Chapter 1

Congruent Figures

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Overleaf:
Tiling with congruent cyber-poles.
1. Congruence and Isometries

In plane geometry we study figures that can be drawn on a flat surface, like a sheet of paper. We say that a figure \( X \) is congruent to a figure \( Y \) (written \( X \cong Y \)) if we can orient one, or a copy of one, on top of the other so that they match exactly. We are free to move, turn or flip over a figure to make it coincide with another figure. In particular, each figure is congruent to itself.

A mature cyber-pole is called a “cyber-frog”. In Figure 1.1 we have six cyber-frogs, labeled \( A, B, C, D, E \) and \( F \), that look somewhat alike.

![Figure 1.1. Six cyber-frogs.](image)

**Problem 1.** Find two cyber-frogs \( A \) to \( F \) that are congruent. Explain in words how you would move one to make it coincide with the other.

How can we tell if two figures are not congruent? If two figures are congruent, then each part of one will be congruent to the corresponding part of the other. We quote this fact as CPCFC: “corresponding parts of congruent figures are congruent”. If some part of one is not congruent to any part of the other, then the two figures must not be congruent.
Problem 2. Find two cyber-frogs $A$ to $F$ that are not congruent. Use CPCFC to explain in words how you know they are not congruent.

Here is a way to tell if a cyber-frog is congruent to any other cyber-frog. Choose one that you are not sure about. Make a drawing of the cyber-frog you chose, perhaps using tracing paper. It doesn’t need to be perfect. It only needs to show you how its parts fit together, and it should be the same size as the one you chose. Cut your drawing out, and then move it around to see if it can be made to coincide with any of the other five cyber-frogs.

In the next two problems you will determine all pairs of congruent cyber-frogs and verify that your answer is complete.

Problem 3. For each two cyber-frogs that are congruent, write down a description of the motions you would make to get one of them to coincide with the other.

Problem 4. For each cyber-frog, find a part of it that it only shares with those cyber-frogs that are congruent to it. This tells you that it is only congruent to the ones you found it congruent to in Problem 3.

A congruence class is a set of one or more figures in which any two are congruent to each other and no figure in the set is congruent to a figure outside the set.

Problem 5. How many different congruence classes of cyber-frogs did you find? Draw a circle for each congruence class, and then list inside each circle the names of the cyber-frogs who are in that congruence class.

To show that a figure is congruent to another figure, we need to move the first one to make them coincide. The motions we use are called isometries (“iso” for “same” and “metry” for “measure”). There are three kinds of isometries that are particularly useful. We call these basic isometries.

- A translation is an isometry that slides the figure in one direction without turning it.
- A rotation is an isometry that holds one point in place and rotates the figure around that point.
- A reflection is an isometry that holds the points along one line in place and flips the entire sheet over that line.

Figure 1.2 shows how to establish that two figures $A$ and $B$ are congruent using basic isometries to make $A$ coincide with $B$: 
1. CONGRUENCE AND ISOMETRIES

Figure 1.2. Basic isometries

- reflect A through line \( \ell \) to produce X,
- translate X so that its circle coincides with the circle in B to produce Y,
- rotate Y around the center of the circle until it coincides with B.

**Problem 6.** Repeat your answer to Problem 1, this time by listing basic isometries that would make them coincide, as illustrated in Figure 1.2.

**Problem 7.** List the basic isometries you would need to show that the two cars in each of the four pairs of Figure 1.3 are congruent. Start by copying one if they are on the same sheet. (Then see if you can do each of these using only reflections.)

Figure 1.3. Four pairs of congruent figures.
In Figure 1.4 we have a more complex problem consisting of twenty-five small figures that look rather similar.

Problem 8. Find a pair of the figures A to Y that are congruent. List the basic isometries you would use to make one coincide with the other.

A twin pair

Figure A has a twin pair in the upper left corner that has no isolated point as a neighbor. By CPCFC, any figure congruent to A must have a twin pair in some corner with no isolated point as a neighbor. For example, this tells us that M is not congruent to A.

