Chapter 1

Preliminaries

> Unsolved problems abound, and additional interesting open questions arise faster than solutions to the existing problems. – F. Harary

The above quote, which appeared in the 1983 article "A Tribute to F. P. Ramsey," is at least as *apropos* today as it was then. In this book alone, which covers only a modest portion of Ramsey theory, you will find a great number of open research problems. The beauty of Ramsey theory, especially Ramsey theory dealing with the set of integers, is that, unlike many other mathematical fields, very little background is needed to understand the problems. In fact, with just a basic understanding of some of the topics in this text, and a desire to discover new results, the undergraduate mathematics student will be able to experience the excitement and challenge of doing mathematical research.

Ramsey theory is named after Frank Plumpton Ramsey and his eponymous theorem, which he proved in 1928 (it was published posthumously in 1930). So, what is Ramsey theory? Although there is no universally accepted definition of Ramsey theory, we offer the following informal description:

> Ramsey theory is the study of the preservation of properties under set partitions.

> In other words, given a particular set S that has a property P, is it true that whenever S is partitioned into finitely many subsets, one of the subsets must also have property P?

To illustrate further what sorts of problems Ramsey theory deals with, here are a few simple examples of Ramsey theory questions.

Example 1.1. Obviously, the equation $x + y = z$ has a solution in the set of positive integers (there are an infinite number of solutions); for example, $x = 1$, $y = 4$, $z = 5$ is one solution. Here's the question: is it true that whenever the set of positive integers is partitioned into a finite number of sets $S_1, S_2, \ldots, S_r$, then at least one of these sets will contain a solution to $x + y = z$? The answer turns out to be yes, as we shall see later in this chapter.

Example 1.2. Is it true that whenever the set $S = \{1, 2, \ldots, 100\}$ is partitioned into two subsets A and B, then at least one of the two subsets contains a pair of integers which differ by exactly two? To answer this question, consider the partition consisting of

$$A = \{1, 5, 9, 13, \ldots, 97\} \cup \{2, 6, 10, \ldots, 98\}$$

and

$$B = \{3, 7, 11, \ldots, 99\} \cup \{4, 8, 12, \ldots, 100\}.$$

We see that neither A nor B contains a pair of integers that differ by two, so that the answer to the given question is no. So, our original set S has the property that it contains 2 integers that differ by two; however, we have presented a partition of S for which none of the partitioned parts inherit this property.

Example 1.3. True or false: if there are 18 people in a group, then there must be either 4 people who are mutual acquaintances or 4 people who are mutual "strangers" (no two of whom have ever met). (You will find the answer to this one later in the chapter.) Here we are partitioning pairs of people based on the relationship between those two people. We want to investigate if one of the partition classes contains all $\binom{4}{2}$ pairs defined between 4 people, for some set of 4 people.

There is a wide range of structures and sets with which Ramsey theory questions may deal, including the real numbers, algebraic structures such as groups or vector spaces, graphs, points in the plane or in n dimensions, and others. This book limits its scope to Ramsey theory on the set of integers. (There is one exception – Ramsey's theorem itself – which is covered in this chapter.)

In this chapter we introduce the reader to some of the most well-known and fundamental theorems of Ramsey theory. We also present some of the basic terminology and notation that we will use.

1.1. The Pigeonhole Principle

Imagine yourself as a mailroom clerk in a mailroom with n slots in which to place the mail. If you have $n + 1$ pieces of mail to place into the n slots, what can we say about the amount of mail that will go into a slot? Well, we can't say much about the amount of mail a particular slot receives because, for example, one slot may get all of the mail (or none of the mail). However, we can say that *some* slot must end up with at least two pieces of mail. To see this, imagine that you are trying to avoid having any slot with more than one piece of mail. By placing one piece of mail at a time into an unoccupied slot you can sort n pieces of mail. However, since there are more than n pieces of mail, you will run out of unoccupied slots before you are done. Hence, at least one slot must have at least two pieces of mail.

This simple idea is known as the pigeonhole principle (or Dirichlet's box principle, although this name is much less common), which can be stated this way:

If more than n pigeons are put into n pigeonholes, then some pigeonhole must contain at least two pigeons.

We now present the pigeonhole principle using somewhat more mathematical language.

Theorem 1.4 (Basic Pigeonhole Principle)**.** *If an n-element set is partitioned into r disjoint subsets where $n > r$, then at least one of the subsets contains more than one element.*

Example 1.5. Consider the well-known 2-player game Tic-Tac-Toe and let S be the set of all people who have ever played Tic-Tac-Toe. We claim that there must be two people in S who have had the same number of opponents. To see this, let there be n people in S. Each of these n people has had at least 1 opponent and no more than $n-1$ opponents. Place each person in S into a category based on the number of other people in S that he/she has had as an opponent. Since we are placing n people (pigeons) into $n-1$ categories (pigeonholes), there must exist 2 people in the same category, thereby proving the statement.

Theorem 1.4 is a special case of the following more general principle.

Theorem 1.6 (Generalized Pigeonhole Principle)**.** *If more than mr elements are partitioned into r sets, then some set contains more than m elements.*

Proof. Let S be a set with $|S| > mr$. Let $S = S_1 \cup S_2 \cup \cdots \cup S_r$ be any partition of S. Assume, for a contradiction, that $|S_i| \leq m$ for all $i = 1, 2, \ldots, r$. Then

$$|S| = \sum_{i=1}^{r} |S_i| \leq mr,$$

a contradiction. Hence, for at least one i, the set S_i contains more than m elements, i.e., $|S_i| \geq m + 1$. □

We see that Theorem 1.4 is a special case of Theorem 1.6 by taking $m = 1$. There are other common formulations of the pigeonhole principle; you will find some of these in the exercises.

