The purpose of this book is to provide a broad introduction to the theory of scattering resonances.

Scattering resonances appear in many branches of mathematics, physics and engineering. They generalize eigenvalues or bound states for systems in which energy can scatter to infinity. A typical state has then a rate of oscillation (just as a bound state does) and a rate of decay. Although the notion is intrinsically dynamical, an elegant mathematical formulation comes from considering meromorphic continuations of Green’s functions or scattering matrices. The poles of these meromorphic continuations capture the physical information by identifying the rate of oscillations with the real part of a pole and the rate of decay with its imaginary part. The resonant state, which is the corresponding wave function, then appears in the residue of the meromorphically continued operator. An example from pure mathematics is given by the zeros of the Riemann zeta function: they are, essentially, the resonances of the Laplacian on the modular surface. The Riemann hypothesis then states that the decay rates for the modular surface are all either 0 or $\frac{1}{4}$. A standard example from physics is given by shape resonances created when the interaction region is separated from free space by a potential barrier. The decay rate is then exponentially small in a way depending on the width of the barrier.
In this book we provide an introduction to mathematical techniques used in the study of scattering resonances, concentrating on the simplest models but providing references to modern literature and indications of what happens in more general situations. Some chapters (such as Chapters 2 and 3) are meant to be easily accessible and others (such as Chapter 5) somewhat more demanding. The rather substantial set of appendices provides detailed accounts of most methods needed in the text. A diagram representing the dependencies of various sections is presented at the end of Chapter 1. The choice of topics is necessarily determined by the research interests of the authors, and many important aspects of the subject are not covered. We also stayed away from exciting but technical developments such as precise asymptotics for shape resonances, fractal Weyl laws, resonance gaps for chaotic systems, or the applications of scattering theory to hyperbolic dynamical systems – see the survey [Zw17] for an overview and references.

SD was introduced to scattering resonances by MZ, who in turn had the good fortune to be introduced to this field by Richard Melrose. We would like to thank him for his generous guidance and insights and for his foundational results on resonance counting and trace formulas.

The viewpoint and many discoveries of Johannes Sjöstrand changed the subject in a profound way. MZ was privileged to maintain a long collaboration with Sjöstrand and would like to thank him for sharing his ideas and expertise over the years.

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