Chapter 5

The Titchmarsh Convolution Theorem

Overview. The Titchmarsh Convolution Theorem characterizes the null spaces of Volterra convolution operators. In this chapter we'll understand the theorem's statement, then reduce its proof to a special case, which we'll tackle in the next chapter.

5.1. Convolution operators

Each g in $C\big([0,\infty)\big)$ induces on that space a "Volterra convolution operator" T_g, defined for each $f \in C\big([0,\infty)\big)$ by

$$(5.1)\qquad (T_g f)(x) = \int_{t=0}^{x} g(x-t)f(t)\,dt \qquad (x \in [0,\infty)).^1$$

Thus T_g is a linear transformation on the vector space $C\big([0,\infty)\big)$. The definition works as well for $C\big([0,a]\big)$ which—thanks to the fact that each function continuous on $[0,a]$ can be extended continuously to $[0,\infty)$—we'll now feel free to think of as consisting of restrictions to $[0,a]$ of functions in $C\big([0,\infty)\big)$.

For f and g in $C\big([0,\infty)\big)$ note that, for $x \in [0,a]$ the integrand on the right-hand side of equation (5.1) depends only on the values these

[1] Note the change of notation from Part 1: x is now the "independent variable," with t the "variable of integration." This signals the fact that, from now on, we won't be discussing initial value problems.

functions take on $[0, a]$. Thus if we start with f and g defined and continuous only on $[0, a]$, and extend both to functions continuously to $[0, \infty)$, the restriction of $T_g f$ to $[0, a]$ is independent of the particular extensions chosen for f and g.

Note also that T_g is, on $C\big([0, a]\big)$, an operator V_κ "of Volterra type" (as defined by equation (3.3) on page 38), with $\kappa(t, s) = g(t-s)$.

Exercise 5.1. Show that $T_g f = T_f g$ for all $f, g \in C\big([0, \infty)\big)$.

Here are some examples of Volterra convolution operators that we've already encountered:

(a) The Volterra operator itself: $V = T_g$ with $g = \mathbf{1}$.[2]

(b) The Volterra powers: $V^n = T_g$ with $g(x) = \frac{x^{n-1}}{(n-1)!}$ (n=2, 3, ...).

(c) Operators providing solutions, in Chapters 1–3, of various integral equations and initial-value problems, in particular:
   (i) Exercise 1.15 on page 13 (concerning the spectrum of V),
   (ii) Exercise 2.27 on page 34 (motion of a mass-spring system),
   (iii) Exercise 3.20 on page 50[3]

Exercise 5.2. Consider once more the initial-value problem (IVP_n) of §3.1 (page 37). Show that whenever the coefficients $p_1, p_2, \ldots, p_n$ are all constant on $[0, \infty)$, the solution is given by a convolution operator.

For any linear transformation, it's of fundamental importance to find the null space. For the Volterra operator $V = T_{\mathbf{1}}$ we know the answer; the operator is one-to-one, so its null space is $\{\mathbf{0}\}$.[4] For general convolution operators the answer is more complicated; it's provided by the namesake of this chapter, the *Titchmarsh Convolution Theorem* [**56**, 1926]. We'll devote the rest of this section to understanding both the statement of Titchmarsh's theorem and its role in determining convolution-operator null spaces. Then we'll begin the (nontrivial) process of proving the theorem—a consciousness-expanding quest that will take us through the next chapter!

Here's the "easy half" of the Titchmarsh Convolution Theorem.

[2]The symbol **1** denotes the function taking only the value 1.
[3]Proposition 1.2, page 5.
[4]The symbol **0** denotes the function taking only the value zero.

Proposition 5.3. *Suppose* $a, b > 0$ *and* $f, g \in C\big([0,\infty)\big)$. *If* $g \equiv 0$ *on* $[0,b]$ *and* $f \equiv 0$ *on* $[0,a]$, *then* $T_g f \equiv 0$ *on* $[0, a+b]$.[5]

Proof. Fix x and t, with $0 \le t \le x \le a+b$. Since $g \equiv 0$ on $[0,b]$ we have $g(x-t) = 0$ for $x - t \le b$, i.e., for $x - b \le t$. Thus if $g(x-t) \neq 0$ then we must have $t < x - b \le (a+b) - b = a$, hence $f(t) = 0$.

Conclusion. $0 \le t \le x \le a+b \Longrightarrow g(x-t)f(t) = 0 \Longrightarrow (T_g f)(x) = 0.$ □

Figure 2 below depicts the proof of Proposition 5.3.

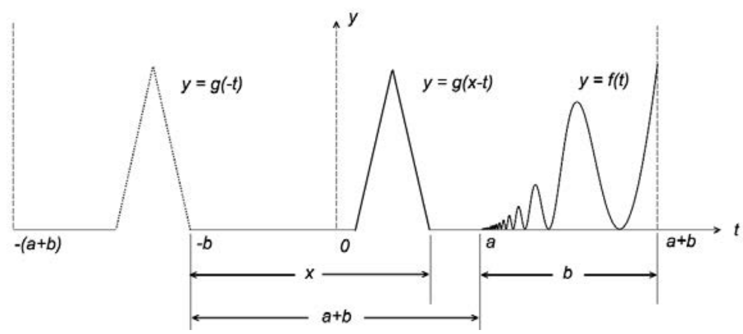

Figure 2. $g(x-t)f(t) = 0$ for $0 \le t \le x \le a+b$

For an efficient statement of this result, define for $f \in C\big([0,\infty)\big)$:

$$\ell(f) = \begin{cases} \inf\{t \ge 0 \colon f(t) \neq 0\} & \text{if } f \neq \mathbf{0} \\ \infty & \text{if } f = \mathbf{0} \end{cases} \tag{5.2}$$

In other words, $\ell(f)$ is the "left-most" point of the support of f.[6] In particular, $f \neq \mathbf{0}$ iff $\ell(f) < \infty$. Note that if $0 < \ell(f) < \infty$, then, since f vanishes identically on the semi-open interval $[0, \ell(f))$, continuity demands that f also vanish at $\ell(f)$. This need not happen if $\ell(f) = 0$ (Example: $f = \mathbf{1}$).

[5] Recall: "$f \equiv 0$ on S" (short for: "f is identically zero on S") means that $f(s) = 0$ for every $s \in S$.

[6] The *support* of a function is the closure of the set of points at which the function does not vanish.

The assignment $a = \ell(f)$ and $b = \ell(g)$ converts Proposition 5.3 into:

Proposition 5.4 (The "Easy Titchmarsh Inequality")**.** *For each pair of functions* $f, g \in C\big([0,\infty)\big)$,

$$\text{(ET)} \qquad \ell(T_g f) \geq \ell(f) + \ell(g).$$

Figure 2 depicts f and g both non-negative, with $a = \ell(f)$, $b = \ell(g)$, and $g > 0$ on an interval with right-hand endpoint $\ell(g)$. It's clear from the picture that under these hypotheses: as soon as x exceeds $\ell(f) + \ell(g)$, the function $t \to g(x-t)f(t)$ ceases to vanish identically on $[0, x]$ so, thanks to positivity, $(T_g f)(x) \neq 0$. Thus for the picture's situation there is actually *equality* in inequality (ET). The Titchmarsh Convolution Theorem asserts that there is *always* equality in (ET), i.e., that there is a companion "Hard Titchmarsh" inequality:

$$\text{(HT)} \qquad \ell(T_g f) \leq \ell(f) + \ell(g) \qquad \big(f, g \in C\big([0,\infty)\big)\big).$$

In concrete terms, (HT) means that: even though there may be certain $x > \ell(f) + \ell(g)$ for which the values of $t \to g(x-t)f(t)$ have enough cancellation to render $(T_g f)(x) = 0$, there must be a sequence (x_n) strictly decreasing to $\ell(f) + \ell(g)$ with $T_g f(x_n) \neq 0$ for each n.

It might appear, upon redrawing Figure 2 for more general f and g, that this ought to be obvious. Not so; the proof of inequality (HT) will occupy the latter part of this chapter, and all of the next one!

5.2. Null spaces

Let's see how Titchmarsh's theorem: "$\ell(T_g f) = \ell(f) + \ell(g)$" determines the null spaces of Volterra convolution operators. The case $C\big([0,\infty)\big)$ is special.

Theorem 5.5. *For each* $g \in C\big([0,\infty)\big)\backslash\{\mathbf{0}\}$, *the null space of* T_g *is* $\{\mathbf{0}\}$, *i.e.,* T_g *is one-to-one on* $C\big([0,\infty)\big)$.