Although the pigeonhole principle is such a simple concept, and seems rather obvious, it is a very powerful result, and it can be used to prove a wide array of not-so-obvious facts.

Here are some examples.

Example 1.7. For each integer $n = 1, 2, \ldots, 200$, let $R(n)$ be the remainder when n is divided by 7. Then some value of $R(n)$ must occur at least 29 times. To see this, we can think of the 200 integers as the pigeons, and the seven possible values of $R(n)$ as the pigeonholes. Then, according to Theorem 1.6, since $200 > 28(7)$, one of the pigeonholes must contain more than 28 elements.

Example 1.8. We will show that within any sequence of n^2+1 integers there exists a monotonic subsequence of length $n+1$. (A sequence $\{x_i\}_{i=1}^{m}$ is called monotonic if it is either nondecreasing or nonincreasing). Let our sequence be $\{a_i\}_{i=1}^{n^2+1}$. For each $i \in \{1, 2, \ldots, n^2+1\}$, let ℓ_i be the length of the longest nondecreasing subsequence starting at (and including) a_i. If $\ell_i \geq n+1$ for some i, then the result clearly holds; hence, we may assume that $\ell_i \leq n$ for all i, $1 \leq i \leq n^2+1$. Since each of the ℓ_i's has a value between 1 and n, by Theorem 1.6 with $m = r = n$, there exists $j \in \{1, 2, \ldots, n\}$ so that $n+1$ of the numbers ℓ_i equal j. Call these $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_{n+1}}$, where $i_1 < i_2 < \cdots < i_{n+1}$. Next, look at the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$. We claim that this is a nonincreasing subsequence of length $n+1$. To see this, assume, for a contradiction, that it is not nonincreasing. Then $a_{i_k} < a_{i_{k+1}}$ for some k. Hence, the nondecreasing subsequence of length j starting with $a_{i_{k+1}}$ creates a nondecreasing subsequence of length $j+1$ by starting with a_{i_k}. This is a contradiction since $\ell_{i_k} = j$.

Next, we give another example for which the pigeonhole principle may not immediately appear to be applicable.

Example 1.9. Color each point in the xy-plane having integer coordinates either red or blue. We show that there must be a rectangle with all of its vertices the same color. Consider the lines $y = 0$, $y = 1$, and $y = 2$ and their intersections with the lines $x = i$, $i = 1, 2, \ldots, 9$. On each line $x = i$ there are three intersection points colored either red or blue. Since there are only $2^3 = 8$ different ways to color three points either red or blue, by the pigeonhole principle two of the vertical lines, say $x = j$ and $x = k \neq j$ must have the identical coloring (i.e., the color of (j, y) is the same as the color of (k, y) for $y = 0, 1, 2$). Using the pigeonhole principle again, we see that two of the points $(j, 0), (j, 1)$, and $(j, 2)$ must be the same color, say (j, y_1) and (j, y_2). Then the rectangle with vertices $(j, y_1), (j, y_2), (k, y_1)$, and (k, y_2) is the desired rectangle.

In the last example, we used colors as the "pigeonholes." Using colors to represent the subsets of a partition in this way is often convenient, and is quite typical in many areas of Ramsey theory.

1.2. Ramsey's Theorem

Ramsey's theorem can be considered a refinement of the pigeonhole principle, where we are not only guaranteed a certain number of elements in a pigeonhole, but we also have a guarantee of a certain relationship between these elements. It is a theorem that is normally stated in terms of the mathematical concept known as a graph. We will define what we mean by a graph very shortly, but before doing so, we consider the following example, known as the Party problem.

Example 1.10. We will prove the following: at a party of six people, there must exist either three people who have all met one another or three people who are mutual strangers (i.e., no two of whom have met). By the pigeonhole principle, we are guaranteed that for each person, there are three people that person has met or three people that person has never met. We now want to show that there are three people with a certain relationship *between* them; namely, three people who all have met one another, or three people who are mutual strangers. First, assign to each pair of people one of the colors red or blue, with a red "line" connecting two people who have met, and a blue "line" connecting two people who are strangers. Hence, we want to show that for any coloring of the lines between people using the colors red and blue, there is either a red triangle or a blue triangle (with the people as vertices). To this end, single out one person at the party, say person X. Since there are five other people at the party, by the pigeonhole principle X either knows at least three people, or is a stranger to at least three people. We may assume, without loss of generality, that X knows at least 3 people at the party. Call these people A, B, and C. So far we know that the lines connecting X to each of A, B, and C are red. If there exists a red line between any of A, B, and C then we are done, since, for example, a red line between A and B would give the red triangle ABX. If the lines connecting A, B, and C are all blue, then ABC is a blue triangle.

Concerning the Party problem, another question we might ask is this: is 6 the lowest number of party members for the property we seek to hold? That is, does there exist a way to have five people at the party and not have either of the types of "triangles" discussed in the above example? To see that we cannot have only five people at the party and *guarantee* the same result, place five people in a circle and assume that each person knows the two people next to him/her, but no one else (draw a sketch to see that there is no red triangle and no blue triangle).

