# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>xi</td>
</tr>
<tr>
<td>Projects</td>
<td>xiv</td>
</tr>
<tr>
<td>Chapter 1. Chain Complexes, Homology, and Cohomology</td>
<td>1</td>
</tr>
<tr>
<td>§1.1. Chain complexes associated to a space</td>
<td>1</td>
</tr>
<tr>
<td>§1.2. Tensor products, adjoint functors, and Hom</td>
<td>8</td>
</tr>
<tr>
<td>§1.3. Tensor and Hom functors on chain complexes</td>
<td>12</td>
</tr>
<tr>
<td>§1.4. Singular cohomology</td>
<td>14</td>
</tr>
<tr>
<td>§1.5. The Eilenberg-Steenrod axioms</td>
<td>19</td>
</tr>
<tr>
<td>§1.6. Projects for Chapter 1</td>
<td>22</td>
</tr>
<tr>
<td>Chapter 2. Homological Algebra</td>
<td>23</td>
</tr>
<tr>
<td>§2.1. Axioms for Tor and Ext; projective resolutions</td>
<td>23</td>
</tr>
<tr>
<td>§2.2. Projective and injective modules</td>
<td>29</td>
</tr>
<tr>
<td>§2.3. Resolutions</td>
<td>33</td>
</tr>
<tr>
<td>§2.4. Definition of Tor and Ext - existence</td>
<td>35</td>
</tr>
<tr>
<td>§2.5. The fundamental lemma of homological algebra</td>
<td>36</td>
</tr>
<tr>
<td>§2.6. Universal coefficient theorems</td>
<td>43</td>
</tr>
<tr>
<td>§2.7. Projects for Chapter 2</td>
<td>49</td>
</tr>
<tr>
<td>Chapter 3. Products</td>
<td>51</td>
</tr>
<tr>
<td>Chapter</td>
<td>Section</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>3</td>
<td>§3.1</td>
</tr>
<tr>
<td></td>
<td>§3.2</td>
</tr>
<tr>
<td></td>
<td>§3.3</td>
</tr>
<tr>
<td></td>
<td>§3.4</td>
</tr>
<tr>
<td></td>
<td>§3.5</td>
</tr>
<tr>
<td></td>
<td>§3.6</td>
</tr>
<tr>
<td>4</td>
<td>§4.1</td>
</tr>
<tr>
<td></td>
<td>§4.2</td>
</tr>
<tr>
<td></td>
<td>§4.3</td>
</tr>
<tr>
<td></td>
<td>§4.4</td>
</tr>
<tr>
<td></td>
<td>§4.5</td>
</tr>
<tr>
<td></td>
<td>§4.6</td>
</tr>
<tr>
<td></td>
<td>§4.7</td>
</tr>
<tr>
<td>5</td>
<td>§5.1</td>
</tr>
<tr>
<td></td>
<td>§5.2</td>
</tr>
<tr>
<td></td>
<td>§5.3</td>
</tr>
<tr>
<td></td>
<td>§5.4</td>
</tr>
<tr>
<td></td>
<td>§5.5</td>
</tr>
<tr>
<td>6</td>
<td>§6.1</td>
</tr>
<tr>
<td></td>
<td>§6.2</td>
</tr>
<tr>
<td></td>
<td>§6.3</td>
</tr>
<tr>
<td></td>
<td>§6.4</td>
</tr>
<tr>
<td></td>
<td>§6.5</td>
</tr>
<tr>
<td></td>
<td>§6.6</td>
</tr>
<tr>
<td></td>
<td>§6.7</td>
</tr>
</tbody>
</table>
Contents

§6.8. Replacing a map by a cofibration 131
§6.9. Sets of homotopy classes of maps 134
§6.10. Adjoint of loops and suspension; smash products 136
§6.11. Fibration and cofibration sequences 138
§6.12. Puppe sequences 141
§6.13. Homotopy groups 143
§6.14. Examples of fibrations 145
§6.15. Relative homotopy groups 152
§6.16. The action of the fundamental group on homotopy sets 155
§6.17. The Hurewicz and Whitehead theorems 160
§6.18. Projects for Chapter 6 163