Proof. We wish to prove that if $f \in C\big([0,\infty)\big)$ and $T_g f = \mathbf{0}$, then $f = \mathbf{0}$. We'll proceed contrapositively, assuming that $f \neq \mathbf{0}$ and proving $T_g f \neq \mathbf{0}$. In "ℓ-language" we're assuming that both $\ell(f)$ and

$\ell(g)$ are $< \infty$, so by inequality (HT): $\ell(T_g f) \leq \ell(f) + \ell(g) < \infty$, i.e., $T_g f \neq \mathbf{0}$. □

For $0 < a < \infty$, convolution-operator null spaces are more complicated.

Theorem 5.6. *If $g \in C\big([0,a]\big)\backslash\{\mathbf{0}\}$, then the null space of T_g on $C\big([0,a]\big)$ is*

$$\{f \in C\big([0,a]\big) \colon \ell(f) \geq a - \ell(g)\}.$$

Proof. We're viewing functions in $C\big([0,a]\big)$ as restrictions to $[0,a]$ of functions in $C\big([0,\infty)\big)$. If $f \in C\big([0,a]\big)$ with $\ell(f) \geq a - \ell(g)$, then $\ell(T_g f) \geq \ell(f) + \ell(g) \geq a$ by (ET). Thus $T_g f$ vanishes identically on $[0,a]$, i.e., f is in the null space of T_g.

We need to show that no other function in $C\big([0,a]\big)$ belongs to this null space. To this end, suppose $f \in C\big([0,a]\big)$ *does* belong to this null space, i.e., that $T_g f \equiv 0$ on $[0,a]$; in ℓ-language, $\ell(T_g f) \geq a$. Combining this with the "Hard Titchmarsh" inequality (HT) we obtain $a \leq \ell(T_g f) \leq \ell(f) + \ell(g)$, so $\ell(f) \geq a - \ell(g)$, as desired. □

In plain language, Theorem 5.6 asserts that $T_g f \equiv 0$ on $[0,a]$ if and only if $f \equiv 0$ on the interval $[0, a - \ell(g)]$ (our assumption that g is not $\equiv 0$ on $[0,a]$ means that $\ell(g) < a$, so this interval is nontrivial). Thus we can think of Theorem 5.5 as the limiting case "$a = \infty$" of this result. Here's another important special case.

Corollary 5.7. *Suppose $g \in C\big([0,a]\big)$. Then T_g is one-to-one on $C\big([0,a]\big)$ if and only if the support of g contains the origin.*

In particular, the operator T_g can still be one-to-one even if $g(0) = 0$; all that's needed is a sequence of points $t_j \searrow 0$ in $[0,a]$ such that $g(t_j) \neq 0$ for each j. We've already seen examples of this phenomenon: for each integer $k \geq 2$ the k-th power V^k of the Volterra operator is one-to-one (because V is), and $V^k = T_g$ where $g(t) = \frac{t^{k-1}}{(k-1)!}$, a function with the origin in its support, even though it vanishes there.

Exercise 5.8. Consider the function g defined by setting $g(0) = 0$ and $g(t) = t\sin(1/t)$ if $t \neq 0$. Show that $g \in C\big([0,\infty)\big)$ and that T_g is one-to-one on $C\big([0,a]\big)$ for every $a > 0$.

The following exercise asks you to establish a prototype of Corollary 5.7 whose proof does not require Titchmarsh's theorem.

Exercise 5.9. Suppose $g \in C\big([0,a]\big)$ is differentiable on $[0,a]$, with $g' \in C\big([0,a]\big)$. Show that if $g(0) \neq 0$ then T_g is one-to-one on $C\big([0,a]\big)$.

Suggestion: Differentiate both sides of the equation $T_g f = \mathbf{0}$, using the Leibniz Rule (equation (2.1), page 23) on the left-hand side. Then use Exercise 3.21 on page 51.

To finish the proof of the Titchmarsh Convolution Theorem it remains to establish inequality (HT). This will be best done by re-casting the operator-theoretic notion of convolution into an algebraic form.

5.3. Convolution as multiplication

We define the *convolution* of two functions f and g in $C\big([0,\infty)\big)$ to be

$$(5.3) \qquad (f * g)(x) = \int_{t=0}^{x} f(x-t)g(t)\,dt \qquad (x \in [0,\infty)).$$

Thus $f * g = T_g f$, so by Exercise 5.1, $f * g = T_f g = T_g f = g * f$, i.e., convolution is a *commutative* operation. For $a > 0$, Proposition 3.3 (page 40) asserts that T_g maps $C\big([0,a]\big)$ into itself, with operator norm $\leq a\,\|g\|$. Thus:

Proposition 5.10. *Suppose $f, g \in C\big([0,\infty)\big)$:*

(a) $f * g \in C\big([0,\infty)\big)$.

(b) *If $\|\cdot\|$ denotes the max-norm of $C\big([0,a]\big)$, then $\|f * g\| \leq a\,\|f\|\,\|g\|$.*

Exercise 5.11. Suppose $g, h \in C\big([0,a]\big)$. Reverting to the notation of the previous chapters, define Volterra kernels κ and μ on the triangle $\Delta_a : 0 \leq s \leq t \leq a$ by:

$$\kappa(t,s) = g(t-s) \text{ and } \mu(t,s) = h(t-s).$$

Show that the "Volterra product" $\kappa \star \mu$, as defined by equation (3.9) on page 45, is induced by the convolution $g * h$, i.e.,

$$(\kappa \star \mu)(t,s) = (g * h)(t-s) \qquad (0 \leq s \leq t \leq a).$$

Proposition 5.12. *For $f, g, h \in C\big([0,\infty)\big)$, and λ a scalar,*

(a) $f * \mathbf{0} = \mathbf{0}$,

(b) $f * g = g * f$,

(c) $\lambda(f * g) = f * (\lambda g)$,

(d) $f * (g + h) = f * g + f * h$,

(e) $f * (g * h) = (f * g) * h$.

Proof. Part (a) is obvious, and we've just discussed (b). Parts (c) and (d) follow from standard properties of integrals. As for (e):

$$\begin{aligned}[f * (g * h)](x) &= \int_{t=0}^{x} f(x-t)(g * h)(t)\,dt \\ &= \int_{t=0}^{x} f(x-t)\left(\int_{s=0}^{t} g(t-s)h(s)\,ds\right) dt\,.\end{aligned}$$

Now interchange the order of integration, then make the change of variable $u = t - s$ in the resulting inner integral:

$$\begin{aligned}[f * (g * h)](x) &= \int_{s=0}^{x}\left(\int_{t=s}^{x} f(x-t)g(t-s)\,dt\right) h(s)\,ds \\ &= \int_{s=0}^{x}\left(\int_{u=0}^{x-s} f(x-s-u)g(u)\,du\right) h(s)\,ds \\ &= \int_{s=0}^{x} [f * g](x-s)h(s)\,ds \\ &= [(f * g) * h](x)\end{aligned}$$

as desired. □

Exercise 5.13. Derive the associative property (e) of convolution from the associative property of linear transformations under composition.

Suggestion: Use Exercise 5.11 above. along with Lemmas 3.11 and 3.12 on pp. 45–46.

Exercise 5.14. Show that convolution operators commute with each other: $T_g T_h = T_h T_g$ for all $h, g \in C\big([0,\infty)\big)$ (Note that by Exercise 3.16 on page 47 this is not true for all operators of Volterra type.).

Properties (a) and (c)–(e) of Proposition 5.12, along with

(d') $(g+h)*f = g*f + h*f \qquad (f,g,h \in C\big([0,\infty)\big))$

show that convolution multiplication turns $C\big([0,\infty)\big)$ into an *algebra* over its scalar field, and (b) asserts that this algebra is *commutative.* Further examples of algebras: Any *field* is an algebra (commutative) over itself. The collection of linear transformations of a vector space $\mathscr{V}$, with composition as multiplication, is an algebra over the scalar field of $\mathscr{V}$ (non-commutative if $\dim \mathscr{V} > 1$). The space of continuous, scalar-valued functions on an interval, with pointwise multiplication or (as we've just seen) convolution multiplication, is a commutative algebra.

An algebra, or more generally a *ring*[7] is an *integral domain* if the only way a product of two elements can be the zero element is for at least one of them to be the zero element. Thus, any field is an integral domain, as is the ring of integers with its usual algebraic operations. On the other hand, the algebra of $n \times n$ real (or complex) matrices is (for $n > 1$) not an integral domain, and the same is true of both $C\big([0,\infty)\big)$ and $C\big([0,a]\big)$ with *pointwise* multiplication. However, thanks to the Titchmarsh Convolution Theorem:

Proposition 5.15. *With* convolution multiplication*,* $C\big([0,\infty)\big)$ *is an integral domain.*

Proof. Suppose f and g belong to $C\big([0,\infty)\big)$. If $f*g = \mathbf{0}$ then f belongs to the null space of the convolution operator T_g, which by Theorem 5.5 is the singleton $\{\mathbf{0}\}$. $\square$

In the other direction, Theorem 5.6 shows that $C\big([0,a]\big)$, with convolution multiplication, is *not* an integral domain. It asserts, for example, that if $f \in C\big([0,a]\big)\backslash\{\mathbf{0}\}$ is $\equiv 0$ on $[0,\frac{a}{2}]$, then $f*f = \mathbf{0}$.