The fact that there is a solution to the Party problem is a special case of what is known as Ramsey's theorem. In order to state Ramsey's theorem we will use a few definitions from graph theory.

Definition 1.11. A *graph* $G = (V, E)$ is a set V of points, called *vertices*, and a set E of pairs of vertices, called *edges*.

Definition 1.12. A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

Definition 1.13. A *complete graph on n vertices*, denoted K_n, is a graph on n vertices, with the property that every pair of vertices is connected by an edge.

Definition 1.14. An *edge-coloring* of a graph is an assignment of a color to each edge of the graph. A graph that has been edge-colored is called a *monochromatic graph* if all of its edges are the same color.

We may now express the solution to the Party problem in graph-theoretical language. It says that every 2-coloring, using the colors red and blue, of the edges of K_6 must admit either a red K_3 (a triangle) or a blue K_3; and furthermore, that there exists a 2-coloring of the edges of K_5 that fails to have this property.

We now state Ramsey's theorem for two colors.

Theorem 1.15 (Ramsey's Theorem for Two Colors)**.** *Let $k, \ell \geq 2$. There exists a least positive integer $R = R(k, \ell)$ such that every edge-coloring of K_R, with the colors red and blue, admits either a red K_k subgraph or a blue K_l subgraph.*

Proof. First note that $R(k, 2) = k$ for all $k \geq 2$, and $R(2, \ell) = \ell$ for all $\ell \geq 2$ (this is easy). We proceed via induction on the sum $k + \ell$, having taken care of the case when $k + \ell = 5$. Hence, let $k + \ell \geq 6$, with $k, \ell \geq 3$. We may assume that both $R(k, \ell - 1)$ and $R(k - 1, \ell)$ exist. We claim that $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$, which will prove the theorem.

Let $n = R(k-1, \ell)+R(k, \ell-1)$. Now choose one particular vertex, v, from K_n. Then there are $n - 1$ edges from v to the other vertices. Let A be the number of red edges and B be the number of blue edges coming out of v. Then, either $A \geq R(k - 1, \ell)$ or $B \geq R(k, \ell - 1)$, since if $A < R(k - 1, \ell)$ and $B < R(k, \ell - 1)$, then $A + B \leq n - 2$, contradicting the fact that $A + B = n - 1$. We may assume, without loss of generality, that $A \geq R(k - 1, \ell)$. Let V be the set of vertices connected to v by a red edge, so that $|V| \geq R(k - 1, \ell)$. By the inductive hypothesis, K_V contains either a red K_{k-1} subgraph or a blue K_ℓ subgraph. If it contains a blue K_ℓ subgraph, we are done. If it contains a red K_{k-1} subgraph, then by connecting v to each vertex of this red subgraph we have a red K_k subgraph (since v is connected to V by only red edges), and the proof is complete. □

The numbers $R(k, \ell)$ are known as the 2-color Ramsey numbers. The solution to the Party problem tells us that $R(3, 3) = 6$. By Ramsey's theorem, we may extend the Party problem in various ways. For example, we know there exists a number n so that if there were n people at a party, then there would have to be either a group of four mutual acquaintances or a group of five mutual strangers. This number n is the Ramsey number $R(4, 5)$.

There are other ways to extend the Party problem. For example, in Exercise 1.11 we consider the case where people either love, hate, or are indifferent to, each other. In this situation we want to find three people who all love one another, three people who all hate one another, or three people who are all indifferent toward one another. Exercise 1.11 states that 17 people at the party will suffice (in fact 17 is the least such number with this property, but you cannot conclude this from Exercise 1.11). This is an example of a 3-color Ramsey number. More generally, Ramsey's theorem for two colors can easily be generalized to $r \geq 3$ colors (this is left as Exercise 1.18), in which case the Ramsey numbers are denoted by $R(k_1, k_2, \ldots, k_r)$. In case $k_i = k$ for $i = 1, \ldots, r$, we use the simpler notation $R_r(k)$. Thus, for example, in the "love-hate-indifferent" problem, we have $R(3, 3, 3) = R_3(3) = 17$.

The existence of the Ramsey numbers has been known since 1930. However, they are notoriously difficult to compute; the only known values are $R(3, 3) = 6$, $R(3, 4) = 9$, $R(3, 5) = 14$, $R(3, 6) = 18$, $R(3, 7) = 23$, $R(3, 8) = 28$, $R(3, 9) = 36$, $R(4, 4) = 18$, $R(4, 5) = 25$, and $R(3, 3, 3) = 17$. (The fact that $R(4, 4) = 18$ answers, in the affirmative, the question posed in Example 1.3.)

Obviously, Ramsey theory is named after Frank Ramsey. However, his famous theorem is the only result of Ramsey's in the field named after him. Unfortunately, Ramsey died of complications due to jaundice in 1930, a month before his 27th birthday, but not before he left his mark.

1.3. Some Notation

In this section we will cover some notation that we will frequently use.

We shall denote the set of integers by $\mathbb{Z}$, the set of positive integers by $\mathbb{Z}^+$, and the set of real numbers by $\mathbb{R}$. Most of our work will be confined to the set of integers. Hence, when speaking about an "interval" we will mean a set of the form $\{a, a+1, \ldots, b\}$, where $a < b$ are integers. Usually we will denote this interval more simply by $[a, b]$.

