Introduction

Amongst the many great advances in mathematics in the last part of the twentieth century, the successes in geometric analysis rank very highly. The most spectacular are the resolutions of the Poincaré conjecture and Thurston's geometrization conjecture, through the work of Perelman and the many people who clarified and extended his ideas. These were proved using Hamilton's Ricci flow, a powerful tool in the subject later used by Brendle and Schoen to prove the differentiable sphere theorem in higher dimensions. Beyond these, we mention the many dramatic advances in the study of minimal submanifolds, harmonic mappings and related variational problems, the deeper understanding of general relativity using tools from PDE, Riemannian and Lorentzian geometry, the use of gauge theory to detect subtle new topological invariants, and the relationship between the spectral behavior of the Laplace-Beltrami operator on a Riemannian manifold and the dynamical properties of the underlying geodesic flow.

It is not easy to give a comprehensive definition of this subject, and the name 'geometric analysis' has only been in common currency for the last 25 years or so. Loosely speaking, this field involves the many interlocking relationships between geometry and partial differential equations. These interconnections go both ways. For example, a 'purely geometric' problem, such as finding the optimal shape of a manifold, can be translated into an equivalent problem which involves solving a PDE. If a solution of that equation can be found, this can then be translated back into a solution of the original geometric problem. A classic instance is the uniformization theorem, where one seeks optimal (constant curvature) metrics on surfaces. There are several different analytic approaches, the earliest involving complex analysis and later ones involving semilinear elliptic PDE's. In the other direction, new perspectives in the field of PDE and many new techniques to solve various classes of equations have been inspired by the geometry underlying these equations. Among the many examples here, deep advances in fully nonlinear elliptic equations originated in the fundamental breakthroughs by Yau and others on Monge-Ampere equations arising in geometry. In a different direction, the entire modern theory of linear partial differential using microlocal analysis, pioneered by Hörmander, Kohn, Nirenberg and others, relies on a new way of viewing linear PDE through a geometric lens and exploiting the deep connections with symplectic geometry.

The research area highlighted in the 2013 session of the Park City Mathematics Institute was geometric analysis. The program of this summer school included lectures by: Michael Eichmair, Fernando Coda Marques, Tristan Riviere, Igor Rodnianski, Peter Topping, Jeff Viaclovsky, Ben Weinkove, Brian White, Steve Zelditch, and the Clay Scholars Gerhard Huisken and Richard Schoen. All were chosen both for the excellence of their mathematical work as well as their expository talents.

This volume collects papers contributed by Huisken, Marques, Riviere, Topping, Viaclovsky, Weinkove, White and Zelditch. The topics covered include general relativity, the proof of the long-standing Willmore conjecture as well as analytic aspects of the Willmore equation, Ricci and Kähler-Ricci flow, critical metrics, minimal surfaces and spectral theory. The papers here represent the lectures for the Graduate Summer School at PCMI, presented to an audience of 80 graduate students and 60 researchers. We also include a paper by the Clay Senior Scholar Gerhard Huisken, loosely based on his public lecture during the summer session. The organizers are grateful to the Clay Mathematics Institute for their sponsorship and support of two Clay Senior Scholars during the 2013 PCMI session: Gerhard Huisken and Richard Schoen.

As with every PCMI volume, this collection of papers is meant to serve as a high level introduction to many of the most important topics in geometric analysis by some of the great experts in the field, and is intended for graduate students or anyone else wishing an entrée into the field. The 2013 session was marked by recollections by the more seasoned researchers of their participation in the previous PCMI session on geometric analysis in 1992, and the lasting influence that workshop had on their careers. We can only hope that the 2013 session of PCMI will have a similarly long-lasting and far-reaching effect in this wonderful field.

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