[7]A ring is a set with binary "addition" and "multiplication" that obeys properties (a), (b), (d), and (e), of Proposition 5.12, but omits any notion of scalar multiplication. Example: The integers, with the usual algebraic operations, is a ring.

Exercise 5.16. Each of the vector spaces $C([0,a])$ and $C([0,\infty))$, when viewed as an algebra with pointwise multiplication, has a *multiplicative identity*, namely the "identically 1" function $\mathbf{1}$. Show that no such identity element exists when the spaces are viewed as algebras with convolution multiplication.

Suggestion: Exercise 3.21 (page 51) makes a stronger assertion.

5.4. The One-Half Lemma

Here are the two components of Titchmarsh's theorem, restated in terms of convolution multiplication. First, there's the "Easy Titchmarsh Inequality," a.k.a. Proposition 5.4 (or in plain language, Proposition 5.3):

$$\text{(ET)} \qquad \ell(f*g) \geq \ell(f)+\ell(g) \qquad (f,g\in C([0,\infty))).$$

Then there is the heart of Titchmarsh's theorem, the yet-to-be-established "Hard Titchmarsh Inequality," which implies that there is *equality* in (ET):

$$\text{(HT)} \qquad \ell(f*g) \leq \ell(f)+\ell(g) \qquad (f,g\in C([0,\infty))).$$

We'll show in this section that the crucial step in establishing (HT) is to prove the special case $f=g$, i.e., the statement: $\ell(f*f)\leq 2\ell(f)$, the plain-language version of which is:

Lemma 5.17 (The "One-Half Lemma")**.** *Suppose* $f\in C([0,\infty))$, $a>0$, *and* $f*f\equiv 0$ *on* $[0,a]$. *Then* $f\equiv 0$ *on* $[0,\frac{a}{2}]$.

The proof of the One-Half Lemma will occupy the next chapter; we'll devote the rest of this one to showing how the lemma implies inequality (HT), and therefore the full Titchmarsh theorem. The argument depends on two subsidiary lemmas, the first of which uncovers an interesting interaction between convolution and the operator M of "multiplication by x," defined for $f\in C([0,\infty))$ by

$$Mf(x) = xf(x) \qquad (0\leq x<\infty). \tag{5.4}$$

Clearly M is a linear transformation on $C([0,\infty))$. What's interesting for us is that, relative to convolution, it's a "derivation."

Sublemma 5.18. *If* f *and* g *belong to* $C([0,\infty))$ *then*

$$M(f*g) = (Mf)*g + f*(Mg). \tag{5.5}$$

Proof. For $x \in [0,\infty)$:

$$\begin{aligned}\big((Mf)*g\big)(x) + \big(f*Mg\big)(x) &= \int_{t=0}^{x}(x-t)f(x-t)g(t)\,dt + \int_{t=0}^{x} f(x-t)tg(t)\,dt \\ &= x\int_{t=0}^{x} f(x-t)g(t)\,dt = x(f*g)(x) = M(f*g)(x)\,,\end{aligned}$$

as we wished to show. □

The last result was routine, but the next one is not. In plain language it states that if $f*g$ vanishes identically on an interval $[0,a]$, then so does $f*Mg$. Although its proof depends only on Sublemma 5.18 and the "Easy Titchmarsh" inequality (ET), the argument is, nevertheless, surprising.

Sublemma 5.19. *For each pair f, g of functions in $C\big([0,\infty)\big)$,*

$$\ell(f*Mg) \geq \ell(f*g)\,.$$

Proof. Let G denote the set of all non-negative numbers γ for which $\ell(f*Mg) \geq \gamma\ell(f*g)$ for all functions $f,g \in C\big([0,\infty)\big)$ with $f*g \neq \mathbf{0}$. Since the condition $f*g \neq \mathbf{0}$ means $\ell(f*g) \neq \infty$, the definition of G involves no "zero times infinity" problem. Clearly $0 \in G$, so G is nonempty.

In fact, $G \subset [0,1]$. To see why, fix $f \in C\big([0,\infty)\big)$ with $0 < \ell(f) < \infty$, and $f > 0$ on the open half-line $(\ell(f),\infty)$. Note that $\ell(f) < \infty$ (since $f \neq \mathbf{0}$). Now $Mf > 0$ on $(\ell(f),\infty)$, and from the discussion centered on Figure 5.1 (page 83) we know that $\ell(f*f) = \ell(f*Mf) = \ell(f) \in (0,\infty)$. Thus if $\gamma > 1$ there exist f and g (in this case $g = f$) in $C\big([0,\infty)\big)$ such that $\ell(f*Mg) < \gamma\,\ell(f*g)$. *Conclusion*: $\gamma \notin G$.

Let $\Gamma = \sup G$, so $0 \leq \Gamma \leq 1$. Our goal is to show that $\Gamma = 1$. First of all, observe that $\Gamma \in G$. To see why, fix f and g in $C\big([0,\infty)\big)$ with $f*g \neq \mathbf{0}$. We wish to show that $\ell(f*Mg) \geq \Gamma\ell(f*g)$. Choose a sequence (γ_n) in G with $\gamma_n \nearrow \Gamma$. Now $\ell(f*Mg) \geq \gamma_n\ell(f*g)$ for each index n (by the definition of G), so this inequality remains true in the limit, i.e., $\ell(f*Mg) \geq \Gamma\ell(f*g)$, so $\Gamma \in G$, as we wished to show.

Now for the surprising part. Suppose, for the sake of contradiction, that Γ is < 1. Since $\Gamma \in G$ we have $\ell(f * Mg) \geq \Gamma\ell(f * g)$ for each pair f, g of functions in $C\big([0,\infty)\big)$ with $f * g \neq \mathbf{0}$. In plain language: *For $a > 0$ and $f, g \in C\big([0,\infty)\big)$ with $f * g \neq \mathbf{0}$*:

$$f * g \equiv 0 \text{ on } [0, a] \implies f * Mg \equiv 0 \text{ on } [0, \Gamma a].$$

Fix $a > 0$, and fix two functions $f, g \in C\big([0,\infty)\big)$ with $f*g$ identically 0 on $[0, a]$, but not on $[0,\infty)$. Since $f * g \neq \mathbf{0}$, the same is true of $M(f * g)$, so by Sublemma 5.18, either $(Mf) * g$ or $f * Mg$ (or both) is $\neq \mathbf{0}$. Upon swapping the names of the functions, if necessary, we may assume that $f * Mg \neq \mathbf{0}$.

Next, consider the convolution product

$$h = (f * Mg) * [(Mf) * g + f * Mg].$$

Thanks to our hypothesis on $f * g$, and the fact that $\Gamma \in G$, the first term in round brackets on the right-hand side is identically 0 on $[0, \Gamma a]$, while the term in square brackets, which by Sublemma 5.18 is just $M(f * g)$, is $\equiv 0$ on $[0, a]$. Thus by the "Easy Titchmarsh" inequality" (ET),

$$h \equiv 0 \quad \text{on} \quad [0, a + \Gamma a]. \tag{5.6}$$

On the other hand, by the algebraic properties of convolution established in parts (b), (d), and (e) of Proposition 5.12 (page 87):

$$h = \underbrace{(f * g) * (Mf * Mg)}_{\text{I}} + \underbrace{(f * Mg) * (f * Mg)}_{\text{II}}$$

In Term I, note that $Mf * Mg$ is $\equiv 0$ on $[0, \Gamma^2 a]$. *Reason*: Since $\Gamma \in G$ we know that $\ell(f * Mg) \geq \Gamma\ell(f * g)$. Now $Mg \in C\big([0,\infty)\big)$ with $f * Mg \neq \mathbf{0}$, so the pair f and Mg satisfy the conditions specified for test functions in the definition of G. Thus $\ell(Mf*Mg) \geq \Gamma\ell(f*Mg) \geq \Gamma^2\ell(f*g)$, i.e., $Mf * Mg \equiv 0$ on $[0, \Gamma^2 a]$. Thanks to this, and the fact that $f * g \equiv 0$ on $[0, a]$, inequality (ET) guarantees that term I is $\equiv 0$ on $[0, a+\Gamma^2 a]$, a smaller interval than the one guaranteed by (5.6) on which $h \equiv 0$. It follows that term II is also $\equiv 0$ on $[0, a + \Gamma^2 a]$, hence by the One-Half Lemma (at last!) $f * Mg \equiv 0$ on $[0, \frac{1}{2}(a + \Gamma^2 a)]$.

