Preface

**Combinatorics is not a field, it’s an attitude.**
Anon

A *combinatorial reciprocity theorem* relates two classes of combinatorial objects via their counting functions: consider a class $\mathcal{X}$ of combinatorial objects and let $f(n)$ be the function that counts the number of objects in $\mathcal{X}$ of size $n$, where size refers to some specific quantity that is naturally associated with the objects in $\mathcal{X}$. Similar to canonization, it requires two miracles for a combinatorial reciprocity to occur:

1. the function $f(n)$ is the restriction of some reasonable function (e.g., a polynomial) to the positive integers, and
2. the evaluation $f(-n)$ is an integer of the same sign $\sigma = \pm 1$ for all $n \in \mathbb{Z}_{>0}$.

In this situation it is only human to ask if $\sigma f(-n)$ has a combinatorial meaning, that is, if there is a natural class $\mathcal{X}^\circ$ of combinatorial objects such that $\sigma f(-n)$ counts the objects of $\mathcal{X}^\circ$ of size $n$ (where size again refers to some specific quantity naturally associated to $\mathcal{X}^\circ$). Combinatorial reciprocity theorems are among the most charming results in mathematics and, in contrast to canonization, can be found all over enumerative combinatorics and beyond.

As a first example we consider the class of maps $[k] \to \mathbb{Z}_{>0}$ from the finite set $[k] := \{1, 2, \ldots, k\}$ into the positive integers, and so $f(n) = n^k$ counts the number of maps with codomain $[n]$. Thus $f(n)$ is the restriction of a polynomial and $(-1)^k f(-n) = n^k$ satisfies our second requirement above. This relates the number of maps $[k] \to [n]$ to itself. This relation is a genuine combinatorial reciprocity but the impression one is left with is that of being
underwhelmed rather than charmed. Later in the book it will become clear that this example is not boring at all, but for now let’s try again.

The term *combinatorial reciprocity theorem* was coined by Richard Stanley in his 1974 paper [162] of the same title, in which he developed a firm foundation of the subject. Stanley starts with an appealing reciprocity that he attributes to John Riordan: For a set $S$ and $d \in \mathbb{Z}_{\geq 0}$, the collection of $d$-subsets\(^1\) of $S$ is

$$\binom{S}{d} := \{ A \subseteq S : |A| = d \} .$$

For $d$ fixed, the number of $d$-subsets of $S$ depends only on the cardinality $|S|$, and the number of $d$-subsets of an $n$-set is

\[
(0.0.1) \quad f(n) = \binom{n}{d} = \frac{1}{d!} \ n(n-1) \cdots (n-d+2)(n-d+1),
\]

which is the restriction of a polynomial in $n$ of degree $d$. From the factorization we can read off that $(-1)^d f(-n)$ is a positive integer for every $n > 0$. More precisely,

$$(-1)^d f(-n) = \frac{1}{d!} \ n(n+1) \cdots (n+d-2)(n+d-1) = \binom{n+d-1}{d},$$

which is the number of $d$-multisubsets of an $n$-set, that is, the number of picking $d$ elements from $[n]$ with repetition but without regard to the order in which the elements are picked. Now this is a combinatorial reciprocity! In formulas it reads

\[
(0.0.2) \quad (-1)^d \binom{-n}{d} = \binom{n+d-1}{d}.
\]

This is enticing in more than one way. The identity presents an intriguing connection between subsets and multisubsets via their counting functions, and its formal justification is completely within the realms of an undergraduate class in combinatorics. Equation (0.0.2) can be found in Riordan’s book [143] on combinatorial analysis without further comment and, charmingly, Stanley states that his paper [162] can be considered as “further comment”. That further comment is necessary is apparent from the fact that the formal proof above falls short of explaining why these two sorts of objects are related by a combinatorial reciprocity. In particular, comparing coefficients in (0.0.2) cannot be the method of choice for establishing more general reciprocity relations.