When dealing with two sets X and Y, we will sometimes use the set $S = X - Y$, which we define to be the set of elements in X that are not in Y. Also, for S a set and a a real number, $a + S$ and aS will denote $\{a + s : s \in S\}$ and $\{as : s \in S\}$, respectively. We may sometimes write $S + a$ instead of $a + S$.

Oftentimes we will find it convenient to use symbols such as $0, 1$, or 2 to stand for different "colors" rather than actual color names such as red or blue. We make this more formal in the following definition.

Definition 1.16. An *r-coloring* of a set S is a function $\chi : S \to C$, where $|C| = r$.

Typically, we will use $C = \{0, 1, \ldots, r-1\}$ or $C = \{1, 2, \ldots, r\}$. We can think of an r-coloring χ of a set S as a partition of S into r subsets $S_1, S_2, \ldots, S_r$, by associating the subset S_i with the set $\{x \in S : \chi(x) = i\}$.

The next definition will be used extensively.

Definition 1.17. A coloring χ is *monochromatic* on a set S if χ is constant on S.

Example 1.18. Let $\chi : [1,5] \to \{0,1\}$ be defined by $\chi(1) = \chi(2) = \chi(3) = 1$ and $\chi(4) = \chi(5) = 0$. Then χ is a 2-coloring of $[1,5]$ that is monochromatic on $\{1,2,3\}$ and on $\{4,5\}$.

We will often find it convenient to represent a particular 2-coloring of an interval as a string of 0's and 1's. For example, the coloring in Example 1.18 could be represented by the string 11100. We may also abbreviate this coloring by writing $1^3 0^2$. We may extend this notation to r-colorings for $r \geq 3$ by using strings with symbols belonging to the set $\{0, 1, 2, \ldots, r-1\}$. For example, define the 3-coloring χ on the interval [1,10] by $\chi(i) = 0$ for $1 \leq i \leq 5$, $\chi(i) = 1$ for $6 \leq i \leq 9$, and $\chi(10) = 2$. Then we may write $\chi = 0000011112$ or, equivalently, $\chi = 0^5 1^4 2$.

Sometimes we will want to describe the magnitude of functions asymptotically. For this purpose we mention two very commonly used symbols, called "Big-O" and "little-o."

Let $f(n)$ and $g(n)$ be functions that are nonzero for all n. We say that $f(n) = O(g(n))$ if there exist constants $c, m > 0$, independent of n, such that $0 < \left|\frac{f(n)}{g(n)}\right| \leq c$ for all $n > m$. In other words, $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| \leq c$, if the limit exists. We say that $f(n) = o(g(n))$ if for all $c > 0$ there exists a constant $m > 0$, independent of n, such that $\left|\frac{f(n)}{g(n)}\right| < c$ for all $n > m$. In other words, $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$.

If f and g are nonzero functions such that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists and is equal to ℓ, where $|\ell| \neq \infty$ and $\ell \neq 0$, then $f(n) = O(g(n))$. If $\ell = 0$ we have $f(n) = o(g(n))$. If $\ell = \infty$, then we have $g(n) = o(f(n))$ by taking the reciprocal of the argument of the limit. Intuitively, if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n)$ and $g(n)$ have a similar growth rate (and we may write $f(n) = \Omega(g(n))$, but this notation won't be used in this book); and if $f(n) = o(g(n))$, then $f(n)$ is insignificant compared to $g(n)$, for large n.

If $f(n)$ and $g(n)$ are functions with the same growth rate, i.e., if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, we may write $f(n) \sim g(n)$.

An example to explain these concepts is in order.

Example 1.19. Let $f(n) = \frac{n^2}{22} + 5n$ and $g(n) = n^2$. Then $f(n) = O(g(n))$, or, equivalently, $f(n) = O(n^2)$. We may also describe $f(n)$'s rate of growth by $f(n) = \frac{n^2}{22}(1 + o(1))$. To see this, we have

$$\frac{n^2}{22}(1 + o(1)) = \frac{n^2}{22} + o(1)\frac{n^2}{22}.$$

Now, since $\frac{5n}{n^2/22} = \frac{110}{n}$ and $\lim_{n\to\infty} \frac{110/n}{1} = 0$, we have $5n = o(1)\frac{n^2}{22}$. We may also write $f(n) \sim \frac{n^2}{22}$ to describe the growth rate of $f(n)$.

We will also use the following functions. For x a real number, we use $\lfloor x \rfloor$ to denote the greatest integer n such that $n \leq x$ (this is often called the "floor" function). The least integer function of a real number x, defined as the least integer n such that $n \geq x$, is denoted by $\lceil x \rceil$ (this is often referred to as the "ceiling function.")

1.4. Three Classical Theorems

Somewhat surprisingly, Ramsey's theorem was not the first, nor even the second, theorem in the area now known as Ramsey theory. The results that are generally accepted to be the earliest Ramsey-type theorems are due, in chronological order, to Hilbert, Schur, and van der Waerden. All of these results, which preceded Ramsey's theorem, deal with colorings of the integers, the theme of this book. Interestingly, even though Ramsey's theorem is a theorem about graphs, we will see later that it can be used to give some Ramsey-type results about the integers.

In this section we introduce three classical theorems concerning Ramsey theory on the integers. We will talk much more about each of these theorems in later chapters.

We start with a reminder of what an arithmetic progression is.

Definition 1.20. A *k-term arithmetic progression* is a sequence of the form $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$, where $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$.