Chapter 7. Obstruction Theory and Eilenberg-MacLane Spaces 165
§7.1. Basic problems of obstruction theory 165
§7.2. The obstruction cocycle 168
§7.3. Construction of the obstruction cocycle 169
§7.4. Proof of the extension theorem 172
§7.5. Obstructions to finding a homotopy 175
§7.6. Primary obstructions 176
§7.7. Eilenberg-MacLane spaces 177
§7.8. Aspherical spaces 183
§7.9. CW-approximations and Whitehead’s theorem 185
§7.10. Obstruction theory in fibrations 189
§7.11. Characteristic classes 191
§7.12. Projects for Chapter 7 192

Chapter 8. Bordism, Spectra, and Generalized Homology 195
§8.1. Framed bordism and homotopy groups of spheres 196
§8.2. Suspension and the Freudenthal theorem 202
§8.3. Stable tangential framings 204
§8.4. Spectra 210
§8.5. More general bordism theories 213
§8.6. Classifying spaces 217
§8.7. Construction of the Thom spectra 219
§8.8. Generalized homology theories 227
§8.9. Projects for Chapter 8 234

Chapter 9. Spectral Sequences 237
§9.1. Definition of a spectral sequence 237
§9.2. The Leray-Serre-Atiyah-Hirzebruch spectral sequence 241
§9.3. The edge homomorphisms and the transgression 245
§9.4. Applications of the homology spectral sequence 249
§9.5. The cohomology spectral sequence 254
§9.6. Homology of groups 261
§9.7. Homology of covering spaces 264
§9.8. Relative spectral sequences 266
§9.9. Projects for Chapter 9 266

Chapter 10. Further Applications of Spectral Sequences 267
§10.1. Serre classes of abelian groups 267
§10.2. Homotopy groups of spheres 276
§10.3. Suspension, looping, and the transgression 279
§10.4. Cohomology operations 283
§10.5. The mod 2 Steenrod algebra 288
§10.6. The Thom isomorphism theorem 295
§10.7. Intersection theory 299
§10.8. Stiefel–Whitney classes 306
§10.9. Localization 312
§10.10. Construction of bordism invariants 317
§10.11. Projects for Chapter 10 319

Chapter 11. Simple-Homotopy Theory 323
§11.1. Introduction 323
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>§11.2. Invertible matrices and $K_1(R)$</td>
<td>326</td>
</tr>
<tr>
<td>§11.3. Torsion for chain complexes</td>
<td>334</td>
</tr>
<tr>
<td>§11.4. Whitehead torsion for CW-complexes</td>
<td>343</td>
</tr>
<tr>
<td>§11.5. Reidemeister torsion</td>
<td>346</td>
</tr>
<tr>
<td>§11.6. Torsion and lens spaces</td>
<td>348</td>
</tr>
<tr>
<td>§11.7. The s-cobordism theorem</td>
<td>357</td>
</tr>
<tr>
<td>§11.8. Projects for Chapter 11</td>
<td>357</td>
</tr>
</tbody>
</table>

Bibliography                                                            359

Index                                                                  363
Preface

To paraphrase a comment in the introduction to a classic point-set topology text, this book might have been titled *What Every Young Topologist Should Know*. It grew from lecture notes we wrote while teaching second–year algebraic topology at Indiana University.

The amount of algebraic topology a student of topology must learn can be intimidating. Moreover, by their second year of graduate studies students must make the transition from understanding simple proofs line-by-line to understanding the overall structure of proofs of difficult theorems.

To help our students make this transition, the material in these notes is presented in an increasingly sophisticated manner. Moreover, we found success with the approach of having the students meet an extra session per week during which they took turns presenting proofs of substantial theorems and writing lecture notes to accompany their explanations. The responsibility of preparing and giving these lectures forced them to grapple with “the big picture” and also gave them the opportunity to learn how to give mathematical lectures, preparing for their participation in research seminars. We have collated a number of topics for the students to explore in these sessions; they are listed as projects in the table of contents and are enumerated below.

Our perspective in writing this book was to provide the topology graduate students at Indiana University (who tend to write theses in geometric topology) with the tools of algebraic topology they will need in their work, to give them a sufficient background to be able to interact with and appreciate the work of their homotopy theory cousins, and also to make sure that they are exposed to the critical advances in mathematics which came about
with the development of topology in the years 1950-1980. The topics discussed in varying detail include homological algebra, differential topology, algebraic K-theory, and homotopy theory. Familiarity with these topics is important not just for a topology student but any student of pure mathematics, including the student moving towards research in geometry, algebra, or analysis.