In this book we develop tools and techniques for handling combinatorial reciprocities. However, our own perspective is firmly rooted in *geometric* combinatorics and, thus, our emphasis is on the geometric nature of the

\(^1\)All our definitions will look like that: incorporated into the text but bold-faced and so hopefully clearly visible.
combinatorial reciprocities. That is, for every class of combinatorial objects we associate a geometric object (such as a polytope or a polyhedral complex) in such a way that combinatorial features, including counting functions and reciprocity, are reflected in the geometry. In short, this book can be seen as further comment with pictures. At any rate, our text was written with the intention to give a comprehensive introduction to contemporary enumerative geometric combinatorics.

A Quick Tour. The book naturally comes in two parts with a special role played by the first chapter: Chapter 1 introduces four combinatorial reciprocity theorems that we set out to establish in the course of the book. Chapters 2–4 are for-the-most-part-independent introductions to three major themes of combinatorics: partially ordered sets, polyhedra, and generating functions. Chapters 5–7 treat more sophisticated topics in geometric combinatorics and are meant to be digested in order. Here is what to expect.

Chapter 1 sets the rhythm. We introduce four functions to count colorings and flows on graphs, order-preserving functions on partially ordered sets, and lattice points in dilations of lattice polygons. The definitions in this chapter are kept somewhat informal, to provide an easy entry into the themes of the later chapters. In all four cases we state a surprising combinatorial reciprocity and we point to some of the relations and connections between these examples, which will make repeated appearances later on. All in all, this chapter is a source of examples and motivation. You should revisit it from time to time to see how the various ways to view these objects shape your perspective.

Chapter 2 gives an introduction to partially ordered sets (posets, for short). Relating posets by means of order-preserving maps gives rise to the order polynomials from Chapter 1. One of the highlights here is a purely combinatorial proof of the reciprocity surrounding order polynomials (and only later will we see that there was geometry behind it). This gives us an opportunity to introduce important machinery, including Möbius inversion, zeta polynomials, and Eulerian posets in a hands-on and nonstandard form.

Geometry enters (quite literally) the picture in Chapter 3, in which we introduce convex polyhedra. Polyhedra are wonderful objects to study in their own right, as we hope to convey here, and much of their combinatorial structure comes in poset-theoretic terms. Our main motivation, however, is to develop a language that enables us to give the objects from Chapters 1 and 2 a geometric incarnation. The main player in Chapter 3 is the Euler characteristic, which is a powerful tool to obtain combinatorial truths from geometry. Two applications of the Euler characteristic, which we will witness
in this chapter, are Zaslavsky’s theorem for hyperplane arrangements and the Brianchon–Gram relation for polytopes.

Chapter 4 sets up the main algebraic machinery for our book: (rational) generating functions. We start gently with natural examples of compositions and partitions, and combinatorial reciprocity theorems appear almost instantly and just as naturally. The second half of Chapter 4 connects the world of generating functions with that of polyhedra and cones, where we develop Ehrhart and Hilbert series from first principles, including Stanley’s reciprocity theorem for rational simplicial cones, which is at the heart of this book. This connection, in turn, allows us to view the first half of Chapter 4 from a new, geometric, perspective.

Chapter 5 is devoted to decomposing polyhedra into simple pieces. In particular, organizing the various pieces automatically suggests to view triangulations and, more generally, subdivisions as posets. Together with the technologies developed in the first part of the book, this culminates in a proof of our main combinatorial reciprocity theorems for polytopes and cones. The theory of subdividing polyhedra is worthy of study in its own right and we only glimpse at it by studying various ways to subdivide polytopes in a geometric, algorithmic, and, of course, combinatorial fashion. A powerful tool is that of half-open decompositions that quite remarkably help us to see some deep combinatorics in a clear way.

In Chapter 6 we give general posets life in Euclidean space as polyhedral cones. The theory of order cones allows us to utilize Chapters 2 5 often in surprisingly interconnected ways, to study posets using geometric means and, at the same time, interesting arithmetic objects derived from posets. Just as interesting are applications of this theory, which include permutation statistics, order polytopes, \( P \)-partitions, and their combinatorial reciprocity theorems.

Chapter 7 finishes the framework that was started in Chapter 1: we develop a unifying geometric approach to certain families of combinatorial polynomials. The last missing piece of the puzzle is formed by hyperplane arrangements, which constitute the main players of Chapter 7. They open a window to certain families of graph polynomials, including chromatic and flow polynomials, and we prove combinatorial reciprocity theorems for both. Hyperplane arrangements also naturally connect to two important families of polytopes, namely, alcoved polytopes and zonotopes.