Problem 9. Look at figure K. Find the congruence class of K. What part or special feature do the figures congruent to K have that none of the others have?

Problem 10. Find other pairs of the figures A to Y that are congruent. Divide the twenty-five figures into congruence classes so that each two figures in the same congruence class are congruent and each two figures
in different congruence classes are not congruent. How many different congruence classes are there? Draw a circle for each congruence class, and then list inside each circle the names of the figures that are in that congruence class.

**Problem 11.** For each congruence class, find a part or special feature that all the figures in that class have but that no others have. This will demonstrate that you have found the right congruence classes.

**Problem 12.** Number the cyber-poles on the cover page of this chapter from 1 to 12, starting at the upper left.

(i) Say that two cyber-poles are Tr-congruent if one can be made to coincide with the other using only translations. List all the distinct Tr-congruence classes.

(ii) Say that two cyber-poles are TrRo-congruent if one can be made to coincide with the other using only translations and rotations. List all the distinct TrRo-congruence classes.

(iii) Say that two cyber-poles are TrRoRe-congruent (that is, congruent,) if one can be made to coincide with the other using any basic isometries. List all the distinct congruence classes.

(iv) Say that two cyber-poles are Re-congruent if one can be made to coincide with the other using only reflections. List all the distinct Re-congruence classes.

2. The Language of Geometry

Many figures, like the ones we have seen here, can be drawn with line segments and circles. We will now turn to a more careful study of these simpler figures. A point is a single location on the flat surface that can be illustrated by a dot drawn with a pencil point. In geometry we say that a point $B$ is between points $A$ and $C$ only if $B$ actually lies on the straight line from $A$ to $C$.

![Figure 1.5. Five points.](image-url)
Problem 13. Without marking in your book, apply your straight edge to Figure 1.5 to find out exactly which points are between which other points.

If $A$ and $B$ are two points, then the segment $AB$ is the set consisting of $A$ and $B$ and all the points between $A$ and $B$. We will use a sharp pencil and straight edge to draw segments. In Figure 1.6 we see a collection of segments. Among them the segments $AB$ and $GH$ are congruent. We express this fact by writing $AB \cong GH$.

![Figure 1.6. Several segments, two of which are congruent.](image)

An important feature of segments is that each one can be measured to give us a length. In order to measure lengths, we need to agree on a single segment $OI$ that we will use as the unit segment. We can then think of the length of a segment $AB$, written $L(AB)$, as the number of unit segments we pass over as we travel from $A$ to $B$.

In order to effectively communicate, it is helpful if people use the same units for measuring lengths. Most countries in the world use metric units: millimeters, centimeters, meters, kilometers and light years. Other standard units are inches, feet, yards, furlongs and miles. Many instruments measure length; for example, micrometers, measuring sticks, tape measures, and odometers. The important facts about length measure are summarized in our first axiom.

An axiom is a fact that we can experimentally verify by looking carefully at physical examples. In this book we use ten axioms about geometry that we will present as they are needed. These ten axioms will be enough to solve all of the problems and prove all of the theorems in this book. You will get these axioms only as you need them. They are all listed together in Appendix A.
2. THE LANGUAGE OF GEOMETRY

Axiom 1: Length Measure. Each segment $AB$ can be assigned a positive number $\mathcal{L}(AB)$, called the length of $AB$, so that the following properties hold.

(i) The length of the unit segment is 1.

(ii) Two segments are congruent if and only if they have the same length.

(iii) If $A$, $B$ and $C$ are three points with $B$ between $A$ and $C$, then $\mathcal{L}(AC) = \mathcal{L}(AB) + \mathcal{L}(BC)$.