The prerequisites for a course based on this book include a working knowledge of basic point-set topology, the definition of CW-complexes, fundamental group/covering space theory, and the construction of singular homology including the Eilenberg-Steenrod axioms. In Chapter 8, familiarity with the basic results of differential topology is helpful. In addition, a command of basic algebra is required. The student should be familiar with the notions of $R$-modules for a commutative ring $R$ (in particular the definition of tensor products of two $R$-modules) as well as the structure theorem for modules over a principal ideal domain. Furthermore, in studying non simply-connected spaces it is necessary to work with tensor products over (in general non-commutative) group rings, so the student should know the definition of a right or left module over such a ring and their tensor products. Basic terminology from category theory is used (sometimes casually), such as category, functor, and natural transformation. For example, if a theorem asserts that some map is natural, the student should express this statement in categorical language.

In a standard first-year course in topology, students might also learn some basic homological algebra, including the universal coefficient theorem, the cellular chain complex of a CW-complex, and perhaps the ring structure on cohomology. We have included some of this material in Chapters 1, 2, and 3 to make the book more self-contained and because we will often have to refer to the results. Depending on the pace of a first-year course, a course based on this book could start with the material of Chapter 2 (Homological Algebra), Chapter 3 (Products), or Chapter 4 (Fiber Bundles).

Chapter 6 (Fibrations, Cofibrations and Homotopy Groups) and Chapter 9 (Spectral Sequences) form the core of the material; any second-year course should cover this material. Geometric topologists must understand how to work with non simply-connected spaces, and so Chapter 5 (Homology with Local Coefficients) is fundamental in this regard. The material in Chapters 7 (Obstruction Theory and Eilenberg-MacLane Spaces) and 8 (Bordism, Spectra, and Generalized Homology) introduces the student to the modern perspective in algebraic topology. In Chapter 10 (Further Applications of Spectral Sequences) many of the fruits of the hard labor that preceded this chapter are harvested. Chapter 11 (Simple-Homotopy theory) introduces the ideas which lead to the subject of algebraic K-theory and to the s-cobordism theorem. This material has taken a prominent role in
research in topology, and although we cover only a few of the topics in this area (\(K_1\), the Whitehead group, and Reidemeister torsion), it serves as good preparation for more advanced courses.

These notes are meant to be used in the classroom, freeing the student from copying everything from the chalkboard and hopefully leaving more time to think about the material. There are a number of exercises in the text; these are usually routine and are meant to be worked out when the student studies. In many cases, the exercises fill in a detail of a proof or provide a useful generalization of some result. Of course, this subject, like any subject in mathematics, cannot be learned without thinking through some exercises. Working out these exercises as the course progresses is one way to keep up with the material. The student should keep in mind that, perhaps in contrast to some areas in mathematics, topology is an example driven subject, and so working through examples is the best way to appreciate the value of a theorem.

We will omit giving a diagram of the interdependence of various chapters, or suggestions on which topics could be skipped, on the grounds that teachers of topology will have their own opinion based on their experience and the interests of the students. (In any case, every topic covered in this book is related in some way to every other topic.) We have attempted (and possibly even succeeded) to organize the material in such a way as to avoid the use of technical facts from one chapter to another, and hence to minimize the need to shuffle pages back and forth when reading the book. This is to maximize its usefulness as a textbook, as well as to ensure that the student with a command of the concepts presented can learn new material smoothly and the teacher can present the material in a less technical manner. Moreover, we have not taken the view of trying to present the most elementary approach to any topic, but rather we feel that the student is best served by learning the high-tech approach, since this ultimately is faster and more useful in research. For example, we do not shrink from using spectral sequences to prove basic theorems in algebraic topology.

Some standard references on the material covered in this course include the books [14], [36], [43], [9], [17] [31], and [7]. A large part of the material in these notes was distilled from these books. Moreover, one can find some of the material covered in much greater generality and detail in these tomes. Our intention is not to try to replace these wonderful books, but rather to offer a textbook to accompany a course in which this material is taught.

We recommend that students look at the article “Fifty years of homotopy theory” by G. Whitehead [44] for an overview of algebraic topology, and look
back over this article every few weeks as they are reading this book. The books a student should read after finishing this course (or in conjunction with this course) are Milnor and Stasheff, Characteristic Classes [30] (every mathematician should read this book), and Adams, Algebraic Topology: A Student’s Guide [1].

