrightarrow \mathbb{P}^1_k$. Counting the closed points on $\mathbb{P}^1_k$ with each residue field shows this is possible only in these cases:

(i) $L = k^5$ with $k = \mathbb{F}_2$ or $k = \mathbb{F}_3$; or

(ii) $L = \mathbb{F}_2 \times \mathbb{F}_4 \times \mathbb{F}_4$ with $k = \mathbb{F}_2$.

In case (i), X is the blowup of $\mathbb{P}^2_k$ at four k-points, so X is birational to $\mathbb{P}^2_k$. In case (ii), $L \simeq L_3 \times L_2$ for étale algebras L_3 of degree 3 and L_2 of degree 2. Fix one closed immersion $\iota\colon \operatorname{Spec} L_3 \to \mathbb{P}^1_k$. Then each $(\operatorname{Aut} \mathbb{P}^1)$-orbit of closed immersions $\operatorname{Spec} L \to \mathbb{P}^1_k$ contains a unique representative whose restriction to $\operatorname{Spec} L_3$ is ι, so X' is isomorphic to the space of closed immersions $\operatorname{Spec} L_2 \to U := \mathbb{P}^1 - \iota(\operatorname{Spec} L_3)$. Thus we have birational equivalences

$$X \sim X' \sim \operatorname{Res}_{L_2/k} U \sim \operatorname{Res}_{L_2/k} \mathbb{A}^1_k \simeq \mathbb{A}^2_k \sim \mathbb{P}^2_k. \qquad \square$$

9.4.8. Degree 4. These X are smooth intersections of two quadrics in $\mathbb{P}^4$ [**Kol96**, Theorem III.3.5.4]. If k is a global field, then the local-global principle can fail (Exercise 8.3), and weak approximation can fail even if X has a k-point [**CTS77**, III, Exemple (a)].

9.4.9. Degree 3. These are nice cubic surfaces in $\mathbb{P}^3$ [**Kol96**, Theorem III.3.5.3]. Mordell [**Mor49**] conjectured that nice cubic surfaces over $\mathbb{Q}$ satisfy the local-global principle, but this turned out to be false [**SD62**, §2]. Selmer [**Sel53**] proved that diagonal cubic surfaces $ax^3+by^3+cz^3+dw^3=0$ in $\mathbb{P}^3_{\mathbb{Q}}$ for nonzero integers a, b, c, d *satisfy* the local-global principle if $ab = cd$ or $|abcd| \leq 500$, but later Cassels and Guy [**CG66**] discovered that the surface

$$5x^3 + 9y^3 + 10z^3 + 12w^3 = 0$$

over $\mathbb{Q}$ violates the local-global principle. Also, weak approximation can fail even if X is minimal and has a rational point [**SD62**, §3] (we ask for X to be minimal since otherwise one could simply blow up a k-point on a degree 4 counterexample to weak approximation). See [**CTKS87**] for many more counterexamples.

9.4.10. Degree 2. The anticanonical map is a degree 2 morphism $X \to \mathbb{P}^2$ ramified along a nice curve of degree 4 in $\mathbb{P}^2$ [**Kol96**, Theorem III.3.5.2]. In other words, X is of degree 4 in a weighted projective space $\mathbb{P}(1,1,1,2)$. If k is a global field, then X need not satisfy the local-global principle: Kresch and Tschinkel [**KT04**] give many counterexamples, including the surface over $\mathbb{Q}$ defined by the weighted homogeneous equation

$$w^2 = -6x^4 - 3y^4 + 2z^4.$$

Colliot-Thélène observed that one can obtain a counterexample also by replacing z^2 by z^4 in Iskovskikh's surface of Section 8.2.5; this results in the surface

$$y^2 = -z^4 + (3w^2 - x^2)(x^2 - 2w^2)$$

over $\mathbb{Q}$.

Weak approximation can fail too, even if X is minimal and has a k-point [**KT04**].

Remark 9.4.30. So far, every time a del Pezzo surface has been found to violate the local-global principle, the violation has been explained by the Brauer–Manin obstruction, as predicted by Conjecture 9.2.27.

9.4.11. Degree 1. Then X is of degree 6 in a weighted projective space $\mathbb{P}(1,1,2,3)$ [**Kol96**, Theorem III.3.5.1]. The common zero locus of any basis s_1, s_2 of the 2-dimensional space $\mathrm{H}^0(X, -K)$ is independent of the choice of basis. This locus consists of a single degree 1 point, since $(-K).(-K) = 1$. In other words, the intersection of any two distinct divisors in $|-K|$ is a canonical k-point! Thus $X(k) \neq \emptyset$.