Problem 14. On a blank sheet of paper mark two points $O$ and $P$. Now mark another point $U$ so that $OP \cong OU$. (By Axiom 1(ii), this is the same as saying that $P$ and $U$ are the same distance from $O$.) Then find another point $V$ so that $OP \cong OV$. Find new points $W$, $X$, $Y$ and $Z$ that each make a segment with $O$ that is congruent to $OP$. Using your pencil, mark all points that make a segment with $O$ that is congruent to $OP$. What does this figure look like?

If $O$ and $P$ are two points, then the set $C$ of all points that make a segment with $O$ that is congruent to $OP$ is called a circle. The point $O$ is called the center of the circle. For each point $X$ on the circle the segment $OX$ is called a radius of the circle. Since radii (plural of radius) are all congruent, they all have the same length. This length is called the radius of $C$. If $OX$ and $OY$ are radii and $O$ is between $X$ and $Y$, then $XY$ is a diameter of $C$ and its length is called the diameter of $C$.

If $A$ and $B$ are two points, the ray starting at $A$ and going through $B$ is the set consisting of all points $X$ such that either

- $X$ is $A$ or $B$,
- $X$ is between $A$ and $B$, or
- $B$ is between $A$ and $X$.

We call this “ray–$A–B$” and write it as $\overrightarrow{AB}$.

The line through $A$ and $B$ consists of all points on either the ray $\overrightarrow{AB}$ or on the ray $\overrightarrow{BA}$ (or both). We call this “line–$A–B$” and write it as $\overrightarrow{AB}$. Unlike segments and circles, rays and lines go on and on so they can never be fit on one piece of paper.

Recall that the union of two sets consists of all things in the first set together with all things in the second set. The intersection consists of
all things in both the first set and the second set. We use the symbols $\cup$ for union and $\cap$ for intersection. For example, in Figure 1.7 the union $AB \cup \overrightarrow{BC}$ is the ray $\overrightarrow{AC}$ and the intersection $AB \cap \overrightarrow{BC}$ contains just the point $B$. We write this in equations as

$$AB \cup \overrightarrow{BC} = \overrightarrow{AC} \quad \text{and} \quad AB \cap \overrightarrow{BC} = \{B\}.$$  

**Problem 15.** Complete each equation.

(i) $BC \cup CD =$ (ii) $BC \cap CD =$ (iii) $\overrightarrow{AC} \cup BD =$  
(iv) $\overrightarrow{CB} \cup \overrightarrow{CD} =$ (v) $\overrightarrow{CB} \cap \overrightarrow{AD} =$ (vi) $\overrightarrow{BC} \cup \overrightarrow{CD} =$

A line $\ell$ divides the plane into two sides. If $C$ and $D$ are two points not on line $\ell$, we say that they are on the **same side** of $\ell$ if the segment $CD$ does not intersect $\ell$. If $E$ and $F$ are also not on $\ell$, then $E$ and $F$ are on **opposite sides** of $\ell$ if $EF$ does intersect $\ell$. If $C$ and $D$ are on the same side of $\ell$ and $D$ and $F$ are on opposite sides of $\ell$, what do you know about $C$ and $F$? If $E$ and $F$ are on opposite sides of $\ell$ and $F$ and $D$ are on opposite sides of $\ell$, what do you know about $E$ and $D$?

If rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ emanate from the same point $B$ but are not contained in the same line, then their union, $\overrightarrow{BA} \cup \overrightarrow{BC}$, is called an **angle**. In Figure 1.9, for example, $\overrightarrow{BA} \cup \overrightarrow{BC}$ is an angle. We call this “angle $A\dot{B}\dot{C}$”, written $\angle ABC$, or just “angle $B$” (written $\angle B$) for
short. A point $D$ is in the interior of $\angle ABC$ if $D$ is on the $A$ side of $\overrightarrow{BC}$ and on the $C$ side of $\overrightarrow{AB}$.

Points that lie on the same line are said to be collinear. If $A$, $B$ and $C$ are three points that are not collinear, then the triangle $A$–$B$–$C$ is the union of the three segments $AB$, $BC$ and $CA$. We will denote this triangle by the symbol $\triangle ABC$. We say that $A$ is a vertex of $\triangle ABC$. The $\triangle ABC$ has three vertices, $A$, $B$ and $C$. It also has three angles, $\angle A$, $\angle B$ and $\angle C$, and three sides, $AB$, $BC$ and $CA$.

