# 0

# *Seeing Arithmetic*

The original numbers are counting numbers. We can use counting numbers to count dots: *two* red dots , *five* blue dots . Counting numbers are answers to questions that begin “How many...?”, asked about things we can see or touch or hear. Mathematicians use the term **natural numbers** to refer to the counting numbers. The natural numbers are zero[1], one, two, three, four, etc..

[1] Is zero a natural number? Some mathematicians include zero in the set of natural numbers, and some do not. It is a matter of taste. In this text, we declare: **0 is a natural number.**

Rearranging dots does not change their number. Here are three red dots and here are three red dots. “Threeness” persists.

If we use numbers to count dots, then we can see equality of numbers by pairing off the dots. Here are some myred dots and here are some blue dots. It is hard, at first glance, to see that there are the same number of red dots as blue dots. But arrange the dots in straight lines, each red dot paired with a blue dot, one over the other, , and see that they have the same number.

Visualizing equality by “pairing off” is the heart of counting. In some time past (long before 3000BCE), a shepherd paired sheep and pebbles: as the first sheep walked by, he placed down the first pebble; the second sheep walked by and down went the second pebble. At the end of the day, the number of pebbles placed down equaled the number of sheep that walked past. Whether sheep and pebbles, or red dots and blue dots, equality of numbers is defined by pairing off[2] the items of two sets.

[2] Such a pairing off is called a **bijection** by mathematicians, a term coined by Nicolas Bourbaki (the pseudonym of a tightly-knit group of mathematicians most active in the mid-twentieth century).

To count things, arrange them in a straight line above the positive numbers. For example, to count the primes between 1 and 20, arrange them:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Primes: | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 |
| | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th | 8th |

There are 8 primes between 1 and 20.

COUNTING will play a central role in this text. Not counting sheep or pebbles, but counting numbers of various flavors. The following examples illustrate how pairing off can help one count with confidence.

**Problem 0.1** How many even numbers are between 1 and 100?

SOLUTION: Arrange the even numbers in a line above the positive numbers.

$$\begin{array}{cccccc} 2 & 4 & 6 & 8 & \cdots & 100 \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & 4^{\text{th}} & \cdots & ?^{\text{th}} \end{array}$$

The top row includes the even numbers between 1 and 100. The bottom row consists of positive numbers, in effect counting the top row. The unknown number ? is the answer to the problem.

Filling in the dots would be tiresome. Instead, observe that each number in the bottom row is precisely half its partner in the top row. 1 is half of 2. 2 is half of 4. 3 is half of 6. Et cetera. The unknown number ? is half of 100 – it is 50. There are 50 even numbers between 1 and 100. ✓

$$\begin{array}{cccccc} 2 & 4 & 6 & 8 & \cdots & 100 \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & 4^{\text{th}} & \cdots & 50^{\text{th}} \end{array}$$ Divide by two.

**Problem 0.2** How many natural numbers are between[3] 37 and 185?

SOLUTION: Arrange the numbers between 37 and 185 in a line above the positive numbers:

$$\begin{array}{cccccc} 37 & 38 & 39 & 40 & \cdots & 185 \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & 4^{\text{th}} & \cdots & ?^{\text{th}} \end{array}$$

Each number in the bottom row is 36 less than its partner in the top row. Therefore, $? = 185 - 36 = 149$. There are 149 natural numbers between 37 and 185. ✓

$$\begin{array}{cccccc} 37 & 38 & 39 & 40 & \cdots & 185 \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & 4^{\text{th}} & \cdots & 149^{\text{th}} \end{array}$$ Subtract 36.

[3] When we write "between," we always mean this in the *inclusive* sense. So "between 37 and 185" means between *and including* 37 and 185.

Many students approach the problem by simply subtracting $185 - 37 = 148$. But 148 is not the correct answer. This exemplifies a common "off-by-one" error that occurs when counting within an interval. Some students are taught to "always add one," to correct this off-by-one error, but that rule does not apply in other circumstances.

It is better to line up the numbers, and never make such mistakes.

**Problem 0.3** How many multiples of three are between 1 and 1000?

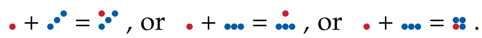
SOLUTION: Arrange the multiples of three in a line above the positive numbers:

$$\begin{array}{cccccc} 3 & 6 & 9 & 12 & \cdots & 999 \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & 4^{\text{th}} & \cdots & ?^{\text{th}} \end{array}$$ Divide by three.

Notice that 999 is the last multiple of three between 1 and 1000. Each number in the bottom row is one-third of its partner in the top row. Therefore, $? = 999 \div 3 = 333$. There are 333 multiples of three between 1 and 1000. ✓

ADDITION is the numerical complement to aggregation. One *aggregates* two collections of dots, by putting them together. Aggregating two red dots with five blue dots yields seven dots. If we care only about the numbers, then we would just write $2 + 5 = 7$.

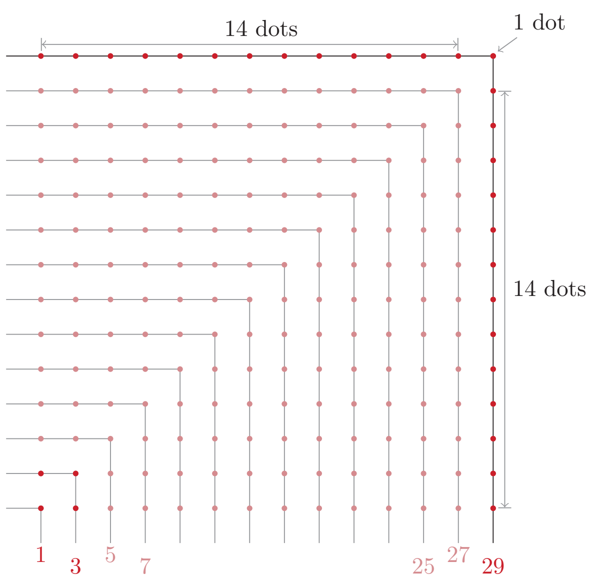
While addition of two numbers has only one correct result, aggregation of two collections has many correct results – there are many ways of arranging the resulting collection. For example the numerical expression $1 + 3 = 4$ might be realized in any of the following ways:




You might say "who cares," since, in the end, there are still four dots (three blue and one red). But the creative arrangement of dots allows one to solve otherwise difficult addition problems.

**Problem 0.4** Add the odd numbers between 1 and 30. In other words, what is $1 + 3 + 5 + \cdots + 25 + 27 + 29$?

SOLUTION: An odd number of dots can be arranged into a ⌉-shaped configuration. Stacking these configurations of dots yields a square configuration.

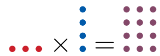



The entire figure contains $1 + 3 + 5 + \cdots + 25 + 27 + 29$ dots, arranged in a pattern of "stacking corners".[4] The square arrangement is 15 dots wide and 15 dots tall, having a total of $15 \times 15 = 225$ dots. Hence $1 + 3 + 5 + \cdots + 25 + 27 + 29 = 225$. ✓

[4] The corner-configuration is the *gnomon* of the square, to use the Pythagorean language.

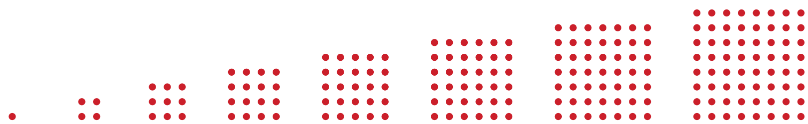
This method of stacking corners allows one to add the odd numbers between 1 and every natural number.

Multiplication of two numbers can be represented by the arrangement of dots into a rectangular array. The numerical expression $3 \times 4 = 12$ can be realized by the following figure:

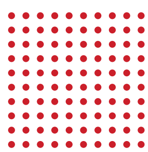


**Square numbers**[5] are those obtained by multiplying a natural number by itself. Below are eight square numbers.

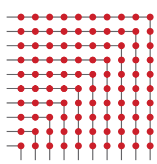
[5] Square numbers are sometimes called "perfect squares". But the word "perfect" has another meaning in number theory, and adds little more than emphasis, so we avoid it.