Since this is true for all $f, g \in C\big([0,\infty)\big)\backslash\{\mathbf{0}\}$ with $f * g \equiv 0$ on $[0, a]$, and all $a > 0$, it follows that $\frac{1}{2}(1 + \Gamma^2)$, which is $> \Gamma$, belongs to G. Thus our assumption that $\Gamma = \sup G$ is < 1 has led to the contradiction that something in G is strictly larger than Γ. *Conclusion*: $\Gamma = 1$, as desired.

So far, we've proved Sublemma 5.19 under the additional assumption that $f * g \neq \mathbf{0}$. To remove this assumption, fix functions f and g in $C\big([0,\infty)\big)$ with $f * g = \mathbf{0}$, i.e., $\ell(f*g) = \infty$. We wish to prove $\ell(f * Mg) \geq \ell(f*g)$, i.e, $\ell(f*Mg) = \infty$, i.e., $f * Mg = \mathbf{0}$.

To this end, and fix $a > 0$. It will be enough to show that $f * Mg \equiv 0$ on $[0,a]$. We may assume for simplicity (and without loss of generality) that the values of both $|f|$ and $|g|$ are ≤ 1 on $[0,a]$. We're thinking of functions in $C\big([0,a]\big)$ as being restrictions to $[0,a]$ of functions in $C\big([0,\infty)\big)$. If needed, make a new extension $\tilde{f}$ of f from $[0,a]$ to $[0,\infty)$, defining $\tilde{f}$ to be $\equiv 1$ on $[2a,\infty)$, and to have graph over $[a,2a]$ that is the line segment connecting the points $(a, f(a))$ and $(2a,1)$. Then on $[0,\infty)$ our newly-defined extension $\tilde{f}$ is continuous, with absolute value ≤ 1 there. Make a similar extension $\tilde{g}$ of g, and note that $\tilde{f} * \tilde{g} = f * g \equiv 0$ on $[0,a]$.

Claim. $\tilde{f} * \tilde{g} \neq 0$.

Once this is claim is established, we'll be able to apply the part of Sublemma 5.19 just proved to conclude that $\tilde{f} * M\tilde{g} \equiv 0$ on $[0,a]$. On $[0,a]$ we know that $\tilde{f} = f$ and $\tilde{g} = g$, so also $M\tilde{g} = g$, hence—thanks to the claim—we'll also have $f * Mg = \tilde{f} * M\tilde{g} \equiv 0$ on $[0,a]$, thus finishing the proof of Sublemma 5.19.

Proof of Claim. Please begin by verifying that: *For* $x \geq 4a$ *and* $t \in [2a, x-2a]$, *both* $f(t)$ *and* $g(x-t)$ *equal* 1. This suggests that for $x \geq 4a$ we break the integral $(\tilde{f} * \tilde{g})(x) = \int_{t=0}^{x} \tilde{f}(x-t)\tilde{g}(t)\,dt$ into three pieces:

- I_1 over the interval $[0,2a]$, where both $|\tilde{f}(t)|$ and $|\tilde{g}(x-t)|$ are ≤ 1,
- I_2 over $[2a, x-2a]$, where $\tilde{f}(t) = \tilde{g}(x-t) = 1$, and
- I_3 over $[x-2a, x]$,where both $|\tilde{f}(t)|$ and $|\tilde{g}(x-t)|$ are ≤ 1.

Consequently, both $|I_1|$ and $|I_3|$ are $\leq 2a$ while $I_2 = x - 4a$, so by the "reverse triangle inequality":

$$(\tilde{f} * \tilde{g})(x) \geq I_2 - |I_1| - |I_3| \geq (x-4a) - 2a - 2a = x - 8a.$$

Thus $(\tilde{f} * \tilde{g})(x) > 0$ for $x > 8a$; in particular $\tilde{f} * \tilde{g} \neq \mathbf{0}$, which—as explained above—finishes the proof of Sublemma 5.19. □

The One-Half Lemma implies the Titchmarsh theorem. So far we have shown that the One-Half Lemma implies Sublemma 5.19. We finish this chapter by using Sublemma 5.19 to derive the full Titchmarsh theorem. For transparency we break the argument into several pieces.

STEP 1. Suppose $a > 0$ and $f, g \in C\big([0,\infty)\big)$ with $f * g \equiv 0$ on $[0,a]$. By Sublemma 5.19 we know that $f * Mg \equiv 0$ on $[0,a]$. Apply Sublemma 5.19 again, now with Mg in place of g, thus obtaining $f * M^2 g \equiv 0$ on $[0,a]$. By induction we have $f * M^k g \equiv 0$ on $[0,a]$ for $k = 0, 1, 2, \ldots$, i.e.,

$$\int_{t=0}^{x} f(x-t)t^k g(t)\,dt = 0 \qquad (0 \le x \le a).$$

Thus, by the linearity of integration, for each polynomial p:

$$\int_{t=0}^{x} f(x-t)g(t)p(t)\,dt = 0 \qquad (0 \le x \le a). \tag{5.7}$$

Now fix $x \in [0,a]$. Thanks to the *Weierstrass Approximation Theorem*[8] (see Appendix C, page 195) there is a sequence (p_n) of polynomials that is uniformly convergent on $[0,x]$ to the function $t \to f(x-t)g(t)$ (or, in the case of complex scalars, to the complex conjugate of this function), hence

$$\begin{aligned} 0 &= \lim_n \int_{t=0}^{x} f(x-t)g(t)p_n(t)\,dt \\ &= \int_{t=0}^{x} \lim_n p_n(t) f(x-t)g(t)\,dt \\ &= \int_{t=0}^{x} |f(x-t)g(t)|^2\,dt\,, \end{aligned}$$

with uniform convergence justifying the interchange of limit and integral in the second line. Since the integrand in the last integral is

[8] This famous theorem asserts that each continuous, scalar-valued function on a finite, closed interval of the real line is the limit, uniformly on that interval, of polynomials. It is universally regarded to be the Fundamental Theorem of Approximation Theory.

continuous and ≥ 0 on the interval of integration, it must be identically zero there, ie.,

$$f(x-t)g(t) = 0 \quad \text{whenever} \quad 0 \leq t \leq x \leq a. \tag{5.8}$$

STEP 2. Suppose $\ell(f*g) = \infty$, i.e., that $f*g \equiv 0$ on $[0,\infty)$. We desire to show that $\ell(f) + \ell(g) = \infty$, i.e., that either $\ell(f)$ or $\ell(g)$ (or both) is infinite. In plain language: we wish to show that at least one of the functions f, g vanishes identically on $[0,\infty)$.

We're assuming that $f*g \equiv 0$ on the interval $[0,a]$ for every $a > 0$, hence by Step 1, equation (5.8) holds for every $a > 0$, i.e., for all $x, t \in [0,\infty)$ with $0 \leq t \leq x$. Suppose f does not vanish identically on $[0,\infty)$, so $f(x_0) \neq 0$ for some $x_0 \in [0,\infty)$. For $x \geq x_0$ set $t = x - x_0$, so $0 \leq t \leq x$, so upon substituting these values into equation (5.8) we see that $0 = f(x_0)g(x-x_0)$, hence $g(x-x_0) = 0$ for all $x \geq x_0$, i.e., $g \equiv 0$ on $[0,\infty)$, as desired.

Conclusion: *If $f, g \in C\big([0,\infty)\big)$ and $f*g \equiv 0$ on $[0,\infty)$, then either f or g (or both) vanishes identically on $[0,\infty)$.* In other words $\ell(f*g) = \ell(f) + \ell(g)$ when either side is infinite.

STEP 3. Finally, suppose $\ell(f*g) < \infty$ (i.e., $f*g$ is not identically zero on $[0,\infty)$). We want to prove inequality (HT): $\ell(f) + \ell(g) \geq \ell(f*g)$. This is trivial if $\ell(f*g) = 0$ (i.e., if 0 is in the support of $f*g$), so suppose $a = \ell(f*g) > 0$.

By inequality (ET): $a \geq \ell(f) + \ell(g)$, so in particular $\ell(f)$ and $\ell(g)$ are both $\leq a$. If $\ell(f) = a$ then we know from (ET) that

$$a = \ell(f*g) \geq \ell(f) + \ell(g) = a + \ell(g)$$

so $\ell(g) = 0$, hence for this case: $\ell(f*g) = \ell(f) + \ell(g)$.