The authors would like to thank Eva-Marie Elliot and Mary Jane Wilcox for typing early versions of the manuscript. Special thanks are due to our colleagues Ayelet Lindenstrauss and Allan Edmonds for their careful proof-reading of our manuscripts; all remaining mistakes and typographical errors are entirely the authors’ fault. The second author would like to thank John Klein for teaching him algebraic topology while they were in graduate school. Special thanks to Marcia and Beth.

Projects

The following is a list of topics to be covered in the extra meetings and lectured on by the students. They do not always match the material of the corresponding chapter but are usually either related to the chapter material or preliminary to the next chapter. Sometimes they form interesting subjects which could reasonably be skipped. Some projects are quite involved (e.g. “state and prove the Hurewicz theorem”), and the students and instructor should confer to decide how deeply to cover each topic. In some cases (e.g. the Hopf degree theorem, the Hurewicz theorem, and the Freudenthal suspension theorem) proofs are given in later chapters using more advanced methods.

- **Chapter 1.**
  1. The cellular approximation theorem.
  2. Singular homology theory.

- **Chapter 2.**
  1. The acyclic model theorem and the Eilenberg-Zilber map.

- **Chapter 3.**
  1. Algebraic limits and the Poincaré duality theorem.
  2. Exercises on intersection forms.

- **Chapter 4.**
  1. Fiber bundles over paracompact bases are fibrations.
  2. Classifying spaces.

- **Chapter 5.**
  1. The Hopf degree theorem.
  2. Colimits and limits.
• **Chapter 6.**
  1. The Hurewicz theorem.
  2. The Freudenthal suspension theorem.

• **Chapter 7.**
  1. Postnikov systems.

• **Chapter 8.**
  1. Basic notions from differential topology.
  2. Definition of $K$-theory.

• **Chapter 9.**

• **Chapter 10.**
  1. Unstable homotopy theory.

• **Chapter 11.**
  1. Handlebody theory and torsion for manifolds.
Index

abstract nonsense, 43
action of $\pi_1(Y,y_0)$ on $[X,Y]_0$, 155–159
acyclic models theorem, 49
Adem relations, 288
adjoining a cell, 3
adjoint, 11, 111, 136, 138
adjoint theorem, 114
based, 136
Alexander duality, 71, 234
polynomial, 348
Alexander–Whitney map, 55, 64–67
algebraic mapping cone, 338
almost complex manifold, 214
aspherical space, 184
associated bundle, 84, 87
associated graded module, 240, 255
attaching map, 4
augmentation, 66, 352

base
of a fiber bundle, 78
based map, 134, 155
Betti number, 48
$BG$, 92, 150, 217
bigraded complex, 238
bilinear map, 9
block sum of matrices, 330
Bockstein operation, 75, 283, 293
bordism, 217, 222, 317
framed, 196
oriented, 225
stably framed, 209
Borel
conjecture, 185
construction, 86, 88, 219

Bott periodicity, 149, 152, 208, 233, 316
boundary, 15
Brown representation theorem, 232