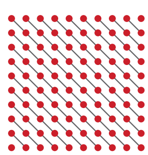

For every partition of the square, there is a corresponding addition fact. Some of these facts are otherwise difficult to find. Consider the following partitions of the square number 100.



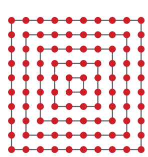
$$100 = 10 \times 10.$$



$$100 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19.$$



$$100 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1.$$



$$100 = 4 + 12 + 20 + 28 + 36$$

Figure 1: Four partitions of 100 dots

We challenge the reader to find other interesting partitions of the square, and corresponding addition facts. In the other direction, when the reader encounters a difficult problem of repeated addition, we recommend searching for a geometric solution – a partition of a square or rectangle.

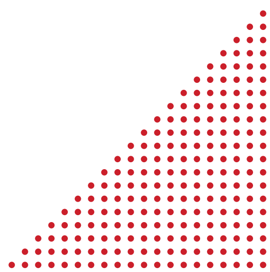
Triangular arrangements of dots can be duplicated to form rectangular arrangements. This principle plays a role in the solution of some addition problems below.

**Triangular numbers** are numbers obtained by summing the first $n$ numbers for some nonzero natural number $n$. The first five triangular numbers are below.

| | | |
|---|---|---|
| 1 | = | 1 |
| 1 + 2 | = | 3 |
| 1 + 2 + 3 | = | 6 |
| 1 + 2 + 3 + 4 | = | 10 |
| 1 + 2 + 3 + 4 + 5 | = | 15 |

**Problem 0.5** What is $1 + 2 + 3 + \cdots + 19 + 20$?

Solution: Assemble dots in a right triangle, with one dot in the first row, two dots in the second row, etc., with 20 dots in the $20^{\text{th}}$ row.



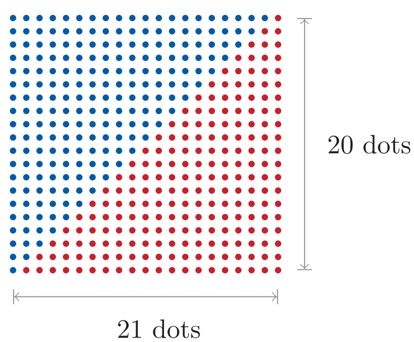
The sum $1 + 2 + 3 + \cdots + 19 + 20$ is the number of dots in this triangular figure. Placing this figure adjacent to a duplicate yields a rectangular[6] figure.

[6] The figure is very slightly but certainly *not* a square. This exhibits a subtle and crucial difference between number theory and geometry. In geometry, one might see the triangle of base 20 and height 20, and compute its area as

$$\mathit{Area} = \frac{1}{2}(20 \times 20) = 200.$$

But when counting dots, the area does not suffice. There are

$$\frac{1}{2}(21 \times 20) = 210 \text{ dots.}$$




There are $21 \times 20 = 420$ dots in the rectangular figure. Half of the dots are red and half of the dots are blue, so there are $420 \div 2 = 210$ red dots. Hence

$$1 + 2 + 3 + \cdots + 19 + 20 = 210.$$

✓

This method of duplicating a triangular arrangement generalizes.

**Proposition 0.6 (Summation up to $N$)** *For every positive integer $N$,*

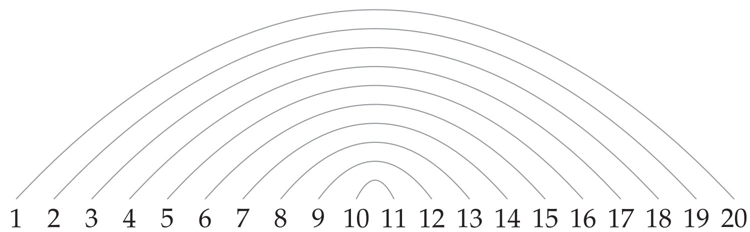
$$\sum_{i=1}^{N} i = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}.$$

For every nonzero natural number $N$, the sum $1 + 2 + 3 + \cdots + N$ equals the number of dots in a right triangle of height and width $N$. Duplicating this triangle yields a rectangle of height $N$ and width $N + 1$. There are $N \times (N+1)$ dots in this rectangle. There are half as many dots in the triangle, whence the formula $N \times (N+1) \div 2$ for the triangular number.

Partnering is a method of considering a sequence in pairs (allowing a few lonely numbers on occasion). Like duplication, partnering can be used to add numbers in arithmetic progression.

**Problem 0.7 (Revisited)** What is $1+2+3+\cdots+19+20$?

Solution: Partner 1 with 20, and partner 2 with 19, and 3 with 18, et cetera. One can see the partnerships in the following diagram.




Notice that $1+20=21$, and $2+19=21$, and $3+18=21$; each number together with its partner sums to 21. The 20 numbers become 10 pairs of numbers, and each pair sums to 21. Hence

$$1+2+3+\cdots+18+19+20 = 10(21) = 210.$$

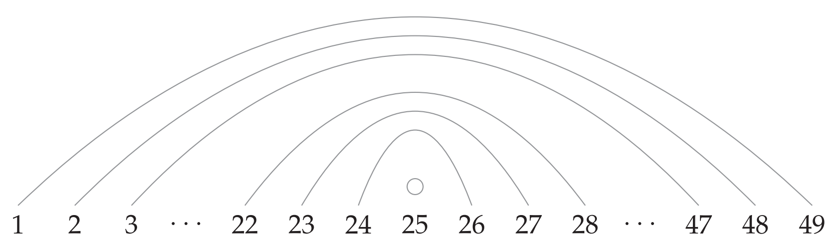
For more steps, we may write

$$\begin{aligned} & 1+2+3+\cdots+18+19+20 \\ = \; & (1+20)+\cdots+(10+11) \\ = \; & (21)+(21)+\cdots+(21) \\ = \; & 10(21). \end{aligned}$$

If we impose partnerships on an odd number of things, one thing is always left alone.

**Problem 0.8** What is $1+2+3+\cdots+47+48+49$?

Solution: Partner 1 with 49, and partner 2 with 48 and 3 with 47, et cetera. Each number, with its partner, sums to 50.




Notice that 24 is partnered with 26 (since they sum to 50). But 25 is left without a partner. In the end, we find 24 pairs – each pair summing to 50 – and 25 left over by itself. Hence

$$1+2+3+\cdots+47+48+49 = 24(50)+25 = 1200+25 = 1225.$$

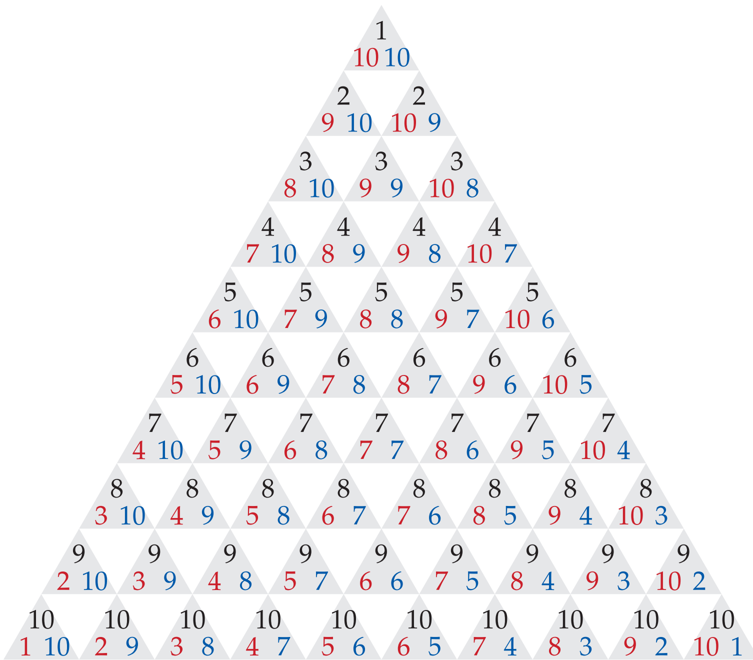
Partnering is an effective method of adding arithmetic progressions. While it may seem redundant after the geometric method of duplication, we will find scenarios later in which partnering is crucial and duplication ineffective.