In particular, the local-global principle holds trivially. On the other hand, there can be a Brauer–Manin obstruction to weak approximation, even if X is minimal [**VA08**, Proposition 7.1].

9.4.12. Summary. The results on the arithmetic of del Pezzo surfaces are summarized in the following table, whose entries answer the following questions about a del Pezzo surface of specified degree over a field k:

- **k-point:** Must the surface have a k-point?
- **k-rational†:** *If the surface has a k-point*, must it be birational to $\mathbb{P}^2_k$?
- **k-unirational†:** *If the surface has a k-point*, must it be dominated by $\mathbb{P}^2_k$?
- **local-global:** If k is a global field, must the surface satisfy the local-global principle?
- **weak appr.:** If k is a global field, must the surface satisfy weak approximation?

Degree	k-point	k-rational†	k-unirational†	local-global	weak appr.
9	NO	YES	YES	YES	YES
8	NO	YES	YES	YES	YES
7	YES	YES	YES	YES	YES
6	NO	YES	YES	YES	YES
5	YES	YES	YES	YES	YES
4	NO	NO	YES	NO	NO
3	NO	NO	YES	NO	NO
2	NO	NO	?	NO	NO
1	YES	NO	?	YES	NO

The daggers † warn that those columns presume the existence of a k-point.

9.5. Rational points on varieties of general type

9.5.1. Curves of genus > 1 over number fields.

(References: [**HS00**], [**BG06**])

Let X be a nice curve over a field K. In Section 2.6.4.2, we stated the following result, conjectured by Mordell in 1922 (for $K = \mathbb{Q}$) [**Mor22**] and proved by Faltings in 1983.

Theorem 9.5.1 ([**Fal83**]). *Let X be a nice curve of genus > 1 over a number field K. Then $X(K)$ is finite.*

We will not give a proof, since the known proofs are very complicated. Faltings's proof uses an idea of Parshin to reduce the problem to proving a conjecture of Shafarevich that for a fixed number field K, a fixed finite set of places S of K, and a fixed $d \geq 0$, there are at most finitely many isomorphism classes of d-dimensional abelian varieties over K with good reduction outside S [**Fal83**]. Vojta [**Voj91**] gave a different proof of Theorem 9.5.1, based

Universes

(Reference: [**SGA** $4_{\mathbf{I}}$, I.Appendice])

The plan is to assume the existence of a very large set, called a universe, such that almost all the constructions we need can be carried out within it. Those constructions that cannot be carried out within it can be carried out in a larger universe.

According to [**SGA** $4_{\mathbf{I}}$, I.Appendice], the theory of universes comes from "the secret papers of N. Bourbaki". According to [**SGA 1**, VI, §1], the details will be given in a book in preparation by Chevalley and Gabriel to appear in the year 3000.

A.1. Definition of universe

Everything is a set. In particular, elements of a set are themselves sets. Given a set x, let $\mathcal{P}(x)$ be the set of all subsets of x.

Definition A.1.1 ([**SGA** $4_{\mathbf{I}}$, I.Appendice, Définition 1])**.** A **universe** is a set $\mathcal{U}$ satisfying the following conditions:

(U.I) If $y \in \mathcal{U}$ and $x \in y$, then $x \in \mathcal{U}$.

(U.II) If $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$.

(U.III) If $x \in \mathcal{U}$, then $\mathcal{P}(x) \in \mathcal{U}$.

(U.IV) If $I \in \mathcal{U}$, and $(x_i)_{i \in I}$ is a collection of elements of $\mathcal{U}$, then the union $\bigcup_{i \in I} x_i$ is an element of $\mathcal{U}$.

A universe $\mathcal{U}$ is not a "set of all sets". In particular, a universe cannot be a member of itself; see Exercise A.2.

A.2. The universe axiom

To the usual ZFC axioms of set theory (the Zermelo-Frenkel axioms with the axiom of choice), one adds the **universe axiom** [**SGA** $4_{\mathbf{I}}$, I.Appendice.§4]:[1]

Every set is an element of some universe.

Suppose that ZFC is consistent. Then it turns out that the *negation* of the universe axiom is consistent with ZFC: given a model of ZFC, one can build another model of ZFC in which the universe axiom *fails*. But it is not known whether the universe axiom itself is consistent with ZFC.

The universe axiom is so convenient that we are going to assume it despite its uncertain status relative to ZFC.

Remark A.2.1. The original proof of Fermat's last theorem made use of constructions relying on the universe axiom! But the proof can probably be redone without this axiom; see Section A.5.