canonical vector bundle, 149
cap product, 62, 67
Cartan formula, 288, 309
Cartan–Hadamard theorem, 185
Čech cochain, 80
cellular
approximation theorem, 22, 147, 166
map, 6
$C_f$, 132
chain complex
acyclic, 36, 334
based, 334
cellular, 4
elementary, 334
projective, 36, 334
simple, 334
of a simplicial complex, 7
singular, 1
chain contraction, 330
Chapman theorem, 345
characteristic
class, 93, 178, 191, 306, 316, 318
map, 3
Chern class, 316
$C$-homomorphism, 269
classifying space, 92, 150, 217, 306
clutching, 82, 150
cobordism, 217
coboundary, 15
cocycle, 15
cofibrant theorem, 186
cofibration, 111, 127, 129, 131, 132
sequence, 135, 141
co-$H$-group, 142
cohomology, 14
with coefficients in a module, 14
with coefficients in a spectrum, 212
compactly supported, 100
of a group, 185
with local coefficients, 98, 106
operation, 283
stable, 285
relative, 17
cohomotopy, 203
coinvariants, 107
colimit, 109
collapse, 325
commutative diagram, 29
commutator subgroup, 6, 328
compact–open topology, 113
compactly generated space, 112, 114
products, 113
cone
reduced, 137
Cone($C$), 338
covering transformations, 97
$CP^n$, 147
cross product, 53, 56
cohomology, 57, 59
homology, 56
cross section problem, 166, 189
cup product, 57, 59, 60, 67, 257
relative, 69, 70
CW-approximation, 186
CW-complex, 4, 109
CW-pair, 324
$CX$, 137
cycle, 15
Cyl($C$), 338
degree of a map, 5, 44
delooping $G$, 218
DeRham cohomology, 16, 49
derivation, 257, 289
differential form, 172
diagonal approximation, 59, 62
difference cochain, 172, 176, 180
differential forms, 15, 16, 88
disjoint union, 109, 130
divisible group, 32
DR-pair, 129
$EG$, 92, 150, 217
EHP sequence, 320
Eilenberg-MacLane
space, 148, 168, 177–184, 192, 210, 284, 286
fundamental class of, 179
spectrum, 211, 232, 288
Eilenberg-Steenrod
axioms, 19, 228, 230
uniqueness theorem, 231
Eilenberg-Zilber
map, 55, 59, 61, 64
theorem, 49, 54
elementary
collapse, 325
matrix, 328
Euler
characteristic, 251
class, 191, 311
number, 82
exact functor, 23
exact sequence
of groups, 143, 184
of sets, 134
exact triangle, 44
excisive pair, 68
expansion, 325
extension problem, 165
$Ext^n_k$, 25, 42
fiber
of a fiber bundle, 78
of a fibration, 116, 119, 138
fiber bundle, 77–85, 92, 115, 145
changing the fiber, 85
morphism, 90
structure group, 78
transition function, 78
fiber homotopy, 123
equivalence, 123, 124
fibration, 92, 111, 115, 116, 145
sequence, 134, 141, 184
filtration
of an $R$-module, 239, 255
flat module, 36
frame bundle, 87
of a manifold, 88
framing, 196, 213
normal, 197
stable, 205
twisting, 197
of a vector bundle, 197
freely homotopic maps, 155
Freudenthal suspension theorem, 152, 163, 203, 261, 281
fundamental group, 6
$G$-bordism group, 216
Gleason theorem, 85, 145
graded
ring, 52
$R$-module, 44, 51
grassmannian, 148, 149
complex, 151
Index