Beyond partnering and duplicating, one can try to add numbers by putting them into groups of three or more. The problem below applies the rare technique of *triplication*.

**Problem 0.9** Add the first ten square numbers. In other words, what is $1 + 4 + 9 + \cdots + 64 + 81 + 100$?

Solution: We may view each square number as the result of repeated addition: $1 = 1$; $4 = 2 + 2$; $9 = 3 + 3 + 3$; et cetera. So, in this way, the sum of the first ten square numbers equals the sum of the numbers in the triangle in the margin.

*Triplicating* this triangle – overlaying it with itself, but rotated by 120° and 240° – yields the diagram below.




1
2 2
3 3 3
4 4 4 4
5 5 5 5 5
6 6 6 6 6 6
7 7 7 7 7 7 7
8 8 8 8 8 8 8 8
9 9 9 9 9 9 9 9 9
10 10 10 10 10 10 10 10 10 10

Figure 2: The sum of the numbers in this triangle equals $1 + 4 + 9 + \cdots + 64 + 81 + 100$.

Add only on the black numbers, and you find the answer to the problem. Add only the red numbers, and again you find the answer to the problem. Add only the blue numbers, and again you find the answer to the problem. So, adding all the numbers yields *three times* the answer to the problem.

But each small triangle, consisting of a black, red, and blue number, sums to 21. There are $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$ small triangles. Hence the sum of *all numbers*, black, red, and blue, equals $55 \times 21 = 1155$. The black numbers sum to $1155 \div 3 = 385$. ✓

Why does each small triangles sum to 21? The top triangle certainly sums to 21 since $10 + 10 + 1 = 21$. Traveling southwest, the blue number in the triangle stays the same, the black increases, and the red decreases, so their sum is unchanged. Traveling southeast, the red number in the triangle stays the same, the black increases, and the blue decreases, so their sum is unchanged. Hence the sum, 21, is unchanged as one travels from triangle to triangle within the large figure.

Counting pairs requires some cleverness, but most importantly a clear definition. Consider the set

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

How many pairs are there, taken from this set? It depends on how one interprets the word "pair"?

An **ordered pair** from $S$ is a pair written $(a, b)$ in which $a \in S$ and $b \in S$. Ordered pairs from $S$ include $(1, 10)$ and $(2, 8)$ and $(8, 2)$ and $(6, 6)$. The adjective "ordered" means that we distinguish $(2, 8)$ from $(8, 2)$.

**Problem 0.10** How many ordered pairs are there from the set $S$?

Solution: Each dot in the 10 by 10 square may be identified by coordinates of the form $(a, b)$ with $a \in S$ and $b \in S$. There are the same number of dots as there are ordered pairs. Hence there are $10 \times 10 = 100$ ordered pairs from the set $S$. ✓

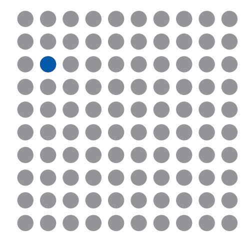


Figure 3: The blue dot has coordinates $(2, 8)$; it is in the 2nd column from the left and 8th row from the bottom. To each dot there corresponds an ordered pair from $S$.

In general, if $S$ is a set with $N$ elements, there are $N \times N$ ordered pairs from $S$.

A **subset** of $S$ is an unordered collection of elements of $S$; an element may appear in a subset once or not at all. For example, a two-element subset of $S$ is $\{3, 5\}$. A four-element subset of $S$ is $\{7, 1, 3, 10\}$. A zero-element subset of $S$ is the empty set, denoted $\emptyset$. Order does not matter: $\{1, 5, 6\} = \{6, 1, 5\} = \{6, 5, 1\}$; they are the same subset. Repetition is not allowed: one would never write $\{2, 2\}$.

**Problem 0.11** How many two-element subsets are there in the set $S$?

Solution: Choose a two-element subset $\{a, b\}$ of $S$, so $a \neq b$. There are two ways to order the subset: as $(a, b)$ or as $(b, a)$. The resulting ordered pairs correspond to dots on the square, excluding the diagonal (since $a$ cannot equal $b$).

The number of such ordered pairs is $(10 \times 10) - 10 = 90$, the number of dots in the square, excluding the diagonal. There are twice as many such ordered pairs as two-element subsets; the ordered pairs $(2, 8)$ and $(8, 2)$ correspond to the same two-element subset $\{2, 8\}$. Hence there are $90 \div 2 = 45$ two-element subsets of $S$. ✓

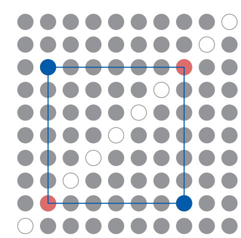


Figure 4: The blue dots have coordinates $(2, 8)$ and $(8, 2)$. These arise from two ways of ordering the same subset $\{2, 8\} = \{8, 2\}$. The red dots are in the 2nd and 8th positions along the diagonal.

The following theorem generalizes the above result.

**Proposition 0.12 (Counting pairs)** *Let $S$ be a set with $N$ elements. The number of ordered pairs from $S$ is $N^2$. The number of two-element subsets[7] of $S$ is equal to*

$$\frac{1}{2}(N \times N - N) = \frac{N^2 - N}{2} = \frac{N(N-1)}{2}.$$

[7] The number of two-element subsets is called "$N$ **choose** 2" and written $\binom{N}{2}$. Thus we find

$$\binom{N}{2} = \frac{N(N-1)}{2}.$$

Two questions have the answer 45:

1. What is $1+2+3+4+5+6+7+8+9$?
2. What is $\binom{10}{2}$, i.e., how many 2-element subsets are there of $\{1,2,3,4,5,6,7,8,9,10\}$?

Why? The figure below connects these questions.[8]

[8] Similar figures have been published. For example, see Roger B. Nelsen, "Visual Gems of Number Theory," from *Math Horizons*, Feb. 2008, p. 7. I had the benefit of learning about this visualization from Juan Felipe Pérez in the Summer of 2011.

