Preface

This book guides mathematics students who have completed solid first courses in linear algebra and analysis on an expedition into the field of functional analysis. At the journey's end they will have captured two famous theorems—often stated in graduate courses, but seldom proved, even there:

(a) *The Titchmarsh Convolution Theorem*, which characterizes the null spaces of Volterra convolution operators, and which implies (in fact, is equivalent to):

(b) *The Volterra Invariant Subspace Theorem*, which asserts that the only closed, invariant subspaces of the Volterra operator are the "obvious ones."

The pursuit of these theorems breaks into three parts. The first part (four chapters) introduces the Volterra operator, while gently inducing readers to reinterpret the classical notion of uniform convergence on the interval $[a, b]$ as convergence in the max-norm, and to reimagine continuous functions on that interval as points in the Banach space $C\big([a, b]\big)$. It exploits, at several levels, this "functions are points" paradigm (often attributed to Volterra himself) in the process of solving integral equations that arise—via the Volterra operator—from the kinds of initial-value problems that students encounter in their beginning differential equations courses. At the conclusion of this part of the book, readers will be convinced (I hope) that even linear transformations can be thought of as "points in a space," and that within this framework the proof that "Volterra-type" integral equations have unique solutions boils down to summation of a geometric series.

In the process of tackling initial-value problems and integral equations we naturally encounter Volterra convolution operators, which form the subject of the second part of the book (two chapters). It's here that the problem of characterizing the null spaces of these operators is introduced, and solved via the Titchmarsh Convolution Theorem. The final step in proving the Titchmarsh theorem involves Liouville's theorem on bounded entire functions, for which just enough complex analysis (using only power series) is developed to give a quick proof.

The final part of the book (four chapters) aims toward using Titchmarsh's theorem to prove the Volterra Invariant Subspace Theorem. Here we encounter a pair of results that lie at the heart of functional analysis: the Hahn-Banach Theorem on separation by bounded linear functionals of closed subspaces from points not in them, and the Riesz representation of the bounded linear functionals on $C\big([a,b]\big)$ by means of Riemann-Stieltjes integrals. The Hahn-Banach theorem is derived from its extension form, which is proved in the usual way: extending by one dimension, then using some form of induction. This is done first for separable spaces, using ordinary mathematical induction, and then in general by transfinite induction, which is carefully introduced.

The Hahn-Banach extension theorem (nonseparable version!) then provides a quick proof of Riesz's representation theorem. Here it's hoped—but not assumed—that the reader has seen the Stieltjes extension of the Riemann integration theory. In any case, an appendix covers much of the standard material on Riemann-Stieltjes integration, with proofs omitted where they merely copy those for the Riemann integral. The book's final chapter completes the proof of the invariant subspace theorem for the Volterra operator.

Each chapter begins with an "Overview" and ends with a section of "Notes" in which the reader may find further results, historical material, and bibliographic references. Exercises are scattered throughout, most of them rather easy, some needed later on. Their purpose is twofold: first, to enhance the material at hand, and second (no less important) to emphasize the necessity of interacting actively with the mathematics being studied.

I hope this book will expand its readers' horizons, sharpen their technical skills, and for those who pursue functional analysis at the graduate level, enhance—rather than duplicate—that experience. In pursuit of this goal the book meanders through mathematics that is algebraic and analytic, abstract and concrete, real and complex, finite and transfinite. In this, it's inspired by the words of the late Louis Auslander: "Mathematics is like a river. You just jump in someplace; the current will take you where you need to go."[1]

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[1]Paraphrased by Anna Tsao in: Notices Amer. Math. Soc., Vol. 45, #3, 1998, page 393.