We now state van der Waerden's theorem, which was proved in 1927.

Theorem 1.21 (Van der Waerden's Theorem)**.** *For all positive integers k and r, there exists a least positive integer $w(k;r)$ such that for every r-coloring of $[1, w(k;r)]$ there is a monochromatic k-term arithmetic progression.*

The numbers $w(k;r)$ are known as the van der Waerden numbers. Let's look at a simple case. Let $k = r = 2$. Hence, we want to find the minimum integer $w = w(2;2)$ so that no matter how we partition the interval $[1, w] = \{1, 2, \ldots, w\}$ into two subsets (i.e., 2-color $[1, w]$), we must end up with at least one of the two subsets containing a pair of elements $a, a+d$, where $d \geq 1$ (i.e., we must end up with a monochromatic 2-term arithmetic progression). Consider a 2-coloring of $\{1, 2\}$ where 1 and 2 are assigned different colors. Obviously, under such a coloring, $\{1, 2\}$ does not contain a 2-term arithmetic progression that is monochromatic. Thus, $w(2;2)$ is greater than 2 (not every 2-coloring of $[1, 2]$ yields the desired monochromatic sequence). Does 3 work? That is, does every 2-coloring of $[1, 3]$ yield a monochromatic 2-term arithmetic progression? The answer is yes, by a simple application of the pigeonhole principle, since any 2-element set of positive integers is a 2-term arithmetic progression. Thus, we have shown that $w(2;2) = 3$.

Finding $w(2;2)$ was rather simple. All the van der Waerden numbers $w(2;r)$ are just as easy to find (we leave this as an exercise in a later chapter). For $k \geq 3$, the evaluation of these numbers very quickly becomes much more difficult. In fact, the only known van der Waerden numbers are $w(3;2) = 9$, $w(3;3) = 27$, $w(3;4) = 76$, $w(4;2) = 35$, $w(5;2) = 178$, and $w(6;2) = 1132$. Besides trying to find exact values of the van der Waerden numbers, there is another open question that has been one of the most difficult, and most appealing, problems in Ramsey theory. Namely, finding a reasonably good estimate of $w(k;r)$ in terms of k and r. We shall talk more about such questions, and the progress that has been made on them, in Chapter 2.

Van der Waerden's theorem has spawned many results in Ramsey theory. For this reason, and because the notion of an arithmetic progression is such a natural and simple concept, a large portion of this book is dedicated to various offshoots, refinements, extensions, and generalizations of van der Waerden's theorem.

The next two main results deal with solutions to equations and systems of equations. Let $\mathcal{E}$ represent a given equation or system of equations. We call $(x_1, x_2, \ldots, x_k)$ a *monochromatic solution to* $\mathcal{E}$ if $x_1, x_2, \ldots, x_k$ are all the same color and they satisfy $\mathcal{E}$.

The next theorem we present, proved by Issai Schur in 1916, is one of the earliest results in Ramsey theory.

Theorem 1.22 (Schur's Theorem)**.** *For any $r \geq 1$, there exists a least positive integer $s = s(r)$ such that for every r-coloring of $[1, s]$ there exists a monochromatic solution to $x + y = z$.*

The numbers $s(r)$ are called the Schur numbers. As a simple example, we look at $s(2)$. Here we want the least positive integer s so that whenever $[1, s]$ is 2-colored, there will exist monochromatic integers x, y, z (not necessarily distinct) satisfying $x + y = z$. Notice that $s(2)$ must be greater than four, because if we take the 2-coloring χ of [1,4] defined by $\chi(1) = \chi(4) = 0$ and $\chi(2) = \chi(3) = 1$, then it is not possible to find x, y, and z all of the same color satisfying $x + y = z$. Meanwhile, every 2-coloring of $[1, 5]$ (there are $2^5 = 32$ of them) does yield such a monochromatic triple (this is proved in Example 8.5). Thus, $s(2) = 5$.

As it turns out, the only Schur numbers that are currently known are $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, and $s(4) = 45$. We will learn much more about Schur's theorem in Chapter 8.

The third classical theorem we mention is Rado's theorem, which is a generalization of Schur's theorem. In fact, Richard Rado was a student of Schur. The idea of Rado's theorem may be described as follows. Thinking of Schur's theorem as a theorem about the homogeneous linear equation $x+y-z = 0$, we ask the following more general question. Which systems, $\mathcal{L}$, of homogeneous linear equations with integer coefficients have the following property: for every $r \geq 1$, there exists a least positive integer $n = n(\mathcal{L}; r)$ such that every r-coloring of $[1, n]$ yields a monochromatic solution to $\mathcal{L}$?

In a series of articles published in the 1930's, Rado completely answered this question. Since Rado's theorem in its most general form is a bit complicated to describe, we will postpone stating the general theorem until Chapter 9, which is devoted to Rado's theorem. Instead, we mention here the special case of Rado's theorem in which the system consists of only a single equation.

We first need the following definition.

Definition 1.23. For $r \geq 1$, a linear equation $\mathcal{E}$ is called *r-regular* if there exists $n = n(\mathcal{E}; r)$ such that for every r-coloring of $[1, n]$ there is a monochromatic solution to $\mathcal{E}$. It is called *regular* if it is r-regular for all $r \geq 1$.

Example 1.24. Using Definition 1.23, Schur's theorem can be stated as "the equation $x + y = z$ is regular."

We now state Rado's theorem for a single equation.