<table>
<thead>
<tr>
<th>Term</th>
<th>Page Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>group action, effective</td>
<td>78</td>
</tr>
<tr>
<td>group action, free</td>
<td>77</td>
</tr>
<tr>
<td>group ring, trivial units in</td>
<td>331</td>
</tr>
<tr>
<td>Gysin sequence</td>
<td>253</td>
</tr>
<tr>
<td>half-smash</td>
<td>209</td>
</tr>
<tr>
<td>handlebody theory</td>
<td>358</td>
</tr>
<tr>
<td>$H$-group</td>
<td>142</td>
</tr>
<tr>
<td>homogenous space</td>
<td>148</td>
</tr>
<tr>
<td>homology with coefficients in a module</td>
<td>13</td>
</tr>
<tr>
<td>homology with coefficients in a spectrum</td>
<td>212</td>
</tr>
<tr>
<td>homology with local coefficients</td>
<td>98, 102, 104, 105</td>
</tr>
<tr>
<td>homology theory</td>
<td>233</td>
</tr>
<tr>
<td>connective</td>
<td>233</td>
</tr>
<tr>
<td>homotopy cofiber</td>
<td>138</td>
</tr>
<tr>
<td>homotopy extension property</td>
<td>127</td>
</tr>
<tr>
<td>homotopy fiber</td>
<td>138</td>
</tr>
<tr>
<td>homotopy group</td>
<td>111, 143</td>
</tr>
<tr>
<td>homotopy lifting property</td>
<td>115</td>
</tr>
<tr>
<td>homotopy problem</td>
<td>165</td>
</tr>
<tr>
<td>Hopf bundle</td>
<td>82</td>
</tr>
<tr>
<td>degree theorem</td>
<td>108, 147, 152, 161, 200</td>
</tr>
<tr>
<td>invariant</td>
<td>320</td>
</tr>
<tr>
<td>horseshoe lemma</td>
<td>39</td>
</tr>
<tr>
<td>$\mathbb{H}P^n$</td>
<td>148</td>
</tr>
<tr>
<td>Hurewicz</td>
<td></td>
</tr>
<tr>
<td>fibration theorem</td>
<td>92, 115</td>
</tr>
<tr>
<td>map, relative</td>
<td>160, 232</td>
</tr>
<tr>
<td>relative theorem</td>
<td>160</td>
</tr>
<tr>
<td>theorem, relative</td>
<td>6, 102, 152, 161, 163, 186, 204, 269</td>
</tr>
<tr>
<td>injective module</td>
<td>32</td>
</tr>
<tr>
<td>intersection</td>
<td></td>
</tr>
<tr>
<td>form, definite</td>
<td>72</td>
</tr>
<tr>
<td>definite</td>
<td>73</td>
</tr>
<tr>
<td>pairing</td>
<td>72</td>
</tr>
<tr>
<td>invariants</td>
<td>108</td>
</tr>
<tr>
<td>J-homomorphism</td>
<td>201, 204</td>
</tr>
<tr>
<td>stable</td>
<td>204, 208</td>
</tr>
<tr>
<td>$k$-invariant</td>
<td>144, 193, 314</td>
</tr>
<tr>
<td>$K_1$</td>
<td>329</td>
</tr>
<tr>
<td>$K(\pi, n)$-space</td>
<td>168, see also Eilenberg–MacLane space</td>
</tr>
<tr>
<td>Kronecker pairing</td>
<td>15, 43, 46, 48, 62, 67, 179, 301</td>
</tr>
<tr>
<td>$K_*$</td>
<td>136</td>
</tr>
<tr>
<td>$K$-theory</td>
<td>31, 233, 234</td>
</tr>
<tr>
<td>$\ell$</td>
<td></td>
</tr>
<tr>
<td>coefficients</td>
<td>212, 229, 231</td>
</tr>
<tr>
<td>reduced</td>
<td>228</td>
</tr>
<tr>
<td>unreduced</td>
<td>231</td>
</tr>
<tr>
<td>ordinary</td>
<td>19, 229, 231</td>
</tr>
<tr>
<td>homotopy cofiber</td>
<td>138</td>
</tr>
<tr>
<td>homotopy extension property</td>
<td>127</td>
</tr>
<tr>
<td>homotopy fiber</td>
<td>138</td>
</tr>
<tr>
<td>homotopy group</td>
<td>111, 143</td>
</tr>
<tr>
<td>long exact sequence of a fibration</td>
<td>144</td>
</tr>
<tr>
<td>relative</td>
<td>152</td>
</tr>
<tr>
<td>long exact sequence of spheres</td>
<td>276–279, 294, see also $\pi_n S^n$</td>
</tr>
<tr>
<td>stable</td>
<td>203, 211</td>
</tr>
<tr>
<td>homotopy lifting property</td>
<td>115</td>
</tr>
<tr>
<td>homotopy problem</td>
<td>165</td>
</tr>
<tr>
<td>Hopf bundle</td>
<td>82</td>
</tr>
<tr>
<td>degree theorem</td>
<td>108, 147, 152, 161, 200</td>
</tr>
<tr>
<td>invariant</td>
<td>320</td>
</tr>
<tr>
<td>mapping</td>
<td></td>
</tr>
<tr>
<td>cone, algebraic</td>
<td>132</td>
</tr>
<tr>
<td>cylinder, algebraic</td>
<td>338</td>
</tr>
<tr>
<td>mapping path</td>
<td></td>
</tr>
<tr>
<td>fibration</td>
<td>124</td>
</tr>
<tr>
<td>space, Mayer–Vietoris sequence</td>
<td>69</td>
</tr>
<tr>
<td>$M_f$</td>
<td>131</td>
</tr>
<tr>
<td>Möbius strip</td>
<td>81</td>
</tr>
<tr>
<td>$n$-connected space</td>
<td>161</td>
</tr>
<tr>
<td>NDR–pair</td>
<td>128, 129</td>
</tr>