Suppose, then, that $\ell(f) < a$. For each positive integer n there exists $x_n \in [\ell(f), \ell(f) + 1/n]$ such that $f(x_n) \neq 0$. For $x \in [x_n, a]$ set $t = x - x_n$ and apply the result of Step 1 to the interval $[0, a - x_n]$. The result is:

$$0 = f(x-t)g(t) = f(x_n)g(x-x_n) \qquad 0 \leq x - x_n \leq a - x_n.$$

Thus $g \equiv 0$ on $[0, a - x_\varepsilon]$. Now let $n \to \infty$, so $x_n \to \ell(f)$, and invoke the continuity of g to conclude that $g \equiv 0$ on $[0, a - \ell(f)]$. Thus $\ell(g) \geq a - \ell(f)$, i.e., $\ell(f) + \ell(g) \geq a = \ell(f*g)$.

Thus the One-Half Lemma implies inequality (HT), and therefore the Titchmarsh Convolution Theorem. □

In the next chapter we'll prove the One-Half Lemma, thereby completing the proof of the Titchmarsh Convolution Theorem.

Notes

Who was Titchmarsh? The British mathematician Edward C. Titchmarsh (1899–1963) taught at Oxford University from 1932 to 1963. In addition to fundamental contributions to the mathematical research of his time, Titchmarsh wrote influential books on topics as diverse as the Riemann zeta function, Fourier transforms, and eigenfunction expansions for solutions of differential equations. His most popular book: "Theory of Functions" (1932) introduced complex analysis and Lebesgue integration to an entire generation of mathematicians; it still makes fascinating reading.

Proofs of the Titchmarsh Convolution Theorem. Titchmarsh proved his theorem in [**56**, 1926], using deep results from complex analysis. Our proof of Lemma 6.8 is based on arguments of Mikusiński from [**35**]. In the next chapter we'll follow in the footsteps of Titchmarsh by using complex variables, albeit in a much more elementary way, to finish the proof of the One-Half Lemma, and therefore of his theorem.

The Weierstrass Approximation Theorem. This result, universally acknowledged to be the "Fundamental Theorem of Approximation Theory," was published by Weierstrass (in [**67**]) when he was seventy years old! The Weierstrass Approximation Theorem is especially remarkable in view of the fact that on any finite, closed interval there exist continuous real-valued functions that are *nowhere differentiable* (see, e.g., [**47**], Theorem 7.18, page 154). Even worse, there exist "space-filling curves": continuous mappings taking $[0, 1]$ onto the unit square (see, e.g., [**47**], page 168, Exercise 14). Nevertheless, Weierstrass assures us that such monstrosities[9] (or, in the case

[9] In an 1893 letter, the French mathematician Charles Hermite wrote: "I turn away with fear and horror from the plague of continuous functions which do not have derivatives." You can find a reference to this letter, as well as a fascinating history of the "plague" it mentions, in Allan Pinkus's beautiful exposition [**39**] of the history, proofs, and importance of the Weierstrass Approximation Theorem.

of space-filling curves, their coordinate functions) can be uniformly approximated, to any desired accuracy, by polynomials.

Chapter 6

Titchmarsh Finale

Overview. The last chapter saw the proof of Titchmarsh's Convolution Theorem "$\ell(f * g) = \ell(f) + \ell(g)$" reduced to the special case $f = g$, a.k.a. the "One-Half Lemma." Here we'll connect the One-Half Lemma with the *Finite Laplace Transform*, for which we'll prove a powerful uniqueness theorem that establishes the Lemma, and with it, the Titchmarsh Theorem. Our proof will take us through several-variable calculus, then into complex plane, where *Liouville's Theorem*—one of the miracles of complex analysis—will provide *coup de grâce*.

6.1. The Finite Laplace Transform

The Laplace transform of $f \in C\big([0,\infty)\big)$ is the integral

$$(\mathscr{L}f)(s) = \int_{t=0}^{\infty} e^{\,s\,t} f(t)\,dt = \lim_{a\to\infty} \int_{t=0}^{a} e^{\,s\,t} f(t)\,dt\,,$$

defined for those values of $s \in \mathbb{R}$ (if there are any) for which the limit on the right exists.[1]

Here we'll be interested only in the integrals under the limit symbol. These are the the "Finite Laplace Transforms," for which we use

[1]In the usual definition, this is the Laplace transform of f at $-s$. The definition given here is more convenient for our purposes.

the symbol $\mathscr{L}_a f$. Officially:

$$(6.1) \qquad (\mathscr{L}_a f)(s) = \int_{t=0}^{a} e^{s\,t} f(t)\,dt \qquad \big(f \in C\big([0,a]\big),\ s \in \mathbb{R}\big).$$

> *Exercise* 6.1. Show that $\mathscr{L}_a$ is a linear transformation taking $C\big([0,a]\big)$ into $C(\mathbb{R})$, the vector space of scalar-valued functions continuous on the whole real line.

Finite Laplace transforms have an important power-series representation.

Lemma 6.2. *For $a > 0$ and $f \in C\big([0,a]\big)$:*

$$(6.2) \qquad (\mathscr{L}_a f)(s) = \sum_{n=0}^{\infty} c_n s^n \qquad (s \in \mathbb{R})$$

where the series on the right converges absolutely, and

$$(6.3) \qquad c_n = \frac{1}{n!}\int_{t=0}^{a} t^n f(t)\,dt \qquad (n = 0, 1, 2, \ldots).$$

Proof. Fix $s \in \mathbb{R}$. We know from calculus that for $x \in \mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

with the series converging uniformly on each compact subinterval of $\mathbb{R}$. Thus, if $s \in \mathbb{R}$ is fixed, the series for e^{st} converges uniformly for $t \in [0,a]$ (exercise). Consequently:

$$(6.4) \qquad f(t)e^{s\,t} = f(t)\sum_{n=0}^{\infty} \frac{s^n}{n!} t^n = \sum_{n=0}^{\infty} \frac{s^n}{n!} f(t) t^n\,,$$

with the series on the right uniformly convergent for $t \in [0,a]$ (another exercise). The desired representation for $(\mathscr{L}_a f)(s)$ then follows upon integrating both sides of equation (6.4) in t over $[0,a]$, and on the right-hand side using uniform convergence to interchange the order of integration and summation. □

> *Exercise* 6.3. Show that for each $f \in C\big([0,a]\big)$ there's a positive constant C for which $|(\mathscr{L}_a f)(s)| \le Ce^{a|s|}$ for $s \in \mathbb{R}$. Then give an example to show that this exponential bound on the growth of $\mathscr{L}_a f$ cannot be improved.

Corollary 6.4. *$\mathscr{L}_a$ is one-to-one on $C\big([0,a]\big)$ for each $a > 0$.*

Proof. Fix $f \in C([0,a])$ with $\mathscr{L}_a f = \mathbf{0}$. By the linearity of $\mathscr{L}_a$ it's enough to show that $f = \mathbf{0}$. By Lemma 6.2 and the uniqueness theorem for power series (see, e.g., [**59**, Corollary 4.5.7, page 263], or Theorem B.8, page 192), all the coefficients c_n of this series must vanish:

$$0 = c_n = \int_{t=0}^{a} t^n f(t)\, dt \qquad (n = 0, 1, 2, \dots).$$

The linearity of integration now implies that $\int_{t=0}^{a} p(t) f(t)\, dt = 0$ for each polynomial p. By the Weierstrass Approximation Theorem there is a sequence (p_n) of polynomials convergent uniformly on $[0,a]$ to f (or, if the scalars are complex, to its complex conjugate), hence

$$0 = \lim_n \int_{t=0}^{a} p_n(t) f(t)\, dt = \int_{t=0}^{a} \lim_n p_n(t) f(t)\, dt = \int_{t=0}^{a} |f(t)|^2\, dt\,,$$

with uniform convergence justifying the interchange of limit and integral. From this we infer, thanks to the continuity and non-negativity of $|f|^2$, that $f \equiv 0$ on $[0,a]$, i.e., that $f = \mathbf{0}$. □

Exercise 6.5. Is $\mathscr{L}_a$ invertible on $C([0,a])$?

Exercise 6.6. Suppose $f \in C([0,a])$ is ≥ 0 on $[0,a]$. Show that if $\mathscr{L}_a f$ is merely *bounded* on $\mathbb{R}$, then $f \equiv 0$ on $[0,a]$.

To prove the One-Half Lemma we'll need to establish the result of Exercise 6.6 without the positivity assumption on f.