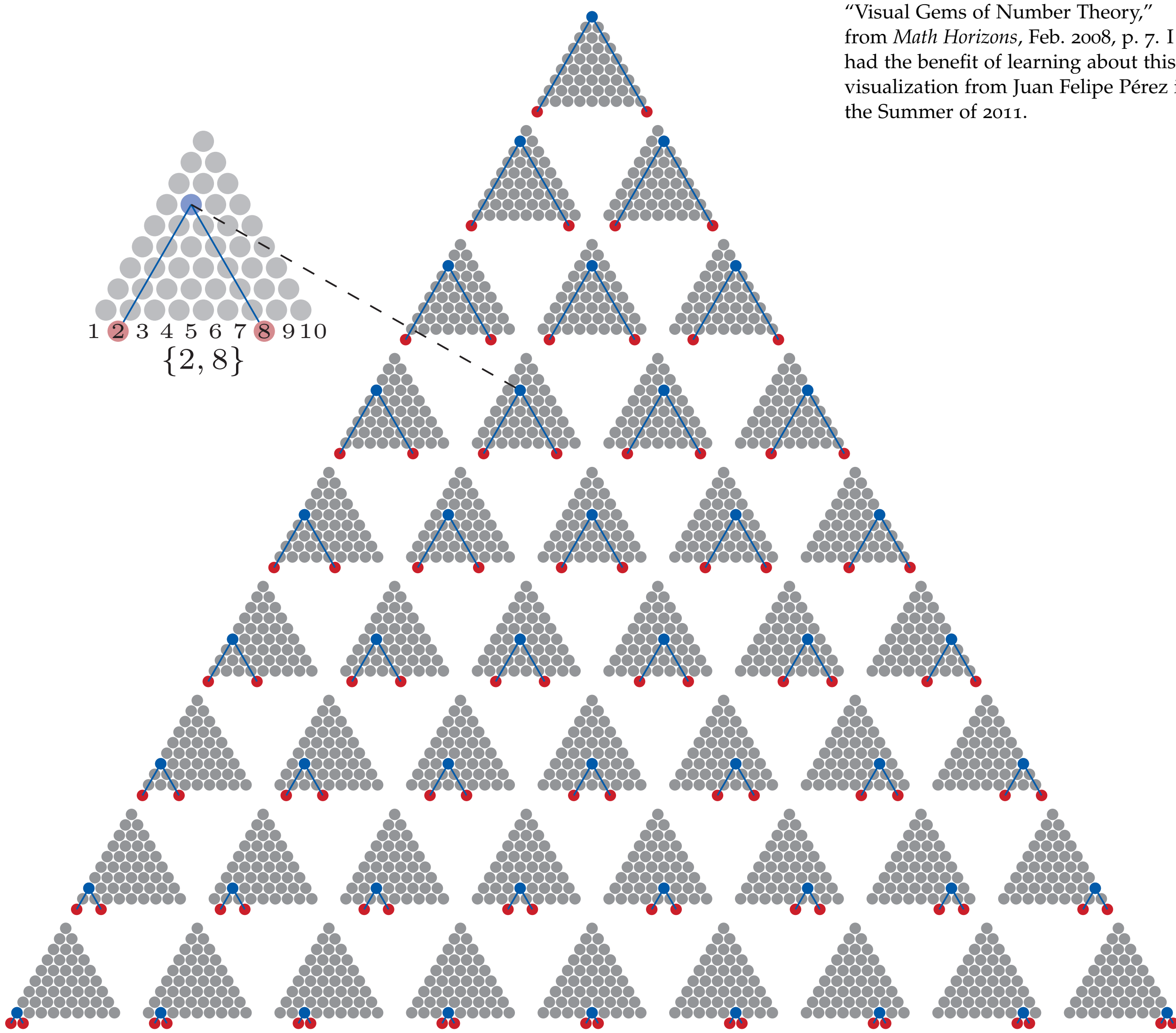



Figure 5: Visualizing 10 choose 2 as the sum of numbers from 1 to 9.

There are $1+2+3+\cdots+8+9$ locations in the gray triangle for a blue dot. To each blue dot, there corresponds a pair of red dots, whose locations are form a two-element subset of $\{1,2,3,4,5,6,7,8,9,10\}$.

**Proposition 0.13 (Summation up to $N-1$)** *If $N \geq 2$ then*

$$1+2+3+\cdots+(N-1) = \binom{N}{2}.$$

ROUNDING is not usually counted among the fundamental operations. We expect rounding to introduce imprecision, to sacrifice accuracy for convenience. But in number theory, the opposite can be true – an imprecise approximate result can be made correct by rounding.

Since one may round down or up, we use two words and notations. The result of rounding down is called the **floor** and the result of rounding up is called the **ceiling**. The floor of 3.2 is 3. The ceiling of 17.5 is 18. Symbolically, we write this[9]

$$\lfloor 3.2 \rfloor = 3, \quad \lceil 17.5 \rceil = 18.$$

The floor is the nearest integer[10] less than or equal to the number.

$$\lfloor \pi \rfloor = 3, \quad \lfloor -7.2 \rfloor = -8, \quad \lfloor 5 \rfloor = 5.$$

The ceiling is the nearest integer greater than or equal to the number.

$$\lceil \pi \rceil = 4, \quad \lceil -7.2 \rceil = -7, \quad \lceil 5 \rceil = 5.$$

The floor equals the ceiling precisely when the number is an integer.

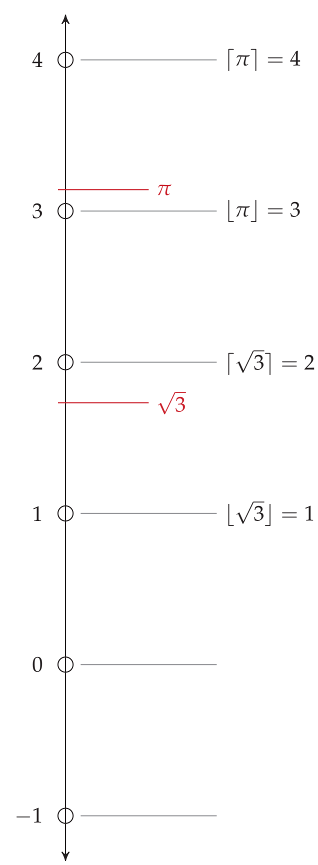
$$\lfloor 5 \rfloor = \lceil 5 \rceil = 5.$$

[9] This wonderful notation for floor and ceiling was introduced by Kenneth Iverson, in his book "A Programming Language" (John Wiley and Sons, 1962).

[10] The **integers** are the whole numbers, positive and negative and zero. The bold letter $\mathbb{Z}$ is used to denote the set of integers.

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

The letter $\mathbb{Z}$ comes from the German word "Zahl" meaning number.




**Proposition 0.14 (Counting multiples)** *Let $N$ and $m$ be positive integers. The number of multiples of $m$ between $1$ and $N$ is equal to $\lfloor N/m \rfloor$.*

PROOF: Arrange the multiples of $m$ on the top row, and positive integers on the bottom.

$$\begin{array}{cccccc} m & 2m & 3m & \cdots & \lfloor N/m \rfloor m & N \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & \cdots & \lfloor N/m \rfloor^{\text{th}} & N/m \end{array} \quad \text{Divide by } m.$$

The quotient $N/m$ is not necessarily a natural number; the natural number just beneath it is the floor, $\lfloor N/m \rfloor$. And that is the number of multiples of $m$ between $1$ and $N$. ∎

Underappreciated, the floor is most convenient for counting.

**Proposition 0.15 (Counting squares)** *Let $N$ be a positive integer. The number of squares between $1$ and $N$ is equal to $\left\lfloor \sqrt{N} \right\rfloor$.*

PROOF: Arrange the squares on the top row, and positive integers on the bottom.

$$\begin{array}{cccccc} 1 & 4 & 9 & \cdots & \left\lfloor \sqrt{N} \right\rfloor^2 & N \\ 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & \cdots & \left\lfloor \sqrt{N} \right\rfloor^{\text{th}} & \sqrt{N} \end{array} \quad \text{Square root.}$$

The square root $\sqrt{N}$ is not necessarily a natural number; the natural number just beneath it is the floor, $\left\lfloor \sqrt{N} \right\rfloor$. And that is the number of squares between $1$ and $N$. ∎

Representing numbers in decimal is second nature. We learn the base-ten positional system for representing natural numbers, and eventually for all real numbers. According to this decimal system, an expression such as 8039 consists of a units digit (9), tens digit (3), hundreds digit (0), and thousands digit (8). The digits belong to the set $\{0,1,2,3,4,5,6,7,8,9\}$

The expression 8093 is a shorthand for a sum,

$$8093 = (3 \times 10^0) + (9 \times 10^1) + (0 \times 10^2) + (8 \times 10^3).$$

Nestled between $10^3$ and $10^4$; the number 8093 has 4 digits.

| | | | | | | |
|---|---|---|---|---|---|---|
| Number: | 1 | 10 | 100 | 1000 | 8093 | 10000 |
| Digits: | 1 | 2 | 3 | 4 | 4 | 5 |

?

The number of decimal digits in $10^k$ is $k+1$. More generally,

**Proposition 0.16 (Counting digits)** *If $N$ is a positive integer, then the number of digits in the decimal representation of $N$ is $\lfloor \log_{10}(N) \rfloor + 1$.*

Binary representations are more natural than decimal expansions, if we care more about numbers than about how humans[11] represent numbers. In the binary (base-two) system, the only digits are the binary digits (**bits**) 0 and 1.

[11] Humans use a base-ten system as most of us have ten fingers.

To avoid confusion, we use a bold **b** to precede a binary representation. For example, an expression such as **b** 1011 consists of a units digit (1), twos digit (1), fours digit (0), eights digit (1). The expression **b** 1011 is a shorthand for a sum,

$$\mathbf{b}\,1011 = 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 = 1 + 2 + 0 + 8 = 11.$$

Note that $\mathbf{b}\,1 = 1$ and $\mathbf{b}\,10 = 2$ and $\mathbf{b}\,100 = 4$ and $\mathbf{b}\,1000 = 8$.

| Decimal | Binary |
|---|---|
| 0 | **b** 0 |
| 1 | **b** 1 |
| 2 | **b** 10 |
| 3 | **b** 11 |
| 4 | **b** 100 |
| 5 | **b** 101 |
| 6 | **b** 110 |
| 7 | **b** 111 |
| 8 | **b** 1000 |

Table 1: Counting up to 8, and counting up to **b** 1000.