Congruent figures are all around us, and play an important role in our lives. The letter keys on your keyboard are all congruent, as are the stars on an American flag. If you make a dress, you will want the left sleeve to be congruent to the right sleeve; at least that is the current fashion. If a baseball goes through the window, we need to replace it with a new window. We count on the window company to make windows that are congruent; otherwise the new one will not fit.

These examples show us that it is often necessary to know that two figures will be congruent in advance, before we construct them. For example, two circles will be congruent provided only that they have the same radius. That is easy. But congruence of triangles is a bit more complicated.

**Problem 16.** Describe a sequence of basic isometries that will make $\triangle ABC$ coincide with $\triangle YXZ$ in Figure 1.9. (Then see if you can do it using only reflections.)

From Problem 16 we see that these two triangles are congruent with $A$ corresponding to $Y$, with $B$ to $X$ and $C$ to $Z$. We express this fact by writing $\triangle ABC \cong \triangle YXZ$. This tells us, using CPCFC, that the six
corresponding parts are congruent:
\[ AB \cong YX, \ BC \cong XZ, \ CA \cong ZY, \ \angle A \cong \angle Y, \ \angle B \cong \angle X, \ \angle C \cong \angle Z. \]

Conversely, if some correspondence between the vertices of two triangles, such as
\[ A \mapsto Y, \ B \mapsto X, \ C \mapsto Z, \]
makes the six corresponding angles and sides congruent, it would appear that the triangles must be congruent. It turns out that it is often enough to check congruence of only three corresponding parts to guarantee that \( \triangle ABC \cong \triangle YXZ \).

**Problem 17.** Which of the following guarantee \( \triangle ABC \cong \triangle YXZ \)?

1. (SSS): \( AB \cong YX, \ BC \cong XZ \) and \( CA \cong ZY \);
2. (AAA): \( \angle A \cong \angle Y, \ \angle B \cong \angle X \) and \( \angle C \cong \angle Z \);
3. (SAS): \( AB \cong YX, \ \angle A \cong \angle Y \) and \( CA \cong ZY \);
4. (SSA): \( AB \cong YX, \ CA \cong ZY \) and \( \angle C \cong \angle Z \);
5. (ASA): \( \angle A \cong \angle Y, \ AB \cong YX \) and \( \angle B \cong \angle X \);
6. (AAS): \( \angle A \cong \angle Y, \ \angle B \cong \angle X \) and \( BC \cong XZ \).

For those that do not guarantee congruence of the triangles, draw an example in which the three stated conditions are true but the triangles are not congruent.

**3. Construction Problems**

Look carefully at the diagram in Figure 1.10.

![Figure 1.10](image_url)

**Figure 1.10.** Does this sculpture belong in the city park?

This drawing was proposed for a large sculpture that children could climb and play on in the city park. While some people thought it would be a great idea, there were others who doubted that such a structure
could ever be built. (To see the problem, imagine the frustration of a little girl starting at $A$ or $C$ and trying to catch her big brother hiding at $B$!) In order to settle the dispute, it was agreed that the large sculpture would be built provided that someone could construct a small one based on the same design. As it turned out, no one could solve this construction problem so the sculpture was never built.

When we study figures in the plane, we face similar dilemmas. Just because we have a description of a particular figure, maybe even a diagram of the figure, this does not guarantee us that the intended figure actually exists. In order to demonstrate that a particular figure exists, we will construct it using three simple tools: a sharp pencil, a straight edge and a compass. The pencil and straight edge will be used to draw segments, lines and rays. The compass will be used to draw circles, and also to copy segments. Using these tools, there are many different figures that you can construct. You will need to do a number of different constructions in what follows. In these problems, construct will always mean to do three things.