The prerequisites for this book are minimal: undergraduate knowledge of linear algebra and combinatorics should suffice. The numerous exercises throughout the text are designed so that the book could easily be used for a graduate class in combinatorics or discrete geometry. The exercises that are needed for the main body of the text are marked by \( \diamond \).
Preface

Acknowledgments. The first (and very preliminary) version of this manuscript was tried on some patient and error-forgiving students and researchers at the Mathematical Sciences Research Institute in Spring 2008 and in a course at the Freie Universität Berlin in Fall 2011. We thank them for their crucial input at the early stages of this book. In particular Lennart Claus, who took the 2011 class and did not see this book finally being finished, is vividly remembered for his keen interest, his active participation, and his Mandelkekse.

Since then, the book has, like its authors, matured (and aged). In particular it has expanded in breadth and depth (and, inevitably, length). We have had the fortune of receiving many valuable suggestions and corrections; we would like to thank in particular Tewodros Amdeberhan, Spencer Backman, Hélène Barcelo, Seth Chaiken, Adam Chavin, Susanna Fishel, Curtis Greene, Christian Haase, Max Hlavacek, Katharina Jochemko, Florian Kohl, Cailan Li, Sebastian Manecke, Jeremy Martin, Tyrrell McAllister, Louis Ng, Peter Paule, Bruce Sagan, Steven Sam, Paco Santos, Miriam Schlöter, Tom Schmidt, Christina Schulz, Matthias Schymura, Sam Sehayek, Richard Sieg, Christian Stump, Ngô Việt Trung, Andrés Vindas Meléndez, Wei Wang, Russ Woodroofe, Tom Zaslavsky, and Günter Ziegler. Richard Stanley does not only also belong to this list, but he deserves special thanks: as one can see in the references throughout this text, he has been the main creative mind behind the material that forms the core of this book.

We thank the organizers and students of several classes, graduate schools, and workshops, in which we could test run various parts of the book: the 2011 Rocky Mountain Mathematics Consortium in Laramie, the 2013 Spring School in Hanoi, a Winter 2014 combinatorics class at the Freie Universität Berlin, and the 2015 Summer School at the Research Institute for Symbolic Computation in Linz.

We are grateful to the editorial staff at the American Mathematical Society, particularly Sergei Gelfand, who was relentlessly cheerful of this book project from its inception to its final polishing; his patience and wit have not only been much appreciated but needed. We thank Ed Dunne, Chris Thivierge, and the Editorial Committee and reviewers for many helpful insights, Mary Letourneau for her meticulous copy-editing, and the AMS {\TeX} gurus, particularly Brian Bartling and Barbara Beeton, for invaluable assistance. David Austin made much of the geometry in this book come to life in the figures featured here; we are big fans of his art.

We thank the US National Science Foundation for their support, San Francisco State University for a presidential award (the resulting sabbatical allowed M.B. to give the above-mentioned lectures at MSRI), and the DFG Collaborative Research Center TRR 109 Discretization in Geometry and
Dynamics (sponsoring M.B.’s guest professorship at Freie Universität in Fall 2014).

M.B. is deeply grateful to Tendai for her love, support, and patience while he tries to turn coffee into theorems, to Kumi for her energy and emotional support, and to his family zuhause and kumusha for their love. The idea for this book was conceived on numerous long trips to spend precious time with his Papa during the last months of his life. He dedicates this book to his memory.

R.S. is eternally grateful to Vanessa and Konstantin for their support, their patience, and, above all, for their love. When living with somebody who often times concentratedly stares at nothing (while figuring something out), all three merits are surely necessary. This book is dedicated to them. R.S. also thanks the Villa people at Freie Universität Berlin in the years 2011–2016, in particular Günter, for sharing the atmosphere, the freedom, and their wisdom (mathematically and otherwise).

San Francisco \hspace{1cm} Matthias Beck
Frankfurt \hspace{1cm} Raman Sanyal
June 2018