Rather than calling **b** 10 "ten", it is better to say "one-zero" to avoid confusion.

Numerically, a bit is a 1 or a 0. But more importantly, a bit is the smallest unit of information. The smallest unit of information is the answer to a yes/no question – yes or no, on or off, 1 or 0. Such an answer is given with a bit. Computers store information in bits, and so computers work in binary.

The number of bits of a number is the amount of information carried by the number, and thus the amount of storage space required to hold the number in memory. Algebraically, the number of bits of a number is related to its base-two logarithm.

**Proposition 0.17 (Counting bits)** *If $N$ is a positive integer, then the number of bits in the binary representation of $N$ is $\lfloor \log_2(N) \rfloor + 1$.*

Division with remainder is not a childhood relic, to be forgotten in favor of grown-up decimal division or fractions. Though the word "remainder" suggests unwanted left-overs and "quotient" suggests a terminus, mathematicians respect remainders as much as quotients since they answer different questions of similar importance.

Let's begin by introducing some new notation for division with remainder. In school, many students are taught to write "$23 \div 4 = 5R3$", to mean that "23, divided by 4, is 5 with a remainder of 3". This again suggests that 5 is the answer, and 3 an undesirable left-over. We recommend the following compact notation: $23 = 5(4) + 3$. It is a statement about multiplication and addition. And that is what division with remainder is: a statement about multiplication and addition.

Why do we write $23 = 5(4) + 3$ when it is numerically the same as $23 = 5 \times 4 + 3$? We ask the reader to interpret two notations for multiplication in two visually different ways.

Interpret $5 \times 4$ as a five-by-four array and say "five times four".

Interpret $5(4)$ as five groups of four and say "five fours".

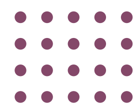


Figure 6: $5 \times 4$ suggests a rectangular arrangement.

Figure 7: $5(4)$ means five fours.

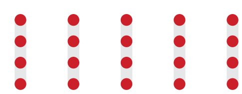


Figure 8: $5(4)$ means five fours. The arrangement within the"fours" is not important.

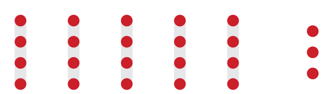


Figure 9: $23 = 5(4) + 3$.

So when we write $23 = 5(4) + 3$, you should think "23 is composed of five groups of four (five fours), with three left over."

This grouping with leftovers *is* division with remainder. Here are some more examples, in both notations:

$$93 \div 80 = 1R13, \quad 93 = 1(80) + 13.$$

$$4 \div 13 = 0R4, \quad 4 = 0(13) + 4.$$

$$1000 \div 4 = 250R0, \quad 1000 = 250(4) + 0.$$

Sometimes division leaves no remainder, as in $35 \div 7 = 5R0$. This common phenomenon can be expressed in a multitude of ways; most common in schools are the following: "35 is a **multiple** of 7", "7 goes into 35", "7 is a **factor** of 35". In ancient Greece, one might say (translated) "7 **measures** 35"; the interpretation is that given a 7-foot-long rod, one can precisely measure a 35-foot-long distance.[12]

[12] Questions about this type of measurability (divisibility) are studied in the latter half of Euclid's *Elements*.

ROUNDING is intimately connected to division with remainder. In fact, properties of the floor demonstrate the existence of quotient and remainder.

**Proposition 0.18 (Traditional division with remainder)** *Let $a$ and $b$ be integers, with $b$ positive. Then there exist integers $q$ and $r$ satisfying*

$$a = q(b) + r \text{ and } 0 \leq r < b.$$

For example, we divide 23 by 4 with remainder. The quotient is

$$q = \lfloor 23 \div 4 \rfloor = \lfloor 5.75 \rfloor = 5.$$

The remainder is what's left:

$$23 = 5(4) + r, \text{ so } r = 3.$$

PROOF: Since $a/b$ is a real number, it is not far above its floor:

$$0 \leq \frac{a}{b} - \left\lfloor \frac{a}{b} \right\rfloor < 1.$$

Multiply through by $b$ and we find

$$0 \leq b \cdot \left( \frac{a}{b} - \left\lfloor \frac{a}{b} \right\rfloor \right) < b$$

and distribute the $b$ to arrive at

$$0 \leq a - b \left\lfloor \frac{a}{b} \right\rfloor < b.$$

Define $q = \lfloor a/b \rfloor$ and $r = a - q(b)$. These satisfy

$$a = q(b) + r \text{ and } 0 \leq r < b. \quad \blacksquare$$

Quotient and remainder may be defined *formulaically* using addition, subtraction, multiplication, division, and the floor. Traditionally, dividing $a$ by $b$ yields a remainder beween 0 and $b - 1$. Another convention gives remainders between $-b/2$ and $b/2$.

For example, we divide 23 by 4 with remainder, allowing negative remainders. The quotient $q$ is the integer *closest* to $23/4 = 5.75$, so $q = 6$. The remainder is what's "left":

$$23 = 6(4) + r, \text{ so } r = -1.$$

In this convention, 23 divided by 4 is 6 with a remainder of $-1$.

**Proposition 0.19 (Division with minimal remainder)** *Let $a$ and $b$ be integers, with $b$ positive. Then there exist integers $q$ and $r$ satisfying*

$$a = q(b) + r \text{ and } -\frac{b}{2} \leq r \leq \frac{b}{2}.$$

PROOF: Let $q$ be an integer *closest* to $a/b$. Thus $q = \lfloor a/b \rfloor$ or $q = \lceil a/b \rceil$, or if $a/b$ is halfway between integers, we may choose either floor or ceiling. Thus

$$-\frac{1}{2} \leq \frac{a}{b} - q \leq \frac{1}{2}.$$

Multiply through by $b$ and distribute to find

$$-\frac{b}{2} \leq a - q(b) \leq \frac{b}{2}.$$

Define $r = a - q(b)$ and observe

$$a = q(b) + r \text{ and } -\frac{b}{2} \leq r \leq \frac{b}{2}. \quad \blacksquare$$

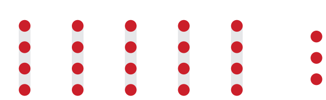


Figure 10: Using positive remainders, we have $23 = 5(4) + 3$.

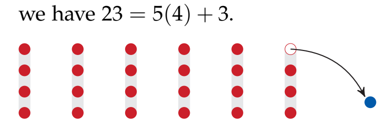


Figure 11: Using minimal remainders, we have $23 = 6(4) - 1$.

DIVISIBILITY is the blanket term for when one number "goes into" another. But "goes into" is a bit informal; mathematicians use the verb "**divides**" instead. Examples of this usage of "divides" are in the margin; read them aloud until they becomes second nature. This may be a shift in perspective; no longer do *people divide numbers*. Now *numbers divide numbers*. Synonyms for "divides" are "goes into" and "is a factor of" and, in ancient Greece, "measures."

2 divides 4. 4 divides 12. 3 divides 99. 13 divides 91. 1000 divides 98000. 5 does not divide 13. 3 does not divide 11.

The universally accepted but most unfortunate[13] notation for "divides" is a single vertical line.

$$x \mid y \text{ means that } x \text{ divides } y.$$

Addition endows the integers with a linear order. Multiplication endows the integers with infinite dimensions of complexity.[14]

[13] One should never use a symmetric symbol for an antisymmetric relation. But overthrowing accepted notation is not the goal of this text. In case of a revolution, $\trianglelefteq$ would be a nice notation for divides, and $\trianglerighteq$ for "is a multiple of".

[14] This is not hyperbole. See Chapter 3.