A.3. Strongly inaccessible cardinals

Definition A.3.1. A cardinal κ is **strongly inaccessible** if the following two conditions hold:

(1) For every $\lambda < \kappa$, we have $2^\lambda < \kappa$.
(2) Whenever $(\lambda_i)_{i \in I}$ is a family of cardinals indexed by a set I such that $\#I < \kappa$ and $\lambda_i < \kappa$ for every $i \in I$, we have $\sum_{i \in I} \lambda_i < \kappa$.

The two smallest strongly inaccessible cardinals are 0 and $\aleph_0$. By (1), any other strongly inaccessible cardinal κ must be larger than all of

$$\begin{aligned} \beth_0 &:= \aleph_0 \\ \beth_1 &:= 2^{\aleph_0} \\ \beth_2 &:= 2^{2^{\aleph_0}} \\ &\vdots \end{aligned}$$

By (2), κ must also be larger than the supremum $\beth_\omega$ of all these. Transfinite induction continues this sequence of cardinals by defining $\beth_\alpha$ for any ordinal α. Then κ must be larger than $\beth_{\omega^\omega}$, $\beth_{\omega^{\omega^\omega}}$, ..., and even $\beth_{\omega_1}$, where ω_1 is the first uncountable ordinal. Identify each cardinal with the first ordinal of

[1] In [**SGA** $4_{\mathbf{I}}$, I.Appendice.§4] one finds an additional axiom (UB) that is present only because Bourbaki's axioms for set theory are different from the usual ZFC axioms. Bourbaki's set theory includes a global choice operator τ: for any 1-variable predicate $P(x)$, the expression $\tau_x P(x)$ represents an element y such that $P(y)$ is true, if such a y exists. Axiom (UB) says that for any 1-variable predicate $P(x)$ and any universe $\mathcal{U}$, if there exists $y \in \mathcal{U}$ such that $P(y)$ is true, then $\tau_x P(x)$ is an element of $\mathcal{U}$. So axiom (UB) says that the elements produced by the global choice operator lie in a given universe whenever possible.

its cardinality. Then $\omega_1 \leq 2^{\aleph_0} = \beth_1$, so $\beth_{\omega_1} \leq \beth_{\beth_1}$. But κ is also larger than $\beth_{\beth_{\omega_1}}$, $\beth_{\beth_{\beth_{\omega_1}}}$, and so on.

Theorem A.3.2. *Within ZFC, the universe axiom is equivalent to the following "large cardinal axiom":*

> *For every cardinal, there is a strictly larger strongly inaccessible cardinal.*

Proof. One direction is easy, because if $\mathcal{U}$ is a universe, then the cardinal $\sup\{\#x : x \in \mathcal{U}\}$ is strongly inaccessible. For the other direction, see [**SGA** $4_{\mathbf{I}}$, I.Appendice.§5], which constructs a universe from a strongly inaccessible cardinal. □

A.4. Universes and categories

We now assume that an uncountable universe $\mathcal{U}$ has been fixed.

Recall that everything is a set. For instance, an ordered pair (x, y) is $\{x, \{y\}\}$. A group is a 4-tuple (G, m, i, e) such that various conditions hold. Even a scheme can be described as a set.

Definition A.4.1. A **small category** is a category in which the collection of objects is a set (instead of a class).

We want all our categories to be small categories. Thus for example, **Sets** will denote not the category of all sets, but the category of *sets that are elements of* $\mathcal{U}$. Similarly, **Groups** will be the category of groups that are elements of $\mathcal{U}$, and so on.

For categories such as these two, the set of objects is a subset of $\mathcal{U}$ having the same cardinality as $\mathcal{U}$, which implies that the set of objects cannot be an *element* of $\mathcal{U}$. This creates a minor problem: the collection of all functors $\mathbf{Schemes}^{\mathrm{opp}} \to \mathbf{Sets}$, say, is a set of cardinality *larger than that of* $\mathcal{U}$! The category of such functors is still a small category, but it lives in a larger universe $\mathcal{U}'$.

A.5. Avoiding universes

Suppose that we want to prove theorems that are not conditional on the universe axiom. Then we cannot define the category **Schemes** as the set of schemes that are elements of a particular universe $\mathcal{U}$. Instead we choose *a* category of schemes that is closed under various operations, and work within that category [**SP**, Tag 020S, Tag 03XB]. Ideally, we should then show that our choice did not matter for our particular objects of study; see [**SP**, Tag 00VY] for an example of this.

Exercises

A.1. Classify all finite universes.

A.2. Let $\mathcal{U}$ be a universe. Prove that $\mathcal{U} \notin \mathcal{U}$.

A.3. Let $P_0 = \emptyset$. For $n \in \mathbb{N}$, inductively define $P_{n+1} := \mathcal{P}(P_n)$. Let $\mathcal{U} = \bigcup_{n\in\mathbb{N}} P_n$. Prove that $\mathcal{U}$ is a universe.