6.2. Stalking the One-Half Lemma

In this section we'll use the Finite Laplace Transform to get "halfway" to the proof of the One-Half Lemma. To avoid fractions, let's replace the interval $[0,a]$ in the statement of the One-Half Lemma (Lemma 5.17, page 89), by the interval parameter $[0,2a]$. Thus our new statement of the One-Half Lemma is:

Lemma 6.7 (The One-Half Lemma redux)**.** *If $a > 0$ and $f \in C([0,\infty))$ with $f * f \equiv 0$ on $[0,2a]$, then $f \equiv 0$ on $[0,a]$.*

For $f \in C([0,\infty))$ and $a > 0$, define f_a on the half-line $(-\infty, a]$ by:

$$f_a(t) = f(a-t) \quad (-\infty < t \leq a). \tag{6.5}$$

Thus f_a is continuous $(-\infty, a]$, in particular on $[0, a]$, so we can form its finite Laplace transform $\mathscr{L}_a f_a$. To show $f \equiv 0$ on $[0, a]$ it's enough to do the same for f_a (since $t \in [0, a]$ iff $a - t \in [0, a]$). Thus, thanks to Corollary 6.4 we can prove the One-Half Lemma by showing that if $f \in C\big([0, \infty)\big)$ with $f * f \equiv 0$ on $[0, 2a]$ then $\mathscr{L}_a f_a \equiv 0$ on $\mathbb{R}$. Here's a down payment:

Lemma 6.8. *Suppose $f \in C\big([0, \infty)\big)$ with $f * f \equiv 0$ on $[0, 2a]$. Then $\mathscr{L}_a f_a$ is bounded on $\mathbb{R}$.*

Proof. Fix $a > 0$ and f satisfying the hypotheses of the Lemma. Without loss of generality we may assume that $|f(t)| \leq 1$ for $t \in [0, 2a]$, so also $|f_a(t)| \leq 1$ for $t \in [-a, a]$.

For $s \leq 0$ we have $e^{s\,t} \leq 1$ for each $t \in [0, a]$, hence

$$\big|(\mathscr{L}_a f_a)(s)\big| = \left|\int_{t=0}^{a} e^{s\,t} f_a(t)\, dt\right| \leq \int_{t=0}^{a} e^{s\,t} |f_a(t)|\, dt \;\leq\; a\,, \tag{6.6}$$

Conclusion: $\mathscr{L}_a f_a$ is bounded on the half-line $(-\infty, 0]$ (regardless of whether or not $f * f \equiv 0$ on $[0, 2a]$).

The goal now is to show that $\mathscr{L}_a f_a$ is bounded on $(0, \infty)$. For this we'll consider the symmetric integral

$$L(s) = \int_{t=-a}^{a} e^{s\,t} f_a(t)\, dt \qquad (s \in \mathbb{R}),$$

the definition of which makes sense because f_a is defined and continuous on $[-a, a]$. To show $\mathscr{L}_a f_a$ is bounded for $s > 0$, it's enough to show this for L: the point being that $\mathscr{L}_a f_a(s)$ is the difference of $L(s)$ and the integral $\int_{t=-a}^{0} e^{s\,t} f_a(t)\, dt$, this latter integral having absolute value $\leq a$ because—just as in the last paragraph—the absolute value of its integrand is ≤ 1 over its interval of integration.

We have not yet used the hypothesis that $f * f \equiv 0$ on $[0, 2a]$. To get this condition into the picture let's note that: to show $L(s)$ is bounded for $s > 0$ it's enough to do the same for

$$
\begin{aligned}
L(s)^2 &= \int_{u=-a}^{a} e^{s\,u} f(a-u)\,du \int_{v=-a}^{a} e^{s\,v} f(a-v)\,dv \\
&= \int_{v=-a}^{a} \left(\int_{u=-a}^{a} e^{s\,u} e^{s\,v} f(a-u) f(a-v)\,du \right) dv \\
&= \iint_S e^{s\,(u+v)} f(a-u) f(a-v)\,du\,dv\,,
\end{aligned}
$$

where S is the square in $\mathbb{R}^2$ of side-length 2a, centered at the origin. The diagonal $u+v=0$ bisects S into an upper triangle Δ_1 for which

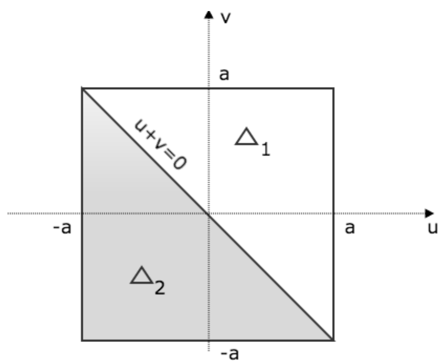

Figure 3. The square S, decomposed into triangles Δ_1 and Δ_2

$u+v$ is ≥ 0, and a lower triangle Δ_2 on which $u+v$ is ≤ 0. Thus $L(s)^2 = I_1(s) + I_2(s)$, where

$$
(6.7)\quad I_j(s) = \iint_{\Delta_j} e^{s\,(u+v)} f(a-u) f(a-v)\,du\,dv \quad (j = 1, 2;\ s \in \mathbb{R}).
$$

Just as in our analysis of $L(s)$ for $s \leq 0$, we see that, since $u+v \leq 0$ on Δ_2, the integrand of $I_2(s)$ has, for each $s > 0$, magnitude ≤ 1 there, hence

$$
(6.8)\qquad |I_2(s)| \leq \text{Area}(\Delta_2) = 2a^2 \quad (s > 0).
$$

For $I_1(s)$ we make the change of variables $u = a - (x - y)$ and $v = a - y$, which takes Δ_1 (in the "u, v-plane") to the triangle Δ_3 (in the "x, y-plane") having vertices at the origin, $(2a, 2a)$, and $(2a, 0)$.

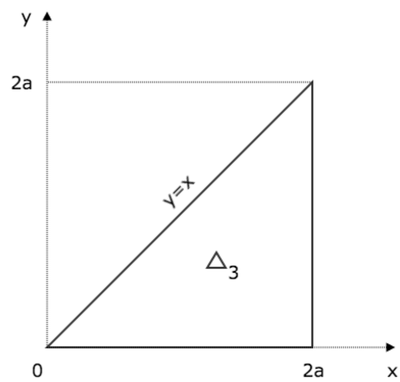

Figure 4. The triangle Δ_3

Since $\frac{\partial(x,y)}{\partial(u,v)} \equiv 1$, the change-of-variable formula for multiple integrals yields for each $s \in \mathbb{R}$:

$$I_1(s) = \iint_{\Delta_3} e^{s(2a-x)} f(x-y)f(y)\,dx\,dy$$
$$= \int_{x=0}^{2a} e^{s(2a-x)} \underbrace{\left(\int_{y=0}^{x} f(x-y)f(y)\,dy\right)}_{=(f*f)(x)=0 \text{ for } 0\le x\le 2a} dx$$

Thus $I_1(s) = 0$, so $L(s)^2 = I_2(s)$ for each $s \in \mathbb{R}$, which, along with inequality (6.8), proves that that $L^2(s)$ is bounded for $s > 0$. Consequently, the same is true of $L(s)$, hence $\mathscr{L}_a f_a$ is bounded on $(0, \infty)$. We previously showed that $\mathscr{L}_a f_a$ is bounded on $(-\infty, 0]$, so it's bounded on $\mathbb{R}$, as promised. □

Exercise 6.9 (One-Half Lemma down payment). Prove the One-Half Lemma under the additional assumption that f is ≥ 0 on $[0, 2a]$.

Suggestion: Lemma 6.8 and Exercise 6.6.

To finish the proof of the One-Half Lemma we'll be leaving the friendly confines of the real line and venturing into wide-open spaces of the complex plane. The next few sections contain the necessary background material which, if you've already had an introduction to complex analysis, should be highly skimmable.

6.3. The complex exponential

First, we need to extend the Finite Laplace Transform from the real line to the complex plane; this requires knowing what e^{st} means when s is a complex number.

The *complex exponential* e^z is defined, for $z \in \mathbb{C}$, by the power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, . \tag{6.9}$$

Proposition 6.10. *The infinite series on the right-hand side of equation* (6.9) *converges absolutely for each $z \in \mathbb{C}$, and uniformly on each bounded subset of $\mathbb{C}$.*

Proof. Suppose B is a bounded subset of $\mathbb{C}$, and choose $R > 0$ so that each point of B has absolute value $< R$. Let $M_n = \frac{R^n}{n!}$. Thus $\frac{|z^n|}{n!} \leq M_n$ for each n, and $\sum_{n=0}^{\infty} M_n = e^R < \infty$. Thus the series defining e^z converges uniformly on B and absolutely at every point of B. □

Exercise 6.11. Show that $e^{i\theta} = \cos\theta + i\sin\theta$ for each $\theta \in \mathbb{R}$.

According to this exercise we're already acquainted with the exponential $e^{i\theta}$ ($\theta \in \mathbb{R}$): it's the complex version of the unit vector $(\cos\theta, \sin\theta) \in \mathbb{R}^2$ that makes an angle of θ radians with the horizontal axis.