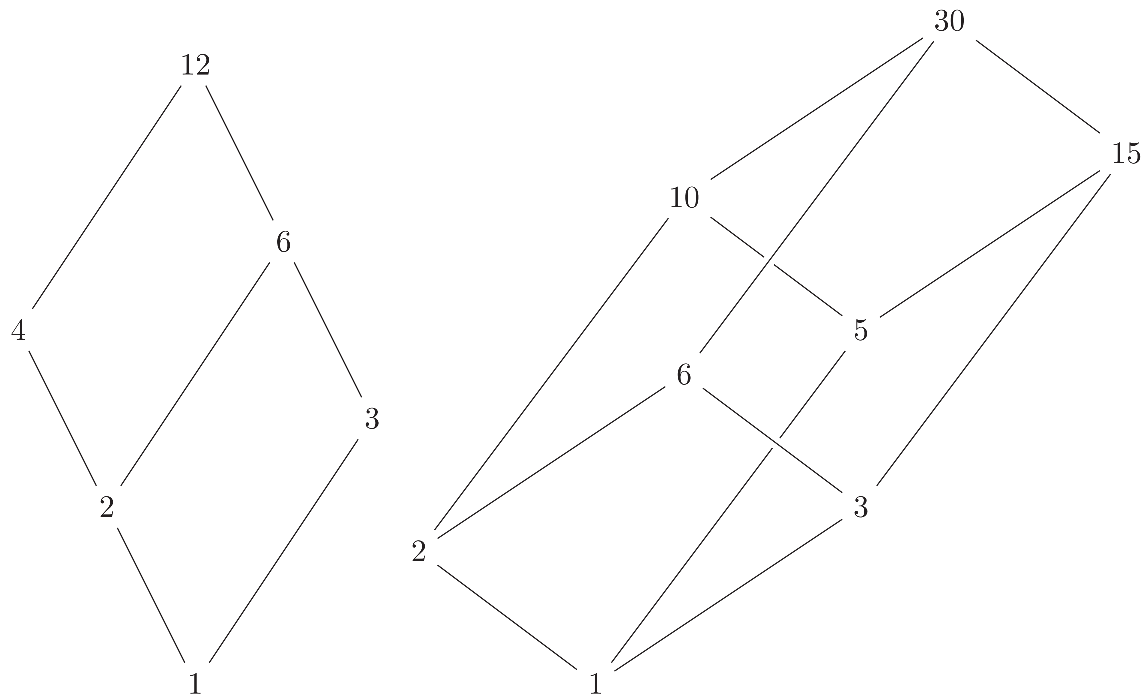



Figure 12: The Hasse diagrams for the divisors of 12 and 30. The Hasse diagram for 30 is best seen in three dimensions, since 30 has three prime factors: 2, 3, and 5.

To see the divisors of a number, we display them in a **Hasse diagram**. Each line segment, from $x$ below to $y$ above, expresses a divisibility $x \mid y$. On the left, observe $1 \mid 2$, $1 \mid 3$, $2 \mid 6$, $3 \mid 6$, $2 \mid 4$, $4 \mid 12$, and $6 \mid 12$. Further, every upwards *path* of line segments gives a chain of divisibilities. Since $2 \mid 6$ and $6 \mid 12$, we also see $2 \mid 12$.

Parallel line segments in the same diagram display proportions; for example, the line segment joining 2 to 6 is parallel to the line segment joining 4 to 12 reflecting the fact that $6/2 = 12/4 = 3$.

To WRITE $x \mid y$ means that $x$ "goes into" $y$, but implicit is a third integer. Namely, $x \mid y$ means that *there exists* an integer $m$ such that $y = mx$. It is crucial to observe that the definition of *divisibility* does not refer at all to the operation *division*. It refers to the existence of an "accesory" integer which satisfies a multiplicative relationship.

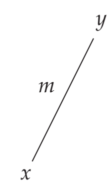



Figure 13: In a Hasse diagram, we express the divisibility $x \mid y$ by a line segment joining $x$ to $y$. To express the accessory integer $m = y/x$, we label the line segment with the number $m$.

**Problem 0.20** Demonstrate that $-3 \mid 12$.

SOLUTION: $-4$ is an integer which satisfies $12 = (-4)(-3)$. $\checkmark$

**Problem 0.21** Demonstrate[15] that $0 \mid 0$.

SOLUTION: Every integer $m$ satisfies $0 = m(0)$. $\checkmark$

[15] The fact that 0 divides 0 is most alarming to novices!

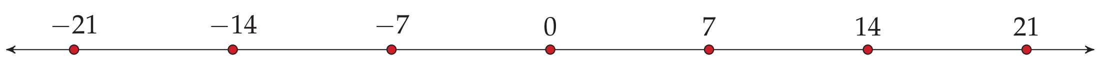
**Problem 0.22** Is it true or false that $0 \mid 12$?

SOLUTION: If $0 \mid 12$ then there would exist an integer $m$ such that $12 = m(0)$. But $m(0) = 0$ and $12 \neq 0$. Hence 0 does not divide 12. $\checkmark$

MULTIPLES of a number are evenly spaced on the number line.

**Problem 0.23** Plot the integers $x$ which satisfy $7 \mid x$.

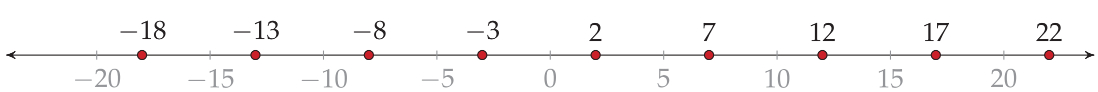
SOLUTION: The multiples of 7 are $\{\ldots, -14, -7, 0, 7, 14, \ldots\}$. $\checkmark$




More generally, arithmetic progressions are evenly spaced.

**Problem 0.24** Plot the integers $x$ which satisfy $5 \mid (x-2)$.

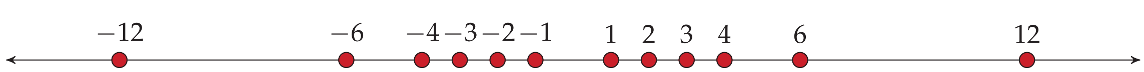
SOLUTION: If $5 \mid (x-2)$ then $x-2$ is among the multiples of 5.




Thus $x$ is two more than a multiple of five, lying among the arithmetic progression $\{\ldots, -8, -3, 2, 7, 12, \ldots\}$ plotted above. $\checkmark$

**Problem 0.25** Plot the integers $x$ which satisfy $x \mid 12$.

SOLUTION: The positive divisors of 12 are $1, 2, 3, 4, 6, 12$. The negative divisors of 12 are $-1, -2, -3, -4, -6, -12$. $\checkmark$



Three properties of $|$ follow below. We leave the reader to consider analagous properties of $\leq$.

**Proposition 0.26 (Reflexive property of $|$)** *For every integer $x$, $x \mid x$.*

Every integer is a multiple of itself. To see that $x \mid x$, it suffices to observe that $x = 1 \times x$.

**Proposition 0.27 (Antisymmetric property of $|$)** *For integers $x, y$, if $x \mid y$ and $y \mid x$ then $x = \pm y$.*

If two numbers are multiples of each other, then they are equal up to sign.

Proof: The hypothesis implies $y = mx$ and $x = ny$ for some integers $m$ and $n$. Hence

$$x = ny = n(mx) = (nm)x. \quad (1)$$

■

If $x = 0$ then $y = mx = 0$, so $x = \pm y$ as claimed.

If $x \neq 0$, then we divide Equation (1) by $x$ to discover that $m$ and $n$ are reciprocals: $nm = 1$. The only integers whose reciprocals are integers are 1 and $-1$. Hence $m = \pm 1$ and $n = \pm 1$, and so $x = \pm y$.

**Proposition 0.28 (Transitive property of $|$)** *If $x, y, z$ are integers, and $x \mid y$ and $y \mid z$, then $x \mid z$.*

The relation of "being a multiple" or "divides" is transitive.

Proof: The hypothesis implies $y = mx$ and $z = ny$ for some integers $m$ and $n$. Substitution yields $z = ny = n(mx) = (nm)x$ and so $x \mid z$. ■

Divisibility is preserved by addition and multiplication in the following two circumstances.

**Proposition 0.29** *If $d$, $x$, and $y$ are integers and $d \mid x$ then $d \mid xy$.*

The product of a multiple of $d$ with an integer is again a multiple of $d$.