Theorem 1.25 (Rado's Single Equation Theorem). *Let $\mathcal{E}$ represent the linear equation $\sum_{i=1}^{n} c_i x_i = 0$, where $c_i \in \mathbb{Z} - \{0\}$ for $1 \leq i \leq n$. Then $\mathcal{E}$ is regular if and only if some nonempty subset of the c_i's sums to 0.*

Example 1.26. The equation $x + y = z$, i.e., $x + y - z = 0$, satisfies the requirements of Theorem 1.25. Hence, as noted before and proved by Schur, $x + y = z$ is regular.

Example 1.27. It follows from Rado's theorem that the equation $3x_1 + 4x_2 + 5x_3 - 2x_4 - x_5 = 0$ is regular, since the sum of the first, fourth, and fifth coefficients is 0.

1.5. A Little More Notation

The three classical theorems mentioned above all have a somewhat similar flavor. That is, they have the following general form: there exists a positive integer $n(r)$ such that for every r-coloring of $[1, n(r)]$ there is a monochromatic set belonging to a particular family of sets.

In one case, the family of sets was the k-term arithmetic progressions; in another case the family consisted of all solutions to a certain equation; and so on. Throughout this book we will be looking at this type of problem, and so it will be worthwhile to have a general notation that can be used for any such problem.

Notation. Let $\mathcal{F}$ be a certain family of sets, and let k and r be positive integers. We denote by $R(\mathcal{F}, k; r)$ the least positive integer, if it exists, such that for any r-coloring of $[1, R(\mathcal{F}, k; r)]$, there is a monochromatic member of $\mathcal{F}$ of size k. In the case where no such integer exists, we say $R(\mathcal{F}, k; r) = \infty$. Because our discussion will often be confined to the situation in which the number of colors is two, we often denote the function $R(\mathcal{F}, k; 2)$ more simply as $R(\mathcal{F}, k)$. If the length of the sequence is understood (as in Schur's theorem), we write $R(\mathcal{F}; r)$.

For certain Ramsey-type functions we deal with, it will be convenient to use a notation other than $R(\mathcal{F}, k; r)$. For example, later in the book we will encounter a type of sequence called a descending wave, for which we will use the notation $DW(k; r)$ rather than something like $R(DW, k; r)$ (where DW would represent the family of all descending waves). Similarly, since the notation $w(k; r)$ is so standard, we will use $w(k; r)$ instead of $R(AP, k; r)$ and $w(k)$ instead of $R(AP, k)$, where AP is the family of all arithmetic progressions. Finally, we remark that the notation $R(k, \ell)$ is reserved for the classical Ramsey numbers defined in Section 1.2 (note the absence of a family $\mathcal{F}$ here).

Throughout this book we will be considering various collections, $\mathcal{F}$, of sets of integers and, as with the three classical theorems of Section 1.4, wanting to know if, for a specified value of r and a particular set $M \subseteq \mathbb{Z}$, every r-coloring of M yields a monochromatic member of $\mathcal{F}$. For the case in which M is the set of positive integers, we have the following definition.

Definition 1.28. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{Z}^+$, and let $r \geq 1$. If for every r-coloring of $\mathbb{Z}^+$ and all $k \geq 1$, there is a monochromatic k-element member of $\mathcal{F}$, then we say that $\mathcal{F}$ is *r-regular.* If $\mathcal{F}$ is r-regular for all r, we say that $\mathcal{F}$ is *regular.*

Sometimes we will replace the phrase "for all $k \geq 1$, there is a monochromatic k-element member of $\mathcal{F}$" by "there are arbitrarily large members of $\mathcal{F}$."

Example 1.29. Let $\mathcal{F} = AP$, the collection of all arithmetic progressions. By van der Waerden's theorem, $\mathcal{F}$ is regular since for every finite coloring of $\mathbb{Z}^+$ there exists, for every $k \geq 1$, a monochromatic k-term arithmetic progression.

Whereas Definition 1.28 pertains to *all* colorings of a set, we will also want to consider whether or not a particular coloring of a set M yields a monochromatic member of the collection $\mathcal{F}$. For this we have the next definition.

Definition 1.30. Let $\mathcal{F}$ be a family of subsets of $\mathbb{Z}$ and let k be a positive integer. Let $r \geq 1$. An r-coloring of a set $M \subseteq \mathbb{Z}$ is called $(\mathcal{F}, k; r)$-*valid* if there is no monochromatic k-element member of $\mathcal{F}$ contained in M.

When the number of colors is understood, we may simply say that a coloring is $(\mathcal{F}, k)$-valid. Also, when there is no possible confusion as to the meaning of $\mathcal{F}$ or the value of k, we may simply say that a coloring is *valid*.

As an example, if $\mathcal{F}$ is the family of sets of even numbers, then the 2-coloring of $[1, 10]$ represented by the binary sequence 1110001110 is $(\mathcal{F}, 4)$-valid since there is no monochromatic 4-term sequence belonging to $\mathcal{F}$ (i.e., there do not exist four even numbers that have the same color).

Let's consider another example.