Proposition 6.12 (The polar decomposition)**.** *For each $z \in \mathbb{C}\backslash\{0\}$ there exists $\theta \in \mathbb{R}$ such that $z = re^{i\theta}$, where $r = |z|$.*

Proof. We have $z = x + iy$ where x and y are real, not both zero. Thus $z = r\omega$, where $|z| = \sqrt{x^2 + y^2}$ and $\omega = z/|z|$. Since

$$\text{Re}\,\omega = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \text{Im}\,\omega = \frac{y}{\sqrt{x^2+y^2}}$$

trigonometry tells us that for θ any value of $\arctan \frac{y}{x}$:

$$\text{Re}\,\omega = \cos\theta \quad \text{and} \quad \text{Im}\,\omega = \sin\theta \, ,$$

i.e., $\omega = e^{i\theta}$. □

Exercise 6.13. Suppose $z, w \in \mathbb{C}$ have polar decompositions $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$. Show that:

(a) $\dfrac{1}{z} = \dfrac{1}{r} e^{-i\theta}$,
(b) $zw = (r\rho)e^{i(\theta+\varphi)}$ (so $|zw| = |z||w|$),
(c) $z^n = r^n e^{in\theta}$ for each $n \in \mathbb{Z}$.

Proposition 6.14 (The Addition Law)**.** *For each pair $z, w \in \mathbb{C}$:*

$$e^{z+w} = e^z e^w \,.$$

Proof. By definition of the complex exponential:

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \,.$$

According to the binomial theorem:

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Thus

$$e^{z+w} \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} z^{n-k} w^k \;=\; \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k}}{(n-k)!} \frac{w^k}{k!} \,.$$

The expression on the right, the *Cauchy product* of the series defining e^z and e^w respectively, converges to $e^z\, e^w$ by Theorem B.3 (page 188). □

Corollary 6.15. $|e^z| = e^{\mathrm{Re}\, z}$ *for any $z \in \mathbb{C}$.*

Proof. Write $z \in \mathbb{C}$ as $x+iy$ where $x, y \in \mathbb{R}$. Thanks to the Addition Law: $e^z = e^x e^{iy}$ with $|e^{iy}| = 1$. Thus $|e^z| = e^x$ by Exercise B.1(d). □

Exercise 6.16. (a) Use the addition law for the complex exponential to prove the addition formulae for the (real) sine and cosine.

(b) Show that the trigonometric addition formulae for the (real) sine and cosine imply the addition law for the complex exponential.

6.4. Complex integrals

The functions in $C\big([0,a]\big)$ and $C\big([0,\infty)\big)$ are "scalar-valued" where the choice of scalars (real or complex) has so far been left to you. For complex-valued functions defined on real intervals, we've asserted that the usual rules of calculus regarding integration and differentiation continue to hold. Now it's time to get (more) serious about this.

Continuity and integral. Suppose $f\colon [a,b] \to \mathbb{C}$. For each $t \in [a,b]$ define $u(t) = \text{Re } f(t)$ and $v(t) = \text{Im } f(t)$, so we can write $f = u + iv$, where u and v are real-valued functions on $[a,b]$. Thanks to Exercise 3.8 (page 42) we know that f is continuous on $[a,b]$ iff u and v are continuous there.

We define the *integral* of f over $[a,b]$ to be

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\,. \tag{6.10}$$

Exercise 6.17. Show that $\displaystyle\int_0^{2\pi} e^{in\theta}\,d\theta = 0$ for each $n \in \mathbb{Z}\backslash\{0\}$.

The usual properties that we learn in calculus for integrals of real-valued functions on intervals remain true for complex-valued ones. For example, the mapping $f \to \int_a^b f(t)\,dt$ is a linear map from the *complex* vector space $C([a,b])$ (consisting of complex-valued functions continuous on $[a,b]$) into $\mathbb{C}$. The additivity of this map follows directly from that of real-valued integrals. As for homogeneity:

Exercise 6.18. For $f\colon [a,b] \to \mathbb{C}$ and $\lambda \in \mathbb{C}$, prove that

$$\lambda\int_a^b f(t)\,dt = \int_a^b (\lambda f)(t)\,dt\,.$$

A property of real-valued integrals that's crucial to estimating their sizes is: "$|\int| \le \int|\cdot|$". This is true for complex-valued integrals as well, but the proof requires a little trick.

Proposition 6.19. *Suppose $f\colon [a,b] \to \mathbb{C}$ is continuous. Then*

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt\,.$$

Proof. We may, without loss of generality, assume that the complex number $c = \int_a^b f(t)\,dt$ is not zero. Let $\omega = |c|/c$, so $|\omega| = 1$. Then invoking successively the definition of ω and Exercise 6.18 we obtain

$$\left|\int_a^b f(t)\,dt\right| = \omega \int_a^b f(t)\,dt = \int_a^b \big(\omega f(t)\big)\,dt\,.$$

Since the left-hand side of this string of equations is real (postive, in fact), so is the right-hand side, hence by the definition of complex-valued integral:

$$\left|\int_a^b f(t)\,dt\right| = \operatorname{Re} \underbrace{\int_a^b (\omega f(t))\,dt}_{>0} = \underbrace{\int_a^b \operatorname{Re}(\omega f(t))\,dt}_{\text{equation (6.10)}}$$
$$\leq \int_a^b |\omega f(t)|\,dt = \int_a^b |f(t)|\,dt\,,$$

where in the second line, the inequality follows from the fact that the absolute value of a complex number dominates the real part of that number, while the final equality follows from Exercise 6.13(b) and the fact that $|\omega| = 1$. □

Corollary 6.20. *Suppose (f_n) is a sequence of complex-valued functions, continuous on the compact interval $[a,b]$ and uniformly convergent there to a (necessarily continuous) function f. Then*

$$\int_a^b f_n(t)\,dt \to \int_a^b f(t)\,dt.$$

Proof. Thanks to Proposition 6.19 above, the proof is word-for-word the same as that of the corresponding one for real-valued functions: Theorem A on page 15. □

The following exercise will be important to us later in this chapter.

Exercise 6.21 (A complex-exponential series). Suppose $(c_n)_0^\infty$ is a complex sequence with $\sum_{n=0}^\infty |c_n| < \infty$. For $\theta \in \mathbb{R}$, let $F(\theta) = \sum_{n=0}^\infty c_n e^{in\theta}$.

(a) Show that the series defining F converges uniformly on the real line, and that its sum is a (complex-valued) function that is continuous on $\mathbb{R}$.

(b) Use Exercise 6.17 to show that $c_n = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{-in\theta}\,d\theta$ for each non-negative integer n.

We'll also have occasion to consider integrals of complex-valued continuous functions defined on compact subsets of the plane (here, just closed squares or triangles). The definition of such integrals in terms of real and imaginary parts, as well as their properties, are the same as the ones described above for integrals over intervals, and the proofs are the same.

6.5. The (complex) Finite Laplace Transform

We can now define the Finite Laplace Transform of a function $f \in C\big([0,a]\big)$ by equation (6.1), with the variable s now allowed to run through the complex plane. For the record:

$$(\mathscr{L}_a f)(s) = \int_{t=0}^{a} e^{s\,t} f(t)\,dt \qquad (s \in \mathbb{C}). \tag{6.1$\mathbb{C}$}$$

The integrand on the right, being the product of two functions continuous on $[0,a]$, is itself continuous, so the integral on the right-hand side of equation (6.1$\mathbb{C}$) exists in the sense described in the last section.

We can now generalize Lemma 6.8 to the complex case.

Proposition 6.22. *Suppose $f \in C\big([0,\infty)\big)$ with $f * f \equiv 0$ on $[0,2a]$. Then $\mathscr{L}_a f_a$ is bounded on $\mathbb{C}$.*

Proof. Just as in the real setting, fix f satisfying the hypotheses of the Proposition, with $|f(t)| \le 1$ for $t \in [0,2a]$. Set $f_a(t) = f(a-t)$, so in particular: $|f_a(t)| \le 1$ for $t \in [-a,a]$.

For $s \in \mathbb{C}$ with $\text{Re } s \le 0$ and $t \ge 0$ we know from Corollary 6.15 that $|e^{st}| = e^{t\,\text{Re } s} \le 1$. Therefore, as in the proof of Lemma 6.8:

$$\begin{aligned}\big|(\mathscr{L}_a f_a)(s)\big| &= \left|\int_{t=0}^{a} e^{s\,t} f_a(t)\,dt\right| \le \int_{t=0}^{a} |e^{s\,t}|\,|f_a(t)|\,dt \\ &= \int_{t=0}^{a} \underbrace{e^{t\ \text{Re } s}\,|f_a(t)|}_{\le 1}\,dt \le a\,,\end{aligned}$$

where the interchange of (complex) integral and absolute value in the first line is justified by Proposition 6.19 (page 105). Thus we've shown, independent of any hypotheses on $f*f$, that $\mathscr{L}_a f_a$ is bounded on the "closed left half-plane" $\{s \in \mathbb{C} \colon \text{Re } s \le 0\}$.