Proof: If $d \mid x$ then there exists an integer $m$ such that $x = md$. Multiplying both sides by $y$ yields $xy = mdy = (my)d$. Hence $xy$ is a multiple of $d$, i.e., $d \mid xy$. ■

**Proposition 0.30** *Let $d$, $x$, and $y$ be integers. If $d \mid x$ and $d \mid y$, then $d \mid (x + y)$ and $d \mid (x - y)$.*

The sum or difference of two multiples of $d$ is another multiple of $d$.

Intuitively, if we have two collections, arranged in clumps of $d$, their union can again be arranged in clumps of $d$. This explains the theorem, at least when $d$ and its two multiples are positive.

Proof: The hypothesis implies $x = md$ and $y = nd$ for some integers $m$ and $n$. Adding or subtracting $x$ and $y$ yields

$$x \pm y = (md) \pm (nd) = (m \pm n)d.$$

Hence $x \pm y$ is a multiple of $d$. ■

We rephrase the previous theorem as a principle.[16]

**Corollary 0.31 (The two out of three principle for divisibility)** *Let a,b,c be integers, satisfying the equation $a + b = c$. Let $d$ be an integer. If two of the numbers from the set $\{a, b, c\}$ are multiples of $d$, then the third number must also be a mutliple of $d$.*

[16] This is a rephrasing. Indeed if $a + b = c$ then $b = c - a$ and $a = c - b$. Hence each term is a sum or difference of the other two. Hence if two terms are multiples of $d$, then the third is a multiple of $d$.

**Problem 0.32** Demonstrate that 2 999 997 is a multiple of three.

SOLUTION: Observe that 3 000 000 is a multiple of three; it is three million, or a million threes:

$$3\,000\,000 = 1\,000\,000(3), \text{ and so } 3 \mid 3\,000\,000.$$

Since $3 \mid 3$ and $3 \mid 3\,000\,000$, we find that 3 goes into the difference:

$$3 \mid (3\,000\,000 - 3) = 2\,999\,997.$$

✓

**Problem 0.33** Find all integers $x$ which satisfy $x \mid (x+6)$.

SOLUTION: Note that $x \mid x$ (always). If $x \mid (x+6)$, then $x$ divides the difference:

$$x \mid ((x+6) - x) = 6.$$

Conversely, if $x \mid 6$, then $x$ divides the sum

$$x \mid (x+6).$$

Hence $x \mid (x+6)$ if and only if $x \mid 6$. The solutions to $x \mid (x+6)$ are the divisors of 6: $x \in \{-6, -3, -2, -1, 1, 2, 3, 6\}$. ✓

**Problem 0.34** Does the equation $7x^2 + 11 = 21y$ have any integer solutions?

SOLUTION: If there were integer solutions $x$ and $y$ to the equation, then $7x^2$ would be a multiple of 7 and $21y$ would be a multiple of 7. By the two out of three principle, 11 would also be a multiple of 7.

But this is untrue; 11 is *not* a multiple of 7 and hence there can be no integer solutions to the equation $7x^2 + 11 = 21y$. ✓

In general, if one is faced with an equation and asked to find integer solutions, one should first check whether there is an obvious *divisibility obstruction*. Perhaps, as the previous problem indicates, no solution can be found because any integer solution would lead to a contradiction of principles of divisibility. Later, we will generalize this to find *congruence obstructions* to solubility of equations.

THIS CHAPTER has introduced the main characters of number theory: the natural numbers and integers, counting, addition, multiplication, rounding, division with remainder, and divisibility. But although these characters may feel like old friends, their mysteries will be exposed in the chapters to follow. When difficulties arise, the reader is advised to return to the techniques of this zeroth chapter.

## *Historical Notes*

TRIANGULAR NUMBERS, those numbers obtained by adding $1 + 2 + 3 + \cdots$ up to some fixed number, have at least two thousand years of history. Here we present some highlights, tracing sources back in time. Teachers and books often repeat a legend about C.F. Gauss (1777–1855CE) as a child, in which a cruel or frustrated teacher asks the boy to sum the numbers between 1 and 100. According to this legend, Gauss quickly obtained 5050, the correct answer. This legend, embellished like any good fishing story, has only a remote connection to textual source, which in turn has questionable connection to fact. The textual source is the 1856 Gauss memorial volume, written by Gauss's colleague at Göttingen, Wolfgang Sartorus. This source mentions an arithmetic problem given to the seven-year-old Gauss, but nothing about the problem except that he solved it quickly, and was confident in the correctness of his solution.[17]

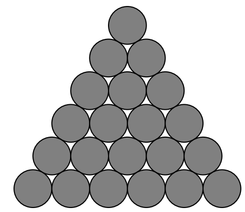


Figure 14: The sixth triangular number,

$$1 + 2 + 3 + 4 + 5 + 6 = 21.$$

Alcuin of York, an English scholar and teacher of the late 8th century CE, presents the problem of adding the numbers from 1 to 100 as Problem 42 in his "Problems to sharpen the young"; we have a copy of his text[18] from the late 9th century. The author presents the reader with a staircase of 100 steps; one pigeon stands on the first step, two pigeons on the second, et cetera, and the question is posed "How many pigeons are there altogether?" The solution given is similar to, but not entirely the same as, our solution by partnering. Alcuin partners 1 and 99, and 2 and 98, etc.., leaving 50 and 100. See the annotated translation[19] by J. Hadley and D. Singmaster for a thorough treatment of this medieval document.

But such problems certainly predate Alcuin. An example is found in a Jewish legal text from before 500CE, the Babylonian Talmud, Menachot folio 106a, in which the numbers from 1 to 60 are correctly summed; the method of summation is not explained, but the commentaries (Tosafot) written between 1000CE and 1400CE explain precisely the method of pairing off we have described in this chapter.[20]

TO ADD PERFECT SQUARES, $1 + 4 + 9 + \cdots$ up to a given square $N^2$, one may use the formula

$$\sum_{i=1}^{N} i^2 = \frac{(N)(N+1)(2N+1)}{6},$$

provable by induction. The result of this summation is called the $N^{\text{th}}$ square pyramidal number. Such a formula occurs in the Ganipāda of Āryabhata (India, 499CE), which reads

[17] For a critical examination of the Gauss legend, see "Gauss's Day of Reckoning," by Brian Hayes, in *American Scientist*, May-June 2006, Vol. 94, No. 3, p. 200.

[18] "Propositiones ad Acuendos Juvenes," by Alcuin of York, c. 800CE. This problem is in XLII, titled "Propositio de scala habente gradus centum," which reads "Est scala una habens gradus C. In primo gradu sedebat columba una: in secundo duae; in tertio tres; in quarto IIII; in quinto V. Sic in omni gradu usque ad centesimum. Dicat, qui potest, quot columbae in totum fuerunt?"

[19] "Problems to Sharpen the Young," by John Hadley and David Singmaster, in *The Mathematical Gazette*, March 1992, Vol. 76, No. 475, pp. 102–126.

[20] For more about this finding in the Babylonian Talmud, see "A Mediaeval Derivation of the Sum of an Arithmetic Progression," by Martin D. Stern, in *The Mathematical Gazette*, June 1990, Vol. 74, No. 468, pp. 157–159.

> The product of the three quantities, the number of terms plus one $[N+1]$, the same increased by the number of terms $[2N+1]$, and the number of terms $[N]$, when divided by six, gives the sum of squares of natural numbers (*varga-citighana*).[21]

A millenium after Āryabhata, Nīlakaṇṭha (c. 1500CE) of the Kerala school in India demonstrates a "stacking-corners" style argument for this formula, using a solid of dimensions $N$ by $N+1$ by $2N+1$. We leave this to the exercises, and refer to K. Plofker's translation and exposition[22] for more details.

In the "Introduction to Arithmetic" by Nicomachus of Gerasa (Roman Syria, now Jordan, c. 100CE), we find the following treatment of cubes:

> But all the products of a number multiplied twice into itself, that is, the cubes... 1,8,27,64,125, and 216, and those that go on analogously, in a simple, unvaried progression as well. For when the successive odd numbers are set forth indefinitely beginning with 1, observe this: The first one makes the potential cube; the next two added together, the second; the next three, the third; the four next following, the fourth; the succeeding five, the fifth; the next six, the sixth; and so on.[23]

The term "Nicomachus's Theorem" today often refers to the result that the sum of the first $N$ cubes equals the square of the $N^{\text{th}}$ triangular number.[24] This can be deduced from Nicomachus's observation above: see the exercises.

