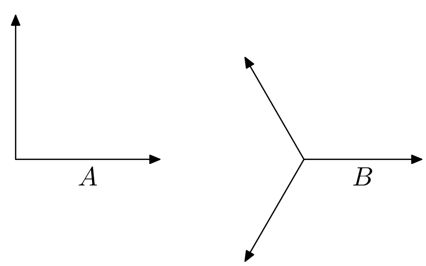
Introduction

Consider two finite sequences of vectors in the plane $\mathbb{R}^2$,

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad B = \left\{ \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\}$$



The sequence A is an orthonormal basis (ONB) for $\mathbb{R}^2$, and it has certain properties which we recall from calculus and linear algebra. The sequence B is a bit different. Let us compare the two.

- Both A and B are spanning sets for $\mathbb{R}^2$, so every $x \in \mathbb{R}^2$ can be written as a linear combination of the vectors in A, and similarly for the vectors in B.
- The vectors in A are linearly independent, so the coefficients in the linear expansion $x = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $c_i \in \mathbb{R}$,

are unique. The vectors in B are not linearly independent, so a linear expansion would not be unique.

- The vectors in A all have length 1. The vectors in B all have length $\sqrt{\frac{2}{3}}$.
- The vectors in A are orthogonal (i.e., their dot product is zero). The vectors in B are not orthogonal.
- The coefficients $\{c_i\}_{i=1}^2$ for a linear expansion in the sequence A can be computed easily using the dot product. Specifically, let $x = c_1e_1 + c_2e_2$ where the vectors in A are denoted e_1 and e_2. The coefficients are computed by the dot products $c_i = x \cdot e_i$ for $i = 1, 2$. This is a property of orthonormal bases. If we check, however, we find that one way to write x as a linear combination of the vectors in B can be found using the coefficients formed by taking the same dot products. In other words, if we denote the vectors of B by f_1, f_2, f_3 and given some $x \in \mathbb{R}^2$ we take $d_i = x \cdot f_i$ for $i = 1, 2, 3$, then $x = d_1f_1 + d_2f_2 + d_3f_3$. Even without being orthogonal or linearly independent, the set B retains one of the extremely useful features of an orthonormal basis.
- Both A and B satisfy a property which is known as Parseval's identity for orthonormal bases. In the case of sequence A, this is a version of the Pythagorean Theorem. Let $\|x\|$ denote the length, or norm, of the vector x. Then,

$$\|x\|^2 = \sum_{i=1}^{2} c_i^2 = \sum_{i=1}^{3} d_i^2.$$

  Again, we see that even though B is not even a basis, and certainly not an orthonormal basis for $\mathbb{R}^2$, it maintains an important characteristic of an ONB.

Both sequences A and B are examples of a particular type of *frame*, called a Parseval frame, for the vector space $\mathbb{R}^2$. Sequence B demonstrates that many of the properties of an orthonormal basis can be achieved by nonbases. This is exactly the motivation behind the study of frames. Frames are more general than orthonormal bases, but can often maintain some of the interesting and useful characteristics of ONBs.

The property that makes orthonormal bases desirable in many applications is that we can find the (unique) expansion coefficients for a vector by taking inner products (dot products in $\mathbb{R}^n$ or $\mathbb{C}^n$). This requires fewer computations than matrix inversion, and is numerically more stable. Let's say you want to send a signal across some kind of communication system, perhaps by talking on your wireless phone or sending a photo to your mom over the internet. We think of that signal as a vector in a vector space. The way it gets transmitted is as a sequence of coefficients which represent the signal in terms of a spanning set. If that spanning set is an ONB, then computing those coefficients just involves finding some dot products of vectors, which a computer can accomplish very quickly. As a result, there is not a significant time delay in sending your voice or the photograph. This is a good feature for a communication system to have, so orthonormal bases are used a lot in such situations.

Orthogonality is a very restrictive property, though. What if one of the coefficients representing a vector gets lost in transmission? That piece of information cannot be reconstructed. It is lost. Perhaps we'd like our system to have some redundancy, so that if one piece gets lost, the information can be pieced together from what does get through. This is where frames come in. Generally, a frame for a finite-dimensional vector space is just a spanning set for the vector space. In particular, it need not be a basis, which would require linear independence. Where frames get interesting is that we can find certain frames that retain that very handy ONB property – that we can find the coefficients for expanding vectors using the dot product instead of matrix inversion. We can retain the quick computation time found in ONBs while not restricting ourselves in number, norms, or linear independence.

By using a frame instead of an ONB, we do give up the uniqueness of the coefficients and the orthogonality of the vectors. In many circumstances, however, these properties are superfluous. If you are sending your side of a phone conversation or a photo, what matters is quickly computing a working set of expansion coefficients, not whether those coefficients are unique. In fact, in some settings the linear independence and orthogonality restrictions inhibit the use of ONBs. Frames can be constructed with a wider variety of characteristics, and can thus be tailored to match the needs of a particular system.

This book gives an introduction to the study of frames to undergraduate students who have had a course in linear algebra and a proof-based course in analysis. The first chapter is a review of some concepts from linear algebra, while the second chapter covers more advanced matrix and finite-dimensional operator theory. The discussion of frames begins in Chapter 3. A student can study the chapters in order, but may wish to jump right into Chapter 3 and flip back to the earlier chapters as needed.

We consider Chapters 7, 8, and 9 to be advanced topics. The first two of these describe more theoretical results in the theory of frames, while the last demonstrates how frames are applied to actual problems in sampling theory.