Preface

This book is based on notes I wrote when teaching an undergraduate seminar on surfaces at Brown University in 2005. Each week I wrote up notes on a different topic. Basically, I told the students about many of the great things I have learned about surfaces over the years. I tried to do things in as direct a fashion as possible, favoring concrete results over a buildup of theory. Originally, I had written 14 chapters, but later I added 9 more chapters so as to make a more substantial book.

Each chapter has its own set of exercises. The exercises are embedded within the text. Most of the exercises are fairly routine, and advance the arguments being developed, but I tried to put a few challenging problems in each batch. If you are willing to accept some results on faith, it should be possible for you to understand the material without working the exercises. However, you will get much more out of the book if you do the exercises.

The central object in the book is a surface. I discuss surfaces from many points of view: as metric spaces, triangulated surfaces, hyperbolic surfaces, and so on. The book has many classical results about surfaces, both geometric and topological, and it also has some extraneous stuff that I included because I like it. For instance, the book contains proofs of the Pythagorean Theorem, Pick's Theorem, Green's Theorem, Dehn's Dissection Theorem, the Cauchy Rigidity Theorem, and the Fundamental Theorem of Algebra.

All the material in the book can be found in various textbooks, though there probably isn't one textbook that has it all. Whenever possible, I will point out textbooks or other sources where you can read more about what I am talking about. The various fields of math surrounding the concept of a surface—geometry, topology, complex analysis, combinatorics—are deeply intertwined and often related in surprising ways. I hope to present this tapestry of ideas in a clear and rigorous yet informal way.

My general view of mathematics is that most of the complicated things we learn have their origins in very simple examples and phenomena. A good way to master a body of mathematics is to first understand all the sources that lead to it. In this book, the *square torus* is one of the key simple examples. A great deal of the theory of surfaces is a kind of elaboration of phenomena one encounters when studying the square torus. In the first chapter of the book, I will introduce the square torus and describe the various ways that its structure can be modified and generalized. I hope that this first chapter serves as a good guide to the rest of the book.

I aimed the class at fairly advanced undergraduates, but I tried to cover each topic from scratch. My idea is that, with some effort, you could learn the material for the whole course without knowing too much advanced math. You should be perfectly well prepared for the intended version of the class if you have had a semester each of real analysis, abstract algebra, and complex analysis. If you have just had the first 2 items, you should still be alright, because I embedded a kind of mini-course on complex analysis in the middle of the book.

Following an introductory chapter, this book is divided into 6 parts. The first 5 parts have to do with different aspects of the theory of surfaces. The 6th part is a collection of several topics, loosely related to the rest of the book, which I included because I really like them. Here is an outline of the book.

Part 1: Surfaces and Topology. In this part, we define such concepts as *surface*, *Euler characteristic*, *fundamental group*, *deck group*, and *covering space*. We prove that the deck group of a surface and its fundamental group are isomorphic. We also prove, under some conditions, that a space has a universal cover.

Part 2: Surfaces and Geometry. The first 3 chapters in this part introduce Euclidean, spherical, and hyperbolic geometry, respectively. (In the Euclidean case, which is so well known, we concentrate on nontrivial theorems.) Following this, we discuss the notion of a Riemannian metric on a surface. In the final chapter, we discuss hyperbolic surfaces, as special examples of Riemannian manifolds.

Part 3: Surfaces and Complex Analysis. In this part, we give a rapid primer on the main points taught in the first semester of complex analysis. Following this, we introduce the concept of a Riemann surface and prove some results about complex analytic maps between Riemann surfaces.

Part 4: Flat Cone Surfaces. In this part, we define what is meant by a *flat cone surface*. As a special case, we consider the notion of a *translation surface*. We show how the "affine symmetry group" of a translation surface, known as the Veech group, leads right back to complex analysis and hyperbolic geometry. We end this part with an application to polygonal billiards.

Part 5: The Totality of Surfaces. In this part, we discuss the basic objects one considers when studying the totality of all flat or hyperbolic surfaces, namely *moduli space*, *Teichmüller space*, and the *mapping class group*. As a warmup for the flat-surface case, we discuss continued fractions and the modular group in detail.

Part 6: Dessert. In this part, we prove 3 classic results in geometry. The Banach–Tarski Theorem says that—assuming the Axiom of Choice—you can cut up a ball of radius 1 into finitely many pieces and rearrange those pieces into a (solid) ball of radius 2. Dehn's Theorem says that you cannot cut up a cube with planar cuts and rearrange it into a regular tetrahedron. The Cauchy Rigidity Theorem says roughly that you cannot flex a convex polyhedron.