(1) Find a method, using only a pencil, straight edge and compass, that you can use to produce the desired result.

(2) Make a numbered list of the steps of your method that anyone else can use to produce the same result.

(3) Explain how you know that this method will produce the desired result.

**Problem 18.** Given a segment $AB$ and a ray $\overrightarrow{CD}$, construct a point $X$ on $\overrightarrow{CD}$ such that $AB \sim CX$.

Let $A$, $B$ and $C$ be three non-collinear points. We say that $\triangle ABC$ is an equilateral triangle if its three sides are all congruent. Remember that neither this definition nor the corresponding diagram in Figure 1.11 guarantee the existence of three points forming an equilateral triangle. To show that they exist, we must construct one.

![Figure 1.11. An equilateral triangle.](image)

**Problem 19.** Construct an equilateral triangle.
Problem 20. Given segments $AB$, $CD$ and $EF$ in Figure 1.12, construct a $\triangle XYZ$ such that $XY \cong AB$, $YZ \cong CD$ and $ZX \cong EF$.

In order to carry out Step 3 of the constructions problems that follow, we will use two additional axioms that will tell us when two triangles are congruent. They will allow us to prove that two segments or two angles are congruent by showing that they are corresponding parts of two triangles that are congruent.

(SSS) Axiom 2: Side-Side-Side. If the three sides of one triangle are congruent to the corresponding three sides of another triangle, then the triangles themselves are congruent.

(SAS) Axiom 3: Side-Angle-Side. If two sides and the included angle of one triangle are congruent to the corresponding two sides and the included angle of another triangle, then the triangles themselves are congruent.

Here are some construction problems for you to do. For each one you will need to follow the three steps required for a construction.
Problem 21. Given triangle $\triangle ABC$, construct a different triangle $\triangle XYZ$ so that $\triangle ABC \cong \triangle XYZ$.

Problem 22. Given angle $\angle ABC$ and a ray $\overrightarrow{EF}$, construct a ray $\overrightarrow{ED}$ so that $\angle ABC \cong \angle DEF$. (Figure 1.13.)

Given $\angle ABC$, we say that the ray $\overrightarrow{BD}$ is the **bisector** of the angle $\angle ABC$ if $D$ is in the interior of $\angle ABC$ and $\angle ABD \cong \angle DBC$.

Problem 23. Construct the bisector of a given angle.

Let $A$, $B$ and $C$ be three collinear points with $B$ between $A$ and $C$, and let $D$ be a point not on the line containing $A$, $B$ and $C$. Then the angles $\angle ABD$ and $\angle CBD$ are called **supplementary angles**. Notice that every angle has two supplements. (Figure 1.14.)

![Figure 1.14. Supplementary angles.](image)

A **right angle** is an angle that is congruent to one of its supplements.

Problem 24. Construct a right angle.

We say that two intersecting lines are **perpendicular** if they form a right angle at their intersection. (Figure 1.15.)

![Figure 1.15. Perpendicular line.](image)

Problem 25. Construct a line perpendicular to a given line through a given point that is on the line.
Given a segment $AB$, we say that a point $M$ between $A$ and $B$ is the **midpoint** of $AB$ if $AM \cong MB$. (Figure 1.16.)

**Problem 26.** Construct the midpoint of a given segment.

**Problem 27.** Construct a line perpendicular to a given line through a given point that is not on the line.

We say that two lines are **parallel** if no point is on both lines. (Figure 1.17.)

**Problem 28.** Given a line $\ell$ and a point $P$ not on $\ell$, construct a line $m$ such that $P$ is on $m$ and $m$ is parallel to $\ell$.

Let $A$, $B$ and $C$ be three non-collinear points. We say that $\triangle ABC$ is a **trisquare** if its three sides are all congruent and its angles are all right angles. (See Figure 1.18.)

We know now what a trisquare is. It remains either to show that trisquares exist by constructing one, or to explain in some way why there can be no trisquares.