EUCLID introduces divisibility and proportion in Book V of the *Elements*. The Definitions of Book V begin with

> Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

Translated,

> A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.[25]

Factors of a number are the "parts" for Euclid, and while we might say "goes into" or "divides", Euclid would write "measures."

[21] Translation from "Development of Calculus in India," by K. Ramasubramanian and M.D. Srinivas, in *Studies in the History of Indian Mathematics*, ed. C. S. Seshadri, published by Hindustan Book Agency, 2010.

[22] "Aryabhatiyabhasya of Nilakantha," translated by K. Plofker and H. White, in Chapter 4 of *The Mathematics of Egypt, Mesopotamia, China, India, and Islam, A Sourcebook*, ed. Victor J. Katz, Princeton University Press, 2007.

[23] From Book 2, Chapter XX, of "Introduction to Arithmetic", by Nicomachus of Gerasa, translated by Martin Luther D'Ooge, published by the Macmillan Company, New York, 1926.

Nicomachus was aware of the following, expressed in a modern table:

| | | |
|---|---|---|
| 1 | = | 1 |
| 8 | = | 3 + 5 |
| 27 | = | 7 + 9 + 11 |
| 64 | = | 13 + 15 + 17 + 19 |
| 125 | = | 21 + 23 + 25 + 27 + 29 |

[24] In other words,

$$\sum_{i=1}^{N} i^3 = \left(\sum_{i=1}^{N} i\right)^2.$$

[25] The Greek text here comes from the 1885 edition of J.L. Heiberg, and the English translation is by Richard Fitzpatrick. Many editions of Euclid, including Fitzpatrick's and Byrne's remarkable illustrated edition of Books I-VI, are freely available online.

## *Exercises*

1. How many multiples of 7 are between 10 and 500?
2. How many numbers between 1 and 100 are not multiples of three?
3. How many numbers between 1 and 100 are mulitples of 2 or[26] multiples of 3? Caution: do not double-count the multiples of 6.
4. Add the even numbers between 1 and 100.
5. Add the numbers in the series $3 + 11 + 19 + 27 + \cdots + 395 + 403$.
6. How many square numbers are between 100 and 10000?
7. Given integers $x$ and $y$, with $x > 0$ and $y < 0$ and $x - y = 5$. What are the possible values of $x$ and $y$? Reason geometrically.
8. Carry out division with *positive* remainder and division with minimal (positive or negative) remainder for the following: $27 \div 7$, $30 \div 6$, $100 \div 3$, $90 \div 13$. Express your answers as equalities of the form $a = q(b) + r$.
9. It is possible to fit three congruent pentagons – all equilateral and equiangular – around a single point without overlapping. For which positive integers $(A, B)$ is it possible to fit $A$ congruent shapes – each equilateral and equiangular with $B$ sides – around a point without overlapping? (This is the crux of Euclid's argument that there are five Platonic solids.)
10. If $x \leq y$ then $x + 5 \leq y + 5$. Is the same true, with $\leq$ replaced by $|$? Prove it or find a counterexample.
11. If $x \leq y$ then $5x \leq 5y$. Is the same true, with $\leq$ replaced by $|$? Prove it or find a counterexample.
12. Draw Hasse diagrams for the positive divisors of 7, 15, 18, and 105.
13. Plot all integers $x$ which satisfy $(x + 1) \mid 14$.
14. Plot integers $x$ which satisfy $4 \mid (x + 1)$.
15. Plot all integers $x$ which satisfy $(x + 4) \mid 2x$.
16. Graph the functions $y = \lfloor x \rfloor$ and $y = \lceil 2x \rceil$.
17. How many bits does the binary expansion of $10^{500}$ have?
18. Describe the effect of the functions $f(x) = \lfloor 10x \rfloor / 10$ and $g(x) = \lfloor 100x \rfloor / 100$.

[26] In mathematical contexts, "or" means the "inclusive or". So "$P$ or $Q$" means "$P$ or $Q$ or both". So the first few numbers which are multiples of 2 or 3 are

$$2, 3, 4, 6, 8, 9, 10, 12, \ldots.$$

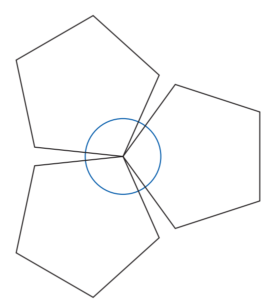


Figure 15: 3 regular 5-gons fit around a point, with a bit of room to spare. So $(3, 5)$ is a possible ordered pair.

19. Write down a formula with one variable $x$, using basic arithmetic operations and the floor function, whose output is the ones digit of $x$ in base 10. Challenge: write down a formula[27] whose output is the *leading* digit of $x$, i.e., if $x = 372.5$ then $f(x) = 3$.

[27] You may use basic arithmetic operations, exponents and logarithms, and the floor function.

20. Prove that
$$\lim_{x\to\infty} \frac{\lfloor x \rfloor}{x} = \lim_{x\to\infty} \frac{\lceil x \rceil}{x} = 1.$$

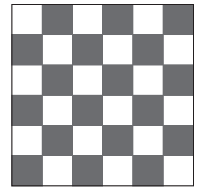


Figure 16: A chessboard with 6 rows and 6 columns.

21. Consider a square chessboard, with $N$ rows and $N$ columns, whose bottom-left corner is black. Find a formula (using floor or ceiling) for the number of black squares on the chessboard.

22. Prove that if $x \mid (x^2+1)$ then $x = \pm 1$.

23. Prove that $6 \mid (x^3 - x)$ for every integer $x$. Hint: among three consecutive integers, one must be a multiple of 3.

24. Let $T(N) = \sum_{i=1}^{N} i$ be the $N^{\text{th}}$ triangular number. When is $T(N)$ even? Prove your answer.

25. Draw a spiral to demonstrate that

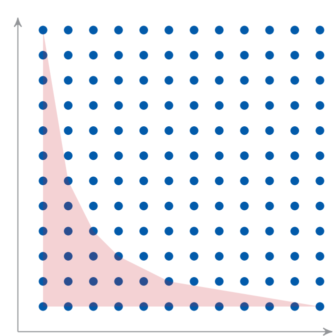
$$100 = 10 + 2(9) + 2(8) + 2(7) + 2(6) + 2(5) + 2(4) + 2(3) + 2(2) + 2(1).$$

26. (Challenge) Let $S(N) = \sum_{i=1}^{N} i^2$ be the sum of the first $N$ perfect squares. Prove that $6N = (N)(N+1)(2N+1)$ by a "stacking corners" argument in a rectangular solid of dimensions $N \times (N+1) \times (2N+1)$.

27. Let $C(N) = \sum_{i=1}^{N} i^3$ be the sum of the first $N$ cubes. Let $T(N) = \sum_{i=1}^{N} i$ be the $N^{\text{th}}$ triangular number. Prove that $C(N) = T(N)^2$ by induction. Can you find a proof using Nicomachus's observation (see the *Notes*) earlier in the chapter)? Can you find a four-dimensional stacking-corners proof?

28. Demonstrate that the equation $3x^{17} + 1111 = 27y + 15z$ has no solution in which $x$, $y$, and $z$ are integers.

29. Let $N$ be a positive integer. Let $\sigma_0(N)$ denote the number of positive divisors of $N$. Prove that $\sigma_0(N)$ is odd if and only if $N$ is a square number. Prove that the *product* of all positive divisors of $N$ equals $N^{\sigma_0(N)/2}$. Hint: Use partnering arguments.



30. Let $A(N)$ be the average of $\sigma_0(1), \sigma_0(2), \sigma_0(3), \ldots, \sigma_0(N)$ – the average number of positive divisors among the numbers up to $N$. Demonstrate that $A(N)$ is approximately $\log(N)$, with error bounded by 1. Hint: Relate $NA(N)$ to the number of dots in a shaded region such as the one on the right.