Example 1.31. Let $\mathcal{F}$ be the family of all subsets of $\mathbb{Z}^+$. We will determine a precise formula for $R(\mathcal{F}, k; r)$. First, let χ be any r-coloring of $[1, r(k-1)+1]$. By the generalized pigeonhole principle, since we are partitioning a set of $r(k-1)+1$ elements into r sets, there must be, for some color, more than $k-1$ elements of that color. Thus, under χ, there is a monochromatic k-element member of $\mathcal{F}$. Since χ is an arbitrary r-coloring, we know that $R(\mathcal{F}, k; r) \leq r(k-1)+1$. On the other hand, there do exist r-colorings of the interval $[1, r(k-1)]$ that are $(\mathcal{F}, k)$-valid. Namely, assign exactly $k-1$ members of the interval to each of the colors. Then no color will have a k-element member of $\mathcal{F}$. Thus, $R(\mathcal{F}, k; r) \geq r(k-1)+1$, and hence $R(\mathcal{F}, k; r) = r(k-1)+1$.

The fact that in Example 1.31 the numbers $R(\mathcal{F}, k; r)$ always exist is not very surprising. After all, $\mathcal{F}$ is so plentiful that it is easy to find a monochromatic member. When the family of sets we are considering is not as "big," the behavior of the associated Ramsey function is much less predictable. For certain $\mathcal{F}$ we will find that $R(\mathcal{F}, k; r) < \infty$ for all k and r, while for others this will only happen (for all k) provided r does not exceed a certain value. There will even be cases where $R(\mathcal{F}, k; r)$ never exists except for a few small values of k and r, even though $\mathcal{F}$ seems reasonably "big."

We will encounter many different results in this book, but the common thread will be an attempt to find answers, to whatever extent we can, to the following two questions.

1. For which $\mathcal{F}$, k, and r does $R(\mathcal{F}, k; r)$ exist?
2. If $R(\mathcal{F}, k; r)$ exists, what can we say about its magnitude?

1.6. Exercises

1.1 A bridge club has 10 members. Every day, four members of the club get together and play one game of bridge. Prove that after two years, there is some particular set of four members that has played at least four games of bridge together.

1.2 Prove that if the numbers $1, 2, \ldots, 12$ are randomly positioned around a circle, then some set of three consecutively positioned numbers must have a sum of at least 19.

1.3 Prove the following versions of the pigeonhole principle.

a) If $a_1, a_2, \ldots, a_n, c$ are real numbers such that $\sum_{i=1}^{n} a_i \geq c$, then there is at least one value of i such that $a_i \geq \frac{c}{n}$.

b) If $a_1, a_2, \ldots, a_n$ are integers, and c is a real number such that $\sum_{i=1}^{n} a_i \geq c$, then there is at least one value of i such that $a_i \geq \left\lceil \frac{c}{n} \right\rceil$.

1.4 With regard to Example 1.8, show that, given a sequence of only n^2 numbers, there need not be a monotonic subsequence of length $n+1$.

1.5 Let $r \geq 2$. Show that there exists a least positive integer $M = M(k;r)$ so that any r-coloring of M integers admits a monochromatic monotonic k-term subsequence. Determine $M(k;r)$. (Note that from Example 1.8, $M(k+1;1) = k^2+1$.)

1.6 Let $r \geq 3$. Let χ be any r-coloring of the set

$$S = \{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Z}\}$$

(the members of S are sometimes called lattice points). Show that, under χ, there must exist a rectangle with all vertices the same color.

1.7 Explain how the Party problem fits the description of Ramsey theory offered on page 1.

1.8 Since $R(3,3) = 6$, we know that any 2-coloring of K_6 must admit at least one monochromatic triangle. In fact, any 2-coloring of K_6 must admit at least two monochromatic triangles. Prove this fact.

1.9 Show that any 2-coloring of K_7 must admit at least four monochromatic triangles.

1.10 Generalize Exercises 1.8 and 1.9 above to K_n. (Hint: Let r_i, $i = 1, 2, \ldots, n$, be the number of red edges connected to vertex i. Show that the number of monochromatic triangles is thus $\binom{n}{3} - \frac{1}{2}\sum_{i-1}^{n} r_i(n-1-r_i)$. Minimize this function to deduce the result.)

1.11 Show that any 3-coloring of K_{17} must admit at least one monochromatic triangle, via an argument similar to the one showing $R(3,3) \leq 6$, and using the fact that $R(3,3) = 6$.

1.12 Prove that $R(k,k)$ exists without making use of the existence of $R(k,\ell)$ for $k \neq \ell$. In other words, prove the following statement *without* introducing another variable: for any $k \in \mathbb{Z}^+$, there exists a minimum number of vertices $R(k)$ such that every 2-coloring of the edges of $K_{R(k)}$ admits a monochromatic K_k.

1.13 Explain why $R(k,\ell) = R(\ell,k)$.

1.14 Prove that $R(k,\ell) < R(k-1,\ell) + R(k,\ell-1)$, if both $R(k-1,\ell)$ and $R(k,\ell-1)$ are even.

1.15 Show that $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$ by showing that the recurrence $R(k,\ell) = R(k-1,\ell) + R(k,\ell-1)$ is satisfied by a certain binomial coefficient.

1.16 We can determine a lower bound for $R(k,k)$ by using the probabilistic method (largely due to Erdős). Show that $R(k,k) > \frac{k}{e\sqrt{2}} 2^{\frac{k}{2}}$ for large k via the following steps.

a) Randomly color the edges of K_n either red or blue, i.e., each edge is colored red with probability $\frac{1}{2}$. Show that for a given set of k vertices of K_n, the probability that the complete graph on these k vertices is monochromatic equals $\frac{2}{2^{\binom{k}{2}}}$.

b) Let p_k be the probability that a monochromatic K_k subgraph exists in our random coloring. Show that

$$p_k \leq \sum_{i=1}^{\binom{n}{k}} 2^{1-\binom{k}{2}} = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

c) Use (b) to show that if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

d) Stirling's formula for the asymptotic behavior of $n!$ says that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Use Stirling's formula to finish the problem.