<tr>
<td>non-degenerate base point</td>
<td>136</td>
</tr>
<tr>
<td>normal bundle</td>
<td>195</td>
</tr>
<tr>
<td>$n$-simple</td>
<td></td>
</tr>
<tr>
<td>pair, 159</td>
<td></td>
</tr>
<tr>
<td>space, 158, 168</td>
<td></td>
</tr>
<tr>
<td>$n$-skeleton</td>
<td>4</td>
</tr>
<tr>
<td>obstruction cocycle</td>
<td>169–171, 190</td>
</tr>
<tr>
<td>obstruction theory</td>
<td>165–183</td>
</tr>
<tr>
<td>$\Omega^G_1(X)$</td>
<td>216</td>
</tr>
<tr>
<td>$\Omega_1 Y$</td>
<td>121</td>
</tr>
</tbody>
</table>
orbit space, 77
orientable
  manifold, 90, 101, 299
  vector bundle, 90
orientation, 214
  character, 101
  double cover, 101
  of a manifold, 299–306
  sheaf, 102, 105
  of a vector bundle, 296
path space, 120
  free, 121
path space fibration, 120
$P_f$, 124
PID, 15
$\pi_*^{S^k}, 204, 207$
$\pi_*^{S^m}, 147, 197, 201, 204, 259, 261, 276–279, 294$
$\pi_*(X, x_0), 6, 143, see also homotopy group$
Poincaré duality, 71, 101, 102, 226
Poincaré–Hopf theorem, 192
Poincaré–Lefschetz duality, 71, 102
Pontrjagin–Thom construction, 196, 204, 210, 222, 227
Postnikov system, 178, 192, 314
$p$-primary subgroup, 268, 277
primary obstruction
  to constructing a homotopy, 177
  to extending a map, 176
  to lifting, 190
principal bundle, 84–86
projective module, 29
projective space, 147, 148
pullback construction, 91, 109, 116
Puppe sequence, 143
pushout construction, 109, 130–132
$P_\gamma Y$, 120
quotient map, 114
rational homotopy, 314
rationalization, 315
Reidemeister torsion, 346
representation, 96, 107
resolution
  projective or injective, 33
$s$, 279
$s$-cobordism theorem, 323, 357
Serre
  class of abelian groups, 267
  exact sequence, 251
Shapiro’s lemma, 100
signature
  of a manifold, 74
  of a symmetric form, 73
simple space, 158
simple-homotopy equivalence, 324, 325, 343
simple-homotopy type, 345
simplicial complex, 6
  geometric realization of, 7
slant product, 63
smash product, 136
Spanier–Whitehead duality, 212, 217, 234
spectral sequence, 237
  Atiyah–Hirzebruch, 228, 229
  cohomology, 254
  collapse of a, 241
  convergence of, 240, 242, 255
  first quadrant, 241
  homology, 240
  Leray–Serre, 239, 257
  Leray–Serre–Atiyah–Hirzebruch, 242, 255, 266
  relative, 266
spectrum, 168, 211, 221
  coefficients of, 213, 225
  $\Omega$-spectrum, 233
  sphere, 211, 232
spin structure, 214
splicing lemma, 34
split surjection or injection, 17, 30
stable $G$-structure, 215
stable $k$-stem, 204, 207
stably free module, 327
Steenrod
  algebra, 287
  squares, 288
Stiefel manifold, 148, 149
  complex, 151
Stiefel–Whitney
  class, 101, 192, 306–311, 318
  number, 317, 318
structure group, 78, 83
  reducing, 89
suspension, 202, 228, 279
  of a chain complex, 338
  of a cohomology operation, 284
  of a framed manifold, 197, 202
  Freudenthal theorem, 152, 163, 203, 261, 281
  reduced, 137
  unreduced, 82
$SX$, 137
tangent bundle, 81
$\tau(C)$, 335, 339, see also Whitehead torsion
$\tau(f)$, 334, see also Whitehead torsion
tensor product, 8, 13, 97
  of chain complexes, 52
$\vartheta^{n+1}(g)$, 169
Thom
  class, 297
isomorphism theorem, 226, 296
space, 220, 295
spectrum, 217, 219, 221, 227, 232
topological group, 77
Tor^R_*, 24, 42
total space
  of a fiber bundle, 78
trace of a homotopy, 199
transgression, 279, 289
triangulation of a space, 7
trivial bundle, 81
tubular neighborhood, 195, 198
twisted
cohomology, 98
homology, 98
universal coefficient theorem, 44, 46–48
vector bundle, 81, 83
  complex, 81, 214
  stable equivalence of, 207
vertical homotopy, 126
Wang sequence, 254
weak homotopy equivalence, 162, 185
  of a chain complex, 339
wedge product, 136
Whitehead
group, 323, 331
  lemma, 328
  theorem, 162, 163, 188, 270, 323
  torsion, 334, 335, 339, 343
Wh(\pi), 331, see also Whitehead group

[X, Y], 134, 155
[X, Y]_0, 134, 155

Y^{-t}, 121

zig-zag lemma, 18