It remains to show that the hypothesis $f * f \equiv 0$ on $[0, 2a]$ will render $\mathscr{L}_a f_a$ bounded on the "open right half-plane" $\{s \in \mathbb{C}\colon \operatorname{Re} s \leq 0\}$. To this end, fix $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$, and note that $|e^{st}| = e^{t \operatorname{Re} s} \leq 1$ for $t < 0$, hence

$$\left| \int_{t=-a}^{0} e^{s\,t} f_a(t)\,dt \right| \leq \int_{t=-a}^{0} \underbrace{|e^{s\,t}|\,|f_a(t)|}_{\leq 1}\,dt \leq a$$

Thus, to show that $\mathscr{L}_a f_a$ is bounded for s in the open right half-plane, it will be enough—just as in our proof of Lemma 6.8—to prove this for the symmetric integral

$$L(s) = \int_{t=-a}^{a} e^{s\,t} f_a(t)\,dt\,.$$

The rest of the proof proceeds exactly as that of Lemma 6.8—once you replace the phrase "$s > 0$" with "$\operatorname{Re} s > 0$"; I leave the details to you. □

6.6. Entire functions

A function $f\colon \mathbb{C} \to \mathbb{C}$ is said to be *entire* if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where the power series converges at every point of the complex plane. By Theorem B.6, the series must therefore converge absolutely at each point of the plane and, for each positive real number r, uniformly on the closed disc $|z| \leq r$ (and so, uniformly on every bounded subset of the plane).

By its definition, the exponential function $z \to e^z$ is entire. By the fundamental properties of convergent series, the set of entire functions, endowed with pointwise addition and scalar multiplication, is a complex vector space. Thus, for example, the "complex hyperbolic functions" $\sinh z = \frac{1}{2}\left(e^z - e^{-z}\right)$ and $\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right)$ are entire, as are the "complex trigonometric functions" $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ and $\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$.

Perhaps more germane to our program is:

Proposition 6.23. *If $f \in C\big([0, \infty)\big)$ and $a > 0$, then $\mathscr{L}_a f$ is entire.*

Proof. The proof of Lemma 6.2 carries over word-for-word to provide the series representation (6.2)–(6.3) for $\mathscr{L}_a f(s)$, where now the series converges for every *complex* number s. □

The complex Laplace transform is one-to-one; indeed, if $\mathscr{L}_a f \equiv 0$ on $\mathbb{C}$ then it's $\equiv 0$ on $\mathbb{R}$, hence by Corollary 6.4 (page 98), $f \equiv 0$ on $[0, a]$. However now that we've been "$\mathbb{C}$-enlightened" we can show that something much stronger is true:

$f \in C\big([0, a]\big)$ is $\equiv 0$ on $[0, a]$ whenever $\mathscr{L}_a f$ is bounded *on $\mathbb{C}$.*[2]

The key to this amazing result is:

Liouville's Theorem. *If an entire function is bounded on the whole complex plane, then it is constant there.*

Proof. We're given an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where the series converges at each point of the complex plane, and are assuming that there exists a positive real number M such that $|f(z)| \leq M$ for each $z \in \mathbb{C}$.

Fix (for the moment) $r > 0$. The series representing f converges absolutely at each point of $\mathbb{C}$, hence it does so for $z = r$, i.e., $\sum_{n=0}^{\infty} |a_n r^n| < \infty$. Let $M_n = |a_n r^n|$ for each index n, so $\sum_n M_n < \infty$. We've noted that $|e^{i\theta}| = 1$ for each $\theta \in \mathbb{R}$, so by Exercise 6.13(b) we have

$$|a_n r^n e^{in\theta}| = |a_n r^n|\,|e^{in\theta}| = |a_n r^n| = M_n.$$

Thus

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n\, r^n e^{in\theta}, \tag{6.11}$$

where the Weierstrass M-test blesses the series on the right with convergence that is uniform for $\theta \in \mathbb{R}$. Exercise 6.21 (page 106), with $c_n = a_n r^n$ and $F(\theta) = f(re^{i\theta})$, now guarantees that for $n = 0, 1, 2, \ldots$

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta}\, d\theta\,,$$

[2] Exercise 6.6 (page 99) has already established a *very* special case of this.

whereupon, finally admitting that $|a_n r^n| = |a_n| r^n$,

$$\begin{aligned}2\pi|a_n|r^n &= \left|\int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta\right| \\ &\leq \int_0^{2\pi} \left|f(re^{i\theta})e^{-in\theta}\right| d\theta \quad \text{(by Prop. 6.19)} \\ &\leq \int_0^{2\pi} \underbrace{|f(re^{i\theta})|}_{\leq M}\, \underbrace{|e^{-in\theta}|}_{\equiv 1}\, d\theta \\ &\leq \int_0^{2\pi} M\,d\theta = 2\pi M\,.\end{aligned}$$

Thus for n a fixed non-negative integer we have $|a_n| \leq \frac{M}{r^n}$ for each $r > 0$. If n is *positive* we see upon "unfixing" r, and letting it $\to \infty$, that $a_n = 0$.

Conclusion: $f(z) \equiv a_0$ on $\mathbb{C}$. ☐

Exercise 6.24. On page 108 we observed that the complex sine and cosine functions are entire. According to Liouville's Theorem, these functions, while bounded on the real line, cannot be bounded on the complex plane. Prove this directly.

Exercise 6.25. Suppose f is an entire function for which there exists $M \in [0, \infty)$ and a positive integer N such that $|f(z)| \leq M|z|^N$ for each $z \in \mathbb{C}$. Show that f is a polynomial of degree at most N.

Corollary 6.26 ("Super-uniqueness" for Finite Laplace Transforms)**.** *Suppose* $f \in C\big([0,\infty)\big)$ *and* $\mathscr{L}_a f$ *is bounded on* $\mathbb{C}$*. Then* $f \equiv 0$ *on* $[0, a]$*.*

Proof. Thanks to Proposition 6.23 and Liouville's Theorem, $\mathscr{L}_a f$ is constant on $\mathbb{C}$, so by the representation of equations (6.2)–(6.3) (page 98) and the uniqueness of power series (Proposition B.8),

$$\int_0^a t^n f(t)\,dt = 0 \quad \text{for} \quad n = 1, 2, \ldots\ .$$

By the linearity of integration, $\int_0^a p(t)tf(t)\,dt = 0$ for every polynomial p. By the Weierstrass Approximation Theorem, there is a sequence p_n of polynomials that converges uniformly on $[0, a]$ to $\overline{f}$, the complex conjugate of the function f. Thus

$$0 = \lim_n \int_{t=0}^{a} p_n(t) t f(t)\, dt = \int_{t=0}^{a} t\overline{f(t)} f(t)\, dt = \int_{t=0}^{a} t|f(t)|^2\, dt\,.$$

Since the function $t \to t|f(t)|^2$, which is continuous and non-negative on $[0, a]$, has integral zero over that interval, it's $\equiv 0$ there. Thus $f \equiv 0$ on $(0, a]$, and therefore by continuity $f(0) = 0$, i.e., $f \equiv 0$ on $[0, a]$, as we wished to show. □

This completes our proof of the One-Half Lemma, and with it, that of the Titchmarsh Convolution Theorem.

Notes

The use of Liouville's Theorem to finish our proof of the One-Half Lemma is due to Yosida and Matsuura [**68**, 1984].

Understanding the Titchmarsh Theorem. At a December 1997 meeting of the American Mathematical Society, the noted American mathematician Gian-Carlo Rota (1932–1999) gave a talk: "Ten Mathematics Problems I will never solve" [**46**]. Rota's "Problem Number Three" was the Titchmarsh Convolution Theorem. Rota claimed that, to the best of his knowledge: "No elementary proof of this theorem has ever been given ... ." He pointed out that a couple of famous mathematicians had already published proofs claimed to be "elementary," but which he'd found to be "neither elementary nor enlightening." Rota would undoubtedly say the same of the proof we've given here, viewing it as neither "elementary" (it appeals to complex analysis), nor "enlightening" (lots of of *ad hoc* trickery).

In Rota's view, the difficulty in finding a good proof of Titchmarsh's theorem resides in our insufficient understanding of the algebraic structure of repeated integration by parts. It's interesting to read further in his paper to better comprehend what he's getting at, and to see an example of how a great mathematician seeks true understanding.