1.17 Consider the following way to color the edges of K_n. Number the vertices of K_n in a counterclockwise fashion from 1 to n. Next, partition the numbers $\{1, 2, \ldots, n-1\}$ into two subsets. Call these sets R and B for red and blue. Now, each edge has two vertices, say i and j. Calculate $|j-i|$ for that edge. If $|j-i| \in R$, then color the edge connecting i and j red. If $|j-i| \in B$, then color the edge connecting i and j blue. Such a coloring is called a *difference coloring.* Since $R(3,4) = 9$, we know that there is an edgewise 2-coloring of K_8 with no red K_3 and no blue K_4. One such coloring is a difference coloring defined as follows. Color an edge red if $|j-i| \in \{1,4,7\}$, and blue if $|j-i| \in \{2,3,5,6\}$. Show that this coloring does indeed prove that $R(3,4) > 8$, i.e., that there is no red K_3 and no blue K_4.

1.18 Prove Ramsey's theorem for r colors, where $r \geq 3$.

1.19 Show that $s(3) \geq 14$, where $s(r)$ is the r-color Schur number.

1.20 Let $r \geq 1$. Show that for any integer a, there exists an integer $M = M(a;r)$ such that, for any r-coloring of $[1,M]$, there is a monochromatic solution to $x + ay = z$. Deduce Schur's theorem from this result.

1.7. Research Problems

Note: In this chapter and the next we present some problems which, although understandable and rather simple to state, are considered to be extremely difficult to solve. We include them primarily for illustrative purposes (and because they are intriguing problems). We suggest that the research problems from Chapters 3 through 10 are more suitable for beginning research in Ramsey theory.

$*$**1.1** For $n \geq 3$, define $g(n)$ to be the least positive integer with the following property. Whenever the set of lines through $g(n)$ points satisfy both (a) no two lines are parallel, and (b) no three lines intersect in the same point, then the set of $g(n)$ points contains the vertices of a convex n-gon. Prove or disprove: $g(n) = 2^{n-2}+1$. It is known to hold for $n = 3,4,5$. It is also known that the existence of $g(n)$ is equivalent to Ramsey's theorem.
References: [**67**], [**126**], [**139**]

$*$**1.2** Prove or disprove the following conjecture proposed by Paul Erdős and V. Sós: $R(3,n+1) - R(3,n) \to \infty$ as $n \to \infty$.
References: [**20**], [**125**], [**230**]

$*$**1.3** Determine $\lim_{n\to\infty} R(n,n)^{1/n}$ if it exists. It is known that if this limit exists, then it is between $\sqrt{2}$ and 4. (The lower bound comes from Exercise 1.16 and the upper bound is deduced from Exercise 1.15 using Stirling's formula, which is given in Exercise 1.16.)
References: [**135**], [**378**], [**379**]

1.8. References

Harary's quote appears in [**194**], which also contains a biographical sketch of Ramsey.

§**1.1**. Example 1.8 is due to Erdős and Szekeres [**137**]. See [**402**] for some recent work in this area.

§**1.2**. Theorem 1.15 is proved in [**317**]. There is a much more general form of Ramsey's theorem (not confined to edge-colorings); for a proof, see, for example, [**175**]. Erdős and Szekeres rediscovered Ramsey's theorem, in an equivalent form, in [**137**] (see Research Problem 1.1). The Ramsey number $R(4,5) = 25$, discussed after Theorem 1.15, is the most recently discovered Ramsey number. See [**282**] for details. The Ramsey numbers $R(3,4)$, $R(3,5)$, $R(4,4)$, and $R(3,3,3)$ were discovered by Greenwood and Gleason [**182**]. Kéry determined $R(3,6)$ in [**224**]. Graver and Yackel [**180**] determine $R(3,7)$ by matching the upper bound given by Kalbfleisch in [**223**]. McKay and Min [**281**] determined $R(3,8)$ by matching the lower bound given by Grinstead and Roberts [**183**]. Grinstead and Roberts [**183**] determined $R(3,9)$ by matching the lower bound given in [**223**]. For a survey of the best bounds to date on small Ramsey numbers see [**315**].

§**1.4**. Hilbert's result is in [**201**]. A proof using more modern language is in [**172**, p. 1368]. Recent work regarding Hilbert's theorem can be found in [**85**], [**187**], and [**338**]. Van der Waerden's theorem and its original proof are in [**394**]. Schur's theorem and its original proof can be found in [**364**]. Rado's theorem was proved in the series of papers [**314**], [**313**], and [**312**]. For a summary of Rado's theorem see [**310**].

Additional References: For a very lively account of Ramsey theory results and history (based on 12 years of archival research) see Soifer's wonderful book [**374**] and essay [**375**]. For specifics on Ramsey himself, see [**376**]. Another brief account of the life and work of Ramsey can be found in [**283**]. The book [**316**] contains an account of Ramsey's work as it pertains to philosophy and logic. A brief history of Ramsey theory is given by Spencer in [**381**]. For a good list of open (and difficult) questions from Ron Graham, see [**170**].